TRIANGULAR DECOMPOSITION OF A POSITIVE DEFINITE MATRIX PLUS A SYMMETRIC DYAD WITH APPLICATIONS TO KALMAN FILTERING

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WILLIAM S. AGEE and ROBERT H. TURNER

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MATHEMATICAL SERVICES BRANCH
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An algorithm for the triangular decomposition of the sum of a positive definite matrix and a symmetric dyad is described. Several applications of the algorithm to the implementation of a square root Kalman filter are given.
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ABSTRACT

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I. INTRODUCTION. Given a positive definite matrix $P$ Choleski's theorem states that there exists a real, non-singular, lower triangular matrix $L$ such that

$$LL^T = P \quad (1)$$

Furthermore if the diagonal elements of $L$ are taken to be positive the decomposition is unique. $L$ is called the square root of $P$. The Choleski algorithm for decomposition of $P$ is presented in [1] and [2]. The Choleski decomposition is very useful in many numerical linear algebra problems. In particular it provides a useful numerical technique in the matrix square root formulation of the Kalman filter [3]. The triangular decomposition of Choleski is extended below to the decomposition of a positive definite matrix $P$ plus a symmetric dyad $cxx^T$.

II. FIRST TRIANGULAR DECOMPOSITION ALGORITHM. Suppose we have a lower triangular decomposition $L$ of a positive definite matrix $P$. The elements of $L$ must satisfy the equations.

$$\sum_{j=1}^i \ell(i,j)\ell(k,j) = P_{ik} \quad k > i \quad (2)$$

$$\sum_{j=1}^i \ell^2(i,j) = P_{ii} \quad (3)$$
Consider the problem of computing the triangular decomposition $L'$ of the modified matrix

$$P' = P + cxx^T$$

(4)

given the decomposition $L$ of $P$. The decomposition $L'$ must satisfy the equations

$$\sum_{j=1}^{i} \ell'(i,j)\ell'(k,j) = \sum_{j=1}^{i} \ell(i,j)\ell(k,j) + cx_k^e, \quad k > i$$

(5)

$$\sum_{j=1}^{i-1} \ell^{-2}(i,j) = \sum_{j=1}^{i} \ell^2(i,j) + cx_i^2$$

(6)

First consider (5) and (6) for the case of $i=1$. In this case

$$\ell'(1,1)\ell'(k,1) = \ell(1,1)\ell(k,1) + cx_k^e, \quad k > 1$$

(7)

$$\ell^{-2}(1,1) = \ell^2(1,1) + cx_1^2$$

(8)

The first column of the modified matrix $L'$ is easily computed from (7) and (8). Now rewrite (5) as

$$\sum_{j=2}^{i} \ell'(i,j)\ell'(k,j) + \ell'(i,1)\ell'(k,1) = \sum_{j=2}^{i} \ell(i,j)\ell(k,j) + \ell(i,1)\ell(k,1)$$

$$+ cx_k^e$$

$$j = 2$$

$$j = 2$$

(9)
The second term on the left of (9) can be computed from (7) as

\[ l'(i,1)l'(k,1) = \frac{\ell^2(1,1)}{\ell^2(1,1)} l(i,1)l(k,1) + \frac{\frac{\ell(1,1)\ell(1,1)}{\ell^2(1,1)}}{c x_i x_k} + \]

\[ \frac{\ell(k,1)\ell(1,1)}{\ell^2(1,1)} c x_i x_k + \frac{c^2 x_1^2}{\ell^2(1,1)} x_i x_k \]

(10)

Substituting (10) into (9) and combining terms gives

\[ \sum_{j=2}^{i} l'(i,j)l'(k,j) = \sum_{j=2}^{i} l(i,j)l(k,j) + c (1) x_i x_k(1) \]

(11)

where

\[ c(1) = \frac{c\ell^2(1,1)}{\ell^2(1,1)} \]

(12)

\[ x_j(1) = left( x_i \frac{\ell(j,1)}{\ell(1,1)} \right) \]

(13)

Similarly, (6) can be written as

\[ \sum_{j=2}^{i} \ell^2(i,j) + \ell^2(i,1) = \sum_{j=2}^{i} \ell^2(i,j) + \ell^2(i,1) + c x_i^2 \]

(14)
Substituting from (7) for the second term on the left of (14) and combining terms gives

\[ \sum_{j=2}^{i} \ell^2(i,j) = \sum_{j=2}^{i} \ell^2(i,j) + c(1)x_1^2 \]

Equations (11) and (15) define a decomposition problem equivalent to the original problem defined by (5) and (6) but with the dimension reduced by one. It is easily seen that the application of the above technique n times, each time reducing the dimension of the decomposition problem by one, solves the original decomposition problem. The following equations summarize the algorithm for the triangular decomposition of \( P + cxx^T \).

\[
\ell^*(i,i) = \left[ \ell^2(j,i) + c(1)x_1(i) \right]^{1/2}, \quad i = 1, n
\]

\[
\ell^*(k,i) = \frac{\ell(i,i)}{\ell^*(1,1)} \ell(k,i) + \frac{c(1)x_1(i)x_k(i)}{\ell^*(1,1)}, \quad 1 < k < n
\]

\[
x_j(i+1) = x_j(i) - \frac{x_1(i)x_j(i)}{\ell^*(1,1)}, \quad 1 < j < n
\]

\[
c(i+1) = c(i) \left( \frac{\ell(i,i)}{\ell^*(1,1)} \right)^2
\]

III. SECOND TRIANGULAR DECOMPOSITION ALGORITHM. An alternate decomposition of positive definite matrix is possible. Let \( L \) be a unit lower triangular matrix, i.e., having ones along the diagonal. Then a positive definite matrix \( P \) can be written as
\[ P = LDL^T \]  

where \( D \) is a diagonal matrix of positive numbers. An algorithm for computing the decomposition

\[ L^{-1}D^{-1}L^{-T} = LDL^T + cxx^T \]  

can be derived in a manner parallel to the previous decomposition algorithm. The algorithm is as follows.

\[ d_i' = d_i + c(i)x_i^2 \quad i = 1, n \]  

\[ x_{(i+1)} = x_i - x_{(i)} \ell(k,i) \]  

\[ \ell'(k,i) = \ell(k,i) + \frac{c(i)x_i}{d_i} x_{(i+1)} \quad i + 1 < k < n \]  

\[ c(i+1) = c(i) \left( \frac{d_i}{d_i'} \right)^2 \] 

IV. APPLICATION TO KALMAN FILTERING I. At a measurement update in a discrete Kalman filter the state estimate is given by

\[ \hat{x}_k = \hat{x}_{k/k-1} + P_{k/k} H R_{k/k}^{-1} \left( z_k - H \hat{x}_{k/k} \right) \]
where the covariance matrix $P_k$ is

$$
P_k = \left( P_{k/k-1}^{\text{T}} H_k^{\text{T}} H_k + R_k^{-1} \right)^{-1}
$$

Consider the measurement $z_k$ to be a scalar since the vector measurement problem can always be reduced to this case, see [3]. In this scalar case $H_k$ is a row vector, $R_k^{-1}$ is a scalar, and $P_k$ is $n \times n$. In the matrix square root formulation of the Kalman filter given in [3], (27) is the inversion of a matrix plus a dyad which is simply expressed by the method of modification given in Householder [4]. This gives

$$
P_k = P_{k/k-1} - \frac{P_{k/k-1}^{\text{T}} H_k^{\text{T}} H_k P_{k/k-1}}{R_k^{-1} + H_k^{\text{T}} P_{k/k-1} H_k}
$$

Since $P_k$ and $P_{k/k-1}$ are positive definite write them as

$$
P_k = L_k^{\text{T}} L_k
$$

and

$$
P_{k/k-1} = L_{k/k-1}^{\text{T}} L_{k/k-1}
$$

Assume that $L_{k/k-1}$ is lower triangular with positive diagonal elements. Substituting in (28)

$$
L_k^{\text{T}} L_k = L_{k/k-1}^{\text{T}} L_{k/k-1} - c_k L_{k/k-1}^{\text{T}} H_k^{\text{T}} H_k L_{k/k-1} L_{k/k-1}^{\text{T}}
$$

(29)
\[ c_k = \frac{1}{R_k + H_k^T P_{k-1} H_k^T} \] (30)

Let

\[ u_k = L_k^T H_k^T \] (31)

and

\[ w_k = L_k H_k^T \] (32)

Then

\[ L_k L_k^T = L_{k-1} L_{k-1}^T - c_k w_k w_k^T \] (33)

Thus (33) is in the proper form for application of the first algorithm. In addition to the above computations the state estimate given by (26) must also be computed. Some manipulation shows that (26) can be written as

\[ \hat{x}_k = \hat{x}_{k-1} + c_k \bar{w}_k \left( z_k - H_k \hat{x}_{k-1} \right) \] (34)

Now consider the numerical efficiency of the application. The execution of (31) and (32) takes \( n(n+1) \) (mult). (30) takes \( n \) (mult) and 1 (div), (34) takes \( n+1 \) (mult). The decomposition algorithm for (33) requires

\[ \frac{3n^2 + 9n - 5}{2} \text{ (mult)} \]
Thus the use of the first decomposition algorithm requires

\[
\left( \frac{5}{2} n^2 + \frac{15}{2} n - 2 \right) \text{(mult)}
\]

2n-1 (div) and n √'s. The technique presented in [3] which we have been using in our Kalman filter program requires about 3n^2 + 2n (mult) but does not generate a triangular square root matrix.

Now consider the numerical efficiency of the second algorithm. A development paralleling (29)-(34) gives

\[
L_k L_k^T = L_{k-1} L_{k-1}^T - C_k \cdot \cdot^T
\]

where

\[
\begin{align*}
\quad u_k &= D_k L_{k-1}^T \\
\quad w_k &= L_{k-1} u_k \\
c_k &= \frac{1}{R_k + H_k P_k H_k^T} \left( R_k + H_k P_k H_k^T \right)
\end{align*}
\]

The use of the second algorithm and execution of (34)-(38) requires 2n^2 + 7n (mult) and n (div).

The application of the second algorithm results in fewer operations than either our present algorithm or the first algorithm presented above plus the benefit of having a covariance square root which is triangular.
V. APPLICATION TO KALMAN FILTERING II. Rather than compute the square root of the covariance matrix the square root of the inverse covariance matrix may be computed. The updating of the inverse covariance matrix is given by (27).

\[
P_{k}^{-1} = P_{k/k-1}^{-1} + H_{k}^{T}R_{k/k}^{-1}H_{k}
\]

(39)

since the inverse covariance is also positive definite, write

\[
P_{k/k-1}^{-1} = L_{k/k-1}^{-1}D_{k/k-1}^{-1}L_{k/k-1}^{T}
\]

and

\[
P_{k}^{-1} = L_{k}^{-1}D_{k}^{-1}L_{k}^{T}
\]

Again considering scalar measurements let \( u_{k} = H_{k}^{T} \) and \( c_{k} = 1/R_{k} \). Then

\[
L_{k}^{-1}D_{k}^{-1}L_{k}^{T} = L_{k/k-1}^{-1}D_{k/k-1}^{-1}L_{k/k-1}^{T} + c_{k}u_{k}u_{k}^{T}
\]

(40)

which is in the form for application of the second algorithm. The computation of the corrected state estimate given by (26) requires the computation of

\[
y_{k} = \frac{P_{k}H_{k}^{T}}{R_{k}} \left( z_{k} - H_{k} \hat{x}_{k/k-1} \right)
\]

(41)
Let $e_k = c_k \left( z_k - H_k \hat{x}_{k-1/k} \right) u_k$ and rewrite (41) as

$$P_k^{-1} y_k = L_k^{-1} D_k L_k^{-T} y_k = e_k$$

(42)

The solution of (42) for $y_k$ requires the solution of two sets of linear equations each with a triangular coefficient matrix. The solution of a triangular set of linear equations is a standard procedure in numerical analysis. The solution of (42) for the vector $y_k$ requires about $n^2 + n$ (mult) and $n$ (div). The solution of the Kalman filter equations (40) and (42) at a measurement update requires about $n^2 + 5n$ (mult) and $n$ (div) using the second decomposition algorithm. The specification of (40) and (42) as the measurement update equations requires that $L_{k/k-1}^{-1}$ be computed at a time update. In the special case where there are no process dynamics, i.e. we are estimating constant parameters, and there is no process noise $L_{k/k-1}^{-1} = L_{k-1}^{-1}$ so that no additional computation is required to obtain $L_{k/k-1}^{-1}$. Although this is a special case, it is of interest in the bias filter portion of the WSMR BET program where the measurement biases are assumed to be constant in time.

VI. APPLICATION TO KALMAN FILTERING III. The use of the second triangular decomposition algorithm in the WSMR BET, which is an extended Kalman filter, has resulted in a significant increase in numerical efficiency, however, the motivating factor for the development and use of this algorithm was for the application described below.

The Kalman filter in the WSMR BET is divided into two filters, the zero-bias filter which produces trajectory state estimates $x_k^*$ and the
bias filter which produces estimates $\hat{b}_k$ of the measurement biases. These two estimates are combined to form the optimal trajectory state estimate $\hat{x}_k$ defined by

$$
\hat{x}_k = x_k^* + T_k \hat{b}_k
$$

(43)

where $T_k$ is a combining matrix which must satisfy equations determined from the orthogonality properties of the state estimates. One property of the filter decomposition given in (43) is the requirement that $x_k^*$ and $b_k^*$ be orthogonal. This requirement is naturally satisfied by the equations governing $x_k^*$, $b_k^*$, and $T_k$ except at a point where a measuring instrument is deleted from the filtering process. An instrument may be dropped because it is no longer taking observations, its bias is too large, or the measurements are chronically inconsistent with their statistics. Let $x_-$, $P_-$ denote the zero-bias state estimate and its covariance just prior to dropping a measurement from the filtering solution and let $x_+$ and $P_+$ be the same quantities immediately after dropping the measurement. Similarly let $\hat{b}_-$ and $\hat{b}_+$ denote the bias state estimates before and after dropping a measurement. $\hat{b}_+$ is formed by deleting the component of $\hat{b}_-$ corresponding to the measurement being dropped. $T_-$ and $T_+$ are the combining matrices before and after. $T_+$ has one less column than $T_-$. $P_-$ and $P_+$ are the bias covariance matrices before and after dropping a measurement. $P_+$ is formed from $P_-$ by deleting the row and column corresponding to the measurement being dropped. The updating equation for $x^*$ is

$$
x^* = x^- + T_1 \left( \hat{b}^- - T_{b^+} \hat{b}_+ \right)
$$

(44)
where \( t_i \) is the column of \( T \) being deleted, \( \hat{b}_i \) is the bias estimate for the measurement being dropped, and \( \ell \) is vector chosen so that \( x^\dagger \) and \( b^\dagger \) will be orthogonal in the usual statistical sense. The updating equation for \( P^\dagger \) is

\[
P^\dagger_+ = P^\dagger_- + t_i t_i^T \left( P_{b-}(i,i) - \ell \ell^T P_{b+} \right)
\]

If

\[
P^\dagger_- = C_D C_D^T
\]

and

\[
P^\dagger_+ = C_D C_D^T
\]

then (45) is in a form for application of the second triangular decomposition algorithm. In addition, let

\[
P_{b-} = C_{b-} D_{b-} C_{b-}^T
\]

and

\[
P_{b+} = C_{b+} D_{b+} C_{b+}^T
\]

Let the \( i \)th row and column \( C_{b-} \) be deleted. Also let the \( i \)th row and column of \( D_{b-} \) be deleted. Call the resulting matrices \( C_{b^-} \) and \( D_{b^-} \).
Then

$$C_{b_+}D_{b_+}C_{b_+}^T = C_{b_-}D_{b_-}^{-1}C_{b_-}^T + D_{i_i}c_i^T$$

(46)

where $c_i$ is the column deleted from $C_{b_-}$ and $d_i$ is the diagonal element deleted from $D_{b_-}$. Thus (46) presents another application for the decomposition algorithm.

VII. REFERENCES:


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