DETECTION SCHEME FOR JUMP OF INPUT WITH KNOWN MEAN AND COVARIANCE MATRIX

Probaj Sanyal and C. H. Shen
Rensselaer Polytechnic Institute
Troy, New York

Abstract

This paper discusses the use of Bayes's rule for the detection of the time of application of an impulse input in a system, along with the Kalman Filter algorithm for estimation of the system parameters.

INTRODUCTION

In the tracking of parameters of a system, there are situations when the system undergoes an abrupt change of parameters, but the exact time of this change is not known a priori. This paper assumes that the mean and covariance of the jump of input is known. We use Bayes rule to detect the time of application of this jump.

PROBLEM STATEMENT

Consider a linear system, the dynamics of which is given by

\[ x_t = F(x_{t-1} + 
\]

where \( \delta_t \) implies that this jump quantity applies at a certain time instant \( t \) which is not known a priori. The observations made on the system are also linear functions of the states and are given by

\[ z_t = H x_t + v_t \]

where \( H \) is a known observation matrix and \( v_t \) is a Gaussian measurement noise vector with zero mean and covariance \( R \). It is assumed that a jump, if any, occurs prior to a measurement.

TRACKING

The Kalman Filtering is used to find the estimate \( \hat{x}_t = E(x_t | z_t) \) where \( z_t \) is the collection of observations to the \( t \)th instant, i.e., \( z_t = \{z_1, z_2, ..., z_t\} \).

A standard Kalman Filter algorithm [1] is used. The estimation is started with a given initial estimate \( \hat{x}_0 \). Based on this quantity, two predictions are made for the states at the next stage, one assuming that there has been no jump and the other assuming that there has been a jump before the \( t \)th observation is made. These will be called \( \hat{x}_t^0 \) and \( \hat{x}_t^1 \) respectively, as shown in Fig. 1. The subscript denotes the stage referred to, while the superscript denotes the stage before which jump is supposed to have occurred. The superscript zero denotes that no jump has occurred so far. The superscript 1 indicates jump just before stage 1. Fig. 1 shows how the different estimates may be obtained by assuming the jump to have occurred at different points. Proceeding in this manner, at every stage \( t \), we shall have \( t+1 \) estimates, one of which is more correct than the others. A number of stages can be considered together, to determine whether there has been a jump. A large number of stages will increase the accuracy of detection. But as a tradeoff between computational load and accuracy, only three stages together will be considered here. Thus, there will be no more than four alternate estimates at any stage.

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** Professor of Mechanical Engineering
PROBABILITY DENSITY FUNCTION OF $x_1$

The observation $x_1$ is a Gaussian random vector\[^{[2]}\].

The mean and covariance are easily shown to be

$$E(x_1) = \mu_1$$

and

$$\text{cov}(x_1) = H_1 \Sigma_{\nu_0} H_1^T + R_1$$

where $\Sigma_{\nu_0}$ is the pre-measurement covariance of $x$.

With the third measurement, consider the joint probabilities $p(x_2, x_3, x_4)$, $p(x_1, x_2, x_3)$, $p(x_1, x_2, x_4)$ and $p(x_1, x_2, x_3, x_4)$, one of which will be highest depending on when the jump occurred or didn't occur at all\[^{[3]}\].

We shall write $x_1 = H_1 \nu_1 + \nu_1$, where the superscript $j$ on $x_j$, $\nu_j$, and $H_j$ denotes that the jump is supposed to have occurred just before the $j^{\text{th}}$ measurement.

JOINT PROBABILITIES AND DETECTION

With the third measurement, consider the joint probabilities $p(x_2, x_3, x_4)$, $p(x_1, x_2, x_3)$, $p(x_1, x_2, x_4)$ and $p(x_1, x_2, x_3, x_4)$, one of which will be highest depending on when the jump occurred or didn't occur at all\[^{[3]}\].

Also, it is expected that the probabilities will be graded according to the nearness of the hypothesis to the true hypothesis. This allows one to compare two adjacent probabilities at a time instead of all four. For example, compare $p(x_1, x_2, x_3)$ and $p(x_1, x_2, x_4)$. If the former is larger, then conclude that the jump was indeed before the 1st measurement. If not, then process one more measurement and make similar comparisons for the next set of joint probabilities which does not include $p(x_1, x_2, x_3)$, and so on.

EVALUATION OF JOINT PROBABILITIES

Since the observations are statistically independent\[^{[3]}\], the joint probabilities are the products of the individual probabilities. We use the logarithm of the probabilities, for convenience in computation. Thus, one needs compute, with each new estimate, the following quantity:

$$C \ln \left( \frac{1}{k=1} (x_k^T H_k H_k^T + R_k)^{-1} (x_k - H_k \nu_k)^T \Sigma_k H_k H_k^T \right)$$

where $C$ is a negative constant for different $j$'s.

The $x_k$ is obtained by Kalman filtering according to Fig. 1. For a certain $j$, the negative of the logarithm of the joint probabilities, i.e., the right side of the above equation is the smallest. That $j$ denotes the time instant of jump. Note, that any $j$, the only $j$'s considered are $j = 1, 2, 3$ and 3, since the likelihood of a jump at an earlier stage has already been discarded in the previous step. Thus at any stage, there are no more than 4 estimates to be made and four sums to be updated. This restricts the size of memory required.

EXAMPLE

A simple example was chosen with $P=x^2$, $r=\tau$, $H=0$, $H=0.02x$ and $x_0(222)$. Kalman filtering was carried out 30 stages. The jump was introduced at one of the 30 stages. The probability of success, as obtained from computer runs, is given in Table 1.

CONCLUSIONS

This paper gives a scheme for detecting a sudden change in a system. A previous paper by Schagiri Prabhu\[^{[3]}\] does not involve any filtering, the present scheme does. A major difference between the two is, thus, that while the former assures known distributions of the measurements, this scheme recalculates the distribution at every stage. The scheme, as tried, was quite successful, though naturally sensitive to initial errors of estimation.

REFERENCES

### TABLE 1

**RESULTS OBTAINED FROM COMPUTER RUNS**

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<th>Sr.No.</th>
<th>Value of $\alpha$</th>
<th>Value of $\Omega$</th>
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**FIGURE 1**

**DIAGRAM SHOWING THE ALTERNATIVE ESTIMATES**