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ON SUMMATION OF SEQUENCES

by

B.I. Golubov

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SUMMARY

On the basis of the one-to-one correspondence of Buck and Pollard (if $x \in (0,1]$ and $x = 0, \alpha_1(x)\,\alpha_2(x)\ldots$ is the expansion of the number x into an infinite dyadic fraction, then $x \sim \{n_k\}$, where the n_k are such that $\alpha_{n_k}(x) = 1$, $\alpha_n(x) = 0$ when $n \neq n_k$ $(k = 1,2,\ldots)$), previous results of various authors are generalized and analogous theorems are formulated for permutations of numerical series and sequences. Given in addition is a number of theorems involving functional series and sequences.

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13. ABSTRACT

On the basis of the one-to-one correspondence of Buck and Pollard (if $x \in (0,1]$ and $x = 0, \alpha_1(x)\,\alpha_2(x)\ldots$ is the expansion of the number x into an infinite dyadic fraction, then $x \sim \{n_k\}$, where the n_k are such that $\alpha_{n_k}(x) = 1$, $\alpha_n(x) = 0$ when $n \neq n_k$ $(k = 1,2,\ldots)$), previous results of various authors are generalized and analogous theorems are formulated for permutations of numerical series and sequences. Given in addition is a number of theorems involving functional series and sequences.

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ON SUMMATION OF SEQUENCES*)

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In [1] Buck and Pollard used the following well-known one-to-one correspondence between numbers of the interval (0,1] and subsequences of a natural series. If $x \in (0,1]$ and $x = 0, \alpha_1(x)\alpha_2(x)\ldots$ is the decomposition of the numbers x into an infinite**) dyadic fraction, then $x \sim \{n_k\}$, where n_k are such that $\alpha_{n_k}(x) = 1$, $\alpha_n(x) = 0$ for $n \neq n_k$ $(k = 1,2,\ldots)$. It is easy to see that the reverse is also true, that is, a certain point $x \in (0,1]$ corresponds, according to this very same law, to every rigorously increasing sequence $\{n_k\}$, and, consequently, the correspondence is one-to-one. In the present note, starting with this correspondence, some results contained in [1-5,12] are generalized and analogous theorems are formulated for permutations of numerical series and sequences. Moreover, a number of theorems are cited in which functional series and sequences are involved.

We introduce the following notation. If $\{s_n\}$ is a certain sequence, then its subsequence, defined by the number $x \in (0,1]$, will be denoted by $\{s(n,x)\}$ and the set of limit points $\{s_n\}$ by $\{s_n\}'$. Furthermore, if $T = \|a_{n,k}\|$

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*) Translated from Izv. VUZ, Mat. [Bull. Higher Educat. Insts., Math.], No. 4(41), pp. 47-55 (1964).

**) By an infinite dyadic fraction is meant a fraction whose signs include an infinite set of ones.

is a regular summation method, we denote $\sigma_n(x) = \sum_{k=1}^{\infty} a_{n,k} s(k, x)$.

The following theorem generalizes the result of Buck and Pollard ([1], theorem 1, and also [7], p. 404, theorem 5.6).

<u>Theorem 1</u>. <u>For every sequence</u> $\{s_n\}$ <u>there is found a set</u> $Q \subset (0,1]$, <u>of complete measure and of the second category</u>*) <u>on</u> $(0,1]$, <u>such that</u>

$$\{s(n, x)\}' = \{s_n\}' \text{ when } x \in Q.$$

<u>Proof</u>. Let $\{n_k\}$ be an arbitrary rigorously increasing sequence of positive numbers outside of which there remains an infinite set of terms of the natural series. We take two real numbers a and b (a < b) and form the sequence $\{u_n\}$, setting $u_n = a$ when $n = n_k$ and $u_n = b$ when $n \neq n_k$ $(k = 1,2,\ldots)$. As follows from the theorem 1 of Buck and Pollard [1], $a = \underline{\lim}_n u(n, x) < \overline{\lim}_n u(n, x) = b$ almost everywhere on $(0,1]$. Hence, whatever the rigorously increasing sequence $\{n_k\}$ that exhausts the whole natural series or that forms a part of it, almost all subsequences of the latter always contain an infinite set of terms from $\{n_k\}$, each its own.

From $\{s_n\}'$ we choose not more than a countable set of points $\{v_i\}$ dense in $\{s_n\}'$. For every point v_i there exists a sequence $\{n_k^{(i)}\}$ such that $\lim_k s_{n_k^{(i)}} = v_i$. As can be seen from the remark made above, there exists a set $A_i \subset (0,1]$ of full measure such that $v_i \in \{s(n,x)\}'$ when $x \in A_i$. We set $A = \prod_i A_i$. Then mes $A = 1$ and $\{s(n,x)\}' \supset \{s_n\}'$ when $x \in A$. In the same way, relying on the theorem of Keogh and Petersen [3] for the case when T is convergence, we find a second-category set $B \subset (0,1]$ such that $\{s(n,x)\}' \supset \{s_n\}'$ when $x \in B$. Setting

*) By a second-category set is meant a set which is the complement of a first-category set.

$Q = A + B$, we complete the proof of the theorem, since the reverse inequality $\{s(n,x)\}' \subset \{s_n\}'$ is obviously valid for all x.

Remark 1. Let us show that the theorem of Buck and Pollard, which is generalized by theorem 1, permits determination of the error of an assertion of Goffman and Petersen contained in their paper [13]. In their article the authors introduce the concept of a submethod of the regular summation method. Namely: if $T = \|a_{n,m}\|$ is the regular method, every method $T\{n_k\}$ defined by the matrix $\|a_{n_k,m}\|$, where $\{n_k\}$ is an arbitrary subsequence of the natural series, is said to be a submethod of T. Thus, every submethod is defined by a subsequence $\{n_k\}$ of the natural series. Consequently, the correspondence $x \sim \{n_k\}$ of Buck and Pollard permits us to establish a one-to-one correspondence between the set of all submethods $T\{n_k\}$ of the method T and the set of points $x \in (0,1]$. We denote $T\{n_k\} = T(x)$, where $x \sim \{n_k\}$. Then theorem 5 from [13] can be formulated in the following way:

Theorem. Let T be the regular method, and let $\{s_n\}$ be a bounded sequence, not summable by the method T. If Q is the set of all points $x \in (0,1]$, for each of which $T(x)$ sums $\{s_n\}$, then the measure of the set Q is always equal to 0 or 1 and both cases are possible.

As a matter of fact, as follows from theorem 1 (or from the weaker theorem of Buck and Pollard), mes $Q = 0$ always. We note also that theorem 3 from the same article [13] of Goffman and Petersen cannot be considered proved either, since there is an error in its proof.

The following is a strengthening of the theorems of Agnew [2] and Keogh and Petersen [3,4].

Theorem 2. If $|s_n| \leq C$ $(n = 1,2,\ldots)$, then for every regular method $T = \|a_{n,k}\|$ there exists a set Q of second category on $(0,1]$ such that

$$\{\sigma_n(x)\}' \supset \{s_n\}' \quad \text{when } x \in Q. \tag{1}$$

But if $\{s_n\}$ is not bounded, the assertion of the theorem is not true.

Proof. Let $|s_n| \le C$ $(n = 1,2,\ldots)$. Then, without loss of generality it can be assumed that the method $T = \|a_{n,k}\|$ is finite-rowed. According to the theorem of Agnew [2], there exists a point $x_0 \in (0,1]$ such that $\{\sigma_n(x_0)\}' \supset \{s_n\}'$. Let us show that the very same inclusion is valid for all points of a certain set $D \subset (0,1]$, everywhere dense in $(0,1]$, i.e. $\{\sigma_n(x)\}' \supset \{s_n\}'$ when $x \in D$. Indeed, let k be an arbitrary positive number and let the point x' be such that $\alpha_n(x') = \alpha_n(x_0)$ when $n > k$, while the first k dyadic signs of the expansion x' are arbitrary, but fixed. Then, by varying the signs of $\alpha_n(x')$ for $k < n \le 2k$, it is possible to get a point x'' at which the total number of ones among the first 2k signs of the expansion coincides with the number of ones among the first 2k signs of the expansion x_0. For this point, obviously, $\{\sigma_n(x'')\}' \supset \{s_n\}'$. Since the natural index k is arbitrary, the set D of all x'' that can be obtained in the indicated manner, starting with the point x_0, is everywhere dense in $(0,1]$.

Now let $\{u_m\}$ be no more than a countable set of points from $\{s_n\}'$, dense in $\{s_n\}'$. We denote by $S^k_{n,m}$ the set of all points $x \in (0,1]$, for each of which there exists a number $\mu = \mu(x) > n$ such that $|\sigma_\mu(x) - u_m| < \frac{1}{k}$. Since the method $T = \|a_{n,k}\|$ is supposed to be finite-rowed, all the sets $S^k_{n,m}$ are open. Obviously, also, all $S^k_{n,m} \supset D$. It is not difficult to show that the set $Q = \prod_{n,m,k} S^k_{n,m}$ satisfies the conditions of the theorem. Going on to the proof of its second part, we take the series $\sum_{k=1}^{\infty} (-1)^{k-1} \alpha_k = 1$, $\alpha_k \downarrow$ *) $\alpha_k > 0$, and the regular method $T = \|a_{n,m}\|$, where $a_{n,m} = 0$ when $m < n$ and $a_{n,m} = (-1)^{m-n} \alpha_{m-n+1}$ when $m \ge n$. If we set $s_n = \alpha_n^{-1}$ $(n = 1,2,\ldots)$, then $\sigma_n(x)$ will not have meaning for any n and x whatever. Indeed, if $x \sim \{n_k\}$, then

$$\sigma_n(x) = \sum_{k=1}^{\infty} a_{n,k} s_{n_k} = \sum_{k=n}^{\infty} (-1)^{k-n} \alpha_{k-n+1} s_{n_k} = \sum_{k=n}^{\infty} (-1)^{k-n} \frac{s_{n-n+1}}{\alpha_{n_k}}.$$

*) The sign $\downarrow(\uparrow)$ means monotonic decrease (increase) of the corresponding sequence.

By virtue of the condition $\alpha_k \downarrow$, the terms of the last series cannot be less than unity in absolute value, i.e. $\sigma_n(x)$ are not meaningful. The theorem is completely proved.

Remark 2. Generally speaking, inclusion (1) cannot be replaced by an equality. Indeed, let us choose an arbitrary diverging bounded sequence $\{s_n\}$ for which $\{s_n\}'$ is not a connected set, and as the method T let us choose the method (C, 1). As is well known [6], the set $\{\sigma_n\}'$ for the method (C, 1) and for the bounded sequences $\{s_n\}$ is connected; and since $\{s_n\}'$ is not connected in the given example, the equality in (1) does not hold.

Remark 3. The measure of the set Q in theorem 2 can be equal to 0 and 1. For example, if T = (C,1) and $\{s_n\}$ is a certain divergent bounded (C,1)-summable sequence, then, as follows from theorem 2 of Buck and Pollard [1], almost all the subsequences $\{s(n,x)\}$ are also (C,1)-summable. Consequently, in this case mes Q = 0. But if T is convergence, then, according to theorem 1, we have mes Q = 1. Whether there exist T methods for which 0 < mes Q < 1, we do not know.

Now let us formulate a result on the summation of sequences of 0's and 1's. If $x \in (0,1]$ and $T = \|a_{n,k}\|$ is a regular method, then we set $\sigma(n, x) = \sum_{k=1}^{\infty} a_{n,k} s_k(x)$.

Theorem 3.*) For every regular method T there exists a set Q of second category on (0,1] such that

$$\{\sigma(n, x)\}' \supset \{1; 0\} \text{ **) when } x \in Q.$$

Proof. According to Agnew's theorem [2], for a certain point $x_0 \in (0,1]$ we have $\{\sigma(n,x_0)\}' \supset \{1;0\}$. This inclusion is preserved if at x_0 we arbitrarily

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*) This theorem generalizes theorem (2.3) of Hill.

**) $\{1;0\}$ means a set of two elements: 1 and 0.

change several first coordinates. Therefore, there is found a set D, everywhere dense on (0,1], such that $\{\sigma(n,x)\}' \supset \{1;0\}$ for $x \in D$. Let $\{u_m\}$ be a diverging sequence of 0's and 1's. We let $S^k_{n,m}$ $(n,m,k = 1,2,\ldots)$ denote the set of all points $x \in (0,1]$, for each of which there exists a $\mu = \mu(x) > n$ such that $|\sigma_\mu(x) - u_n| < 1/k$. Obviously, $S^k_{n,m}$ are open (since, without loss of generality, the method T can be considered finite-rowed) and $S^k_{n,m} \supset D$ $(n,m,k = 1,2,\ldots)$. Then, $Q = \prod_{m,k} S^k_{n,m}$ is the desired second-category set. The theorem is proved.

Remark 4. The measure of the set Q in the given theorem can be only 1 or 0, since this set is homogeneous (the definition of a homogeneous set is given in [1], see also [7], p. 403). The first case is realized, e.g., when the method T is convergence, and the second when T = (C,1), since in this case, according to Borel's theorem [8], we have $\{\sigma(n,x)\}' = \{1/2\}$ almost everywhere on (0,1].

Remark 5. According to Khinchin's law of the iterated logarithm [9], on the set $E_0 \subset (0,1]$ of complete measure $\overline{\lim_n} \frac{\left|\nu_n(x) - \frac{n}{2}\right|}{\sqrt{(n \lg\lg n)/2}} = 1$, where $\nu_n(x)$ is the number of zeros among the first n signs of the dyadic expansion x. But $\nu_n(x) = n - \sum_{k=1}^{n} \alpha_k(x)$ and, applying the preceding theorem with T = (C,1), we get

$$\underline{\lim_n} \frac{\nu_n(x) - \frac{n}{2}}{n} = -\frac{1}{2}, \quad \overline{\lim_n} \frac{\nu_n(x) - \frac{n}{2}}{n} = \frac{1}{2}$$

for $x \in Q$, where Q is a certain second-category set on (0,1]. Hence it follows that the set E_0 in Khinchin's theorem is of the first category*), although it is of complete measure.

Introduced in the definition of the concept of density of a sequence of

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*) Note added in proof. Recently it was made known to the author that this result follows from theorem 2 of [14].

a natural series in the quantity $d_n\{n_k\} = \frac{N(n)}{n}$, where N(n) is the number of terms of the given sequences $\{n_k\}$ that do not exceed n. If $\lim a_n\{n_k\} = d$ exists, then d is called the density of the sequence $\{n_k\}^n$ in the natural series. If the latter limit does not exist, then the upper density $\overline{d} = \overline{\lim_n}\, d_n\{n_k\}$ and the lower density $\underline{d} = \underline{\lim_n}\, d_n\{n_k\}$ are introduced. Using the correspondence introduced above, $\{n_k\} \sim x \in (0,1]$, we get $d_n\{n_k\} \equiv d_n(x) = \frac{1}{n}\sum_{k=1}^{n} a_k(x)$.

According to Borel's theorem [8], we have $\lim_n d_n(x) = 1/2$ almost everywhere on (0,1], that is, almost all the subsequences of a natural series have density, and it is equal to 1/2 for almost all of them. Insofar as the category is concerned, the situation is different. Namely: $\{d_n(x)\}' = [0,1]$ for $x \in Q$, where $Q \subset (0,1]$ is a certain second-category set. In fact, according to theorem 3, if we set T = (C,1) in it, we have $\{d_n(x)\}' \supset \{1;0\}$ for $x \in Q \subset (0,1]$, where Q is of the second category. But, as was already noted in remark 2, the set $\{\sigma_n\}'$ is connected for the method (C,1) applied to bounded sequences; therefore, $\{d_n(x)\}' \supset [0,1]$ for $x \in Q$. But since $0 \le d_n(x) \le 1$, we find that $\{d_n(x)\}' = [0,1]$ for $x \in Q$, which is what was stated. Thus, a second-category set of subsequences of a natural series does not have density; moreover, $d(x) = 0$, $\overline{d}(x) = 1$ on a second-category set.

Now let A(x) be a subsequence, defined by the number $x \in (0,1]$, of a natural series. Let A(x)A(y) denote a sequence (in increasing order) whose terms are all elements common to A(x) and A(y). The sequence A(x)A(y) may prove to be finite or empty, that is, not having even one term. Let $d_n(A(x)A(y)/A(x))$ denote the number of terms of the sequence A(x)A(y) among the first n terms of the sequence A(x), divided by n, and let $d_n(A(x))$ be the number of terms, divided by n, of the sequence A(x) among the first n positive numbers. The quantities $d_n(A(x)A(y)/A(x))$ characterize the "distribution" of the sequence A(x)A(y) in A(x) and the quantities $d_n(A(x))$ the "distribution" of the sequence A(x) itself in a natural series.

Theorem 4. For any point $x_0 \in (0,1]$ there exists a set Q of complete measure on (0,1] such that

$$\{d_n(A(x)A(x_0)/A(x))\}' = \{\sigma_n(A(x_0))\}' \quad \text{when} \quad x \in Q. \tag{2}$$

Proof. It is not difficult to see that

$$\{d_n A(x) A(x_0)/A(x))\}' = \left\{ \frac{\sum_{k=1}^{n} \alpha_k(x_0)\alpha_k(x)}{\sum_{k=1}^{n} \alpha_k(x)} \right\}'.$$

But $\alpha_k(x) = \frac{1}{2}[1 - r_k(x)]$ almost everywhere, where $r_k(x)$ $(k = 1,2,\ldots)$ are Rademacher functions ([10], p. 55), and the preceding equality can be represented as

$$\{d_n(A(x)A(x_0)/A(x)\}' = \left\{ \frac{\frac{1}{n}\sum_{k=1}^{n} \alpha_k(x_0) - \frac{1}{n}\sum_{k=1}^{n} \alpha_k(x_0) r_k(x)}{1 - \frac{1}{n}\sum_{k=1}^{n} r_k(x)} \right\}'. \tag{3}$$

According to lemma 5 of Buck and Pollard [1], $\lim_n \frac{1}{n}\sum_{k=1}^{n} s_k r_k(x) = 0$ always almost everywhere provided $\sum_{k=1}^{\infty} \frac{s_k^2}{k^2} < \infty$. Therefore, there exists a set $Q \subset (0,1]$ of complete measure such that

$$\lim_n \frac{1}{n}\sum_{k=1}^{n} \alpha_k(x_0) r_k(x) = 0, \quad \lim_n \frac{1}{n}\sum_{k=1}^{n} r_k(x) = 0 \quad \text{when } x \in Q. \tag{4}$$

If we take into account the fact that $d_n(A(x_0)) = \frac{1}{n}\sum_{k=1}^{n} \alpha_k(x_0)$, then (2) follows easily from (3) and (4). The theorem is proved.

Roughly speaking, the meaning of this theorem is that the "distribution" of terms of the fixed sequence $\{n_k\}$ in almost all subsequences of a natural series is the same as the "distribution" of the sequence $\{n_k\}$ itself in a natural series.

Let us now apply ourselves to functional sequences. If $\{s_n(t)\}$ is a sequence of functions given on a certain set, then $\{s_n^x(t)\}$ denotes its subsequence, defined by the number $x \in (0,1]$.

Theorem 5. If the sequence $\{s_n(t)\}$ of functions measurable and almost everywhere finite on the set*) E diverges almost everywhere on E, there exists a second-category set $Q \subset (0,1]$ such that the sequences $\{s_n^x(t)\}$ converge almost everywhere on E for every $x \in Q$.

Proof. We introduce the notation

$$\overline{\lim_k}\, s_k^x(t) = \lim_n \lim_m \sup_{m>k>n} s_k^x(t) \equiv \lim_n \lim_m F_m^x(t) \equiv \lim_n F_n^x(t) \equiv F^x(t),$$
$$\underline{\lim_k}\, s_k^x(t) = \lim_n \lim_m \inf_{m>k>n} s_k^x(t) \equiv \lim_n \lim_m f_m^x(t) \equiv \lim_n f_n^x(t) \equiv f^x(t). \tag{5}$$

We shall assume that $F^1(t)$ and $f^1(t)$ are finite almost everywhere on E. Without this assumption, only several technical complications arise in the proof.

Let $\varepsilon_k \to 0$ and $\varepsilon_k > 0$. According to the condition of the theorem, $F^1(t) - f^1(t) > 0$ almost everywhere on E; therefore, for any ε_k there exists an $\alpha_k > 0$ such that for the set $E_k = E\{t: F^1(t) - f^1(t) > \alpha_k\}$ we have

$$\operatorname{mes}(E - E_k) < \varepsilon_k. \tag{6}$$

We denote by R_N^k $(N,k = 1,2,\ldots)$ the set of all $x \in (0,1]$ for each of which there exists a set

$$E_{N,k}^x \subset E_k, \quad \text{where} \quad \operatorname{mes}(E_k - E_{N,k}^x) < \varepsilon_N \tag{7}$$

and indexes $\xi(x)$, $\eta(x)$, $\mu(x)$, $\nu(x) \geq N$, $(\xi \geq \eta,\ \mu \geq \nu)$ such that

$$F_{\xi\eta}^x(t) - f_{\mu\nu}^x(t) > \frac{\alpha_k}{2} \quad \text{for } t \in E_{N,k}^x. \tag{8}$$

Obviously, every set R_N^k is open. Let us show that

$$R_N^k \supset R \quad (N, k = 1, 2, \ldots). \tag{9}$$

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*) We note that the divergence of a sequence at a given point is understood in the sense that the upper and lower limits are not equal to each other. But if they coincide, although being equal to $+\infty$ $(-\infty)$, the sequence is considered to be convergent.

where R is the set of all dyadic rational points from (0,1]. Let us fix arbitrary N and k, and also $x \in R$. According to (5) and Egorov's theorem, we find a set $E'_{N,k} = E'_{N,k}(x)$ and an index $\eta = \eta(x) \geq N$ for which

$$\operatorname{mes}(E - E'_{N,k}) < \frac{\varepsilon_N}{4}, \text{ and } |F^x_\eta(t) - F^x(t)| < \frac{\alpha_k}{8} \quad \text{for } t \in E'_{N,k}. \tag{10}$$

In exactly the same way it is possible to find a set $E''_{N,k} = E''_{N,k}(x)$ and an index $\xi = \xi(x) \geq N$, $\xi \geq \eta$, such that

$$\operatorname{mes}(E_k - E''_{N,k}) < \frac{\varepsilon_N}{4}, \text{ and } |F^x_{\xi\eta}(t) - F^x_\eta(t)| < \frac{\alpha_k}{8} \quad \text{for } t \in E''_{N,k}. \tag{11}$$

As follows from (10) and (11)

$$|F^x_{\xi\eta}(t) - F^x(t)| < \frac{\alpha_k}{4} \text{ for } t \in E'_{N,k} E''_{N,k} \equiv \hat{E}_{N,k}, \quad \text{and}$$

$$\operatorname{mes}\left(E_k - \hat{E}_{N,k}\right) < \frac{\varepsilon_N}{2}. \tag{12}$$

Analogously it is possible to construct a set $\check{E}_{N,k} = \check{E}_{N,k}(x) \subset E_k$, such that for several indexes $\mu(x)$, $\nu(x) \geq N$, $\mu \geq \nu$, we will have

$$|f^x_{\mu\nu}(t) - f^x(t)| < \frac{\alpha_k}{4} \quad \text{for} \quad t \in \check{E}_{N,k}, \quad \text{and } \operatorname{mes}(E_k - \check{E}_{N,k}) < \frac{\varepsilon_N}{2}. \tag{13}$$

From (12) and (13) we get

$$F^x_{\xi\eta}(t) - f^x_{\mu\nu}(t) \geq F^x(t) - f^x(t) - \frac{\alpha_k}{2} \quad \text{for} \quad t \in \hat{E}_{N,k} \check{E}_{N,k} \equiv E^x_{N,k}, \tag{14}$$

where mes $(E_k - E^x_{N,k}) < \varepsilon_N$. Taking into account the equality $E_k = E\{t: E^x(t) - f^x(t) > \alpha_k\}$ for all $x \in R$, where $E^x_{N,k} \subset E_k$, from (14) we get

$$F^x_{\xi\eta}(t) - f^x_{\mu\nu}(t) > \frac{\alpha_k}{2} \quad \text{for} \quad t \in E^x_{N,k}, \quad \text{and} \quad \operatorname{mes}(E_k - E^x_{N,k}) < \varepsilon_N.$$

By this very fact inclusion (9) is proved.

Thus, all sets R^k_N are open and everywhere dense on (0,1]. Hence, the set $Q = \prod_{N,k} R^k_N$ is of second category in (0,1]. Let us show that the set Q satisfies the conditions of the theorem. Let $x_0 \in Q$, and $E^{x_0} = \overline{\lim_k}\, \overline{\lim_N}\, E^{x_0}_{N,k}$. According to (6),(7) and the condition $\varepsilon_k \to 0$, we have mes E^{x_0} = mes E. Let us show that $\{s^{x_0}_k(t)\}$ diverges on E^{x_0}. Let $t_0 \in E^{x_0}$ and $\overline{\lim_N}\, E^{x_0}_{N,k} = E^{x_0}_k$. Then

there is found an index $k_0 = k_0(t_0)$ satisfying the condition $t_0 \in E^{x_0}_{k_0}$. From this, in turn, follows the existence of a sequence of indexes $\{N_i\} = \{N_i(t_0)\}$ such that $t_0 \in E^{x_0}_{N_i,k_0}$ $(i = 1,2,\ldots)$. But this inclusion, in conjunction with (8), yields $F^{x_0}(t_0) - f^{x_0}(t_0) \geq \frac{\alpha_{k_0}}{2}$. Since $x_0 \in Q$ and $t_0 \in E^{x_0}$ are arbitrary, while mes E^{x_0} = mes E, the theorem is proved.

Theorem 5 admits of the following equivalent formulation adjoining Tauber-type theorems.

<u>Theorem 5′. If the sequence $\{s_n(t)\}$ of functions measurable and almost everywhere finite on the set E is such that there exists a set $Q \subset (0,1]$ of second category and that the sequences $\{s^x_n(t)\}$ converge almost everywhere on E for every $x \in Q$, then $\{s_n(t)\}$ converges almost everywhere on E.</u>

It is useful to note that here the convergence set of the sequence $\{s_n(t)\}$ depends, generally speaking, on x. The theorem of Keogh and Petersen [3] may suggest that theorem 5′ admits of such a generalization. If the sequence $\{s_n(t)\}$ of functions measurable and almost everywhere finite on the set E and the regular method $T = \|a_{n,k}\|$ are such that there exists a second-category set $Q \subset (0,1]$ and that the sequences $\{s^x_n(t)\}$ are summed by the T method almost everywhere on E for every $x \in Q$, then $\{s_n(t)\}$ converges almost everywhere on E. But such a generalization is not valid. This follows from the theorem of P.L. Ul'yanov, given in the review to the Russian translation of the book of R. Cooke "Infinite Matrices and Sequence Spaces" ([7], p. 402, theorem 5.4).

Now let us consider rearrangements. Let $y' = \{n'_k\}$ and $y'' = \{n''_k\}$ be two rearrangements of the numbers of a natural series. We determine the distance between them by means of the Fréchet metric:

$$\rho(y', y'') = \sum_{k=1}^{\infty} \frac{1}{2^k} \frac{|n''_k - n'_k|}{1 + |n''_k - n'_k|}.$$

Then the set P of all rearrangements will be a metric Baire space (not complete) of the second category [11]. For the regular method $T = \|a_{n,k}\|$ and the sequence $\{s_n\}$ we set

$$\sigma_n(y) = \sum_{k=1}^{\infty} a_{n,k} s_{n_k},$$

if $y = \{n_k\}$ is the rearrangement of the natural series. The following is an analog of theorem 2.

Theorem 6. If $T = \|a_{n,k}\|$ is a regular method, and $|s_n| \le C$ $(n = 1,2,\ldots)$, then there exists a set U of the second category in P such that $\{\sigma_n(y)\}' \supset \{s_n\}'$ when $y \in U$.

This theorem partially generalizes theorem 4 of [12].

The following is an analog of theorem 5 for series.

Theorem 7. If the series $\sum_{n=1}^{\infty} u_n(t)$, where $u_n(t)$ are measurable functions, almost everywhere finite on the set E, is such that after some rearrangement of terms it diverges almost everywhere on E, there exists a set U of the second category in P such that the series $\sum_{k=1}^{\infty} u_{n_k}(t)$ diverges almost everywhere on E for every $\{n_k\} = y \in U$.

Theorem 7 can be stated in the following equivalent form:

Theorem 7'. If the series $\sum_{n=1}^{\infty} u_n(t)$, (all $u_n(t)$ are measurable and almost everywhere finite on E) is such that there exists a set U of the second category in P and the series $\sum_{k=1}^{\infty} u_{n_k}(t)$ converge almost everywhere on E for every $\{n_k\} = y \in U$, then the series $\sum_{n=1}^{\infty} u_n(t)$ after any rearrangement of terms converges almost everywhere on E, i.e. $\sum_{n=1}^{\infty} u_n(t)$ converges unconditionally almost everywhere on E.

For the rearrangements it is possible to introduce quantities $d_n(y)$ analogous to $d_n(x)$ for the subsequences. We denote by $N(n,y)$ the number of terms of the rearrangement $y = \{n_k\}$ among the first n of its terms that do not exceed n and we set $d_n(y) = \frac{N(n-y)}{n}$.

Theorem 8. *There exists a set U of the second category in P such that when* $y \in U$ *we have* $\{d_n(y)\}' = [0,1]$.

We shall omit the proofs of theorems 6, 7 and 8, in view of their complete analogy with the proofs of the corresponding assertions for subsequences.

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