AD 740377

Office of Naval Research
Contract N00014-67-A-0298-0006 NR-372-012

OPTIMAL STOCHASTIC LINEAR SYSTEMS WITH EXPONENTIAL PERFORMANCE CRITERIA AND THEIR RELATION TO DETERMINISTIC DIFFERENTIAL GAMES

By
D. H. Jacobson

April 1972

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Technical Report No. 631

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The research reported in this document was made possible through support extended the Division of Engineering and Applied Physics, Harvard University by the U S Army Research Office, the U. S Air Force Office of Scientific Research and the U S Office of Naval Research under the Joint Services Electronics Program by Contracts N00014-67-A-0298-0006, 0005, and 0008.

Division of Engineering and Applied Physics

Harvard University · Cambridge, Massachusetts

Unclassified

Security Classification

DOCUMENT CONTROL DATA - R & D

Security classification of title, body of abstract and indexing annotation must be entered when the overall report is classified

1. ORIGINATING ACTIVITY (Corporate author)

Division of Engineering and Applied Physics
Harvard University
Cambridge, Massachusetts

2a. REPORT SECURITY CLASSIFICATION

Unclassified

2b. GROUP

3. REPORT TITLE

OPTIMAL STOCHASTIC LINEAR SYSTEMS WITH EXPONENTIAL PERFORMANCE CRITERIA AND THEIR RELATION TO DETERMINISTIC DIFFERENTIAL GAMES

4. DESCRIPTIVE NOTES (Type of report and inclusive dates)

Interim technical report

5. AUTHOR(S) (First name, middle initial, last name)

D. H. Jacobson

6. REPORT DATE

April 1972

7a. TOTAL NO. OF PAGES

35

7b. NO. OF REFS

2

8a. CONTRACT OR GRANT NO.

N00014-67-A-0298-0006

b. PROJECT NO.

c.

d.

9a. ORIGINATOR'S REPORT NUMBER(S)

9b. OTHER REPORT NO(S) (Any other numbers that may be assigned this report)

10. DISTRIBUTION STATEMENT

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11. SUPPLEMENTARY NOTES

12. SPONSORING MILITARY ACTIVITY

Office of Naval Research

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Finally, some stability properties of the asymptotic controllers are discussed.

DD FORM 1 NOV 65 1473 (PAGE 1)

S/N 0102 014-6700

Unclassified

Security Classification

D-23822

14 KEY WORDS	LINK A ROLE	LINK A WT	LINK B ROLE	LINK B WT	LINK C ROLE	LINK C WT
Stochastic Optimal Control						
Linear Quadratic Gaussian Problems						
Exponential Performance Criteria						
Certainty Equivalence Principle						

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ABSTRACT

In this report two stochastic optimal control problems are solved whose performance criteria are the expected values of exponential functions of quadratic forms. The optimal controller is linear in both cases but depends upon the covariance matrix of the additive process noise so that the Certainty Equivalence Principle does not hold. The controllers are shown to be equivalent to those obtained by solving a cooperative and a noncooperative quadratic (differential) game, and this leads to some interesting interpretations and observations.

Finally, some stability properties of the asymptotic controllers are discussed.

1. Introduction

The so called LQG problem* of optimal stochastic control [1] possesses a number of interesting features. First, the optimal feedback controller is a linear (time varying) function of the state variables. Second, this linear controller is identical to that which is obtained by neglecting the additive gaussian noise and solving the resultant deterministic LQP** (Certainty Equivalence Principle). Thus the controller for the stochastic system is independent of the statistics of the additive noise. This is appealing for small noise intensity, but for large noise (large covariance) one has the intuitive feeling that perhaps a different controller would be more appropriate.

In this paper we consider optimal control of linear systems disturbed by additive gaussian noise, whose associated performance criteria are the expected values of exponential functions of negative semi-definite and positive semi-definite quadratic forms. We shall refer to the former case as the $LE^{-}G$ problem and the latter as the $LE^{+}G$ problem and to their deterministic counterparts as $LE^{-}P$ and $LE^{+}P$ respectively. In the deterministic cases, $LE^{\pm}P$, the solutions are identical to that for the LQP (the natural logarithm of the exponential performance criteria yield quadratic forms). However, when noise is present, $LE^{\pm}G$ problems, the

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*Problem with linear dynamics disturbed by additive gaussian noise, together with a performance criterion which is the expected value of a positive semi-definite quadratic form.

**Same as LQG problem but with noise set to zero.

optimal controllers are different from that of the LQG problem. In particular, though as in the case of the LQG problem these are linear functions of the state variables, they depend explicitly upon the covariance matrices of the additive gaussian noise. For small noise intensity (small covariance) the solutions of the $LE^{\pm}G$ and LQG problems are close, but for large noise intensity there is a marked difference. In particular, as the noise intensity tends to infinity the optimal gains for the $LE^{-}G$ problem tend to zero; intuitively this implies that if the random input is "very wild" little can be gained (in the sense of reducing the value of this particular performance criterion) by controlling the system. In the $LE^{+}G$ problem the optimal controller ceases to exist if the noise intensity is sufficiently large (that is, the performance criterion becomes infinite, regardless of the control input).

These new controllers, which retain the simplicity of the solution of the LQG problem, could prove to be attractive in certain applications.

In addition to formulating and solving the $LE^{\pm}G$ problems we demonstrate that their solutions are equivalent to the solutions of cooperative and noncooperative linear-quadratic zero-sum (differential) games. These equivalences provide interpretations for the stochastic controllers in terms of solutions of deterministic zero-sum games, and vice versa. It is hoped that these equivalences will aid in the quest for new formulations and (proofs of existence of) solutions of stochastic nonlinear systems and nonlinear differential games.

We investigate briefly the infinite time version of the $LE^{\pm}G$ problems and point out that the steady state optimal controller for the $LE^{-}G$ problem is not necessarily stable. On the other hand the steady state optimal controller for the $LE^{+}G$ problem, if it exists, is stable. Thus the $LE^{+}G$ formulation may be preferable in the infinite time case.

2. Formulation of Discrete Time LE$^{\pm}$G Problems

2.1 The LE^{-}G Problem

a) Dynamics

We shall consider a linear discrete time dynamic system described by

$$x_{k+1} = A_k x_k + B_k u_k + \Gamma_k \alpha_k \quad ; \quad k=0,\ldots,N-1,\ x_o \text{ given,} \tag{1}$$

where the "state" vector $x_k \in R^n$, the control vector $u_k \in R^m$ and the gaussian noise input $\alpha_k \in R^q$. The matrices A_k, B_k, Γ_k have appropriate dimensions and depend upon the time k.

b) Noise

The noise input is a sequence $\{\alpha_k\}$ of independently distributed gaussian random variables having probability density

$$p_\alpha(\alpha_o,\ldots,\alpha_{N-1}) = \prod_{k=0}^{N-1} p(\alpha_k;k) \tag{2}$$

where $p_\alpha : R^q \times R^N \to R^+$ and $p : R^q \times I^+ \to R^+$ is given by

$$p(\alpha_k;k) = \frac{1}{\sqrt{(2\pi)^q |P_k^{-1}|}} \exp\left\{-\frac{1}{2}\alpha_k^T P_k \alpha_k\right\} \tag{3}$$

with

$$P_k > 0 \quad \text{(positive-definite)} \quad ; \quad k=0,\ldots,N-1 \tag{4}$$

Note that

$$\mathscr{E}[\alpha_k] = 0, \quad \mathscr{E}[\alpha_k \alpha_k^T] = P_k^{-1} \quad ; \quad k=0,\ldots,N-1 \tag{5}$$

where $\mathscr{E}$ denotes expectation.

c) Performance Criterion

The performance of the stochastic linear system is measured by the criterion

$$V^-(x_o) \triangleq -\mathscr{E}|_{x_o} \prod_{k=0}^{N-1} \mu_x^-(x_k;k)\mu_u^-(u_k;k)\mu_x^-(x_N;N) \tag{6}$$

where $V^- : R^n \to [-1,0]$ and $\mu_x^- : R^n \times I^+ \to [0,1]$, $\mu_u^- : R^m \times I^+ \to [0,1]$ are given

$$\mu_x^-(x_k;k) = \exp\{-\frac{1}{2} x_k^T Q_k x_k\} \quad ; \quad k=0,\ldots,N \tag{7}$$

$$\mu_u^-(u_k;k) = \exp\{-\frac{1}{2} u_k^T R_k u_k\} \quad ; \quad k=0,\ldots,N-1 \tag{8}$$

and

$$Q_k \geq 0 \qquad \text{(positive semi-definite)} \quad ; \quad k=0,\ldots,N \tag{9}$$

$$R_k > 0 \qquad \text{(positive definite)} \quad ; \quad k=0,\ldots,N-1 \tag{10}$$

Note that (6) can be written as

$$V^-(x_o) = -\mathscr{E}|_{x_o} \exp\{-\frac{1}{2}[\sum_{k=0}^{N-1} (x_k^T Q_k x_k + u_k^T R_k u_k) + x_N^T Q_N x_N]\} \tag{11}$$

d) Problem

We are required to find a policy

$$u_k^- = C_k^-(X_k) \quad ; \quad k=0,\ldots,N-1 \; ; \; X_k \triangleq \{x_o, x_1, \ldots, x_k\} \tag{12}$$

which minimizes performance criterion (11). Thus the problem is identical to the LQG problem except that the performance criterion is the negative of the expected value of an exponential function of a negative semi-definite quadratic form.

Note that $V^-(x_o)$ for arbitrary controls $\{u_k\}$ is bounded as follows

$$-1 \leq V^-(x_o) \leq 0 \tag{13}$$

2.2 <u>The LE^+G Problem</u>

The formulation is the same as the LE^-P except for the performance criterion which is

$$V^+(x_o) \triangleq \mathscr{E}|_{x_o} \prod_{k=0}^{N-1} \mu_x^+(x_k;k)\mu_u^+(u_k;k)\mu_x^+(x_N:N) \tag{14}$$

where $V^+ : R^n \to [1,\infty]$, and $\mu_x^+ : R^n \times I^+ \to [1,\infty)$, $\mu_u^+ : R^m \times I^+ \to [1,\infty)$ are given by

$$\mu_x^+(x_k;k) = \exp\{\tfrac{1}{2}\, x_k^T Q_k x_k\}; \quad k=0,\ldots,N \tag{15}$$

$$\mu_u^-(u_k;k) = \exp\{\tfrac{1}{2}\, u_k^T R_k u_k\}; \quad k=0,\ldots,N-1 \tag{16}$$

with Q_k, R_k as in (9), (10).

Note that (14) can be written as

$$V^+(x_o) = \mathscr{E}|_{x_o} \exp\{\tfrac{1}{2}[\sum_{k=0}^{N-1} (x_k^T Q_k x_k + u_k^T R_k u_k) + x_N^T Q_N x_N]\} \tag{17}$$

The problem is to find a policy

$$u_k^+ = C_k^+(X_k) \quad : \quad k = 0,\ldots,N-1 \; ; \; X_k \triangleq \{x_o, x_1, \ldots, x_k\} \tag{18}$$

which minimizes performance criterion (14). Again this problem is identical to the LQG problem except that the performance criterion is the expected value of an exponential function of a positive semi-definite quadratic form.

Note that $V^+(x_o)$, for arbitrary controls $\{u_k\}$, satisfies

$$0 \leq V^+(x_o) \leq \infty$$

3. Formulation of $LE^{\pm}P$

If no noise is present

$$\alpha_k \equiv 0 \quad ; \quad k = 0,\ldots,N-1 \tag{20}$$

Minimization of (11) and (17) is equivalent to minimization of

$$\frac{1}{2}\left[\sum_{k=0}^{N-1} (x_k^T Q_k x_k + u_k^T R_k u_k) + x_N^T Q_N x_N\right] \tag{21}$$

subject to

$$x_{k+1} = A_k x_k + B_k u_k \quad ; \quad k = 0,\ldots,N-1 \tag{22}$$

which is a standard LQP. Thus LE^-P and LE^+P are equivalent and both will be referred to as LEP. As the solution of the LQP is well known, we state it now without proof.

The optimal controller for the LEP(LQP) is

$$u_k = -D_k x_k \quad ; \quad k = 0,\ldots,N-1 \tag{23}$$

where

$$D_k = (R_k + B_k^T M_{k+1} B_k)^{-1} B_k^T M_{k+1} A_k \tag{24}$$

and

$$M_k = Q_k + A_k^T [M_{k+1} - M_{k+1} B_k (R_k + B_k^T M_{k+1} B_k)^{-1} B_k^T M_{k+1}] A_k \tag{25}$$

with

$$M_N = Q_N \tag{26}$$

In view of our assumptions (9), (10) it is easy to show that

$$M_k \geq 0 \quad ; \quad k = 0,\ldots,N \tag{27}$$

so that

$$(R_k + B_k^T M_{k+1} B_k) > 0 \quad ; \quad k = 0,\ldots,N-1 \tag{28}$$

4. Solution of Discrete Time LE$^{\pm}$G Problems

4.1 The LE^{-}G Problem

We define

$$J^{-}(X_k;k) = -\min_{u_k \cdots u_{N-1}} \mathcal{E}_{|X_k} \prod_{i=k}^{N-1} \mu_x^{-}(x_i;i)\mu_u^{-}(u_i;i)\mu_x^{-}(x_N;N) \tag{29}$$

given that the minimizing optimal policy must be of the form

$$u_i^{-} = C_i^{-}(X_i) \quad ; \quad i = k,\ldots,N-1 \tag{30}$$

At time k+1, then,

$$J^{-}(X_{k+1};k+1) = -\min_{u_{k+1} \cdots u_{N-1}} \mathcal{E}_{|X_{k+1}} \prod_{i=k+1}^{N-1} \mu_x^{-}(x_i;i)\mu_u^{-}(u_i;i)\mu_x^{-}(x_N;N) \tag{31}$$

so that

$$J^{-}(X_k;k) = \min_{u_k} [\mu_x^{-}(x_k;k)\mu_u^{-}(u_k;k)\mathcal{E}_{|X_k} J^{-}(X_{k+1};k+1)] \tag{32}$$

where

$$x_{k+1} = A_k x_k + B_k u_k + \Gamma_k \alpha_k \quad ; \quad x_k \text{ given} \tag{33}$$

Because of the Markov property of (33) which is due to the independence of $\{\alpha_k\}$ it is clear from (29) that $J^{-}(X_k;k)$ can be written as $J^{-}(x_k;k)$ so that (32) becomes†

$$J^{-}(x_k;k) = \min_{u_k} [\mu_x^{-}(x_k;k)\mu_u^{-}(u_k;k)\int_{-\infty}^{\infty} p(\alpha_k;k)J^{-}(x_{k+1};k+1)d\alpha_k] \tag{34}$$

and

$$J^{-}(x_N;N) = -\exp\{-\frac{1}{2} x_N^T Q_N x_N\} . \tag{35}$$

We now show that

$$J^{-}(x_k;k) \triangleq -F_k^{-} \exp\{-\frac{1}{2} x_k^T W_k^{-} x_k\} \tag{36}$$

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†Alternatively, the development could be continued using (32) and identical results would be obtained.

which is defined for $k = 0,\ldots,N$ solves (34) where

$$W_k^- \geq 0 \quad : \quad k = 0,\ldots,N \tag{37}$$

is given by

$$W_k^- \triangleq Q_k + A_k^T[\tilde{W}_{k+1}^- - \tilde{W}_{k+1}^- B_k(R_k + B_k^T\tilde{W}_{k+1}^- B_k)^{-1} B_k^T \tilde{W}_{k+1}^-]A_k \tag{38}$$

where

$$\tilde{W}_{k+1}^- \triangleq W_{k+1}^- - W_{k+1}^- \Gamma_k(P_k + \Gamma_k^T W_{k+1}^- \Gamma_k)^{-1}\Gamma_k^T W_{k+1}^- \tag{39}$$

and

$$W_N^- = Q_N \ . \tag{40}$$

In addition we have that

$$F_k^- = F_{k+1}^- \sqrt{\frac{|(P_k + \Gamma_k^T W_{k+1}^- \Gamma_k)^{-1}|}{|P_k^{-1}|}} \quad : \quad F_N^- = 1 \tag{41}$$

and the optimal policy is

$$u_k^- = -C_k^- x_k \tag{42}$$

where

$$C_k^- \triangleq (R_k + B_k^T \tilde{W}_{k+1}^- B_k)^{-1} B_k^T \tilde{W}_{k+1}^- A_k \quad ; \quad k = 0,\ldots,N-1. \tag{43}$$

In order to prove that (36) and (42) solve (34) we need the following, probably well known but underexploited,

<u>Lemma 1</u> : If $(P_k + \Gamma_k^T W_{k+1}^- \Gamma_k) > 0$, then

$$\int_{-\infty}^{\infty} \frac{1}{\sqrt{(2\pi)^d |P_k^{-1}|}} \exp\{-\tfrac{1}{2}\alpha_k^T P_k \alpha_k\} \cdot \exp\{-\tfrac{1}{2} x_{k+1}^T W_{k+1}^- x_{k+1}\} d\alpha_k$$

$$= \sqrt{\frac{|(P_k + \Gamma_k^T W_{k+1}^- \Gamma_k)^{-1}|}{|P_k^{-1}|}} \exp\{-\tfrac{1}{2}(A_k x_k + B_k u_k)^T \tilde{W}_{k+1}^- (A_k x_k + B_k u_k)\} \tag{44}$$

where $\tilde{W}^-_{k+1}$ is defined in (39).

Proof: See Appendix.

Substituting (36) into (34) and using the Lemma and (41) we obtain

$$-\exp\{-\frac{1}{2}x_k^T W_k^- x_k\} = \min_{u_k} -\mu_x^-(x_k;k)\mu_u^-(u_k;k)\exp\{-\frac{1}{2}(A_k x_k+B_k u_k)^T \tilde{W}^-_{k+1}(A_k x_k+B_k u_k)\} \tag{45}$$

which, upon taking logarithms is equivalent to

$$\frac{1}{2}x_k^T W_k^- x_k = \frac{1}{2}\min_{u_k}[x_k^T Q_k x_k+u_k^T R_k u_k+(A_k x_k+B_k u_k)^T \tilde{W}^-_{k+1}(A_k x_k+B_k u_k)] \tag{46}$$

Equation (46) is satisfied by (38), (42), (43) so that the LE$^-$Q problem is indeed solved. As in the LEP (LQP) it is easy to verify that, under assumptions (4), (9), (10), W_k^- and $\tilde{W}_k^-$ are positive semi-definite for $k = 0,\ldots,N$ so that

$$(P_k+\Gamma_k^T W^-_{k+1}\Gamma_k) > 0, \quad (R_k+B_k^T \tilde{W}^-_{k+1}B_k) > 0 \tag{47}$$

which ensures that (38), (39), (41), (43) are well defined.

4.2 The LE$^+$G Problem

Here we define

$$J^+(X_k;k) \triangleq \min_{u_k\ldots u_{N-1}} \mathcal{E}_{|X_k} \prod_{j=k}^{N-1} \mu_x^+(x_i;i)\mu_u^+(u_i;i)\mu_x^+(x_N;N) \tag{48}$$

given that the minimizing optimal policy must be of the form

$$u_i^+ = C_i^+(X_i) \quad ; \quad i = k,\ldots,N-1 \tag{49}$$

so that proceeding as in Section 4.1, we obtain

$$J^+(x_k;k) = \min_{u_k}[\mu_x^+(x_k;k)\mu_u^+(u_k;k)\int_{-\infty}^{\infty} p(\alpha_k;k)J^+(x_{k+1};k+1)d\alpha_k] \tag{50}$$

and

$$J^+(x_N;N) = \exp\{\tfrac{1}{2}\, x_N^T Q_N x_N\} \tag{51}$$

The solution of (50), which is analogous to (36) is

$$J^+(x_k;k) \triangleq F_k^+ \exp\{\tfrac{1}{2}\, x_k^T W_k^+ x_k\} \tag{52}$$

which is defined for $k = 0,\ldots,N$, where

$$W_k^+ = Q_k + A_k^T[\tilde{W}_{k+1}^+ - \tilde{W}_{k+1}^+ B_k (R_k + B_k^T \tilde{W}_{k+1}^+ B_k)^{-1} B_k^T \tilde{W}_{k+1}^+] A_k \tag{53}$$

where

$$\tilde{W}_{k+1}^+ = W_{k+1}^+ + W_{k+1}^+ \Gamma_k (P_k - \Gamma_k^T W_{k+1}^+ \Gamma_k)^{-1} \Gamma_k^T W_{k+1}^+ \tag{54}$$

and

$$W_N^+ = Q_N \tag{55}$$

In addition, we have that

$$F_k^+ = F_{k+1}^+ \sqrt{\frac{|(P_k - \Gamma_k^T W_{k+1}^+ \Gamma_k)^{-1}|}{|P_k^{-1}|}} \quad ; \; F_N^+ = 1 \tag{56}$$

and the optimal policy is

$$u_k^+ = -C_k^+ x_k$$

where

$$C_k^+ \triangleq (R_k + B_k^T \tilde{W}_{k+1}^+ B_k)^{-1} B_k^T \tilde{W}_{k+1}^+ A_k \quad ; \quad k=0,\ldots,N-1 \tag{57}$$

In order to verify that (52)-(57) solve (50) (which we will not do here because the procedure is almost identical to that for the LEG problem) it is necessary to use Lemma 2, which we state below, which is useful only if

$$P_k - \Gamma_k^T W_{k+1}^+ \Gamma_k > 0 \ , \quad k = 0,\ldots,N-1 \tag{58}$$

If (58) is not satisfied, then (52)-(57) do not constitute a meaningful solution for (50) since it follows from Lemma 2 that

$$J^+(x_k;k) \text{ is infinite.} \tag{59}$$

<u>Lemma 2</u> If

$$(P_k - \Gamma_k^T W_{k+1}^+ \Gamma_k) > 0, \tag{60}$$

then

$$\int_{-\infty}^{\infty} \frac{1}{\sqrt{(2\pi)^q |P_k^{-1}|}} \exp\{-\frac{1}{2}\alpha_k^T P_k \alpha_k\} \cdot \exp\{\frac{1}{2} x_{k+1}^T W_{k+1}^+ x_{k+1}\} d\alpha_k$$

$$= \sqrt{\frac{|(P_k - \Gamma_k^T W_{k+1}^+ \Gamma_k)^{-1}|}{|P_k^{-1}|}} \exp\{\frac{1}{2}(A_k x_k + B_k u_k)^T W_{k+1}^+ (A_k x_k + P_k u_k)\} \tag{61}$$

Moreover, if

$$P_k - \Gamma_k^T W_{k+1}^+ \Gamma_k \not> 0 \tag{62}$$

then the left hand side of (61) is infinite.

<u>Proof</u>: See Appendix.

5. Properties of Solutions of Discrete Time $LE^{\pm}G$ Problems

5.1 The LE^-G Problem

The optimal feedback controller for the LE^-G problem is a linear function of the system state,

$$u_k^- = -C_k^- x_k \quad ; \quad k = 0,\ldots,N-1 \tag{63}$$

where C_k^- depends upon the solution of a Riccati type difference equation (38). The main difference between this and the feedback law for the LQG problem is that C_k^- depends upon P_k^{-1}, the covariance matrix of the gaussian additive disturbance α_k. In the LQG case the optimal feedback law is independent of the covariance of the input noise and, indeed, is the same as that for the deterministic LQP (so called Certainty Equivalence Principle). Here, in the case where our criterion is the expected value of minus an exponential function of a negative semi-definite quadratic form, the Certainty Equivalence Principle does not hold.

It is interesting to investigate two limiting cases; the first in which $\lambda_{min}(P_k) \to \infty$ (input $\alpha_k \equiv 0$, $k = 0,\ldots,N-1$) and the second in which $\lambda_{min}(P_k^{-1}) \to \infty$ (input "infinitely wild").

i) $\underline{\lambda_{min}(P_k) \to \infty \quad ; \quad k = 0,\ldots,N-1.}$

In this case it is clear[+] from (36), (38), (39) that

$$C_k^- \to D_k \quad ; \quad k = 0,\ldots,N-1 \tag{64}$$

the optimal gains for the LQP(LEP). Note, from (36) and (41) that

$$J^-(x_k;k) \to -\exp\{-\tfrac{1}{2} x_k^T W_k^- x_k\} \quad , \quad k = 0,\ldots,N \tag{65}$$

Thus for small noise intensities (P_k^{-1} small, $k=0,\ldots,N-1$) the solution of the LE$^-$G problem is close to that of the LEP, LQP, and LQG problem.

ii) $\underline{\lambda_{min}(P_k^{-1}) \to \infty \quad : \quad k = 0,\ldots,N-1.}$

Here we shall assume that

$$\Gamma_k^T Q_{k+1} \Gamma_k > 0 \quad ; \quad k = 0,\ldots,N-1 \tag{66}$$

[+] These limiting cases can be argued rigorously; the arguments are straightforward and are left to the reader.

so that, from (37) - (39),

$$\Gamma_k^T W_{k+1}^- \Gamma_k > 0 \quad ; \quad k = 0,\ldots,N-1 \tag{67}$$

as $P_k \to 0$, then, we have

$$\tilde{W}_{k+1}^- \to W_{k+1}^- - W_{k+1}^- \Gamma_k (\Gamma_k^T W_{k+1}^- \Gamma_k)^{-1} \Gamma_k^T W_{k+1}^- \quad ; \quad k = 0,\ldots,N-1 \tag{68}$$

and, from (36) and (41),

$$J^-(x_k;k) \to 0 \quad ; \quad k = 0,\ldots,N-1. \tag{69}$$

Note that if Γ_k has rank n for $k = 0,\ldots,N-1$, that

$$\tilde{W}_{k+1}^- \to 0 \quad ; \quad k = 0,\ldots,N-1 \tag{70}$$

so that

$$C_k^- \to 0 \quad ; \quad k = 0,\ldots,N-1 \ . \tag{71}$$

An explanation for (71) is that if <u>all</u> components of γ_k are disturbed by an "infinitely wild" additive noise then there is no point (as far as performance criterion (6) is concerned) in exercising control to try and counteract these infinite unpredictable disturbances.

Of major interest are the cases in which

$$0 \leq P_k^{-1} < \infty \quad ; \quad k = 0,\ldots,N-1 \tag{72}$$

for which the new controller (42) offers an alternative to the standard LQG solution.

5.2 <u>The LE^+G Problem</u>

As in the LE^-G problem the Certainty Equivalence Principle does not hold because C_k^+ depends upon the covariance of the additive process

noise. We again consider the two limiting cases of zero noise and "infinite" noise.

i) $\underline{\lambda_{min}(\nu_k) \to \infty \quad ; \quad k = 0,\ldots,N-1.}$

In this case, as the covariance matrix tends to zero, we obtain from (52) - (57) that

$$C_k^+ \to D_k \quad ; \quad k = 0,\ldots,N-1 \tag{73}$$

and

$$J^+(x_k;k) \to \exp\{\tfrac{1}{2}\, x_k^T W_k^+ x_k\} \ , \quad k = 0,\ldots,N-1 \tag{74}$$

so that for small noise intensity the solution of the LE^+G problem is close to that of the LEP, LQP, LQG problem.

ii) $\underline{\lambda_{min}(P_k^{-1}) \to \infty \quad ; \quad k = 0,\ldots,N-1.}$

For P_k sufficiently small (i.e. large covariance) the solution of (50) can cease to exist (indeed (48) can become infinite). To see this, let us assume that

$$\Gamma_k^T Q_{k+1} \Gamma_k > 0 \quad ; \quad k = 0,\ldots,N-1 \ , \tag{75}$$

and that

$$P_j - \Gamma_j^T W_{j+1}^+ \Gamma_j > 0 \quad ; \quad j = k+1,\ldots,N-1. \tag{76}$$

From (75), (76), (53), (54) we have that

$$\Gamma_k^T W_{k+1}^+ \Gamma_k > 0 \tag{77}$$

so that for P_k sufficiently small

$$P_k - \Gamma_k^T W_{k+1}^+ \Gamma_k \not> 0 \tag{78}$$

which implies from Lemma 2, that the left hand side of (60) is infinite. Clearly, then, from (50)

$$J^{+}(x_k;k) \text{ is infinite .} \tag{79}$$

Since k is arbitrary, $k \in \{0,\ldots,N-1\}$, we can conclude that if the noise covariance is sufficiently large, the performance criterion (14) is infinite, regardless of the choice of controls $\{u_k\}$. We shall have more to say about this interesting case when we treat the continuous time $LE^{+}G$ problem in Section 8.

6. The Discrete Time $LE^{\pm}G$ Problems and Deterministic Games

6.1 The $LE^{-}G$ Problem

The solution of the $LE^{-}G$ problem is, by inspection (or short calculation), equivalent to the solution of the following cooperative deterministic game (LQP).

$$\underset{\{u_k\},\{\alpha_k\}}{\text{Minimize}} \left[\frac{1}{2}\sum_{k=0}^{N-1} (x_k^T Q_k x_k + u_k^T R_k u_k + \alpha_k^T P_k \alpha_k) + \frac{1}{2} x_N^T Q_N x_N\right] \tag{80}$$

subject to the dynamic constraint

$$x_{k+1} = A_k x_k + B_k u_k + \Gamma_k \alpha_k \quad ; \quad k = 0,\ldots,N-1, \quad x_o \text{ given} \tag{81}$$

It turns out that

$$\frac{1}{2} x_k^T W_k^- x_k = \min_{\{u_i\},\{\alpha_i\}} \left[\frac{1}{2}\sum_{i=k}^{N-1} (x_i^T Q_i x_i + u_i^T R_i u_i + \alpha_i^T P_i \alpha_i) + \frac{1}{2} x_N^T Q_N x_N\right] \tag{82}$$

Note that in the above formulation we determine optimal control laws

$$u_k^- = -C_k^- x_k, \quad \alpha_k^- = -\Delta_k^- x_k \quad ; \quad k = 0,\ldots,N-1 \tag{83}$$

We now have a new interpretation for the linear-quardatic game: If player u_k assumes that player α_k will cooperate in minimizing the quadratic criterion (even though u_k knows that α_k behaves like a gaussian random variable), then the feedback controller (policy) that is obtained for u_k, upon solving (80) and (81), namely

$$u_k^- = -C_k^- x_k \quad : \quad k = 0,\ldots,N-1 \tag{84}$$

is optimal also for the LE^-G problem. Thus the policy for u_k obtained by treating α_k as a cooperative player makes sense when interpreted as the solution of the stochastic LE^-G problem.

6.2 The LE^+G Problem

Here, the deterministic game that has an equivalent solution is non-cooperative, namely,

$$\min_{\{u_k\}} \max_{\{\alpha_k\}} \left[\frac{1}{2} \sum_{k=0}^{N-1} (x_k^T Q_k x_k + u_k^T R_k u_k - \alpha_k^T P_k \alpha_k) + \frac{1}{2} x_N^T Q_N x_N \right] \tag{85}$$

subject to (81), where u_k^+ and α_k^+ are determined as feedback laws (policies)

$$u_k^+ = -C_k^+ x_k, \quad \alpha_k^+ = -\Delta_k^+ x_k \quad : \quad k = 0,\ldots,N-1. \tag{86}$$

It is well known that if

$$P_k - \Gamma_k^T W_{k+1}^+ \Gamma_k > 0 \quad ; \quad k = 0,\ldots,N-1 \tag{87}$$

then

$$\frac{1}{2} x_k^T W_k^+ x_k = \min_{\{u_i\}} \max_{\{\alpha_i\}} \left[\frac{1}{2} \sum_{i=k}^{N-1} (x_i^T Q_i x_i + u_i^T R_i u_i - \alpha_i^T P_i \alpha_i) + \frac{1}{2} x_N^T Q_N x_N \right] \tag{88}$$

If the determinant of the left hand side of (87) is nonzero but the matrix fails to be positive definite then as is well known, (85) ceases to be bounded. However, if the left hand side of (87) is singular for some values of $k \in \{0,\ldots,N-1\}$ then (85) may exist. Thus, provided

$$|P_k - \Gamma_k^T W_{k+1}^+ \Gamma_k| \neq 0 \quad ; \quad k = 0,\ldots,N-1 \tag{89}$$

we have, from Lemma 2 and (87), (88), that (48) is finite (for $k = 0$) if and only if (85) is finite.

Out interpretation of the above noncooperative deterministic game is as follows: If player u_k assumes that α_k will not cooperate in minimizing the quadratic criterion (even though u_k knows that α_k behaves like a gaussian random variable) then the feedback controller (policy) that is obtained for u_k, upon solving (85), namely

$$u_k^+ = -C_k^+ x_k \quad ; \quad k = 0,\ldots,N-1 \tag{90}$$

is optimal for the LE^+G problem. Thus this rather conservative game formulation in which the noise α_k is treated as a noncooperative player gives rise to a control policy which solves the LE^+G stochastic control problem. When looked at from this viewpoint the min-max game solution for u_k ("worst case design") does not appear to be too pessimistic, since the performance criterion of the LE^+G problem is rather appealing.

7. Formulation of Continuous Time LE^+G Problems

7.1 The LE^-G Problem

In continuous time, the LE^-G problem takes the form

$$\underset{u(.,.)}{\text{Minimize}} \; -\mathcal{E}_{|x_o} \exp\{-[\tfrac{1}{2}\int_{t_o}^{t_f}(x^TQx+u^TRu)dt + \tfrac{1}{2}x(t_f)Q_f x(t_f)]\} \tag{91}$$

subject to

$$\dot{x} = Ax + Bu + \Gamma\alpha \quad : \quad x(t_o) \text{ given} \tag{92}$$

where, for notational simplicity, time dependence of the variables has been suppressed* and where $\alpha(.)$ is a gaussian white noise process having

$$\mathscr{E}[\alpha(t)] = 0 \quad ; \quad t \in [t_o, t_f] \tag{93}$$

$$\mathscr{E}[\alpha(t)\alpha^T(s)] = P^{-1}\delta(t-s) \quad ; \quad t,s \in [t_o, t_f] \tag{94}$$

where δ is the dirac delta function.

Note that in solving (91) we seek an optimal control policy

$$u^-(X,t) = C^-(X,t) \quad ; \quad t \in [t_o, t_f] \quad ; \quad X \triangleq \{x(\tau); \tau \in [t_o, t]\} \tag{95}$$

where $C^- : \mathscr{C} \times R^1 \to R^m$ is a measurable function of its arguments.

7.2 The LE^+G Problem

Here, the performance criterion to be minimized is

$$\mathscr{E}|_{x_o} \exp\{\frac{1}{2}\int_{t_o}^{t_f} (x^TQx + u^TRu)dt + \frac{1}{2}x^T(t_f)Q_f x(t_f)\} \tag{96}$$

and the required control policy is

$$u^+(X,t) = C^+(X,t) \quad : \quad t \in [t_o, t_f] \tag{97}$$

8. Solution of Continuous Time $LE^{\pm}G$ Problem and Relation to Differential Games

8.1 Solution of $LE^{\pm}G$ Problems

We can solve the continuous time $LE^{\pm}G$ problems either by formally

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*Note that $Q \geq 0$, $R > 0$, $P > 0$ for all $t \in [t_o, t_f]$, and $Q_f \geq 0$.

taking the limit of the solutions for the discrete time cases or by solving the "generalized" Hamilton-Jacobi-Bellman equation (see appendix for derivation)

$$-\frac{\partial J^{\sigma}}{\partial t}(x,t) = \min_{u}\{\frac{1}{2}\sigma(x^{T}Qx+u^{T}Ru)J^{\sigma}(x,t)+[J_{x}^{\sigma}(x,t)]^{T}(Ax+Bu)$$

$$+\frac{1}{2}\,\mathrm{tr}[J_{xx}^{\sigma}(x,t)\Gamma P^{-1}\Gamma^{T}]\} \tag{98}$$

where

$$\sigma = \begin{cases} - & ; \quad \text{for } LF^{-}G \text{ problem} \\ + & ; \quad \text{for } LE^{+}C \text{ problem} \end{cases} \tag{99}$$

which is satisfied by

$$J^{\sigma}(x,t) \triangleq \sigma\mathscr{E}_{|x(t)}\exp\{\sigma[\frac{1}{2}\int_{t}^{t_f}(x^{T}Qx+u^{\sigma T}Ru^{\sigma})dt+\frac{1}{2}x^{T}(t_f)Q_f x(t_f)]\}, \tag{100}$$

where

$$u^{\sigma}(x,\tau) = C^{\sigma}(x,\tau) \quad ; \quad \tau\in[t,t_f] \tag{101}$$

is the optimal policy.

Using either method we find that

$$u^{\sigma}(x,t) = -R^{-1}B^{T}S^{\sigma}x \quad ; \quad t\in[t_o,t_f] \tag{102}$$

and

$$J^{\sigma}(x,t) = \sigma F^{\sigma}\exp\{\sigma\frac{1}{2}x^{T}S^{\sigma}x\} \tag{103}$$

where

$$-\dot{S}^{\sigma} = Q+S^{\sigma}A+A^{T}S^{\sigma}-S^{\sigma}(BR^{-1}B^{T}-\sigma\Gamma P^{-1}\Gamma^{T})S^{\sigma} \quad ; \quad S^{\sigma}(t_f) = Q_f \tag{104}$$

and

$$-\dot{F}^{\sigma} = \frac{n}{2}\sigma F^{\sigma}\mathrm{tr}(S^{\sigma}\Gamma P^{-1}\Gamma^{T}) \quad ; \quad F^{\sigma}(t_f) = 1\ . \tag{105}$$

8.2 Relation to Continuous Time Differential Games

By inspection we see that the optimal controller for the LE^-G problem (σ negative) is obtained from the solution of the following cooperative differential game

$$\underset{u(.),\alpha(.)}{\text{Minimize}} \int_{t_o}^{t_f} \frac{1}{2}\,(x^TQx+u^TRu+\alpha^TP\alpha)dt + \frac{1}{2}x^T(t_f)Q_fx(t_f) \tag{106}$$

subject to

$$\dot{x} = Ax + Bu + \Gamma\alpha \quad ; \quad x(t_o) \text{ given} \tag{107}$$

where we require the optimal controls in feedback (policy) form

$$u^-(t) = -C^-(t)x \ , \quad \alpha^-(t) = -\Delta^-(t)x(t) \ ; \ t \in [t_o,t_f] \tag{108}$$

which results in

$$\frac{1}{2}\,x^TS^-(t)x(t) = \min_{u(.),\alpha(.)} \left[\int_t^{t_f} \frac{1}{2}\,(x^TQx+u^TRu+\alpha^TP\alpha)dt+ \frac{1}{2}\,x^T(t_f)Q_fx(t_f)\right] \tag{109}$$

Because of our assumptions of positive (semi)-definiteness of Q, R, P and Q_f, it is known that $S^-(t)$ exists for all $t \in [t_o,t_f]$ so that (91) is well posed.

In the case of the LE^+G problem the appropriate differential game is noncooperative, namely

$$\underset{u(.)}{\text{Min}}\ \underset{\alpha(.)}{\text{Max}} \int_{t_o}^{t_f} \frac{1}{2}\,(x^TQx+u^TRu-\alpha^TP\alpha)dt+ \frac{1}{2}\,x^T(t_f)Q_fx(t_f) \tag{110}$$

subject to (107). The optimal feedback laws are

$$u^+(t) = -C^+(t)x \quad , \quad \alpha^+(t) = -\Delta^+(t)x \quad ; \quad t \in [t_o,t_f] \tag{111}$$

and

$$\frac{1}{2}x^T(t)S^+(t)x(t) = \min_{u(.)}\max_{\alpha(.)}\left[\int_t^{t_f}\frac{1}{2}(x^TQx+u^TRu-\alpha^TP\alpha)dt + \frac{1}{2}x^T(t_f)Q_fx(t_f)\right] \tag{112}$$

provided that

$$-\dot{S}^+ = Q+S^+A+S^+A-S^+(BR^{-1}B^T-\Gamma P^{-1}\Gamma^T)S^+ \quad ; \quad S^+(t_f) = Q_f \tag{113}$$

has a solution in $[t,t_f]$.

Note that by standard results on Riccati differential equations, (113) has a solution for all $t \in [t_o,t_f]$ if

$$(BR^{-1}B^T-\Gamma P^{-1}\Gamma^T) \geq 0, \quad t \in [t_o,t_f] \tag{114}$$

and so (114) guarantees existence of $J^+(x,t)$; $t \in [t_o,t_f]$. If (114) is not satisfied (say for $\lambda_{min}(P^{-1})$ sufficiently large) then (113) may exhibit a finite escape time ($S(t) \to \infty$ for some $t \in [t_o,t_f]$) which would imply that (110) is unbounded and also that $J^+(x_o;t_o)$ is unbounded.

9. Properties of the Solutions of the Continuous Time $LE^{\pm}G$ Problems

9.1 The LE^-G Problem

As in the discrete time case we have that as $P^{-1} \to 0$; $t \in [t_o,t_f]$ $\{\lambda_{min}(P) \to \infty;\ t \in [t_o,t_f]\}$ the optimal controller tends to that for the LQG problem. As $\lambda_{min}(P^{-1}) \to \infty$; $t \in [t_o,t_f]$ problem (106) becomes singular and care must be taken in studying the limit - see [2] for a careful treatment of the singular case. Using arguments very similar to those given in [2] it is possible to show that as $\lambda_{min}(P^{-1}) \to \infty$: $t \in [t_o,t_f]$,

the limit of S^- must exist, $t \in (t_o,t_f]$. Now if we make the assumption that

$$\Gamma \text{ has rank } n \; ; \; t \in [t_o,t_f] \tag{115}$$

then from (104), (with σ negative) and the fact that the limit of S^-: $t \in (t_o,t_f]$ must exist, it follows that

$$\text{Lim } S^- = 0 \quad ; \quad t \in (t_o,t_f] \tag{116}$$

which tells us that

$$R^{-1}B^TS^- \to 0 \quad ; \quad t \in (t_o,t_f] \tag{117}$$

which is analogous to the discrete time case (71).

9.2 <u>The LE^+G Problem</u>

As $\lambda_{min}(P) \to \infty$, $t \in [t_o,t_f]$ we have that the solution of the LE^+G problem, as in the LE^-G case, tends to the solution of the LQG problem. As noise intensity increases, $\lambda_{min}(P^{-1}) \to \infty$; $t \in [t_o,t_f]$, (114) will cease to be satisfied, and ultimately (113) will exhibit a finite escape time signifying that $J^+(x_o,t_o)$ has ceased to exist; i.e., for sufficiently large noise intensity, performance criterion (96) is unbounded. Note that contrary to the LE^-G case, (117), we have that

$$R^{-1}B^TS^+ \to \infty \quad ; \quad t \in [t_o,t_f] \tag{118}$$

as

$$\lambda_{min}(P^{-1}) \to \infty \; ; \; t \in [t_o,t_f] \tag{119}$$

10. Some Stability Properties of Undisturbed Linear System Controlled by Solution of $LE^{\pm}G$ Problems

In this section we assume that all parameters are time invariant and we investigate, briefly, stability of the system

$$\dot{x} = (A-BC_{\infty}^{\sigma})x \quad ; \quad \sigma \text{ negative or positive .} \tag{120}$$

10.1 Stability Properties of C_{∞}^{-}

Here we assume that the pair (A,B) is controllable and that $Q > 0$. These assumptions guarantee the existence of S_{∞}^{-}, the unique positive definite steady state solution of the Riccati equation. That is, $S_{\infty}^{-} > 0$ satisfies

$$Q+S_{\infty}^{-}A+A^{T}S_{\infty}^{-} - S_{\infty}^{-}(BR^{-1}B^{T}+\Gamma P^{-1}\Gamma^{T})S_{\infty}^{-} = 0 \tag{121}$$

and we have the steady state feedback gain

$$C_{\infty}^{-} = R^{-1}B^{T}S_{\infty}^{-} \tag{122}$$

We now define

$$L^{-} \triangleq \frac{1}{2}\, x^{T}S_{\infty}^{-}x \tag{123}$$

which is positive definite. Along trajectories of (120), we have

$$\dot{L}^{-} = \frac{1}{2}\, x^{T}(S_{\infty}^{-}A+A^{T}S_{\infty}^{-})x - x^{T}S_{\infty}^{-}BR^{-1}B^{T}S_{\infty}^{-}x \tag{124}$$

which, upon using (121), is

$$\dot{L} = -\frac{1}{2}\, x^{T}[Q+S_{\infty}^{-}(BR^{-1}B^{T}-\Gamma P^{-1}\Gamma^{T})S_{\infty}^{-}]\, x \tag{125}$$

Now if

$$BR^{-1}B^{T}-\Gamma P^{-1}\Gamma^{T} \geq 0 \tag{126}$$

we have

$$\dot{L}^{-} < 0, \quad \text{for all } x \neq 0 \tag{127}$$

and system (120), with controller C_∞^-, is asymptotically stable.

Note that simple examples show that (120) can be <u>unstable</u> if condition (126) is violated.

10.2 <u>Stability Properties of C_∞^+</u>

In this case we assume condition (114), namely

$$BR^{-1}B^T - \Gamma P^{-1}\Gamma^T \geq 0 \tag{128}$$

and also that $Q > 0$. Note that because of (128) we can write

$$NN^T \triangleq BR^{-1}B^T - \Gamma P^{-1}\Gamma^T \tag{129}$$

If we assume now that the pair (A,N) is controllable then it follows that there exists a unique positive-definite matrix S_∞^+ which satisfies

$$Q + S_\infty^+ A + A^T S_\infty^+ - S_\infty^+ (BR^{-1}B^T - \Gamma P^{-1}\Gamma^T) S_\infty^+ = 0 \tag{130}$$

and

$$C_\infty^+ = R^{-1}B^T S_\infty^+ \tag{131}$$

Define

$$L^+ \triangleq \frac{1}{2} x^T S_\infty^+ x \ . \tag{132}$$

Along trajectories of (120) we have that

$$\dot{L}^+ = \frac{1}{2} x^T (S_\infty^+ A + A^T S_\infty^+) x - x^T S_\infty^+ BR^{-1}B^T S_\infty^+ x \tag{133}$$

which, upon using (130), is

$$\dot{L}^+ = -\frac{1}{2} x^T [Q + S_\infty^+ (BR^{-1}B^T + \Gamma P^{-1}\Gamma^T) S_\infty^+] x \tag{134}$$

$$< 0 \text{ for all } x \neq 0 \ . \tag{135}$$

Here, L^+ is a Liapunov function and (120) with controller C_∞^+ is asymptotically stable. Note the interesting point that (126) is sufficient to guarantee asymptotic stability of (120) with controllers C_∞^- or C_∞^+. In the first case, (126) is used to guarantee negativity of $\dot{L}^-$ while in the second it is used to guarantee existence of S_∞^+.

11. Interpretation of Stability Results in Terms of Infinite Time $LE^{\pm}G$ Problems

Clearly, from (103), (105)

$$J^-(x,t) \to 0 \text{ as } t \to -\infty \tag{136}$$

and

$$J^+(x,t) \to \infty \text{ as } t \to -\infty \tag{137}$$

In order for $LE^{\pm}G$ problems to make sense, therefore, we define our infinite time criterion as

$$\lim_{t\to-\infty}\left[\underset{u(.)}{\sigma\mathrm{Min}}\ \mathscr{E}_{|x(t)}\sigma\exp\{\sigma[\ \frac{1}{2}\int_t^{t_f}(x^TQx+u^TRu)dt+x^T(t_f)Q_fx(t_f)]\}\right]^{\frac{1}{\sigma(t_f-t)}} \tag{138}$$

Note that from (103), (105) (138) is equal to

$$\exp\{\frac{n}{2}\mathrm{tr}(S_\infty^\sigma\Gamma P^{-1}\Gamma^T)\}. \tag{139}$$

In the case where σ is negative and the noise intensity is large an unstable control law may be optimal because in (138) the quantity whose expected value is calculated is bounded below by minus one and above by zero regardless of the control that is applied.

Note that when σ is positive an unstable control law cannot be optimal because the quantity whose expected value is calculated is

unbounded; this is confirmed by (135) which indicates that if an optimal controller exists for the infinite time $LE^{+}G$ problem it must be stable.

12. Conclusion

In this paper we have presented explicit (modulo solution of Riccati difference or differential equations) solutions of stochastic control problems having linear dynamics, additive gaussian noise and exponential objective functions. These solutions are linear feedback control policies which depend upon the covariance matrix of the additive process noise so that the Certainty Equivalence Principle of LQG theory does not hold. In certain applications these new controllers may be preferable, especially perhaps in economics where multiplicative objective functions are of intrinsic interest.

By demonstrating certain equivalences between our stochastic control formulations and deterministic differential games we are able to give a stochastic interpretation to min-max ("worst case") design of linear systems. This suggests that the "pessimistic" min-max design is not unattractive since it corresponds, in a stochastic setting, to minimization of the expected value of an exponential function of a quadratic form, which is quite an appealing criterion. Another significant result of these equivalences is that existence of solutions of the stochastic control problems implies and is implied by existence of solutions of the differential games. Hopefully these notions can be extended to provide existence results for nonlinear stochastic control problems and nonlinear differential games.

Certain stability properties of the steady state solutions of the stochastic control problem are also investigated. In particular, we point out that the steady state controller for the LE^-G problem can result in an unstable dynamic system while the steady state controller for the LE^+G problem, if it exists, always stabilizes the dynamic system. In this sense, the LE^+G formulation is preferable.

Note that we have not considered in this paper the more complex problem in which noisy measurements of the state are made, viz.,

$$z_k = H_k x_k + \beta_k \quad ; \quad k = 0,\ldots,N-1. \tag{140}$$

where $\{\beta_k, \alpha_k, x_o\}$ are independent gaussian random variables. In this case the optimal controls are restricted to be of the form

$$u_k^\sigma = C_k^\sigma(\mathcal{F}_k) \quad ; \quad k = 0,\ldots,N-1, \tag{141}$$

where is - or + and where

$$\mathcal{F}_k \triangleq \{z_o, z_1, \ldots, z_k\} \qquad ; k = 0,\ldots,N-1 \ . \tag{142}$$

The appropriate performance criterion is

$$V^\sigma(z_o) \triangleq \sigma \mathcal{E}_{|z_o} \exp\{\sigma \frac{1}{2} [\sum_{k=0}^{N-1} (x_k^T Q_k x_k + u_k^T R_k u_k) + x_N^T Q_N x_N]\} \tag{143}$$

The above problem appears to be intrinsically much harder than the perfect state case, and could be the topic of a future paper.

References

1. M. Athans, "The Role and Use of the Stochastic Linear-Quadratic-Gaussian Problem in Control System Design," IEEE Trans. Automatic Control, Special Issue on Linear-Quadratic-Gaussian Problem, Vol. 16, Dec. 1971, pp. 529-552.

2. D.H. Jacobson and J.L. Speyer, "Necessary and Sufficient Conditions for Optimality for Singular Control Problems: A Limit Approach," J. Math. Anal. Appl. 34, 1971, pp. 239-265.

13. Acknowledgements

I wish to thank Lotfi Zadeh for stimulating discussions, during the Spring of 1971, on fuzzy set theory. These discussions contributed to the development of certain results in this paper. Critical comments from David Mayne, Larry Ho and Jason Spever are appreciated.

Appendix

Lemma 1 If $P_k+\Gamma_k^T W_{k+1}^- \Gamma_k > 0$, then

$$\int_{-\infty}^{\infty} \frac{1}{\sqrt{(2\pi)^q |P_k^{-1}|}} \exp\{-\tfrac{1}{2}\alpha_k^T P_k \alpha_k\} \cdot \exp\{-\tfrac{1}{2} x_{k+1}^T W_{k+1}^- x_{k+1}\} d\alpha_k$$

$$= \sqrt{\frac{|(P_k+\Gamma_k^T W_{k+1}^- \Gamma_k)^{-1}|}{|P_k^{-1}|}} \exp\{-\tfrac{1}{2} (A_k x_k+B_k u_k)^T \tilde{W}_{k+1}^- (A_k x_k+B_k u_k)\} \qquad (A.1)$$

where $\tilde{W}_{k+1}^-$ is defined in (39).

Proof: The left hand side of (A.1) is, using (1), equal to

$$\int_{-\infty}^{\infty} \frac{1}{\sqrt{(2\pi)^q |P_k^{-1}|}} \exp\{-\tfrac{1}{2} [(A_k x_k+B_k u_k+\Gamma_k \alpha_k)^T W_{k+1}^- (A_k x_k+B_k u_k+\Gamma_k \alpha_k)+\alpha_k^T P_k \alpha_k]\} d\alpha_k$$

$$= \sqrt{\frac{|(P_k+\Gamma_k^T W_{k+1}^- \Gamma_k)^{-1}|}{|P_k^{-1}|}} \exp\{-\tfrac{1}{2} (A_k x_k+B_k u_k)^T \tilde{W}_{k+1}^- (A_k x_k+B_k u_k)\} \cdot$$

$$\int_{-\infty}^{\infty} \frac{1}{\sqrt{(2\pi)^q |(P_k + \Gamma_k^T W_{k+1}^- \Gamma_k)^{-1}|}} \exp\{-\tfrac{1}{2} (\alpha_k - \bar{\alpha}_k)^T (P_k+\Gamma_k^T W_{k+1}^- \Gamma_k)(\alpha_k-\bar{\alpha}_k)\} d\alpha_k \qquad (A.2)$$

where

$$\bar{\alpha}_k \triangleq -(P_k+\Gamma_k^T W_{k+1}^- \Gamma_k)^{-1} \Gamma_k^T W_{k+1}^- (A_k x_k+B_k u_k) \qquad (A.3)$$

The Lemma is proved by (A.2) since the integrand is a probability density function having mean $\bar{\alpha}_k$ and covariance

$$(P_k+\Gamma_k^T W_{k+1}^- \Gamma_k)^{-1} . \tag{A.4}$$

<u>Lemma 2</u>: i) If $(P_k-\Gamma_k^T W_{k+1}^+ \Gamma_k) > 0$ then

$$\int_{-\infty}^{\infty} \frac{1}{\sqrt{(2\pi)^q |P_k^{-1}|}} \exp\{-\frac{1}{2}\alpha_k^T P_k \alpha_k\} . \exp\{\frac{1}{2} x_{k+1}^T W_{k+1}^+ x_{k+1}\} d\alpha_k$$

$$= \sqrt{\frac{|(P_k-\Gamma_k^T W_{k+1}^+ \Gamma_k)^{-1}|}{|P_k^{-1}|}} \exp\{\frac{1}{2}(A_k x_k+B_k u_k)^T \tilde{W}_{k+1}^+ (A_k x_k+B_k u_k)\} \tag{A.5}$$

where $\tilde{W}_{k+1}^+$ is defined in (54).

ii) If $(P_k-\Gamma_k^T W_{k+1}^+ \Gamma_k) \not> 0$, (A.6)

then the left hand side of (A.5) is infinite.

<u>Proof</u>: i) The proof is the same as that of Lemma 1 with W_{k+1}^- replaced by $-W_{k+1}^+$.

ii) We have that

$$\exp\{-\alpha_k^T P_k \alpha_k\} . \exp\{\frac{1}{2} x_{k+1}^T W_{k+1}^+ x_{k+1}\}$$

$$= \exp\{-\frac{1}{2}[\alpha_k^T P_k \alpha_k-(A_k x_k+B_k u_k+\Gamma_k \alpha_k)^T W_{k+1}^+ (A_k x_k+B_k u_k+\Gamma_k \alpha_k)]\} \tag{A.7}$$

and we note that because of (A.6) there exists a direction α_k^* such that the right hand side of (A.7) does not go to zero as $||\alpha_k^*|| \to \infty$. Clearly this implies divergence of the integral on the left hand side of (A.5).

Generalized Hamilton-Jacobi-Bellman Equation

Here we derive equation (98).

From (100) we have that

$$J^{\sigma}(x,t) = \sigma\mathcal{E}|_{x(t)}\exp\{\tfrac{1}{2}\sigma\int_t^{t+\delta}(x^TQx+u^{\sigma T}Ru^{\sigma})dt\}. \tag{A.8}$$

$$\exp\{\tfrac{1}{2}\sigma[\int_{t+\delta}^{t_f}(x^TQx+u^{\sigma T}Ru^{\sigma})dt+x^T(t_f)Q_fx(t_f)]\}$$

$$= \mathcal{E}|_{x(t)}\exp\{\tfrac{1}{2}\int_t^{t+\delta}(x^TQx+u^{\sigma T}Ru^{\sigma})dt\}J^{\sigma}(x+\delta x,t+\delta)$$

$$= \mathcal{E}|_{x(t)}\ [1+\sigma\tfrac{1}{2}(x^TQx+u^{\sigma T}Ru^{\sigma})\delta+\ldots][J^{\sigma}(x,t)+J_x^{\sigma T}(Ax+Bu+\Gamma\alpha)\delta+J_t^{\sigma}\delta$$

$$+\tfrac{1}{2}(Ax+Bu+\Gamma\alpha)^TJ_{xx}^{\sigma}(Ax+Bu+\Gamma\alpha)\delta^2+\ldots] \tag{A.9}$$

Upon taking the expectation and the limit as $\delta \to 0$, we obtain, formally,

$$-\frac{\partial J^{\sigma}}{\partial t}(x,t) = \{\tfrac{1}{2}\sigma(x^TQx+u^{\sigma T}Ru^{\sigma})J^{\sigma}(x,t)+[J_x^{\sigma}(x,t)]^T(Ax+Bu)$$

$$+\tfrac{1}{2}\operatorname{tr}[J_{xx}^{\sigma}(x,t)\Gamma P^{-1}\Gamma^T]\} \tag{A.10}$$

or

$$-\frac{\partial J^{\sigma}}{\partial t}(x,t) = \min_u\{\tfrac{1}{2}\sigma(x^TQx+u^TRu)J^{\sigma}(x,t)+[J_x^{\sigma}(x,t)]^T(Ax+Bu)$$

$$+\tfrac{1}{2}\operatorname{tr}[J_{xx}^{\sigma}(x,t)\Gamma P^{-1}\Gamma^T]\} \tag{A.11}$$

which is equation (98).