

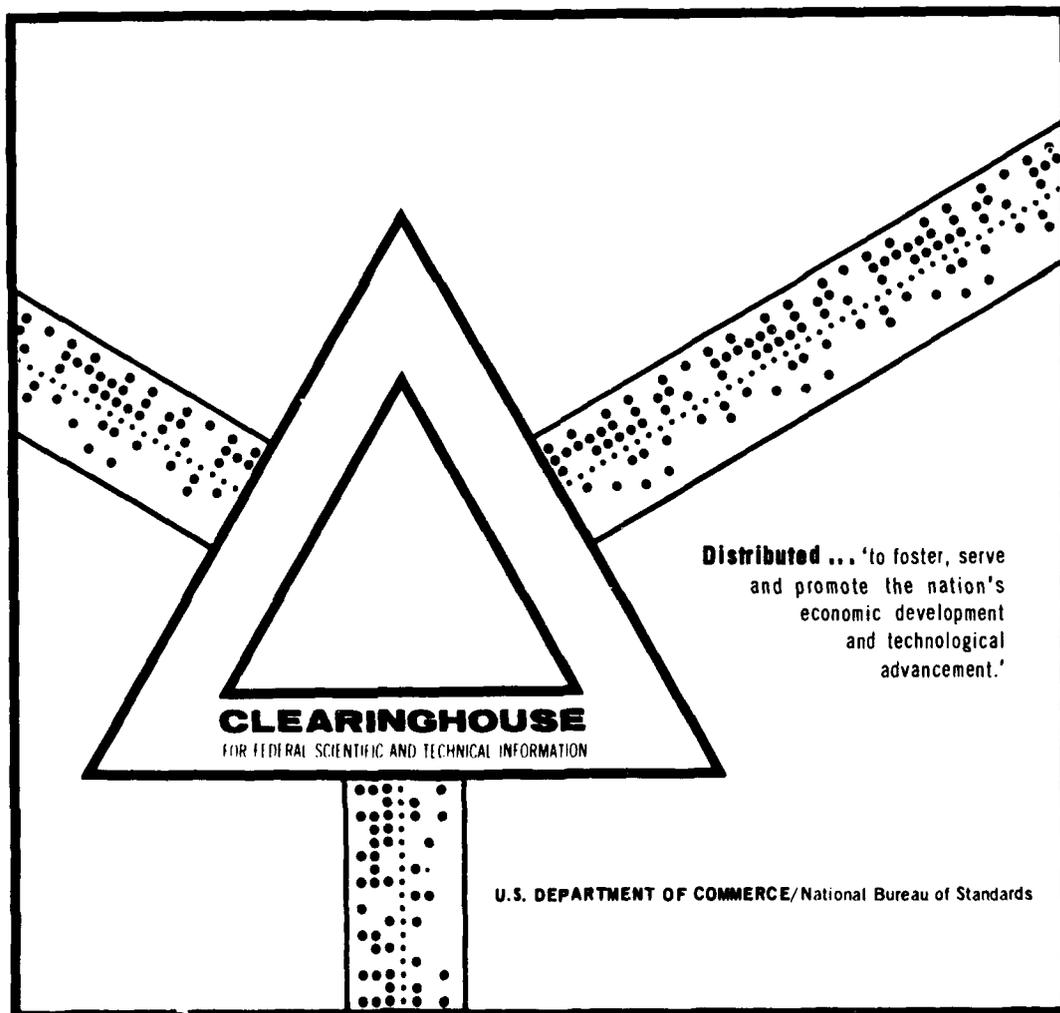
AD 700 353

SMOOTHNESS OF SOLUTIONS OF DEGENERATING ELLIPTIC EQUATIONS

M. F. Freidlin

Johns Hopkins University  
Silver Spring, Maryland

22 September 1969



This document has been approved for public release and sale.



THE JOHNS HOPKINS UNIVERSITY  
**APPLIED PHYSICS LABORATORY**

8621 Georgia Avenue • Silver Spring, Maryland 20910

TRANSLATIONS

CLB 3 - T604

22 September 1969

AD 700353

SMOOTHNESS OF SOLUTIONS OF DEGENERATING ELLIPTIC EQUATIONS

by

M. F. Freidlin

translated from

Akademiya Nauk SSSR, Izvestiya, Seriya matematicheskaya  
[Academy of Sciences USSR, Bulletin, Mathematics Series]  
Vol. 32, No. 6, pp. 1391-1413 (1968)

by

A. I. Schidlovsky

D D C  
RECEIVED  
APR 1 1970  
RECEIVED  
C

Summary

In this study it is proven that the generalized solution of boundary-value problems for degenerating second-order equations satisfies Holder's boundary condition under broad assumptions. An example is given, showing that a greater smoothness of the solutions cannot be achieved only by increasing the smoothness of the data given in the problem. Conditions for the existence of derivatives of the generalized solution are clarified. The investigation is carried out by means of probability methods.

Reproduced by the  
CLEARINGHOUSE  
for Scientific, Technical, & Information  
Reports, Springfield, VA 22151

This document has been approved for public release and sale.  
Its distribution is unlimited.

# SMOOTHNESS OF SOLUTIONS OF DEGENERATING ELLIPTIC EQUATIONS\*

by

M. F. Freidlin

## Introduction

The present study is concerned with boundary-value problems for the degenerating elliptic equation

$$Lu(x) + c(x)u(x) = \frac{1}{2} \sum_{i,j=1}^n a_{ij}(x) \frac{\partial^2 u}{\partial x_i \partial x_j} + \sum b_i(x) \frac{\partial u}{\partial x_i} + c(x)u(x) = 0. \quad (1)$$

It is assumed that the coefficients are defined over all spaces  $R_n$ , are bounded and have bounded first-order derivatives

$$\max_{1 \leq i, j, k \leq n, x \in R^n} \left\{ \left| \frac{\partial a_{ij}(x)}{\partial x_k} \right|, \left| \frac{\partial b_i(x)}{\partial x_k} \right|, \left| \frac{\partial c(x)}{\partial x_k} \right| \right\} < K < \infty;$$

the form  $\sum_{i,j=1}^n a_{ij}(x) \lambda_i \lambda_j$  is considered non-negative. The study of boundary-value problems for degenerating equations has been the object of numerous investigations. References [1] and [11] can be mentioned, where equations degenerating on the boundary of a region are studied. Special cases of degeneration inside a region are examined in references [12-15]. The boundary-value problem for an equation with degeneration of a general form was examined for the first time by Fichera (see [4,5]). However, a single class of functions, in which the problem would have a unique generalized solution, was not constructed in these studies. In a subsequent study [12], a uniqueness class was constructed and conditions were

---

\* Translated from Akad. Nauk SSSR, Izv., Ser. mat. [Acad. Sci. USSR, Bull., Math. Ser.], Vol. 32, No. 6, pp. 1391-1413 (1968).

established for equations with degeneration of a rather special form, under which a generalized solution will be smooth. In the general case, the existence and uniqueness of a generalized solution in one class were proven in [17] and [18] (see also [9]). Note that in Fichera's study [4] and others associated with it the assumption is made that  $c(x) < -c_0 < 0$ . This assumption removes certain interesting effects on the boundary (pasting part of the boundary) at one point [11] and makes the problem of uniqueness more simple. If this assumption is rejected, in order to ensure the uniqueness of the generalized solution it is necessary, in general, to prescribe boundary conditions on a certain "inner boundary" (see [22]). This "inner boundary" arises naturally when stabilization of the solution of the mixed problem is studied as  $t \rightarrow \infty$ .

In [20] we find conditions which guarantee that the generalized solution, in the case of degeneracy of a general form, is smooth and satisfies the Hölder condition. Proof of the statements formulated in [20] constitutes precisely the basic content of the present study. The smoothness of generalized solutions was also studied in references [9] and [10].

Generally speaking, the generalized solution of the Dirichlet problem for equation (1) will be discontinuous. It was found that, under very general assumptions in regard to the coefficients and the boundary function, the Hölder character of the generalized solution can be guaranteed (see example and Theorem 1). In order that the solution should have derivatives, it is no longer sufficient to require only that the data of the problem be smooth, but it is also necessary to impose inequality-type conditions on the coefficients and on their first derivatives. In

order to obtain a priori estimates which ensure smoothness of the generalized solution we shall use a representation of the solution of the Dirichlet problem as the mean value of a certain functional of trajectories of a random Markov process controlled by the operator  $L$  (see [2], [8]). We believe that such an approach to the problem is a geometrically descriptive and natural one. In particular, the conditions of the existence of a priori estimates of derivatives of the generalized solution acquire a simple geometrical meaning. In the present study we shall use the same notation as that used in reference [18].

### 1. Generalized Solution of the Dirichlet Problem

By  $\sigma(x)$  we shall denote a matrix such that  $\{a_{ij}(x)\} = a(x) = \sigma(x) \sigma^*(x)$  (the star means transposition). Let us assume that the matrix  $\sigma(x)$  can be chosen in such a way that all its elements are differentiable, and let

$$\max \left| \frac{\partial \sigma_{ij}(x)}{\partial x_k} \right| < K.$$

According to the condition,  $a(x)$  is a symmetric positive-definite matrix, and therefore we will always find a  $\sigma(x)$  satisfying the relation  $\sigma(x) \sigma^*(x) = a(x)$ . Our assumptions refer only to the smoothness of the functions  $\sigma_{ij}(x)$  with respect to  $x$ . This problem was investigated in [25], where it was shown, in particular, that if  $a_{ij}(x) \in C^2(\mathbb{R}^n)$  then, regardless of the form of degeneracy, the matrix  $\sigma(x)$  can always be chosen in such a way that its elements satisfy the Lipschitz condition, where the Lipschitz constant is estimated in terms of the norm  $a_{ij}(x)$  in  $C^2(\mathbb{R}^n)$ .

If the rank of  $\{a_{ij}(x)\}$  is constant, the elements of  $\sigma(x)$  have the same smoothness as the functions  $a_{ij}(x)$ .

We shall examine the stochastic differential equation (see [6,8])

$$dx_t(\omega) = \sigma(x_t) d\xi_t(\omega) + b(x_t) dt, \quad (2)$$

where  $\xi_t(\omega)$  is an  $n$ -dimensional Wiener process (see [2],  $b(x) = (b_1(x), \dots, b_n(x))$ ). This equation, together with the initial condition  $x_0(\omega) = x \in \mathbb{R}^n$ , has the unique solution  $x_t^x(\omega)$ , which defines the Markov process  $X = \{x_t, P_x\}$  (see [2]). The process  $X$  is said to be governed by operator  $L$ . The properties of solutions of boundary-value problems for operator  $L$  are closely associated with the behavior of the trajectories of process  $X$ .

Let  $D$  be a bounded region in  $\mathbb{R}^n$  with the boundary  $\Gamma$ . The point  $x_0 \in \Gamma$  will be called regular if

$$\lim_{\epsilon \rightarrow 0} P_x\{x_{\tau_\epsilon} \in \Gamma\} = 1$$

for any  $\epsilon > 0$ , where  $\tau_\epsilon$  is the moment at which the trajectory of process  $X$  first reaches the boundary of the set  $U_\epsilon(x_0) \cap D$ . Here, as well as in what follows, we denote by  $U_\epsilon(x_0)$  the  $\epsilon$ -neighborhood of the point  $x_0$ . We can prove the following lemma (see [11]):

Lemma 1. In order that point  $x_0 \in \Gamma$  be regular for the process  $X$ , controlled by operator  $L$ , it is sufficient that there exist a function  $v(x)$  (barrier) such that  $v(x)$  is continuous in a certain neighborhood  $U$  of the point  $x_0$  and that  $v(x_0) = 0$ ,  $v(x) > 0$  and  $Lv(x) \leq 0$  for  $x \in U \setminus x_0$ .

The point  $x_0$  will be called  $(\lambda_1, \lambda_2, h)$ -regular if there exists a function  $v(x)$  with properties specified in lemma 1 such that

$$c_1|x - x_0|^{\lambda_1} \leq v(x) \leq c_2|x - x_0|^{\lambda_2} \text{ for } x \in U_\lambda(x_0).$$

The following lemma gives sufficient conditions of  $(\lambda_1, \lambda_2, h)$ -regularity in terms of the coefficients.

Lemma 2. Let us assume that at least one of the following conditions is fulfilled:

1) the coefficients of the operators and the direction cosines  $\{n_i(x)\}$  of the normal to  $\Gamma$  belong to class  $C^2$  in the neighborhood of the point  $x_0$  and

$$\sum_{i,j=1}^n a_{ij}(x_0) n_i(x_0) \times n_j(x_0) > 0;$$

2) the point  $x_0$  can be touched by the half-space from outside the region D and  $(b(x_0), n(x_0)) > 0$ , where  $n(x_0)$  is the outward normal to the support hyperplane.

Then, the point  $x_0 \in \Gamma$  is  $(\lambda_1, \lambda_2, h)$ -regular for some  $\lambda_1, \lambda_2, h > 0$ .

Proof. Let the first condition be fulfilled. Let us introduce the coordinates  $x'_1, \dots, x'_n$  in such a way that the boundary of region D in the neighborhood of  $x_0$  be described by equation  $x'_0 = 0$  and that region D lie in the half-space  $x'_1 < 0$ . In this coordinate system  $a'_{11}(x_0) > 0$ . The function  $v(x)$  can be chosen in the form

$$v(x) = -ax'_1 - bx'_1 + c \sum_{i=2}^n (x'_i)^2.$$

It is easy to verify that the function  $v(x)$  is the barrier being sought in case of a proper selection of coefficients  $a, b, c$  in a sufficiently small neighborhood of the point  $x_0$ :

If condition 2) is fulfilled, we perform a linear transformation of variables in which the half-space, by means of which the point  $x_0$  can be touched from the

outside, will be described by the equation  $x_1^1 = 0$ . Since in the case of linear transformation of independent variables  $b(x) = \{b_1(x), \dots, b_n(x)\}$  is transformed as a vector, we have  $b_1^1(x_0) > 0$ . In this case, the function  $v(x)$  can be chosen in the form

$$v(x) = -ax_1^1 + b \sum_{i=2}^n (x_i^1)^2.$$

Note that in both cases  $\lambda_1 = \lambda_2 = 1$ .

Remark 1. If the boundary and the coefficients in the neighborhood of point  $x_0$  are smooth, the requirement that point  $x_0$  can be touched by a half-space from outside the region  $D$  can be replaced by the assumption that point  $x_0$  belongs to the closure of the set open on  $\Gamma$ , in which

$$\sum_{i,j=1}^n a_{ij}(x) n_i(x) n_j(x) = 0.$$

The point  $x_0 \in \Gamma$  is called normally regular if  $M_x \tau \leq |x - x_0|$ , where  $\tau$  is the instant of the first exit from region  $D$ :  $\tau = \inf \{t: x_t \in D\}$ .

Lemma 3. If the conditions of lemma 2 are fulfilled, then the point  $x_0$  is normally regular.

For the proof we can make use of the barriers introduced in lemma 2. Indeed, if we denote by  $\alpha$  the characteristic operator of process  $X$  (see [2]), we have

$$\mathbb{M}_x \tau = -1, \quad \mathbb{M}_x \tau |_{x \in \Gamma} = 0, \quad \mathbb{M} v(x) = Lv(x) = -c < 0.$$

Then  $\frac{A}{c} v(x) - M_x \tau \geq 0$  on the boundary of region  $U_h(x_0)$  for sufficiently large  $A$  and

$$\mathbb{M} \left( \frac{A}{c} v(x) - M_x \tau \right) = -A + 1.$$

Consequently, by virtue of the maximum principle

$$M_{x\tau} < \frac{A}{c} v(x) < c_1 |x - x_0|.$$

If  $M_{x\tau} < c_1 |x - x_0|$  for any point  $x_0 \in \gamma \subset \Gamma$ , we shall call the set  $\gamma$  uniformly normally regular.

The set  $\gamma \subset \Gamma$  is called repelling if

$$P_x \left\{ \lim_{t \rightarrow \tau} \rho(x_t, \gamma) > 0 \right\} = 1$$

for any  $x \in D$ ,  $\tau = \inf \{t: x_t \in \bar{D}\}$ . It is possible to give sufficient conditions in order that the set  $\gamma \subset \Gamma$  be repelling in terms of the coefficients (see [11]).

Let us recall the definition of the generalized solution (see [18]). Let  $\tilde{X} = \{\tilde{x}_t, \tilde{P}_x\}$  be the Markov process obtained on  $X$  by a stop on the boundary  $\Gamma$  of region  $D$ , and let  $\tilde{T}_t$  be the semi-group of operators which acts in the space  $B$  of bounded measurable functions on  $D \cup \Gamma$  according to the formula

$$T_t f(x) = M_x f(\tilde{x}_t).$$

By  $\tilde{\Gamma}$  we shall denote a part of the boundary  $\Gamma$  of region  $D$  such that  $\Gamma \setminus \tilde{\Gamma}$  is a repelling set. We shall assume that all points of  $\tilde{\Gamma}$  are regular.

As a generalized solution of the problem

$$Lu(x) = 0, \quad u(x)|_{x \in \tilde{\Gamma}} = \psi(x) \quad (3)$$

we shall call the function  $u(x)$ , which assumes boundary values at points of  $\tilde{\Gamma}$  where

$\psi(x)$  is continuous and which satisfies the relation (see Note)  $\tilde{T}_t u(x) = u(x)$  for all  $t \geq 0$ . The correctness of such a definition is proven in [18].

It can be verified (see, for example, [18]) that the function  $u(x) = M_x \psi(x_\tau)$  ( $\tau = \inf \{t: x_t \notin D\}$ ) satisfies the relation  $\tilde{T}_t u = u$  and boundary conditions on  $\tilde{\Gamma}$ . Without additional assumptions, this solution, generally speaking, is not unique.

It can be proven (see [18]) that for uniqueness it is necessary and sufficient that  $P_x \{\tau = \infty\} = 0$ . Everywhere in what follows, we assume the fulfillment of a somewhat stronger condition of a uniformly rapid exit of the trajectories onto the boundary:

$$\lim_{t \rightarrow \infty} P_x \{\tau > t\} = 0$$

uniformly with respect to  $x \in D$ . The latter equality will be fulfilled, for example, if at least one coefficient of operator  $L$  is different from zero in the entire region  $D \cup \Gamma$  (see [18]).

Lemma 4. If the trajectories of process  $X$  leave region  $D$  with a uniform speed, then an  $\alpha_{L,D} > 0$  can be found such that

$$P_x \{\tau > t\} < ce^{-\alpha_{L,D} t}$$

for any  $x \in D$ .

Proof. Since the trajectories reach the boundaries at a uniformly rapid speed, for a certain  $t_0$  we have, for all  $x \in D$ ,

$$P_x \{\tau > t_0\} \leq \beta < 1.$$


---

Note: This relation is equivalent to the equality  $Au(x) = 0$ , where  $A$  is an infinitesimal operator of the semi-group  $T_t$ .

We denote

$$a(n) = \sup_{x \in D} P_x \{ \tau > nt_0 \}. \quad (4)$$

Using the Markov property of process X, we get

$$\begin{aligned} P_x \{ \tau > nt_0 \} &= M_x \chi_{\tau > t_0}(\omega) \cdot M_{x_{t_0}} \chi_{\tau > t_0(n-1)} \leq \\ &\leq (M_x \chi_{\tau > t_0} \cdot M_x (M_{x_{t_0}} \chi_{\tau > t_0(n-1)})^2)^{1/2} \leq \\ &\leq (P_x \{ \tau > t_0 \} \cdot a^2(n-1))^{1/2} \leq \sqrt{\beta a(n-1)}, \end{aligned}$$

where  $\chi_{\tau > s}(\omega)$  is the characteristic function of the set  $\{\omega: \tau(\omega) > s\}$ . From (4) we conclude that  $a(n) < \sqrt{\beta a(n-1)}$ , consequently  $a(n) < e^{n \ln \sqrt{\beta}}$ . From the latter inequality follows the statement of the lemma for

$$\alpha_{L,D} = -\frac{1}{2} \ln \beta > 0.$$

Corollary. If the conditions of lemma 4 are fulfilled, the random variable  $\tau$  has all moments and

$$M_{x\tau^k} < c \int_0^\infty x^k e^{-\alpha_{L,D} x} dx. \quad M_x \exp \{ \lambda \tau \} < \infty \text{ for } \lambda < \alpha_{L,D}.$$

It is not difficult to find the lower bound for the constant  $\alpha_{L,D}$  in terms of the coefficients of the operator and the dimensions of region D. For example, if the diameter of region D is equal to d,  $b_1(x) > b > 0$ ,  $0 \leq a_{11}(x) < a < \infty$ , then

$$\begin{aligned} P_x \{ \tau > t \} &\leq P_x \left\{ \sum_{i=1}^t \sigma_{11}(x_i) d \xi_i^1 > bt + d \right\} \leq \\ &\leq P \left\{ \int_0^t a_{11}(x_s) ds > bt + d \right\} \leq P \left\{ \xi_{bt} > \frac{bt+d}{2} \right\} \leq ce^{-\frac{b}{2a} t}. \end{aligned}$$

Note that in this case  $\alpha_{L,D}$  is estimated by a constant which does not depend on the diameter of the region.

If  $a_{11}(x) > a > 0$  everywhere in  $D$  and  $|b_1(x)| < b < \infty$ , then, during the time  $t = 1$ , the trajectories originating from any point  $x \in D$  leave the region  $D$  with a probability greater than  $\beta = P\{\xi_a > b + d\} = P\{\xi_1 > \frac{b+d}{\sqrt{a}}\}$ . Hence it follows (see proof of lemma 4) that

$$\alpha_{L,D} \geq -\frac{1}{2} \ln \beta.$$

If, for a certain  $i$ ,  $a_{ii}(x) \equiv 0$  everywhere in region  $D$ ,  $b_i(x) \neq 0$  in  $D \cup \Gamma$ , and region  $D$  lies between the planes  $x_i = a$ ,  $x_i = a + h$ , then  $\alpha_{L,D} = \infty$ . This follows from the fact that, in this case, a determinate motion takes place along the axis  $x_i$  with a nonvanishing speed.

In concluding this section we shall note that in the nondegenerate case the limit

$$\lim_{t \rightarrow \infty} \frac{1}{t} \ln P_x^{-1}\{\tau > t\}$$

exists, does not depend on  $x$  and is equal to the first eigenvalue of the problem. In the degenerating case,

$$\lim_{t \rightarrow \infty} \frac{1}{t} \ln P_x^{-1}\{\tau > t\}$$

has a similar meaning but, for sufficiently strong degeneration, this limit depends on the initial point  $x$ . The assumption that trajectories leave the region with a uniform speed ensures the inequality

$$\inf_{x \in D} \lim_{t \rightarrow \infty} \frac{1}{t} \ln P_x^{-1}\{\tau > t\} > 0.$$

## 2. Example. Idea of the Proof

In this section we shall give an example showing that the estimates of the smoothness of the generalized solution, which will be presented in subsequent sections, cannot be improved substantially. Using this example, it is also possible to grasp those ideas which are used in studying smoothness in a general case.

Let region  $D$  be a square:  $D = \{x, y: |x| < 1, |y| < 1\}$ . By  $\varphi(x, y)$  let us denote an infinitely differentiable function, which is even with respect to  $y$  and equal to zero outside the  $\epsilon$ -neighborhood of the boundary  $\Gamma$  of region  $D$ . Let us examine the Dirichlet problem

$$\begin{aligned} Lu = \alpha \frac{\partial^2 u}{\partial x^2} + \beta y \frac{\partial u}{\partial y} + \varphi^2(x, y) \left( \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} \right) &= 0, \\ u(x, y)|_{\Gamma} &\equiv y. \end{aligned} \quad (5)$$

Operator  $\mathcal{L}$  does not degenerate in the neighborhood of the boundary, and the coefficient  $\alpha > 0$  in  $D \cup \Gamma$ ; therefore, problem (5) has a unique generalized solution. This solution is continuous in the closed region  $D \cup \Gamma$  (see [18]). Since operator  $\mathcal{L}$  does not change when  $y$  is replaced by  $-y$ , and the boundary function is multiplied by  $-1$ , we have  $u(x, y) = -u(x, -y)$  and  $u(x, 0) = u(0, 0) = 0$ . From the maximum principle and the regularity of boundary points it follows that  $\text{sign } u(x, y) = \text{sign } y$  and  $u(x, y) > \frac{99}{100}$  ( $< -\frac{99}{100}$ ) for  $1 - y \leq \epsilon$  (when  $1 + y \leq \epsilon$ ) and for a sufficiently small  $\epsilon > 0$ .

By  $z_t = (x_t, y_t)$  let us denote the trajectories of the Markov process in  $\mathbb{R}^2$  controlled by operator  $\mathcal{L}$  and by  $D_\epsilon$  the region obtained from  $D$  by deleting the

$\epsilon$ -neighborhood of the boundary. The trajectories  $z_t$  can be constructed with the aid of the stochastic equations

$$x_t - x = \int_0^t \sqrt{\alpha + \varphi^2(x_s, y_s)} d\xi_s^1, \quad y_t - y = \int_0^t \varphi(x_s, y_s) d\xi_s^2 + \int_0^t \beta y_s ds, \quad (6)$$

where  $\xi_s^1, \xi_s^2$  are independent Wiener processes and  $(x, y)$  is the initial point. The solution  $u(x, y)$  of problem (5) satisfies the relation  $u(x, y) = M_{(x, y)} u(z_{\tau_\epsilon})$ , where  $\tau_\epsilon$  is the instant of the first exit of trajectories  $z_t$  from the region  $D_\epsilon$ . Let  $y_0 > 0$ . Since  $P_{x, y_0} \{y_{\tau_\epsilon} > 0\} = 1$ , we have

$$u(0, y_0) = M_{(0, y_0)} u(z_{\tau_\epsilon}) \geq \frac{99}{100} P_{0, y_0} \{y_{\tau_\epsilon} = 1 - \epsilon\}. \quad (7)$$

Let us estimate the expression on the right-hand side of inequality (7). The motion along axis  $y$  in  $D_\epsilon$  is determinate. Integrating equation  $\frac{dy}{dt} = \beta y$  with the initial condition  $y(0) = y_0$  we find that the time

$$t(y_0) = \ln \left( \frac{1 - \epsilon}{y_0} \right)^{1/\beta}$$

is required in order to reach the point  $y = 1 - \epsilon$ .

Note that

$$P_{(0, y_0)} \{y_{\tau_\epsilon} = 1 - \epsilon\} = P_{(0, y_0)} \left\{ \sup_{s < t(y_0)} |x_s| < 1 - \epsilon \right\}.$$

Since  $x_s$  in  $D_\epsilon$  is a one-dimensional Markov process with the derivative operator  $\alpha \frac{d^2}{dx^2}$ , the function

$$v(t, x) = P_x \left\{ \sup_{s < t} |x_s| < 1 - \epsilon \right\}$$

is the solution of the mixed problem

$$\frac{\partial v}{\partial t} = \alpha \frac{\partial^2 v}{\partial x^2}, \quad v(t, x) |_{|x|=1-\epsilon} = 0, \quad v(0, x) = 1.$$

Solving this problem (for example, by Fourier's method) we obtain that

$$v(0, t) = c_1 \exp \left\{ - \frac{\alpha \pi^2 t}{(1 - \epsilon)^2} \right\}$$

and, consequently,

$$P_{(0, y_0)} \left\{ \sup_{s < t(y_0)} |x_s| < 1 - \varepsilon \right\} = v(t(y_0), 0) = \\ = c_1 \exp \left\{ - \frac{\alpha \pi^2}{(1 - \varepsilon)^2} \ln \left( \frac{1 - \varepsilon}{y_0} \right)^{1/\beta} \right\} = c_2 y_0^{\frac{\alpha \pi^2}{\beta(1 - \varepsilon)^2}},$$

where  $c_1$  and  $c_2$  do not depend on  $y_0$ . Comparing the equality obtained with (7) we conclude that

$$u(0, y_0) > c_2 \cdot \frac{q_0}{100} \cdot y_0^{\frac{\alpha \pi^2}{\beta(1 - \varepsilon)^2}}.$$

From the latter inequality, if we take into account that  $u(0, 0) = 0$ , it follows that the function  $u(x, y)$  may not have derivatives at the point  $(0, 0)$  without additional assumptions concerning the magnitude of  $\alpha/\beta$ , and that for any  $\gamma > 0$  it is possible to find an  $\alpha/\beta$  so small that the function  $u(x, y)$  will not satisfy Hölder's condition with the index  $\gamma$ .

We have introduced the term  $\varphi^2(x, y)\Delta$  into the operator  $\ell$  only in order to exclude the possibility of disturbing the smoothness at the expense of degeneration on the boundary. If we set  $\varphi(x, y) \equiv 0$ , all estimates for  $|u(0, y) - u(0, 0)|$  will remain in this case and it is only necessary to substitute  $\varepsilon = 0$  into them.

We now obtain the upper bound for  $|u(0, y_0) - u(0, 0)|$ . Since, for  $\varphi(x, y) = 0$ , motion along axis  $y$  is determinate in the entire region  $D$ , we have

$$|u(0, y_0) - u(0, 0)| = |u(0, y_0)| = M_{0, y_0} |y_{\tau_D}| \leq \\ \leq M_{0, y_0} |y_{\tau}| = \int_0^{\infty} y_{\tau}(y_0) p_{\tau}(t) dt, \quad (8)$$

where  $y_{\tau}(y_0)$  is the solution of equation  $dy/dt = \beta y$  with the condition  $y(0) = y_0$ , while  $p_{\tau}(t)$  is the density of the random variable  $\tau = \inf\{t: |x_t| = 1\}$ , starting

from the point  $x = 0$ ,  $\tau_D = \inf\{t: z_t \in D\}$ . This density exists since  $x_t$  differs from the Wiener process only by a factor. From the formulas given earlier it follows that

$$y_t(y_0) = y_0 \exp\{\beta t\}, \quad p_t(t) = -\frac{\partial v}{\partial t}(0, t) = c_2 e^{-\alpha \pi^2 t}.$$

Substituting these formulas into (8) we get

$$|u(0, y_0) - u(0, 0)| \leq c_3 y_0 \int_0^{\infty} e^{\beta t - \alpha \pi^2 t} dt.$$

If  $\beta - \alpha \pi^2 < 0$ , the integral on the right-hand side of the last inequality converges and the function  $u(x, y)$  at the point  $(0, 0)$  satisfies the Lipschitz condition with respect to  $y$ . If  $\beta - \alpha \pi^2 > 0$ , then  $u(x, y)$  does not satisfy at zero Hölder's condition with any index  $\kappa < \min(1, \frac{\alpha \pi^2}{\beta})$ . Indeed, since  $|y_{\tau_D}| < 1$ , we will get, for  $\kappa < 1$

$$\begin{aligned} |u(0, y_0) - u(0, 0)| &= M_{(0, y_0)} |y_{\tau_D}| \leq M_{(0, y_0)} |y_{\tau_D}|^{\kappa} \leq \\ &\leq M_{(0, y_0)} |y_{\tau_D}|^{\kappa} = \int_0^{\infty} y_t^{\kappa}(y_0) p_t(t) dt = c_3 y_0^{\kappa} \int_0^{\infty} e^{(\kappa \beta - \alpha \pi^2) t} dt \leq c_4 c_3 y_0^{\kappa}, \end{aligned}$$

where

$$c_4 = \int_0^{\infty} e^{t(\kappa \beta - \alpha \pi^2)} dt < \infty \quad \text{for } \kappa < \frac{\alpha \pi^2}{\beta}.$$

Thus,

$$c_2 y_0^{\frac{\alpha \pi^2}{2}} < |u(0, y_0) - u(0, 0)| < c_5 y_0^{\kappa}$$

for any

$$\kappa < \min\left(1, \frac{\alpha \pi^2}{\beta}\right).$$

Let us note those qualitative considerations which were used in the above example and which can be carried over to the general case. We consider that  $u(x, y) \equiv 0$ .

1. Let us consider the two trajectories  $z_t(x_1, y_1)$  and  $z_t(x_2, y_2)$ , originating respectively from the points  $(x_1, y_1)$  and  $(x_2, y_2)$ , for the same Wiener trajectories  $(\xi_t^1, \xi_t^2)$ . The trajectories  $z_t$  are constructed with respect to  $(\xi_t^1, \xi_t^2)$  with the aid of equations (6). Then, after the time  $t$  the difference

$$z_t(x_1, y_1, \omega) - z_t(x_2, y_2, \omega) = x_1 - x_2$$

does not increase, while

$$|y_t(x_1, y_1, \omega) - y_t(x_2, y_2, \omega)| = |y_1 - y_2| e^{\beta t}$$

and, consequently,

$$|z_t(x_1, y_1, \omega) - z_t(x_2, y_2, \omega)|^2 = |x_1 - x_2|^2 + |y_1 - y_2|^2 e^{2\beta t}$$

grows at an exponential rapid rate with the index  $2\beta$ . In the general case, the trajectories of the diffusion process with coefficients satisfying the Lipschitz condition, constructed by one and the same Wiener process and originating from the points  $a, b \in \mathbb{R}^n$ , resp., disperse at not greater than an exponential rate

$$M|x_t^a(\omega) - x_t^b(\omega)|^2 < |a - b|^2 e^{cKt},$$

where the constant  $K$  is determined from the Lipschitz constant of the coefficients (see lemma 5). In our case  $c = 2$ ,  $K = \beta$ .

2. If by  $\tau$  we denote the time needed by process  $Z$  to reach the boundary of region  $D$ , then

$$P_x\{\tau > t\} = f_x(t) < c_4 e^{-a_4 t}$$

decreases at an exponential rate. In the general case, if the assumption that the trajectories reach the boundary at a uniform speed is fulfilled, this probability also decreases exponentially (see lemma 4).

3. We shall explain our further course of reasoning by means of an example describing how an estimate for the first derivative is obtained. In order not to be concerned with boundary estimates, we shall assume that the process does not

degenerate in the neighborhood of the boundary. Let

$$\bar{\tau} = \min(\tau_x, \tau_y), \quad \tau_x(\omega) = \inf\{t: x_t^x(\omega) \in \bar{D}\},$$

$$\tau_y(\omega) = \inf\{t: x_t^y(\omega) \in \bar{D}\}.$$

Then, using the probability representation of the solution of Dirichlet's problem, we will get

$$|u(x) - u(y)|^2 \leq M |u(x_{\bar{\tau}}^x) - u(x_{\bar{\tau}}^y)|^2 \leq cM |x_{\bar{\tau}}^x - x_{\bar{\tau}}^y|^2,$$

where  $c$  is a constant depending on the Lipschitz constant for the function  $u(x)$  in the neighborhood of the boundary and on the width of the non-degeneracy region. In order to obtain the estimate of the first derivative of interest to us, it is sufficient to show that  $M|x_{\bar{\tau}}^x - x_{\bar{\tau}}^y|^2 < A|x - y|^2$ . If the non-random variable  $t$  were present instead of the random variable  $\tau$ , such an estimate would be guaranteed by item 1 above. In order to obtain an estimate for a random  $\tau$ , we must use the properties given in items 1 and 2. If the exponent derived in item 2 "exceeds" the exponent of point 1 (in our example,  $\alpha \pi^2 > \beta$ ) the estimate being sought exists. Otherwise, there may be no estimate. In the following sections we will obtain a priori estimates according to the plan outlined here.

In concluding this section we shall note that if the coefficients of the equation with degenerations satisfy only Hölder's condition, the generalized solution does not have to be continuous, even if the operator does not degenerate near the boundary and the statement of lemma 4 is fulfilled. An example confirming this remark can be obtained if we examine the equation

$$\alpha \frac{\partial^2 u}{\partial x^2} + \varphi^2(x, y) \Delta u + \beta y^\nu \frac{\partial u}{\partial y} = 0$$

in the square  $D$ . In this case, the trajectories scatter faster than exponentially, and an exponentially rapid exit from the region cannot "balance" this scattering.

### 3. Continuity of the Hölder-generalized solution

Theorem 1. Let us assume that all points of the set  $\tilde{\Gamma}$  are  $(\lambda_1, \lambda_2, h)$ -regular for certain  $\lambda_1, \lambda_2, h > 0$ , which are the same for all  $x \in \tilde{\Gamma}$ . In addition, let the trajectories of process  $X$  go out at a uniform speed to the boundary of region  $D$  and let the boundary function satisfy Hölder's condition on  $\Gamma$ :

$$|\psi(x) - \psi(y)| < K_1 |x - y|^\nu.$$

Then there exists a  $\gamma_1 > 0$  such that the generalized solution of problem (3) satisfies Hölder's condition with the index  $\gamma_1$ .

Proof. Let  $x, y \in D$  and let  $\bar{x}_t$  and  $\bar{y}_t$  be the solutions of equations (2) with the initial conditions  $\bar{x}_0 = x, \bar{y}_0 = y$ . We set

$$\tau_x = \inf\{t: \bar{x}_t \in \bar{D}\}, \quad \tau_y = \inf\{t: \bar{y}_t \in \bar{D}\}, \quad \bar{\tau} = \min(\tau_x, \tau_y).$$

The quantity  $\bar{\tau}$  does not depend on the future (see Note); therefore, the generalized solution  $u(x)$  of problem (3) satisfies the relations

$$u(x) = Mu(\bar{x}_{\bar{\tau}}), \quad u(y) = Mu(\bar{y}_{\bar{\tau}})$$

(see [2]). Let us estimate the difference  $u(x) - u(y)$ :

$$\begin{aligned} |u(x) - u(y)| &\leq M |u(\bar{x}_{\bar{\tau}}) - u(\bar{y}_{\bar{\tau}})| = \\ &= M |\psi(\bar{x}_{\bar{\tau}}) - \psi(\bar{y}_{\bar{\tau}})| \chi_x(\omega) + M |u(\bar{x}_{\bar{\tau}}) - \psi(\bar{y}_{\bar{\tau}})| \chi_y(\omega) = \\ &= M |\psi(\bar{x}_{\bar{\tau}}) - \psi(\bar{y}_{\bar{\tau}})| \chi_x(\omega) + M |M_{\bar{x}_{\bar{\tau}}} \psi(\bar{\tau}) - \psi(\bar{y}_{\bar{\tau}})| \chi_y(\omega), \end{aligned} \quad (9)$$

where  $\chi_x(\omega)$  is the characteristic function of the set  $\{\omega: \bar{\tau} = \tau_x\}$ ,  $\chi_y(\omega) = 1 - \chi_x(\omega)$ .

Let  $a \in D, b \in \tilde{\Gamma}$ . Then, if by  $d$  we denote the diameter of region  $D$ , we get

$$\begin{aligned} M_a |\psi(x_\tau) - \psi(b)| &\leq K_1 M_a |x_\tau - b|^\nu \leq K_1 d^\nu M_a \left| \frac{x_\tau - b}{d} \right|^\nu \leq \\ &\leq \begin{cases} K_1 d^\nu M_a \left| \frac{x_\tau - b}{d} \right|^{\lambda_1} = K_1 d^{\nu-\lambda_1} M_a |x_\tau - b|^{\lambda_1}, & \text{if } \lambda_1 < \nu, \\ K_1 (M_a |x_\tau - b|^{\lambda_1})^{\nu/\lambda_1}, & \text{if } \lambda_1 \geq \nu. \end{cases} \end{aligned} \quad (10)$$

Note. The non-negative random variable  $\beta(\omega)$  is said to be independent of the future if the event  $\{\omega: \beta(\omega) < t\}$  for any  $t \geq 0$  belongs to a  $\sigma$ -algebra generated by the events  $\{x_s(\omega) \in A\}$  for  $s < t$ , where  $A$  is a Borel set in  $R^n$  (see [2]).

In the last inequality we made use of the fact that

$$(M|\xi|^\nu)^{\lambda_1} \leq (M|\xi|^{\lambda_1})^\nu, \text{ if } \lambda_1 \geq \nu.$$

By  $\alpha$  we shall denote the characteristic operator of process X. It is verified that the function

$$v_1(x) = M_x |x_\tau - b|^{\lambda_1}$$

satisfies the equation  $\alpha v_1 = 0$  in D and the boundary condition

$$v_1(x)|_{x \in \Gamma} = |x - b|^{\lambda_1}.$$

Since all points of the set  $\tilde{\Gamma}$  are  $(\lambda_1, \lambda_2, h)$ -regular, a neighborhood  $U_h(b)$  will be found in which the barrier  $v(x)$  is defined and  $\alpha v(x) = Lv < 0$  in this neighborhood.

We shall choose  $c_1$  so great that  $c_1 v(x) > v_1(x)$  on the boundary of the set  $D \cap U_h(b)$ . This can be done since the above-mentioned inequality is fulfilled on  $\tilde{\Gamma} \cap U_h(b)$ . On the set  $D \cap U_h(b)$   $\alpha (c_1 v - v_1) < 0$ ; therefore, by virtue of the maximum principle, everywhere in  $D \cap U_h(b)$

$$v_1(x) = M_x |x_\tau - b|^{\lambda_1} < c_1 v(x) < \tilde{c}_2 |x - b|^{\lambda_1}. \quad (11)$$

Obviously, if  $\tilde{c}$  is chosen sufficiently great, we can consider that (11) is fulfilled everywhere in D.

We set

$$\lambda = \max\left(\lambda_2, \frac{\lambda_2 \nu}{\lambda_1}\right).$$

From (10) and (11) it follows that

$$M_a |\psi(x_\tau) - \psi(b)| \leq c |a - b|^\lambda.$$

From the latter inequality and (9) we conclude that

$$|u(x) - u(y)| < c_3 M |x_\tau - \bar{y}_\tau|^\lambda. \quad (12)$$

Let us now estimate the right-hand side of inequality (12). Since according to the condition of the theorem the trajectories leave the region at a uniform speed, then, according to lemma 4, it will be possible to find an  $\alpha = \alpha_{L,D} > 0$  such that

$$P\{\bar{\tau} > t\} \leq \max_{x \in D} P_x\{\tau > t\} \leq c_3 e^{-\alpha t}.$$

Let  $0 < \kappa \leq 1$  and let  $\chi_n(\omega)$  be the characteristic function of the set  $\{\omega: n \leq \bar{\tau}(\omega) < n+1\}$ . Since  $|\frac{x_{\bar{\tau}} - y_{\bar{\tau}}}{d}| < 1$ , we have

$$\begin{aligned} M \left| \frac{x_{\bar{\tau}} - y_{\bar{\tau}}}{d} \right|^\lambda &\leq d^\lambda M \left| \frac{x_{\bar{\tau}} - y_{\bar{\tau}}}{d} \right|^{2\lambda} = d^{\lambda(1-\kappa)} M |x_{\bar{\tau}} - y_{\bar{\tau}}|^{2\lambda} \leq \\ &\leq c_4 \sum_{n=0}^{\infty} M |x_{\bar{\tau}} - y_{\bar{\tau}}|^{2\lambda} \chi_n(\omega) \leq c_4 \sum_n M \left\{ \sup_{n \leq s < n+1} |x_s - y_s|^{2\lambda} \chi_n(\omega) \right\} \leq \\ &\leq c_4 \sum_n \left( M \left( \sup_{n \leq s < n+1} |x_s - y_s| \right)^{2\lambda} \cdot P\{\bar{\tau} > n+1\} \right)^{1/2} \leq \\ &\leq c_5 \sum_{n=0}^{\infty} e^{-\frac{\alpha n}{2}} \cdot \left( M \sup_{n \leq s < n+1} |x_s - y_s|^2 \right)^{\frac{\lambda}{2}} \leq \\ &\leq c_5 |x - y|^{2\lambda} \sum_{n=0}^{\infty} \beta_1(n) e^{-\frac{\alpha n}{2}} e^{\frac{\alpha_1 n \cdot \lambda}{2}}. \end{aligned} \quad (13)$$

In the latter inequality we used the corollary from lemma 5. Now, let  $\kappa$  be chosen in such a way that  $\alpha_1 \lambda \kappa < \alpha$ . Then, the series in (13) converges and we obtain finally

$$|u(x) - u(y)| < c_6 |x - y|^{2\lambda}.$$

Lemma 5. Let

$$K = \max_{i, j, k; x \in R^n} \left\{ \left| \frac{\partial \sigma_{ij}(x)}{\partial x_k} \right|, \left| \frac{\partial b_i(x)}{\partial x_k} \right| \right\},$$

and let  $\bar{x}_t$  and  $\bar{y}_t$  be the solutions of equation (2) with the initial conditions  $\bar{x}_0 = x, \bar{y}_0 = y$ . Then

$$M |\bar{x}_t - \bar{y}_t|^{2m} \leq |x - y|^{2m} e^{\alpha_m t},$$

$$\alpha_m = 2m(m-1)K^2 n^2 + mn^3 K^2 + 2mnK.$$

Proof. Let us apply to the function  $\rho^{2m}(\bar{x}_t, \bar{y}_t) = \left( \sum_{i=1}^n (\bar{x}_t - \bar{y}_t^i)^2 \right)^m$  the formula of K. Ito [7]:

$$\begin{aligned}
 & \rho^{2m}(\bar{x}_t, \bar{y}_t) = \\
 & = \rho^{2m}(x, y) + \int_0^t 2m\rho^{2m-2}(\bar{x}_s, \bar{y}_s) \left( \sum_{i=1}^n (\bar{x}_s^i - \bar{y}_s^i) (b_i(\bar{x}_s) - b_i(\bar{y}_s)) \right) ds + \\
 & + \int_0^t 2m\rho^{2m-2}(\bar{x}_s, \bar{y}_s) \left( \sum_{i=1}^n (\bar{x}_s^i - \bar{y}_s^i) \sum_{j=1}^n [\sigma_{ij}(\bar{x}_s) - \sigma_{ij}(\bar{y}_s)] d\bar{\xi}_s^j \right) + \\
 & + \frac{1}{2} \int_0^t 4m(m-1)\rho^{2m-4}(\bar{x}_s, \bar{y}_s) ((\sigma(\bar{x}_s) - \sigma(\bar{y}_s))(\bar{x}_s - \bar{y}_s), \\
 & \quad (\sigma(\bar{x}_s) - \sigma(\bar{y}_s))(\bar{x}_s - \bar{y}_s)) ds + \\
 & + \frac{1}{2} \int_0^t 2m\rho^{2m-2}(\bar{x}_s, \bar{y}_s) \sum_{i,k=1}^n (\sigma_{ik}(\bar{x}_s) - \sigma_{ik}(\bar{y}_s))^2 ds.
 \end{aligned} \tag{14}$$

From (14), using the properties of stochastic integrals and the boundedness of the derivatives of the coefficients, we obtain the inequality

$$M\rho^{2m}(\bar{x}_t, \bar{y}_t) \leq \rho^{2m}(x, y) + \alpha_m \int_0^t M\rho^{2m}(\bar{x}_s, \bar{y}_s) ds.$$

The statement of the lemma follows from the latter inequality (see, for example, [18]).

Corollary. For any  $T > 0$  the following inequality is valid:

$$\begin{aligned}
 & M \sup_{t < T} |\bar{x}_t - \bar{y}_t|^2 \leq c \left[ |x + y|^2 + M \sup_{t < T} \left| \int_0^t [\sigma(\bar{x}_s) - \sigma(\bar{y}_s)] d\bar{\xi}_s \right|^2 + \right. \\
 & + M \left( \int_0^T |b(\bar{x}_s) - b(\bar{y}_s)| ds \right)^2 \leq c \left[ |x - y|^2 + \int_0^T M |\sigma(\bar{x}_s) - \sigma(\bar{y}_s)|^2 ds + \right. \\
 & \quad \left. + T \int_0^T M |x_s - y_s|^2 ds \leq \beta(t) e^{\alpha t}.
 \end{aligned}$$

In estimating the stochastic integral we made use of the fact that

$$\left| \int_0^t [\sigma(x_s) - \sigma(y_s)] d\bar{\xi}_s \right|^2$$

is a semi-martingale and, consequently, according to theorem 3.4 of Chapt. 7 [3]

$$M \sup_{t < T} \left| \int_0^t [\sigma(x_s) - \sigma(\bar{y}_s)] d\bar{x}_s \right|^2 \leq \int_0^T M |\sigma(\bar{x}_s) - \sigma(\bar{y}_s)|^2 ds.$$

Note that  $\beta(t)$  increases with respect to  $t$  not faster than linearly.

#### 4. A Priori Estimates of Derivatives of the Generalized Solution

Theorem 2. Let us assume that  $\sigma_{ij}(x)$ ,  $b_i(x) \in C^1(\mathbb{R}^n)$ , that all points of the set  $\bar{\Gamma}$  are uniformly normally regular and that  $\psi(x) = v(x)|_{\bar{\Gamma}}$ , where  $v(x) \in C^2(\mathbb{R}^n)$ .  
Then, the generalized solution satisfies the Lipschitz condition if only  $\alpha_{L,D} > \alpha_1$ .

Proof. We shall use the same notation as that used in the proof of theorem 1. From (9) it follows that, in order to prove theorem 2, it is sufficient to estimate the terms occurring on the right-hand side of inequality (9). In order to do this, we shall apply to the function  $v(x)$  the formula of K. Ito

$$v(x_t) = v(x) + \int_0^t (\text{grad } v(x_s), \sigma(x_s) d\bar{x}_s) + \int_0^t Lv(x_s) ds.$$

From the latter equality it follows (see Note) that

$$|M_t v(x_t) - v(x)| = \left| M \int_0^t Lv(x_s) ds \right| < \|Lv(x)\| M_t \tau. \quad (15)$$

For the right-hand side of inequality (9) we obtain the estimate

$$\begin{aligned} M |\psi(x_{\tau_x}) - M_{\bar{y}_{\tau_x}} \psi(x_{\tau})| \chi_x(\omega) &\leq M \chi_x(\omega) |v(x_{\tau_x}) - M_{\bar{y}_{\tau_x}} v(x_{\tau})| \leq \\ &\leq M \chi_x(\omega) |v(x_{\tau_x}) - v(\bar{y}_{\tau_x})| + M \chi_x(\omega) |v(\bar{y}_{\tau_x}) - M_{\bar{y}_{\tau_x}} v(x_{\tau})| \leq \\ &\leq \|\text{grad } v(x)\| M |x_{\tau_x} - \bar{y}_{\tau_x}| \chi_x(\omega) + \|Lv\| M (M_{\bar{y}_{\tau_x}} \tau) \chi_x(\omega) \leq c (\|\text{grad } v(x)\| + \\ &\quad + \|Lv\|) M |x_{\tau_x} - \bar{y}_{\tau_x}| \chi_x(\omega). \end{aligned}$$

(16)

---

Note: By  $\|f(x)\|$  we denote  $\max_{x \in \mathbb{R}^n} |f(x)|$ .

In the derivation of this inequality we used the strict Markov property of process X and the uniform normal regularity of the points of set  $\tilde{\Gamma}$ . A similar inequality also holds for the second term on the right-hand side of (9).

From (9) and (16) follows the inequality

$$|u(x) - u(y)| < c_2 M |\bar{x}_\tau - \bar{y}_\tau|. \quad (17)$$

Further, after using the corollary of lemma 5 we get

$$\begin{aligned} M |\bar{x}_\tau - \bar{y}_\tau| &\leq \sum_{n=0}^{\infty} M \sup_{n \leq s < n+1} |\bar{x}_s - \bar{y}_s| \chi_n(\omega) \leq \\ &\leq \sum_{n=0}^{\infty} (M \sup_{s < n+1} |\bar{x}_s - \bar{y}_s|^2 \cdot M \chi_n(\omega))^{1/2} \leq \\ &\leq c_3 \left( \sum_n \beta_n e^{\frac{\alpha}{2}n} e^{-\frac{n}{2}\alpha_{L,D}} \right) |x - y| < c |x - y|. \end{aligned} \quad (18)$$

In the latter inequality

$$\sum \beta_n e^{-\frac{n}{2}(\alpha_{L,D} - \alpha)} < \infty,$$

since  $\alpha_{L,D} > \alpha_1$  according to the condition of the theorem. From (17) and (18) follows the statement of theorem 2.

Let us now consider the estimates of the higher derivatives of the generalized solution. We shall define the m-th difference for the function f(x) by means of the equations

$$\begin{aligned} \delta_1^h f(x) &= f(x+h) - f(x), \\ \delta_m^{h_1, \dots, h_m} f(x) &= \delta_{m-1}^{h_1, \dots, h_{m-1}} f(x+h_m) - \delta_{m-1}^{h_1, \dots, h_{m-1}} f(x). \end{aligned}$$

If the function f(x) is sufficiently smooth, by selecting the increments  $h_1, \dots, h_m \in \mathbb{R}^n$ ,  $|h_i| = h$  it is possible to obtain that

$$\lim_{h^m} \frac{1}{h^m} \delta_m^{h_1, \dots, h_m} f(x) = \frac{\partial^m f(x)}{\partial x_1^{i_1} \dots \partial x_n^{i_n}}.$$

for any  $i_1, \dots, i_n$  summing to  $m$ . By  $H = H(x, h_1, \dots, h_m)$  we shall denote the set of points  $x + h_{i_1} + \dots + h_{i_r}$  for  $i_1, \dots, i_r \leq m$ ,  $0 \leq r \leq m$ ; by  $H_f$  - the convex hull of the values of function  $f(y)$  for  $y \in H$ .

We shall use the following simple inequality:

$$|\delta_m u(f(x))| \leq |(\text{grad } u(x), \delta_m f(x))| + \|D_m u\|_{H_f} \sum_{\substack{\sum i_r = m, \\ i_r \leq n}} |\delta_{i_1} f(x)|^{m_1} \dots |\delta_{i_r} f(x)|^{m_r}, \quad (19)$$

where

$$\|D_m u\|_A = \max_{\substack{\sum i_r = m, \\ x \in A}} \left| \frac{\partial^m u(x)}{\partial x_1^{m_1} \dots \partial x_n^{m_n}} \right|$$

Let us define the sequence of the numbers  $\alpha_{k,m}$  by means of the following relations:

$$\begin{aligned} \alpha_{1,m} &= \alpha_m = (2m(m-1)n^3 + mn^3)K^2 + 2mnK \\ \alpha_{k,1} &= 2Kn + 2K^2n^3 + 1 + \alpha_{k-1,k} \quad \text{for } k > 1, \\ \alpha_{k,m} &= 2m(nk + 1) + m(2n^3K^2 + 1) + 2m(m-1)n^3K^2 + 1 + \alpha_{k-1,km} \\ &\quad \text{for } k > 1. \end{aligned}$$

It is easy to check that  $\alpha_{k,m}$  grows monotonically with respect to each index.

Lemma 6. For all integral  $k$  and  $m$  the following inequality\* is valid:

$$M(\delta_k^{h_1, \dots, h_k} x_i^x)^{2m} \leq C_{k,m} h^{2hm} \alpha_{k,m}^m, \quad h = |h_i|.$$

The proof of this lemma is carried out by induction. For  $k = 1$ , lemma 6 reduces to lemma 5. We will show that the statement of the lemma is preserved when  $k$  is increased by unity. Since  $\delta_k$  is a linear operator, we have

$$\delta_k x_i^x = \int_0^1 \delta_k \sigma(x_s^x) d\xi_s + \int_0^1 \delta_k b(x_s^x) ds.$$

---

\* The operator  $\delta_k$  is applied to  $x_i^x(\omega)$  as in the function of the initial point  $x = x_0^x(\omega)$ .

Let us apply the formula of K. Ito to the function  $\mu_k^2(t) = \sum_1^n (\delta_k x_t^x)^2$ :

$$\begin{aligned} \mu_k^2(t) &= \mu_k^2(0) + 2 \int_0^t \sum_{i=1}^n \left( \delta_k x_s^x \sum_{j=1}^n \delta_k \sigma_{ij}(x_s^x) d\xi_s^j \right) + \\ &+ 2 \int_0^t \sum_{i=1}^n \delta_k x_s^x \delta_k b_i(x_s^x) ds + \int_0^t \sum_{i,j=1}^n (\delta_k \sigma_{ij}(x_s^x))^2 ds. \end{aligned} \quad (20)$$

From (19) and (20) we get

$$\begin{aligned} m_k(t) &= M \mu_k^2(t) \leq m_k(0) + 2 \int_0^t \sum_{i=1}^n |\delta_k x_s^x| \left[ |(\text{grad } b_i(x_s), \delta_k x_s^x)| + \right. \\ &+ \|D_k u\| \sum_{\substack{\sum m_i = k \\ l_i < k}} |\delta_{l_1} x_s^x|^{m_1} \dots |\delta_{l_r} x_s^x|^{m_r} \left. \right] ds + 2 \int_0^t \left[ \sum_{i,j=1}^n |(\text{grad } \sigma_{ij}(x_s), \delta_k x_s^x)| + \right. \\ &\quad \left. + \|D_k u\| \sum_{\substack{\sum m_i = k \\ l_i < k}} |\delta_{l_1} x_s^x|^{m_1} \dots |\delta_{l_r} x_s^x|^{m_r} \right] ds \leq \\ &\leq m_k(0) + (2Kn + 1) \int_0^t m_k(s) ds + 2n^2 K^2 \int_0^t m_k(s) ds + \\ &\quad + c \int_0^t \sum_{\substack{\sum m_i = k \\ l_i < k}} M |\delta_{l_1} x_s^x|^{2m_1} \dots |\delta_{l_r} x_s^x|^{2m_r} ds. \end{aligned} \quad (21)$$

We shall now make use of the fact that

$$M |\delta_{l_1} x_s^x|^{2m_1} \dots |\delta_{l_r} x_s^x|^{2m_r} \leq M \sum_{l < k} \delta_l^{2n/l} x_s \leq \sum_{l < k} (M \delta_l^{2n} x_s^x)^{1/l}.$$

From (21) we get

$$m_k(t) \leq m_k(0) + (2Kn + 2K^2 n^2 + 1) \int_0^t m_k(s) ds + c \int_0^t \sum_{l < k} (M |\delta_l x_s^x|^{2n})^{1/l} ds.$$

According to the assumption of induction for  $l < k$

$$M |\delta_l x_t^x|^{2n} \leq C_{l,h} \rho^{2n} h^{2n} \leq C_{h-1,h} \rho^{2n} h^{2n},$$

since  $\alpha_{l,k}$  increases with increasing  $l$ . From (22) follows the inequality

$$m_k(t) \leq m_k(0) + (2Kn + 2K^2n^3 + 4) \int_0^t m_k(s) ds + C_{k-1,k} e^{\alpha_{k-1,k} t} h^{2k}.$$

From the last inequality we conclude that

$$m_k(t) \leq C_{k,1} e^{\alpha_{k,1} t} h^{2k}.$$

In order to check the validity of the lemma when the second index increases, it is necessary to apply Ito's formula to the function  $\left( \sum_1^m (\delta_k x_t^i)^2 \right)^m$ . This calculation is completely analogous to lemma 5 and we omit it.

Lemma 7. Let  $f(s, \omega)$  be a bounded measurable function. Then,

$$M \left( \int_0^T f(s, \omega) d\xi_s \right)^{2m} \leq c_m T^{m-1} \int_0^T M |f(s, \omega)|^{2m} ds.$$

Proof. Applying Ito's formula to the function  $\left( \int_0^T f(s, \omega) d\xi_s \right)^{2m}$  we get

$$g_m(T) = M \left( \int_0^T f(s, \omega) d\xi_s \right)^{2m} = m(2m-1) \int_0^T M \left( \int_0^t f(s, \omega) d\xi_s \right)^{2(m-1)} f^2(t) dt.$$

From this formula it follows that the function  $g_m(T)$  increases monotonically with respect to  $T$ . From Hölder's inequality, taking monotonicity into account, it follows that

$$\begin{aligned} M \left( \int_0^T f(s, \omega) d\xi_s \right)^{2m} &< \\ &\leq m(2m-1) \left[ \int_0^T M \left( \int_0^t f(s, \omega) d\xi_s \right)^{2m} dt \right]^{\frac{m-1}{m}} \left[ \int_0^T M f^{2m}(t, \omega) dt \right]^{1/m} \leq \\ &\leq m(2m-1) T^{\frac{m-1}{m}} \left[ M \left( \int_0^T f(s, \omega) d\xi_s \right)^{2m} \right]^{\frac{m-1}{m}} \left[ \int_0^T M f^{2m}(t, \omega) dt \right]^{1/m}. \end{aligned}$$

From the last inequality we conclude that

$$M \left[ \int_0^T f(s, \omega) d\xi_s \right]^{2m} \leq [m(2m-1)]^m T^{m-1} \int_0^T M f^{2m}(s, \omega) ds.$$

Lemma 8. For any positive integral  $k$  and  $m$  the following inequality holds:

$$M \sup_{s < t} |\delta_h^{h_1, \dots, h_k} x_s^x|^{2m} < \tilde{C}_{k,m}(t) h^{2km} \cdot e^{-\alpha_{k,m} t},$$

where  $\tilde{C}_{k,m}(t)$  increases with increasing  $t$  not faster than  $t^{2m-1}$ ,  $|h_i| = h$ .

This lemma is proven in the same way as the corollary of lemma 5. It is only necessary to make use of the fact that  $\left( \int_0^t f(s, \omega) d\xi_s \right)^{2m}$  is a semi-martingale and to use lemma 7 to estimate the  $2m$ -th power of the stochastic integral. Let

$$\tau^y(\omega) = \inf\{t: x_t^y \in \bar{D}\}, \quad \bar{\tau} = \min_{y \in H(x, h_1, \dots, h_k)} \tau^y.$$

It is obvious that

$$P\{\bar{\tau} > t\} \leq P_x\{\tau > t\} \leq ce^{-\alpha_{L,D} t}.$$

Lemma 9. If  $\alpha_{k,m} < \alpha_{L,D}$   $u|h_i| = h$ , then

$$M |\delta_h^{h_1, \dots, h_k} x_t^x|^m \leq h^{km} \tilde{C}.$$

Proof. Let  $\chi_n(\omega)$  be the characteristic function of the set  $\{\omega: n \leq \bar{\tau} n + 1\}$ .

Then, using the preceding lemma we obtain

$$\begin{aligned} M |\delta_h^{h_1, \dots, h_k} x_t^x|^m &\leq \sum_{n=0}^{\infty} M \sup_{n \leq s < n+1} |\delta_h^{h_1, \dots, h_k} x_s^x|^m \chi_n(\omega) \leq \sum_{n=0}^{\infty} (M \sup_{n \leq s < n+1} |\delta_h^{h_1, \dots, h_k} x_s^x|^{2m} \cdot M \chi_n(\omega))^{1/2} < \\ &\leq \sum_{n=0}^{\infty} C_{k,m}^{(n+1)} \cdot e^{\frac{n}{2} \alpha_{k,m}} \cdot e^{-\frac{n}{2} \alpha_{L,D}} h^{km} \leq \tilde{C} h^{km}. \end{aligned}$$

The series in the last inequality converges because  $C_{k,m}^{(n)}$  increases exponentially with respect to  $n$  and  $\alpha_{L,D} > \alpha_{k,m}$  according to the condition of the lemma.

Lemma 10. Assume that  $\alpha_{2k, 2k} < \alpha_{L, D}$ . Then,

$$\|D_h u\|_D < C \|D_h u\|_{\bar{\tau}}.$$

Proof. Let us choose  $h_1, \dots, h_m, |h_i| = h, m \leq k$ , in such a way that

$$\lim_{h \rightarrow 0} \frac{1}{h^m} \delta_m^{h_1, \dots, h_m} u(x) = \frac{\partial^m u}{\partial x_1^{h_1} \dots \partial x_n^{h_n}}.$$

Let

$$\bar{\tau} = \min_{y \in H(x, h_1, \dots, h_m)} \tau^y(y).$$

The quantity  $\bar{\tau}$  does not depend on the future, and therefore  $u(x) = Mu(x_{\bar{\tau}}^x)$ . Con-

sequently, for  $\delta_h u(x)$  we get the inequality

$$\begin{aligned} |\delta_m u(x)|^2 &\leq (M |\delta_m u(x_{\bar{\tau}}^x)|)^2 \leq [ \|D_m u\|_{U_{\epsilon_1}(\bar{\tau})}^2 M (|\delta_m x_{\bar{\tau}}^x|^2 + \\ &+ \sum_{\substack{\sum_{i=1}^m i_i = m \\ i_i < m}} |\delta_{i_1} x_{\bar{\tau}}^x|^{2m_1} \dots |\delta_{i_r} x_{\bar{\tau}}^x|^{2m_r}) (1 - \epsilon_2) + \epsilon_2 \|D_m u\|_D^2 M (|\delta_m x_{\bar{\tau}}^x|^2 + \\ &+ \sum_{\substack{\sum_{i=1}^m i_i = m \\ i_i < m}} |\delta_{i_1} x_{\bar{\tau}}^x|^{2m_1} \dots |\delta_{i_r} x_{\bar{\tau}}^x|^{2m_r}) ] C_1. \end{aligned} \quad (22)$$

Here we have used the fact that, for sufficiently small  $h$ , the points  $x_{\bar{\tau}}^y (y \in H(x, h_1, \dots, h_m))$  lie in the  $\epsilon_1$ -neighborhood of  $\bar{\tau}$  with a probability greater than  $1 - \epsilon_2$ .

With the aid of lemma 9 we can derive the inequalities

$$M |\delta_m x_{\bar{\tau}}^x|^2 < \bar{C} h^{2m}, \quad M |\delta_{i_1} x_{\bar{\tau}}^x|^{2m_1} \dots |\delta_{i_r} x_{\bar{\tau}}^x|^{2m_r} < \sum_{i < m} (M |\delta_i x_{\bar{\tau}}^x|^{2i})^{1/l} < h^{2m} \bar{C}. \quad (23)$$

Here we make use of the fact that the sequence  $\alpha_{l, r}$  is monotonic with respect to each index and also of the condition  $\alpha_{L, D} > \alpha_{2k, 2k}$ .

From (22) and (23) it follows that

$$\left| \frac{1}{h^m} \delta_m u(x) \right|^2 \leq C (\|D_m u\|_{U_{\epsilon_1}(\bar{\tau})}^2 \cdot (1 - \epsilon_2) + \epsilon_2 \|D_m u\|_D^2).$$

Now let  $h$  tend to zero. Together with  $h$ ,  $\epsilon_1$  and  $\epsilon_2$  also tend to zero. From the last inequality we obtain the statement of the lemma.

Theorem 3. Let  $\sigma_{ij}(x)$ ,  $b_i(x) \in C^{(k)}(\mathbb{R}^n)$ ,  $\psi(x) \in C(\tilde{\Gamma})$ , let the operator  $L$  not degenerate in the neighborhood of the set  $\tilde{\Gamma}$ , and let  $\alpha_{L,D} > \alpha_{2k,2k}$ . Then, the generalized solution of problem (3) has  $k - 1$  continuous derivatives and the  $(k - 1)$ -th derivatives satisfy the Lipschitz condition.

Proof. First, let us assume that  $\tilde{\Gamma}$  is a smooth manifold of class  $C^k$  and  $\psi(x) \in C^k(\tilde{\Gamma})$ . We shall examine the operator  $L^\epsilon$ , whose coefficients are determined by the equalities

$$\{a_{ij}^\epsilon(x)\} = (\sigma(x) + \epsilon E \cdot \varphi(x)) \cdot (\sigma^*(x) + \epsilon \varphi(x) E), \quad b_i^\epsilon(x) = b_i(x).$$

We shall choose the infinitely differentiable function  $\varphi(x)$  in such a way that it vanishes in the neighborhood of the boundary and is positive outside the  $\delta$ -neighborhood of the boundary. As follows from [18], there exists a unique generalized solution of the problem

$$L^\epsilon u^\epsilon(x) = 0 \quad \text{for } x \in D; \quad u^\epsilon(x)|_{\tilde{\Gamma}} = \psi(x).$$

This generalized solution is continuous. The operator  $L^\epsilon$  does not degenerate outside the  $\delta$ -neighborhood of the boundary, and therefore  $u^\epsilon(x) \in C^{(k)}(D \setminus U_\delta(\tilde{\Gamma}))$ . Let

$$\alpha_{L,D} - \alpha_{2k,2k} = \lambda.$$

Then, for a sufficiently small  $\epsilon$

$$|\alpha_{L^\epsilon,D} - \alpha_{L,D}| < \frac{\lambda}{2}$$

and the coefficient  $\alpha_{2k,2k}^\epsilon$  for operator  $L^\epsilon$  differs from  $\alpha_{2k,2k}$  by not more than  $\lambda/2$ . Therefore,  $\alpha_{L,D}^\epsilon - \alpha_{2k,2k}^\epsilon > 0$  and, consequently, the first  $k$ -derivatives of function  $u^\epsilon(x)$  are estimated according to lemma 10 in terms of derivatives on the boundary  $\tilde{\Gamma}$ .

In the neighborhood of  $\tilde{\Gamma}$ , the operator  $L^\epsilon$  does not depend on  $\epsilon$ ,  $|u^\epsilon(x)| < \max_{x \in \tilde{\Gamma}} |\psi(x)|$ , so that  $\|D_m^\epsilon u^\epsilon\|_{\tilde{\Gamma}}$  is estimated uniformly with respect to  $\epsilon$ , and, therefore, according to lemma 10, the derivatives are estimated uniformly with respect to  $\epsilon$  also inside region  $D \setminus U_\delta(\Gamma)$ . It is proven (see [21]) that

$$\lim_{\epsilon \rightarrow 0} u^\epsilon(x) = u(x)$$

uniformly with respect to  $x \in D$ .

From the last statement and the a priori estimates uniform with respect to  $\epsilon$  the statement of the theorem can be derived in a standard manner. If  $\tilde{\Gamma} \in C^k$ , then it is necessary to replace  $\tilde{\Gamma}$  by the surface  $\tilde{\tilde{\Gamma}} \in C^k$  lying in the region where operator  $L$  is non-degenerate. The function  $u(x)$  on  $\tilde{\tilde{\Gamma}}$  will be sufficiently smooth. This follows from the usual intrinsic estimates for uniformly elliptic equations.

Theorem 4. Assume that  $\sigma_{ij}(x)$ ,  $b_i(x) \in C^k(\mathbb{R}^n)$ , that the boundary  $\Gamma$  of region  $D$  has the direction cosines  $\{n_i(x)\}$  of class  $C^k$ , that  $\psi(x) \in C^k(\Gamma)$  and that  $\alpha_{L,D} > \alpha_{2k,2k}$ . Moreover, let the diffusion along the normal to the boundary  $(\sum_{ij} a_{ij}(x)n_i(x)n_j(x))$  be different from zero on the closure of  $\tilde{\Gamma}$ , let it be equal to zero on  $\Gamma \setminus \tilde{\Gamma}$ , and let the vector  $b(x)$  on  $\tilde{\Gamma}$  be directed towards the exterior of region  $D$ . Then,  $u(x) \in C^{k-1}(D)$  and the  $(k-1)$ -th derivatives satisfy the Lipschitz condition.

Proof. If  $\Gamma = \tilde{\Gamma}$ , then the statement of the theorem follows from lemma 10 and lemma 1.4 of reference [12]. If  $\Gamma \setminus \tilde{\Gamma}$  is not empty, we extend the operator  $L$  across  $\tilde{\Gamma}$  into region  $D \cup \tilde{D}$  in such a way that everywhere on the boundary  $\tilde{\Gamma} \cup \tilde{\tilde{\Gamma}}$  of the region  $D \cup \tilde{D}$  the extended operator  $\tilde{L}$  has a non-zero diffusion along the normal and the maximum of the modulus of the derivatives of the coefficients does not increase.

We extend the function  $\psi(x)$  onto  $\tilde{\Gamma}$  in an arbitrary smooth manner. This is possible since  $\rho(\tilde{\Gamma}, \tilde{\Gamma}) > 0$ . For the operator  $\tilde{L}$  diffusion along the normal does not degenerate,

$$|\alpha_{\tilde{L}, D \cup \tilde{D}} - \alpha_{L, D}| < \frac{\alpha_{L, D} - \alpha_{2k, 2k}}{2}$$

for a sufficiently small extension, the constants  $\alpha_{2k, 2k}$  for operator  $\tilde{L}$  will remain the same and, therefore, as was noted at the beginning of the proof, the assertion of the theorem is fulfilled for the solution  $\tilde{u}(x)$  of the problem in  $D \cup \tilde{D}$ . Therefore, the theorem is also fulfilled for the solution of problem (3) since the solutions of the extended problem and of problem (3) coincide in the region  $D$ . The latter assertion follows from the fact that the trajectories of the process governed by the extended operator  $\tilde{L}$  with the probability 1, originating from points  $x \in D$ , leave  $D$  across  $\tilde{\Gamma}$  since  $\Gamma \setminus \tilde{\Gamma}$  is a repelling boundary.

## 5. Some Remarks

### 1. The parabolic equation

$$\begin{aligned} \frac{\partial u}{\partial t} &= \frac{1}{2} \sum a_{ij}(t, x) \frac{\partial^2 u}{\partial x_i \partial x_j} + \sum b_i(t, x) \frac{\partial u}{\partial x_i} + c(t, x) u(t, x) = \\ &= Lu + c(t, x)u \end{aligned}$$

can be considered as a degenerating elliptic equation. The processes governed by the operator  $\mathcal{L} = -\frac{\partial}{\partial t} + L$  are remarkable in that along one of the coordinates (along  $t$ ) a determinate motion with a unit velocity takes place in the direction of the plane  $t = 0$  (if the original point is in the region  $t > 0$ ). Hence, it follows that the generalized solution of the problem  $\mathcal{L}u = 0$  in the region lying above the plane  $t = t_0$  is unique for some  $t_0$ . In particular, the generalized solution of the Cauchy problem and of the first boundary value problem in a cylinder is always unique. If the region in which the equation  $\mathcal{L}u = 0$  is considered lies above the plane  $t = t_0$ , then, starting from any point lying below the plane  $t = T$ , the

trajectories will reach the boundary not later than in a time  $T - t_0$ . Since the trajectories reach the boundary in finite time, the parameter  $\alpha_{L,D}$  can be considered equal to infinity for the region lying above the plane  $t = t_0$ . Thus, in the case of a boundary value problem for operator  $\mathcal{L}$  in the region lying above the plane  $t = t_0$ , the smoothness of the generalized solution depends only on boundary estimates if only the coefficients of the equation are sufficiently smooth. In particular, in the case of the Cauchy problem for an equation with smooth coefficients, the same smoothness as in the initial function is transferred inside the region.

2. In [19] the problem with a directional derivative was studied for the equation  $Lu = f(x)$ . The following theorem is derived from the results obtained in this article and reference [19].

Theorem 5. In the region D with a smooth boundary  $\Gamma$  let us consider the problem

$$Lu = f(x), \quad \left. \frac{\partial u}{\partial l(x)} \right|_{\Gamma} = 0. \quad (24)$$

It is assumed that the coefficients of the operator and the field  $l(x)$  belong to class  $C^3$ , the field  $l(x)$  does not touch the boundary, the operator L does not degenerate in the neighborhood of  $\Gamma$ , and the trajectories of process X reach the boundary at a uniformly rapid rate. Then, there exists in D the unique probability measure  $\mu(\cdot)$ , namely a solution of the conjugate problem  $\bar{L}^* \mu = 0$ , such that problem (24) has a continuous generalized solution for any continuous  $f(x)$  for which

$\int_D f(x) \mu(dx) = 0$ . This generalized solution is unique with an accuracy up to the constant term. If  $\sigma_{ij}(x), b_i(x), f(x) \in C^m(\mathbb{R}^n)$  and  $\alpha_{L,D} > \alpha_{m,m}$ , the solution has

partial derivatives up to order  $m - 1$  inclusively and the  $m$ -th derivatives satisfy the Lipschitz condition.

The definition of the generalized solution of problem (24) can be found in [19], where an explanation of what  $L^*$  is, is also given. If we discard the assumption of non-degeneracy near the boundary and we assume only that diffusion along the normal is different from zero, then the solution of problem (24) will, generally speaking, not be unique. If we consider the problem for the operator  $Lu - c(x)u = 0$ ,  $c(x) > c > 0$ , we can limit ourselves to the assumption of non-degeneracy along the normal. In this case, a generalized solution exists for any right-hand side of  $C$ . This solution is unique without any assumptions on a uniformly rapid exit from the region. By making use of the results given in [19] it is also possible to study the smoothness of the generalized solution of the second mixed problem for a parabolic equation with degeneration.

3. Let us consider the equation  $Lu(x) - c(x)u(x) = 0$  in the region  $D$  with the boundary conditions  $u|_{\tilde{\Gamma}} = \psi(x)$ , where  $\Gamma \setminus \tilde{\Gamma}$  is a repelling set for operator  $L$  and the points of  $\tilde{\Gamma}$  are assumed to be uniformly and normally regular. If, as was done before, we denote by  $\lambda = \{x_t, P_x\}$  the process which is governed by operator  $L$ , then the generalized solution of the stated problem is given by the formula

$$u(x) = M_x \psi(x_t) \exp \left\{ - \int_0^{\tau} c(x_s) ds \right\}.$$

If  $c(x) \equiv 0$  the written mathematical expectation exists and, consequently, a generalized solution also exists. It is proven that the function  $u(x)$  takes on boundary conditions at normally regular points of the boundary and that the generalized

solution is unique if  $c(x) > c > 0$  in  $D$  without any assumptions whatsoever concerning the exit of trajectories from the region.

Let us consider the question of smoothness of the function  $u(x)$ . Let at first  $c(x) = c > 0$ . Then,  $u(x) = M_x e^{-c\beta} u(x_\beta)$  for any  $\beta(\omega)$  not dependent on the future.

In particular, if  $\beta = \bar{\tau}$  (see lemma 9), then

$$\begin{aligned} |\partial_k u(x)| &= |M e^{-c\bar{\tau}} \partial_k u(x_{\bar{\tau}})| \leq c_1 M e^{c\bar{\tau}} |\partial_k x_{\bar{\tau}}|^k \leq \\ &\leq c_1 \sum_{n=1}^{\infty} e^{-c(n-1)} M \max_{s \leq n} |\partial_k x_s|^k \cdot \chi_n(\omega) \leq \\ &\leq c_2 \sum \tilde{C}_{k,k}(n) e^{n(\alpha_{2k,2k} - \alpha_{L,D} - c)} h^k. \end{aligned}$$

Consequently, for the existence of an estimate of the  $k$ -th derivative it is sufficient that  $\alpha_{2k,2k} < \alpha_{L,D} + c$ . Thus, choosing  $c$  sufficiently large, it is possible to achieve an arbitrarily high smoothness if, of course, the coefficients are sufficiently smooth and there are boundary estimates.

The case when  $c(x) > c > 0$  is reduced to the one examined above by dividing the equation by  $c(x)/c$ .

4. The conditions reported here for the existence of derivatives do not have a local character: smoothness at the point  $x$  depends, in general, on the geometry of the region and the behavior of the coefficients of the operator in the entire region. Simple examples show that such conditions reflect the actual situation in the class of all degenerating equations. However, special classes of degenerating equations do exist in which smoothness has a local character. For example, the

equation of Brownian motion with inertia is such an equation. Definite success in the study of such classes has been achieved in references [13] and [16].

5. In conclusion, we should note that the methods used here can be utilized for the study of degenerating quasilinear equations. These problems are considered in references [23,24].

Received  
13 September 1967

#### References

1. M. V. Keldysh. Some cases of degeneration of equations of the elliptic type on the boundary of a region. Akad. Nauk SSSR, Dokl.; 77, No. 2, 181-183 (1951).
2. E. B. Dynkin. Markov processes; Academic Press, N. Y. (1965).
3. J. L. Doob. Stochastic processes; John Wiley Inc., N. Y. (1953).
4. G. Fichera. On elliptic-parabolic linear differential equations of the second order, Atti Accad. Naz. Linc. Mem.; VIII, No. 6 (1956).
5. G. Fichera. On a single theory of elliptic-parabolic second-order equations, Collection of translations "Matematika"; 7, No. 6, 99-120 (1963).
6. K. Ito. On stochastic differential equations, Ibid.; 1, No. 1, 78-116 (1957).
7. K. Ito. On a formula concerning stochastic differentials, Ibid.; 3, No. 5, 31-41 (1959).
8. A. V. Skorokhod. Studies in the theory of random processes; Addison-Wesley, Reading, Mass. (1965).
9. O. A. Oleinik. On the smoothness of solutions of degenerate elliptic and parabolic equations, Sov. Math., Dokl.; 6, 972-976 (1965).
10. O. A. Oleinik. On linear second-order equations with non-negative characteristic form, Mat. Sbornik; 69 (111), No. 1, 111-139 (1966).
11. R. Z. Khasminskii. Diffusion processes and elliptic differential equations degenerating at the boundary of the domain, Theory of Probability and Its Applications; 3, No. 4, 400-419 (1958).

12. A. M. Il'in. Degenerating elliptic and parabolic equations, Mat. Sborn.; 50 (92), 443-498 (1960).
13. A. M. Il'in. On a class of ultraparabolic equations, Sov. Math., Dokl.; 5, 1673-1676 (1964).
14. M. M. Smirnov. Vyrozhdayushchiesya ellipticheskie i giperbolicheskie uravneniya [Degenerating elliptic and parabolic equations]; Moscow, Izd-vo. "Nauka" (1966).
15. N. S. Piskunov. Boundary-value problems for equations of the elliptic-parabolic type, Mat. Sborn.; 7 (49), 385-424 (1940).
16. I. M. Sonin. On a class of degenerate diffusion processes, Theory of Probability and Its Applications; 12, No. 3, 490-496 (1967).
17. M. I. Freidlin. On some applications of Ito's stochastic equations to differential equations, Ibid.; 4, No. 3, 432-433 (1959).
18. M. I. Freidlin. On Ito's stochastic equations and degenerate elliptic equations, Akad. Nauk SSSR, Izv., Ser. mat.; 26, 653-676 (1962).
19. M. I. Freidlin. Diffusion processes with reflection and problems with a directional derivative on a manifold with a boundary, Theory of Probability and Its Applications; 8, No. 1, 75-83 (1963).
20. M. I. Freidlin. A priori estimates of solutions of degenerating elliptic equations, Sov. Math., Dokl.; 5, 1231-1234 (1964).
21. M. I. Freidlin. A note on the generalized solution of the Dirichlet problem, Theory of Probability and Its Applications; 10, No. 1, 161-164 (1965).
22. M. I. Freidlin. On the formulation of boundary-value problems for degenerating elliptic equations, Sov. Math., Dokl.; 7, 1204-1207 (1966).
23. M. I. Freidlin. Quasilinear parabolic equations and measures in function space, Functional Analysis and Its Applications (USSR); 1, No. 3, 234-240 (1967).
24. M. I. Freidlin. Probability approach to the theory of elliptic quasilinear equations, Uspekhi mat. nauk; 22, No. 5, 183-184 (1967).
25. M. I. Freidlin. On the factorization of non-negative definite matrices, Theory of Probability and Its Applications; 13, No. 2, 354-356 (1968).

Distribution of this document has been made to members of the APL staff in accordance with a list on file in the Distribution Project of the APL/JHU Technical Reports Group. The only distribution external to APL is to the Defense Documentation Center in Alexandria, Virginia, and to the Translations Center of the Special Libraries Association which is located in the John Crerar Library in Chicago, Illinois.

UNCLASSIFIED

Security Classification

DOCUMENT CONTROL DATA - R & D		
<small>Security classification of title, body of abstract and indexing annotation must be entered when the overall report is classified</small>		
1. ORIGINATING ACTIVITY (Corporate author) The Johns Hopkins University, Applied Physics Lab. 8621 Georgia Avenue Silver Spring, Md.		2a. REPORT SECURITY CLASSIFICATION Unclassified
		2b. GROUP
3. REPORT TITLE Smoothness of Solutions of Degenerating Elliptic Equations		
4. DESCRIPTIVE NOTES (Type of report and inclusive dates)		
5. AUTHOR(S) (First name, middle initial, last name) M. F. Freidlin		
6. REPORT DATE 22 September 1969	7a. TOTAL NO. OF PAGES	7b. NO. OF REFS
8a. CONTRACT OR GRANT NO. NOW 62-0604-c	9a. ORIGINATOR'S REPORT NUMBER(S) CLB-3 T-604	
8b. PROJECT NO.	9b. OTHER REPORT NO(S) (Any other numbers that may be assigned this report)	
10. DISTRIBUTION STATEMENT This document has been approved for public release and sale; its distribution is unlimited.		
11. SUPPLEMENTARY NOTES	12. SPONSORING MILITARY ACTIVITY NAVPLANTREPO Naval Ordnance Systems Command	
13. ABSTRACT In this study it is proven that the generalized solution of boundary-value problems for degenerating second-order equations satisfies Holder's boundary condition under broad assumptions. An example is given, showing that a greater smoothness of the solutions cannot be achieved only by increasing the smoothness of the data given in the problem. Conditions for the existence of derivatives of the generalized solution are clarified. The investigation is carried out by means of probability methods.		

DD FORM 1473  
1 NOV 65

UNCLASSIFIED  
Security Classification

13.

KEY WORDS

Math

1/4artoa; Differential Equations

Requester: M. Williams