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# A STABILITY THEORY FOR NON-LINEAR DISTRIBUTED NETWORKS

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## FOREWORD

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## ABSTRACT

Electrical networks consisting of lumped linear and memoryless nonlinear elements and an arbitrary number of lossless transmission lines are considered. It is shown that a large class of such networks may be described by a system of functional-differential equations having the form

$$\dot{\bar{x}}(t) = \bar{f}(\bar{x}_t)$$

where the state of the system at time $t \geq 0$ is represented by $\bar{x}_t$, a point in the space $C_H((-\infty, 0], E^n)$ of bounded continuous functions mapping the interval $(-\infty, 0]$ into $E^n$, with the compact open topology, and the function $\bar{f}$ mapping $C_H((-\infty, 0], E^n)$ into $E^n$ is continuous and locally Lipschitzian. A Lyapunov functional is presented and used to obtain several theorems concerning the stability and instability of the equilibrium solution, $\bar{x} = \bar{0}$, of the network. Several examples of the theory are presented.

TABLE OF CONTENTS

# Chapter I

## INTRODUCTION

In this dissertation several theorems are presented which state sufficient conditions to ensure that an equilibrium state of a nonlinear distributed network is stable, asymptotically stable, completely stable, or unstable. We use the name "nonlinear distributed network" to refer to an electrical network consisting of lumped linear and memoryless nonlinear elements and an arbitrary number of lossless transmission lines. The technical literature abounds with such theory for lumped nonlinear networks [1,2]; while, for linear networks, both lumped and distributed, the problem is simply that of locating the roots of the network's characteristic equation[1]. To the author's best knowledge, relatively little has been written concerning the more general case of nonlinear distributed networks.

There appear to be at least two ways of obtaining stability criteria for nonlinear distributed networks: One way is to write the partial differential equations which govern the distributed elements of the network and then consider as boundary conditions or constraints, the algebraic and ordinary differential equations which arise by

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[1]Much has been written on methods for determining the location of the zeros of exponential sums, that is, functions of the complex variable z of the form: $\Phi(z) = \sum_{k=0}^{n} A_k(z)e^{C_k z}$, where the $A_k$ are polynomials in z, and the $C_k$ are constants. The characteristic equations for linear networks containing lossless transmission lines are of this form. The reader is referred to references [3,4,5,6], and especially Chapters 12 and 13 of reference [7], where further references are to be found. For certain kinds of linear distributed networks, several authors have obtained sufficient conditions to ensure that all roots of the characteristic equation have negative real parts. See, for example, [8].

applying Kirchoff's laws to the lumped portion of the network. Then, applying the Lyapunov theory for dynamical systems [9] (of which the boundary value problem is a particular kind) stability criteria may be obtained. This approach has been considered by Brayton and Miranker [10]. On the other hand, one may treat each distributed element in the network as a two-port and obtain mathematical expressions which show the manner in which the electrical variables at the ports are related. These relations, along with the current-voltage relations of the lumped elements, may then be introduced into the Kirchoff's voltage and current law equations to obtain a system of functional-differential equations which describes the behavior of the network (functional-differential equations are obtained because the expressions which relate the electrical variables at the ports of the distributed elements are functional equations). The Lyapunov theory for functional-differential equations may then be applied to this system to obtain stability criteria. This second approach is the one that we shall consider.

Functional-differential equations, and the application of Lyapunov's second method for determining the stability of solutions of these equations, have been treated by several authors in the mathematical literature [11,12,13]. Recently, J.K. Hale has published several theorems [12,13] which we have found to be particularly suitable for the kind of functional-differential equations which describe a large class of nonlinear distributed networks. We make use of three of these theorems in our work and will state them in the next chapter.

Let us now consider a simple example which will serve to demonstrate the purpose and scope of this work. We start with the lumped network of Figure 1.1, which we shall, at first, assume to be linear. We also assume $R, C > 0$. The resistor $r$ may be negative. Obviously, $v = 0$ is an equilibrium state for this network. If we wish to determine

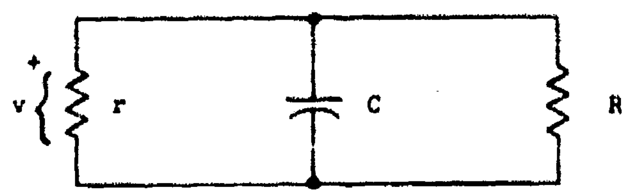


Figure 1.1. A simple lumped linear network.

whether or not this is a stable equilibrium, the procedure is very simple: we compute the value of conductance for the parallel combination of resistors; and, if it is positive, the equilibrium is stable. Letting $g = 1/r$ and $G = 1/R$ we have $g \| G = g + G$ and hence our stability criterion is: If $g > -G$, the equilibrium state is stable.

Let us now add a lossless transmission line to our network, as in Figure 1.2 where $L_t$, $C_t$, and $\ell$ denote, respectively, the inductance per unit length, the capacitance per unit length, and the length of the transmission line. With this modification we may find that a

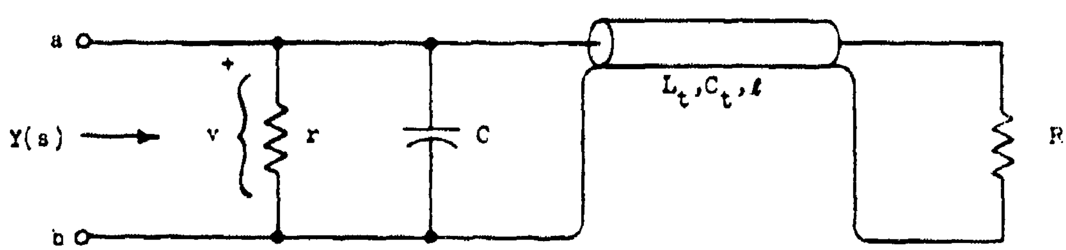


Figure 1.2. A simple distributed linear network.

stable equilibrium state has become unstable. For example, let $r = -1.78$, $R = 1.42$, $C = 1/9$, $L_t = 81/37$, $C_t = 37/81$, $\ell = \pi/6$. We see that $r < -R$, which implies that $g > -G$, and therefore the equilibrium state $v = 0$ for the lumped network of Figure 1.1 is stable. Clearly $v = 0$ is also an equilibrium state for the distributed network. We shall now show that this equilibrium state is unstable. Since the network is linear, what must be shown is that there exists at least one root of the network's characteristic equation which has a positive real part. It is a well-known fact [14 pp.90-94, 15 pp.262-264] that the zeros of the admittance $Y(s)$ seen at the port a-b in Figure 1.2 are roots of this network's characteristic equation. In fact, due to the particularly simple nature of this network, these are the only roots of the characteristic equation. The admittance $Y(s)$ is given by the formula

$$Y(s) = \frac{1}{r} + sC + \frac{1}{Z_o}\left[\frac{(R + Z_o)e^{\tau s} - (R - Z_o)e^{-\tau s}}{(R + Z_o)e^{\tau s} + (R - Z_o)e^{-\tau s}}\right],$$

where $Z_o = \sqrt{L_t/C_t} = \frac{81}{37} \approx 2.19$ and $\tau = a\ell = \sqrt{L_t C_t}\,\ell = \frac{\pi}{6}$. Now, $Y(s)$ has zeros at approximately $s = \frac{1}{2} \pm j1$:

$$Y(\tfrac{1}{2} \pm j1) \approx -\frac{1}{1.78} + (\tfrac{1}{2} \pm j1)\,\frac{1}{9} + \frac{37}{81}\left[\frac{3.61e^{(\frac{\pi}{12} \pm j\frac{\pi}{6})} - 0.77e^{-(\frac{\pi}{12} \pm j\frac{\pi}{6})}}{3.61e^{(\frac{\pi}{12} \pm j\frac{\pi}{6})} + 0.77e^{-(\frac{\pi}{12} \pm j\frac{\pi}{6})}}\right]$$

$$= -0.561 + 0.056 \pm j0.111 + 0.457[1.104 \mp j0.244]$$

$$\approx (-0.561 + 0.056 + 0.505) \pm j(0.111 - 0.111)$$

$$\approx 0$$

Thus, for the distributed network, the equilibrium state $v = 0$ is unstable.

It need not always happen that the addition of lossless transmission lines to a lumped network with a stable equilibrium state causes the resulting distributed network to have an unstable equilibrium. In particular, if C = 1 in our example, both the lumped and distributed networks have stable equilibria v = 0. It is obvious that changing the capacitor's value has no effect on the stability of the lumped network. For the distributed network, however, all of the zeros of Y(s) now have negative real parts. This fact is proved in Appendix A.

It is interesting to note that if the lumped network of Figure 1.1 has an unstable equilibrium then the distributed network of Figure 1.2 also has an unstable equilibrium. That is, the addition of a lossless transmission line to our lumped network, when it is unstable, cannot make it stable. This fact is proved in Appendix A.

The stability criterion for the linear lumped network of Figure 1.1 may be expressed graphically as in Figure 1.3a. A straight line is drawn in the i-v plane (wher i and v denote, respectively, the current

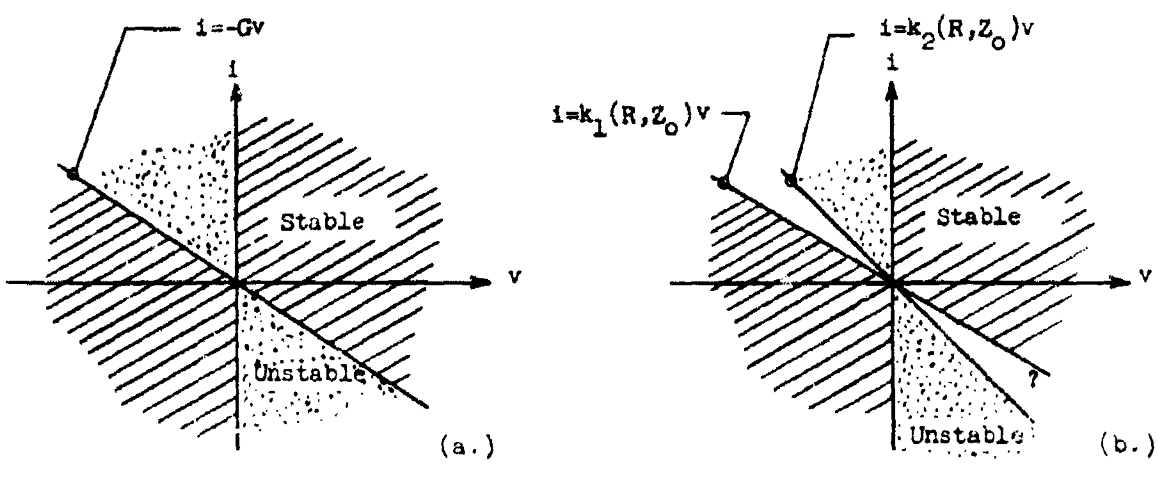


Figure 1.3. Stability criteria for lumped and distributed networks.

through and the voltage across the resistor r) whose location is determined by the parameter R ($G = 1/R$). This line and the line $v = 0$ divide the plane into two regions. If the i-v curve of the resistor r (which in our case is the straight line $i = gv$) lies in the region which is labeled "stable", then the network of Figure 1.1 has a stable equilibrium state $v = 0$. If the i-v curve of the resistor r lies in the region labeled "unstable", then the network of Figure 1.1 has an unstable equilibrium state.

Some of the stability criteria which we shall derive will be similar to this simple criterion. When our theory is applied to the distributed network of Figure 1.2, two lines are determined in the i-v plane. The position of these lines depends only on the parameters R and $Z_o$. These lines, together with the line $v = 0$, divide the plane into three regions as shown in Figure 1.3b. We allow the resistor r to have, in fact, almost any reasonable (nonlinear) i-v curve. Our results are (in part):

1) If the i-v curve for the resistor r lies in the region labeled "stable" in Figure 1.3b, then the equilibrium state $v = 0$ for the distributed network of Figure 1.2 is completely stable[1].

2) If the i-v curve for the resistor r lies in the region labeled "unstable" in Figure 1.3b, then the equilibrium state $v = 0$ for the distributed network of Figure 1.2 is unstable.

3) If the i-v curve of the resistor r lies in the remaining region of Figure 1.3b, then it is uncertain whether or not the equilibrium state $v = 0$ for the distributed network of Figure 1.2 is stable.

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[1]Complete stability, sometimes referred to as asymptotic stability in the large, is asymptotic stability where the region of asymptotic stability is comprised of all the points from which a motion, or trajectory, may originate [16 p. 8, 17 pp.56-66].

Several important features of our results should be emphasized at this point. The following remarks refer particularly to our example; they apply, however, for the most part, to all networks which are members of the rather large class for which the theory is applicable. First of all, since our system is nonlinear, it is significant that the theory can guarantee complete stability. If the system were linear it would be satisfactory to guarantee only asymptotic stability since linearity would then imply complete stability. Since all systems, however, are to some extent nonlinear, it is really complete stability (or at least asymptotic stability with some knowledge of the extent of asymptotic stability) that is needed for practical applications [17 p.57]. Next, we should not be too surprised to find that there exists a region in the i-v plane for which the stability question is left unresolved. The criteria are based upon a knowledge of only the values of the resistor R and the characteristic impedance $Z_o$. In the numerical example it was found that, with all other parameters fixed, a network could be made stable or unstable by adjusting the value of the capacitor C. It is, therefore, rather surprising that regions can indeed be found where complete stability and instability can be assured regardless of the value of so many parameters, e.g., C, a, $\ell$. On the other hand, it is clear that if all of the parameter values were considered the stability criteria that one might develop would not be nearly so easy to apply as our results. In many instances it might be significant that our criteria do not depend upon the length of the transmission line. For instance, if one were designing circuits which would be interconnected by transmission lines, it might be important to know that the resulting network would be stable regardless of the actual length of these lines. We should also point out at this time that the stability and instability regions obtained by our methods are not necessarily the best regions which might be found. For example, it is obvious from physical considerations that the region labeled "stable" in Figure 1.3b should always contain the first and third quadrants. For certain values of R and $Z_o$, however, the line $i = k_1(R,Z_o)v$ has positive slope. This unfortunate circumstance does not occur in what is hoped will be the more usual situations; that is, when the transmission line is terminated in a resistor R whose value is somewhat close to the characteristic impedance of the line. If, in fact, $0.5 < R/Z_o < 2.0$ it turns out that the stable region will include the first and third quadrants. Another characteristic of our results is that the regions labeled "stable" and "unstable" in Figure 1.3b are always contained in the corresponding regions in Figure 1.3a. This, of course, is to be expected since, as was pointed out earlier, the theory does not take into account the length of the transmission line. If transmission lines of infinitesimal length were present in a network of lumped elements, it is obvious from physical considerations that the behavior of the network should approximate that of the corresponding lumped network. If, in our example, $R/Z_o \rightarrow 1$, the regions labeled "stable" and "unstable" in Figure 1.3b approach the corresponding regions in Figure 1.3a. This is a satisfying result since, when $Z_o = R$, the distributed network is equivalent, for most purposes, to the lumped network. Thus, our results numbered 1) and 2) above apply also to the lumped network if the references to Figures 1.3b and 1.2 are changed to read Figures 1.3a and 1.1 respectively. Since we have shown (in Appendix A) that when the networks of

Figures 1.1 and 1.2 are linear, the distributed network is unstable whenever the lumped network is unstable, one might expect the same to apply as well when r is a nonlinear resistor. Our results bear this out whenever $R \leq Z_0$, however, we do not guarantee this property (although it might still be true) when $R > Z_0$.

## Chapter II

## LYAPUNOV STABILITY THEORY FOR FUNCTIONAL-DIFFERENTIAL EQUATIONS

In this chapter we give precise definitions of the terms which will be used in our application of the stability theory of functional-differential equations. Most of the terms used are standard ones in the mathematical literature; see, for example, [11,18,19]. We also state three theorems of J.K. Hale [12,13], upon which much of our work is based. For proof of these theorems the reader is referred to Hale's paper, reference [12].

### 1. Basic Definitions and Notation

The real n-dimensional Euclidean space is denoted by $E^n$, and $\|\bar{x}\| = (x_1^2 + \dots + x_n^2)^{1/2}$ denotes the norm of an element $\bar{x}$ in $E^n$. The elements $\bar{x}$ in $E^n$ are taken to be column vectors and $\bar{x}^t$ denotes the corresponding row vector. Similarly, if M is an $m \times n$ matrix mapping $E^n$ into $E^m$ then $M^t$ denotes the transpose of M; also, $\|M\|$ denotes the norm of M, defined by $\|M\| = \sup\{\|M\bar{x}\|: \bar{x} \in E^n, \|\bar{x}\| = 1\}$. With the above definition for $\|\bar{x}\|$, it can be shown [20 pp.59-60] that $\|M\| = \sqrt{\lambda}$, where $\lambda$ is the largest eigenvalue of the matrix $M^tM$. If $\bar{x}, \bar{y} \in E^n$ then $\langle \bar{x},\bar{y} \rangle = x_1y_1 + \dots + x_ny_n$ denotes the scalar product of $\bar{x}$ and $\bar{y}$.

If f is a function mapping a set X into a set Y then, for every $x \in X$, $f(x)$ denotes that point in Y into which x is mapped by f; if A is a subset of X then $f[A]$ denotes the collection of points in Y defined by $f[A] = \{y: y \in Y, y = f(x) \text{ for some } x \in A\}$.

We denote by $C((-\infty,0], E^n)$, or sometimes by C, the space of continuous functions mapping the interval $(-\infty,0]$ into $E^n$. The topology on C is taken to be the compact open topology.[1] It is fairly easy to show (see Appendix B) that the topological space C with the compact open topology is metrizable, with metric $\rho$ defined as follows: For a fixed real number b, $0 < b < 1$, and for a sequence of points $\{t_k\}$, $0 = t_0 < t_1 < \ldots < t_k < t_{k+1} < \ldots$, with $\lim_{k\to\infty} t_k = \infty$, define, for every $\bar{\varphi},\bar{\psi}$ in C, $\rho(\bar{\varphi},\bar{\psi}) = \sum_{k=0}^{\infty} m_k$, where $m_k = \min\{b^k, \sup\{\|\bar{\varphi}(t) - \bar{\psi}(t)\|: -t_{k+1} \leq t \leq -t_k\}\}$ C is a complete linear metric space.[2] It is clear that convergence in the compact open topology is equivalent to uniform

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[1]The compact open topology for C is constructed as follows: For each subset K of $(-\infty,0]$ and each subset U of $E^n$, let A(K,U) denote the set of all members of C which carry K into U; that is, $A(K,U) = \{\bar{\varphi}: \bar{\varphi} \in C, \bar{\varphi}[K] \subset U\}$. Let the family $\mathcal{A}$ of all sets of the form A(K,U), for K a compact subset of $(-\infty,0]$ and U open in $E^n$, be a subbase for a topology for C. The topology for C which is uniquely determined by this subbase (that is, the smallest topology for C which contains $\mathcal{A}$) is called the compact open topology. It is denoted by $\mathfrak{J}_c$. The family of finite intersections of members of $\mathcal{A}$ is then a base for the compact open topology; each member of this base is of the form $\bigcap_{i=1}^{m} A(K_i,U_i)$, where each $K_i$ is a compact subset of $(-\infty,0]$ and each $U_i$ is an open subset of $E^n$. For a complete discussion of this topology the reader is referred to reference [21] and Chapter 7 of reference [22].

[2]One might hope that a norm could be defined on C so that C would be a complete linear normed space (a Banach space). A natural attempt to define a norm might be made as follows: Let $\|\bar{\varphi}\|_c = \rho(\bar{\varphi},\bar{0}) \; \forall \; \bar{\varphi} \in C$. This, however, does not define a norm, since the homogeneity condition, $\|\alpha\bar{\varphi}\|_c = |\alpha| \cdot \|\bar{\varphi}\|_c$ is not always satisfied. Take, for example, $\varphi(t) = 0$ for $-\infty < t \leq -t_1$ and $\varphi(t) = (2/t_1)t + 2$ for $-t_1 \leq t \leq 0$, and let $\alpha = 1/2$. Then $\|\varphi\|_c = 1$, but $\|\alpha\,\varphi\|_c = 1 \neq 1/2$. That a norm cannot be defined for C is shown by Arens [21] to follow from the fact that the domain of the functions in C, $\{t: -\infty < t \leq 0\}$, is not compact. The concept of a linear metric space is a specialization of the concept of a linear topological space. The concept of a linear normed space is a further specialization. Fortunately, for our purposes it will be of no real consequence that $C((-\infty,0], E^n)$ is not a Banach space. See also, [18 pp. 49-50 and pp. 396-397].

convergence on all compact subsets of $(-\infty,0]$; in fact, a sequence $\bar{\varphi}_n \to \bar{\varphi}$ in C if and only if for every nonnegative integer N, $\max\{ \|\bar{\varphi}_n(t) - \bar{\varphi}(t)\|: -N \leq t \leq 0 \} \to 0$ as $n \to \infty$. For a given positive constant H, we use the notation $C_H( (-\infty,0], E^n )$ to denote the set $\{\bar{\varphi}: \bar{\varphi} \in C, \sup\{ \|\bar{\varphi}(t)\|: -\infty < t \leq 0 \} < H \}$. Again, we shall abbreviate the notation to $C_H$ when the meaning is clear.

Let $A > -\infty$, and let $\bar{x}$ be a continuous function mapping the interval $(-\infty,A)$ into $E^n$. Then, for every t, $-\infty < t < A$, we denote by $\bar{x}_t$ the translation to the interval $(-\infty,0]$, of the restriction of $\bar{x}$ to the interval $(-\infty,t]$; that is, $\bar{x}_t$ is an element of C, defined by $\bar{x}_t(\sigma) = \bar{x}(t + \sigma)$ for $-\infty < \sigma \leq 0$. In other words, the graph of $\bar{x}_t$ is the graph of $\bar{x}$ on $(-\infty,t]$ shifted to the interval $(-\infty,0]$.

If r is a real number, if f is a function mapping $C_H$ into $E^n$, and if $\dot{\bar{x}}(t)$ denotes the derivative of $\bar{x}$ at $t \geq r$, we consider the following autonomous functional-differential equation:

$$\dot{\bar{x}}(t) = \bar{f}(\bar{x}_t), \quad t \geq r. \tag{2-1}$$

Equation (2-1) is called a functional-differential equation because each element of the vector $\dot{\bar{x}}(t)$ is determined by the value of a functional on $C_H$. We say that $\bar{x}(r,\bar{\varphi})$ is a solution of Equation (2-1) with initial condition $\bar{\varphi} \in C_H$ at $t = r$ if there exists some $A > r$ such that $\bar{x}(r,\bar{\varphi})$ is a mapping from $(-\infty,A)$ into $E^n$ with $\bar{x}_t(r,\bar{\varphi})$ in $C_H$ for $r \leq t < A$, $\bar{x}_r(r,\bar{\varphi}) = \bar{\varphi}$, and if $\bar{x}(r,\bar{\varphi})$ satisfies Equation (2-1) for $r \leq t < A$.

The concept of a functional-differential equation is more general than that of an ordinary differential equation. Consider any ordinary differential equation, for example $\dot{x}(t) = 2x(t) + x^3(t)$. If $x(t)$ is a solution of this equation, the derivative of x at some point t may be computed if one knows only the value of x at the point t. For a functional-differential equation, the value of the derivative of a solution at some point t depends upon the values that the function x assumes over an interval for which the point t is the right-hand end. Clearly, the concept of a differential-difference equation is also a special case of a functional-differential equation. From this point of view it is obvious that an initial condition for a functional-differential equation and also the state at some time t of a system which is governed by a functional-differential equation should be specified by a point in some space of functions.

In a manner similar to that used for ordinary differential equations and differential-difference equations (see references [23] and [24]), one may prove the following existence and uniqueness theorem: If $\bar{f}$ is continuous in $C_H$, then for any $\bar{\varphi}$ in $C_H$ there is a solution of Equation (2-1) with initial condition $\bar{\varphi}$ at $t = r$. If $\bar{f}$ is locally Lipschitzian on $C_H$; that is, if for any $H_1 < H$, there exists a constant $L(H_1)$ such that $\|\bar{f}(\bar{\varphi}) - \bar{f}(\bar{\Psi})\| \leq L(H_1)\rho(\bar{\varphi},\bar{\Psi})$ for all $\bar{\varphi},\bar{\Psi}$ in $C_H$ with $\rho(\bar{\varphi},\bar{0}) \leq H_1$, $\rho(\bar{\Psi},\bar{0}) \leq H_1$, then there is only one solution with initial condition $\bar{\varphi}$ at $t = r$ and the solution $\bar{x}(r,\bar{\varphi})$ depends continuously upon $\bar{\varphi}$. Also, $\bar{f}(\bar{\varphi})$ locally Lipschitzian in $\bar{\varphi}$ implies that the solution can be extended in C until the boundary of $C_H$ is reached.

A set M in C is called an invariant set if, for any $\bar{\varphi} \in M$ there exists a function $\bar{x}$ defined on $(-\infty,\infty)$ with $\bar{x}_t \in M$ for every t in $(-\infty,\infty)$ and $\bar{x}_0 = \bar{\varphi}$, such that, for every $\sigma$ in $(-\infty,\infty)$, if $\bar{x}^*(\sigma,\bar{x}_\sigma)$ is the solution of Equation (2-1) with initial condition $\bar{x}_\sigma$ at $\sigma$, then

$\bar{x}^*_t = \bar{x}_t$ for $t \geq \sigma$.

If V is a continuous functional on $C_H$, and if $\bar{x}(0,\bar{\varphi})$ is the unique solution of Equation (2-1) with initial condition $\bar{\varphi}$ at $t = 0$, we define $\dot{V}_{(2-1)}(\bar{\varphi})$ and $\dot{V}^*_{(2-1)}(\bar{\varphi})$by:

$$\dot{V}_{(2-1)}(\bar{\varphi}) = \overline{\lim_{h\to 0^+}} \frac{1}{h}( V(\bar{x}_h(0,\bar{\varphi}) - V(\bar{\varphi}) ),$$

$$\dot{V}^*_{(2-1)}(\bar{\varphi}) = \underline{\lim_{h\to 0^+}} \frac{1}{h}( V(\bar{x}_h(0,\bar{\varphi}) - V(\bar{\varphi}) ).$$

2. Stability Theory for Functional-Differential Equations

If $\bar{f}(\bar{0}) = \bar{0}$, then the solution $\bar{x} = \bar{0}$ of Equation (2-1) is said to be stable if for every $\epsilon > 0$ there exists a $\delta > 0$ such that $\bar{\varphi} \in C_H$ and $\rho(\bar{\varphi},\bar{0}) < \delta$ implies that $\bar{x}_t(0,\bar{\varphi})$ exists for all $t \geq 0$, is in $C_H$, and $\rho(\bar{x}_t(0,\bar{\varphi}), \bar{0}) < \epsilon$ for all $t \geq 0$. If, in addition, there exists a $\delta > 0$ such that $\rho(\bar{\varphi},\bar{0}) < \delta$ implies that $\bar{x}_t(0,\bar{\varphi})$ is in $C_H$ for all $t \geq 0$ and $\bar{x}_t(0,\bar{\varphi}) \to \bar{0}$ as $t \to \infty$, then the solution $\bar{x} = \bar{0}$ is said to be asymptotically stable. If the solution $\bar{x} = \bar{0}$ is asymptotically stable for all $H > 0$ and all $\delta > 0$, then the solution $\bar{x} = \bar{0}$ is said to be completely stable. The solution $\bar{x} = \bar{0}$ is said to be unstable if it is not stable.

It should be noted that if $\bar{x}$ is a continuous function from $(-\infty,\infty)$ to $E^n$ then $\lim_{t\to\infty} \|\bar{x}(t)\| = 0$ if and only if $\lim_{t\to\infty} \rho(\bar{x}_t,\bar{0}) = 0$ since convergence in the compact open topology is equivalent to uniform convergence on all compact subsets of $(-\infty,0]$, in particular the set $\{0\}$. Thus, defining stability, etc., in terms of the compact open topology yields the desired properties.

We now state three theorems due to J. K. Hale [12] concerning the stability of equilibrium solutions of functional-differential equations. As Hale points out, these theorems generalize the results of LaSalle [25,26] for ordinary differential equations. The proofs are also extensions of the ones given by LaSalle and are to be found in Hale's paper. The proofs given by Hale in [12] are actually stated for functional-differential equations on the space $C([-r,0], E^n)$, where r is a positive real number; however, as is pointed out in the last section of his paper, the theory applies as well to functional-differential equations on $C((-\infty,0], E^n)$. In some places the wording of these theorems has been changed slightly; and in Theorem 3 (Theorem 4 of Hale's paper) a trivial change has been made in condition 1). In all of the following theorems we assume that the function $\bar{f}$ in Equation (2-1) is continuous and Lipschitzian on $C_H$, for $H > 0$.

Theorem 1. Let $C^* = \bigcup_{0<\gamma<\infty} C_\gamma$. Let V be a continuous functional on $C^*$. If $U_\ell$ designates the region of $C^*$ where $V(\bar{\varphi}) < \ell$, suppose there exists a non-negative constant $K \ni \|\bar{\varphi}(0)\| < K$, $V(\bar{\varphi}) \geq 0$, and $\dot{V}_{(2-1)}(\bar{\varphi}) \leq 0$ for all $\bar{\varphi} \in U_\ell$. If R is the set of all points in $U_\ell$ where $\dot{V}_{(2-1)}(\bar{\varphi}) = 0$ and M is the largest invariant set in R, then every solution of Equation (2-1) with initial condition in $U_\ell$ approaches M as $t \to \infty$.

Theorem 2. Suppose $\bar{f}(\bar{0}) = \bar{0}$, and let the continuous functional V be defined on $C^* = \bigcup_{0<\gamma<\infty} C_\gamma$ such that $V(\bar{0}) = 0$. Let $U_\ell$ denote that region of $C^*$ where $V(\bar{\varphi}) < \ell$. Assume that there exists K such that $\|\bar{\varphi}(0)\| < K$ for all $\bar{\varphi} \in U_\ell$. Let u(s) be a function, continuous and increasing on $[0,K)$, where $u(0) = 0$. If $C_H \subset U_\ell$, if $u(\|\bar{\varphi}(0)\|) \leq V(\bar{\varphi})$, and if $\dot{V}_{(2-1)}(\bar{\varphi}) \leq 0$ for all $\bar{\varphi} \in U_\ell$, then the solution $\bar{x} = \bar{0}$ of Equation (2-1) is stable.

**Theorem 3.** Suppose $f(\bar{0}) = \bar{0}$ and V is a bounded continuous functional[1] on $C_H$ and there exists a $\gamma$ and an open set U in C such that the following conditions are satisfied:

1) $V(\bar{\varphi}) > 0$ on $U \cap C_\gamma$, $V(\bar{\varphi}) = 0$ on that part of the boundary of U in $C_\gamma$;

2) $\bar{0}$ belongs to the closure of $U \cap C_\gamma$;

3) $V(\bar{\varphi}) \leq u(\|\bar{\varphi}(0)\|)$ on $U \cap C_\gamma$; where u(s) is continuous, nonnegative and nondecreasing on $[0,H)$, and $u(0) = 0$;

4) $\dot{V}^*_{(2-1)}(\bar{\varphi}) \geq 0$ on the closure of $U \cap C_\gamma$ and the set R of $\bar{\varphi}$ in the closure of $U \cap C_\gamma$ such that $\dot{V}^*_{(2-1)}(\bar{\varphi}) = 0$ contains no invariant set of Equation (2-1) except $\bar{\varphi} = \bar{0}$.

Under these conditions, the solution $\bar{x} = \bar{0}$ of Equation (2-1) is unstable and the trajectory of each solution of Equation (2-1) with initial condition in $U \cap C_\gamma$ must leave $C_\gamma$ in some finite time.

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[1]By bounded we mean here that there exists $\ell > 0 \ni V(\bar{\varphi}) \leq \ell$ for all $\bar{\varphi} \in C_H$.

Chapter III

# TRANSMISSION LINE FUNCTIONAL REPRESENTATION

In this chapter we shall develop several pairs of functional equations which describe the electrical behavior of certain simple two-port networks which contain a lossless transmission line. The two-port networks which we shall be concerned with are constructed by connecting a resistor, either in series or in parallel, at each end of a lossless transmission line. We consider such two-port networks, rather than simply treating the transmission line itself as a two-port because, if it is required that at least one of the resistors have a finite value, greater than zero (which shall be the case in the application of these results), then the linear functionals which occur in our resulting equations will be continuous. That the functionals have this property, in fact, that the functionals satisfy a Lipschitz condition, will be proved in the final section of this chapter.

## 1. The Transmission Line

We define a lossless transmission line to be a distributed electrical two-port network, as shown in Figure 3.1, which is characterized by three parameters: $\ell$, the length of the transmission line; C, the distributed capacitance per unit length; and L, the distributed inductance per unit length. We always assume $\ell, C, L > 0$. It is convenient to define two additional parameters which may be used, along with $\ell$, to give an alternate method of characterizing the line. We define

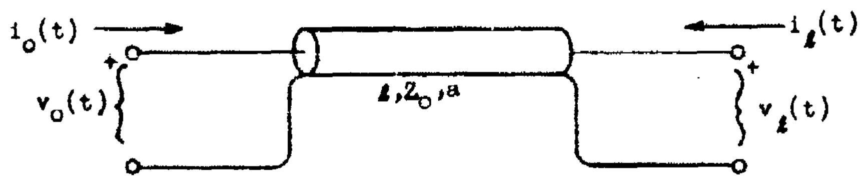


Figure 3.1. A typical lossless transmission line.

$$Z_o = \sqrt{L/C} ,$$

$$a = \sqrt{LC} .$$

$Z_o$ is called the characteristic impedance of the transmission line and a is the reciprocal of the line's phase velocity. We also define $Y_o$, the line's characteristic admittance, by $Y_o = 1/Z_o$.

If x is a point in the interval $[0,\ell]$ and t is a point in the interval $(-\infty,T]$, for some $T > -\infty$, then at any time t and at any point x the transmission line's voltage and current are denoted by $v(x,t)$ and $i(x,t)$ respectively. Thus, v and i are real valued functions defined on the set $M_T$:

$$M_T = \{ (x,t) : 0 \leq x \leq \ell, \quad -\infty < t \leq T \}$$

Note that $v_o(t) = v(0,t)$, $i_o(t) = i(0,t)$, $v_\ell(t) = v(\ell,t)$, and $i_\ell(t) = -i(\ell,t)$. We assume that for each lossless transmission line in our distributed networks there exist functions $f_1$ and $f_2$ such that

$$v(x,t) = f_1(ax - t) + f_2(ax + t),$$

$$i(x,t) = \frac{1}{Z_o} [ f_1(ax - t) - f_2(ax + t) ],$$

for all $(x,t) \in M_T$. From these equations we obtain:

$$v(0,t) = f_1(-t) + f_2(t), \tag{3-1}$$

$$i(0,t) = \frac{1}{Z_0} f_1(-t) - \frac{1}{Z_0} f_2(t), \tag{3-2}$$

$$v(\ell,a\ell-t) = f_1(t) + f_2(2a\ell-t), \tag{3-3}$$

$$i(\ell,a\ell-t) = \frac{1}{Z_0} f_1(t) - \frac{1}{Z_0} f_2(2a\ell-t). \tag{3-4}$$

From Equations (3-3) and (3-4) we find

$$f_1(t) = \tfrac{1}{2}[\ v(\ell,\ a\ell-t) + Z_0 i(\ell,\ a\ell-t)\ ], \tag{3-5}$$

and from Equations (3-1) and (3-2) we have

$$f_2(t) = \tfrac{1}{2}[\ v(0,t) - Z_0 i(0,t)\ ]. \tag{3-6}$$

Substituting Equations (3-5) and (3-6) into Equation (3-4) and replacing t by $2a\ell-t$, then substituting Equations (3-5) and (3-6) into Equation (3-1) and replacing t by $t-a\ell$ gives

$$v(0,t) - Z_0 i(0,t) = v(\ell,t-a\ell) - Z_0 i(\ell,t-a\ell)$$

and

$$v(\ell,t) + Z_0 i(\ell,t) = v(0,t-a\ell) + Z_0 i(0,t-a\ell).$$

Thus, letting $\tau = a\ell$ and recalling that $i(\ell,t) = -i_\ell(t)$,

$$v_0(t) - Z_0 i_0(t) = v_\ell(t-\tau) + Z_0 i_\ell(t-\tau),$$

$$v_\ell(t) - Z_0 i_\ell(t) = v_0(t-\tau) + Z_0 i_0(t-\tau). \tag{3-7}$$

Equations (3-7) are used as the starting point in the next section where the functional equations which govern the behavior of the various two-ports are derived.

2. The Transmission Line Two-Ports

In this section Equations (3-7) are used to derive the functional equations which describe the electrical behavior at the ports of the three two-port networks shown in Figure 3.2. That is, for example,

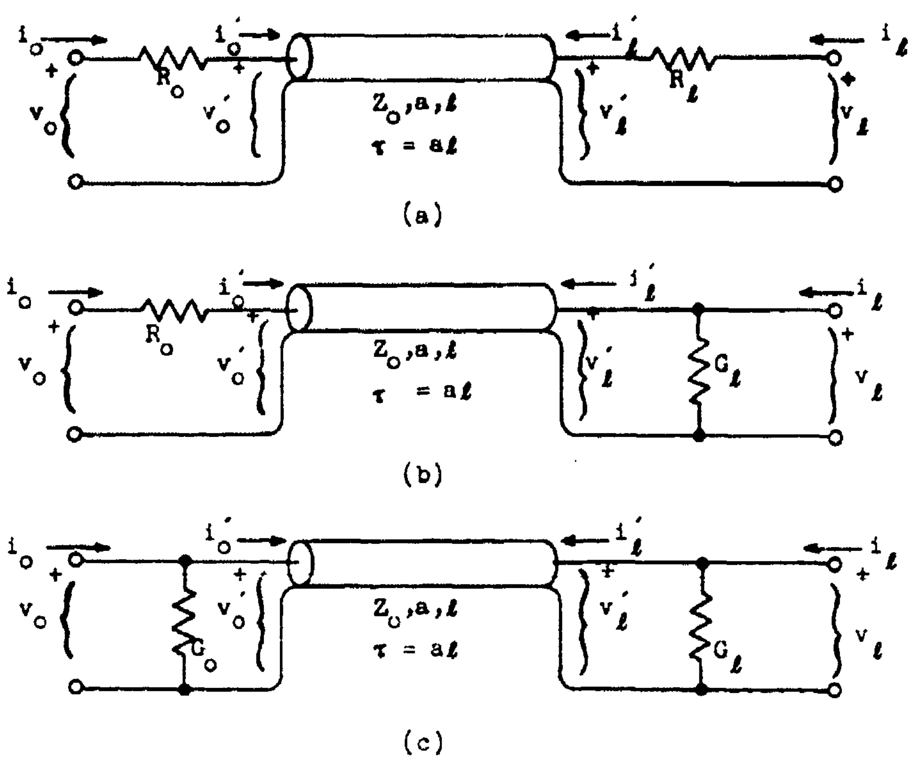


Figure 3.2. Transmission line two-ports

for any $t \in (-\infty, T]$, $T > -\infty$, we will show, for the network of Figure 3.2a, how $i_o(t)$ and $i_\ell(t)$ may be expressed as functionals whose arguments are the functions ( of $\sigma$, $-\infty < \sigma \leq 0$), $v_o(t+\sigma)$ and $v_\ell(t+\sigma)$. The resulting functional equations are:

For Figure 3.2a:

$$i_o(t) = \frac{1}{R_o + Z_o} v_o(t) - 2 \frac{Z_o \Gamma_\ell}{(R_o + Z_o)^2} \sum_{k=o}^{\infty} (\Gamma_o \Gamma_\ell)^k v_o(t - 2(k+1)\tau)$$

$$- 2 \frac{Z_o}{(R_o + Z_o)(R_\ell + Z_o)} \sum_{k=o}^{\infty} (\Gamma_o \Gamma_\ell)^k v_\ell(t - (2k+1)\tau),$$

$$i_\ell(t) = \frac{1}{R_\ell + Z_o} v_\ell(t) - 2 \frac{Z_o \Gamma_o}{(R_\ell + Z_o)^2} \sum_{k=o}^{\infty} (\Gamma_o \Gamma_\ell)^k v_\ell(t - 2(k+1)\tau)$$

$$- 2 \frac{Z_o}{(R_\ell + Z_o)(R_o + Z_o)} \sum_{k=o}^{\infty} (\Gamma_o \Gamma_\ell)^k v_o(t - (2k+1)\tau).$$

For Figure 3.2b:

$$i_o(t) = \frac{1}{R_o + Z_o} v_o(t) - 2 \frac{Z_o \Gamma_\ell}{(R_o + Z_o)^2} \sum_{k=o}^{\infty} (\Gamma_o \Gamma_\ell)^k v_o(t - 2(k+1)\tau.$$

$$- 2 \frac{1}{(R_o + Z_o)(G_\ell + Y_o)} \sum_{k=o}^{\infty} (\Gamma_o \Gamma_\ell)^k i_\ell(t - (2k+1)\tau.,$$

$$v_\ell(t) = \frac{1}{G_\ell + Y_o} i_\ell(t) + 2 \frac{Y_o \Gamma_o}{(G_\ell + Y_o)^2} \sum_{k=o}^{\infty} (\Gamma_o \Gamma_\ell)^k i_\ell(t - 2(k+1)\tau;$$

$$+ 2 \frac{1}{(G_\ell + Y_o)(R_o + Z_o)} \sum_{k=o}^{\infty} (\Gamma_o \Gamma_\ell)^k v_o(t - (2k+1)\tau).$$

For Figure 3.2c:

$$v_o(t) = \frac{1}{G_o + Y_o} i_o(t) + 2 \frac{Y_o \Gamma_\ell}{(G_o + Y_o)^2} \sum_{k=o}^{\infty} (\Gamma_o \Gamma_\ell)^k i_o(t - 2(k+1)\tau.$$

$$+ 2 \frac{Y_o}{(G_o + Y_o)(G_\ell + Y_o)} \sum_{k=o}^{\infty} (\Gamma_o \Gamma_\ell)^k i_\ell(t - (2k+1)\tau.,$$

$$v_\ell(t) = \frac{1}{G_\ell + Y_o} i_\ell(t) + 2 \frac{Y_o \Gamma_o}{(G_\ell + Y_o)^2} \sum_{k=o}^{\infty} (\Gamma_o \Gamma_\ell)^k i_\ell(t - 2(k+1)\tau)$$

$$+ 2 \frac{Y_o}{(G_\ell + Y_o)(G_o + Y_o)} \sum_{k=o}^{\infty} (\Gamma_o \Gamma_\ell)^k i_o(t - (2k+1)\tau). \qquad (3\text{-}8)$$

In Equations (3-8) $\Gamma_o$ and $\Gamma_\ell$ are the usual reflection coefficients at each end of the transmission lines when the independent port variables are set equal to zero; e.g., for Figure 3.2a, $\Gamma_o = (R_o - Z_o)/(R_o + Z_o)$ and $\Gamma_\ell = (R_\ell - Z_o)/(R_\ell + Z_o)$. We assume for each network that both resistors are nonnegative and at least one has a positive real value (hence $|\Gamma_o \Gamma_\ell| < 1$).

Equations (3-7), of course, apply to the primed variables in Figure 3.2a. The primed variables, however, are related to the unprimed variables by the following equations:

$$i_o' = i_o,$$

$$v_o' = v_o - R_o i_o,$$

$$i_\ell' = i_\ell,$$

$$v_\ell' = v_\ell - R_\ell i_\ell.$$

Thus, Equations (3-7) may be written in terms of the unprimed variables as

$$v_o(t) - R_o i_o(t) - Z_o i_o(t) = v_\ell(t-\tau) - R_\ell i_\ell(t-\tau) + Z_o i_\ell(t-\tau),$$

$$v_\ell(t) - R_\ell i_\ell(t) - Z_o i_\ell(t) = v_o(t-\tau) - R_o i_o(t-\tau) + Z_o i_o(t-\tau),$$

and hence,

$$i_o(t) = \frac{1}{R_o + Z_o} v_o(t) - \frac{1}{R_o + Z_o} v_\ell(t-\tau) + \frac{R_\ell - Z_o}{R_o + Z_o} i_\ell(t-\tau),$$

$$i_\ell(t) = \frac{1}{R_\ell + Z_o} v_\ell(t) - \frac{1}{R_\ell + Z_o} v_o(t-\tau) + \frac{R_o - Z_o}{R_\ell + Z_o} i_o(t-\tau). \qquad (3\text{-}9)$$

Replacing t by $t-\tau$ in Equations (3-9), we have

$$i_o(t-\tau) = \frac{1}{R_o + Z_o} v_o(t-\tau) - \frac{1}{R_o + Z_o} v_\ell(t-2\tau) + \frac{R_\ell - Z_o}{R_o + Z_o} i_\ell(t-2\tau),$$

$$i_\ell(t-\tau) = \frac{1}{R_\ell + Z_o} v_\ell(t-\tau) - \frac{1}{R_\ell + Z_o} v_o(t-2\tau) + \frac{R_o - Z_o}{R_\ell + Z_o} i_o(t-2\tau). \qquad (3\text{-}10)$$

Substituting Equations (3-10) into Equations (3-9) gives

$$i_o(t) = \frac{1}{R_o + Z_o} v_o(t) - \frac{(R_\ell - Z_o)}{(R_o + Z_o)(R_\ell + Z_o)} v_o(t-2\tau)$$
$$- \frac{1}{R_o + Z_o}\left[1 - \frac{R_\ell - Z_o}{R_\ell + Z_o}\right] v_\ell(t-\tau) + \frac{(R_\ell - Z_o)(R_o - Z_o)}{(R_o + Z_o)(R_\ell + Z_o)} i_o(t-2\tau),$$

$$i_\ell(t) = \frac{1}{R_\ell + Z_o} v_\ell(t) - \frac{(R_o - Z_o)}{(R_\ell + Z_o)(R_o + Z_o)} v_\ell(t-2\tau)$$
$$- \frac{1}{R_\ell + Z_o}\left[1 - \frac{R_o - Z_o}{R_o + Z_o}\right] v_o(t-\tau) + \frac{(R_o - Z_o)(R_\ell - Z_o)}{(R_\ell + Z_o)(R_o + Z_o)} i_\ell(t-2\tau),$$

and hence,

$$i_o(t) = \frac{1}{R_o + Z_o} v_o(t) - \frac{1}{R_o + Z_o} \Gamma_\ell v_o(t-2\tau)$$
$$- 2\frac{Z_o}{(R_o + Z_o)(R_\ell + Z_o)} v_\ell(t-\tau) + \Gamma_o\Gamma_\ell i_o(t-2\tau),$$

$$i_\ell(t) = \frac{1}{R_\ell + Z_o} v_\ell(t) - \frac{1}{R_\ell + Z_o} \Gamma_o v_\ell(t-2\tau)$$
$$- 2\frac{Z_o}{(R_\ell + Z_o)(R_o + Z_o)} v_o(t-\tau) + \Gamma_o\Gamma_\ell i_\ell(t-2\tau). \qquad (3\text{-}11)$$

We now note that the second of Equations (3-11) may be obtained from the first by simply exchanging the o and $\ell$ subscripts on the symbols i, v, R and $\Gamma$. Due to this symmetry we need now consider only the first of these equations. If $\Gamma_0 \neq 0$ and $\Gamma_\ell \neq 0$ then, for $k = 0, 1, 2, \ldots$ ,

$$(\Gamma_0\Gamma_\ell)^k\, i_0(t-2k\tau) = \frac{1}{R_0 + Z_0}(\Gamma_0\Gamma_\ell)^k\, v_0(t-2k\tau) - \frac{1}{R_0 + Z_0}\Gamma_0^k\Gamma_\ell^{k+1} v_0(t-2(k+1)\tau)$$

$$-2\,\frac{Z_0}{(R_0 + Z_0)(R_\ell + Z_0)}(\Gamma_0\Gamma_\ell)^k\, v_\ell(t-(2k+1)\tau) + (\Gamma_0\Gamma_\ell)^{k+1}\, i_0(t-2(k+1)\tau);$$

therefore, for $p = 1, 2, \ldots$ ,

$$\sum_{k=0}^{p}(\Gamma_0\Gamma_\ell)^k\, i_0(t-2k\tau) = \frac{1}{R_0 + Z_0}\sum_{k=0}^{p}(\Gamma_0\Gamma_\ell)^k\, v_0(t-2k\tau)$$

$$-\frac{1}{R_0 + Z_0}\sum_{k=0}^{p}\Gamma_0^k\Gamma_\ell^{k+1}\, v_0(t-2(k+1)\tau)$$

$$-2\,\frac{Z_0}{(R_0 + Z_0)(R_\ell + Z_0)}\sum_{k=0}^{p}(\Gamma_0\Gamma_\ell)^k\, v_\ell(t-(2k+1)\tau)$$

$$+\sum_{k=0}^{p}(\Gamma_0\Gamma_\ell)^{k+1}\, i_0(t-2(k+1)\tau).$$

But,

$$\sum_{k=0}^{p}(\Gamma_0\Gamma_\ell)^{k+1}\, i_0(t-2(k+1)\tau) = \sum_{k=1}^{p+1}(\Gamma_0\Gamma_\ell)^k\, i_0(t-2k\tau);$$

therefore,

$$i_0(t) + \cancel{\sum_{k=1}^{p}(\Gamma_0\Gamma_\ell)^k\, i_0(t-2k\tau)} = \frac{1}{R_0 + Z_0}\sum_{k=0}^{p}(\Gamma_0\Gamma_\ell)^k\, v_0(t-2k\tau)$$

$$-\frac{1}{R_0 + Z_0}\sum_{k=0}^{p}\Gamma_0^k\Gamma_\ell^{k+1}\, v_0(t-2(k+1)\tau) - 2\,\frac{Z_0}{(R_0 + Z_0)(R_\ell + Z_0)}\sum_{k=0}^{p}(\Gamma_0\Gamma_\ell)^k v_\ell(t-(2k+1)\tau)$$

$$+\cancel{\sum_{k=1}^{p}(\Gamma_0\Gamma_\ell)^k\, i_0(t-2k\tau)} + (\Gamma_0\Gamma_\ell)^{p+1}\, i_0(t-2(p+1)\tau).$$

Also

$$\sum_{k=0}^{p}(\Gamma_o\Gamma_\ell)^k\, v_o(t-2k\tau) = v_o(t) + \sum_{k=0}^{p}(\Gamma_o\Gamma_\ell)^{k+1}\, v_o(t-2(k+1)\tau)$$

$$- (\Gamma_o\Gamma_\ell)^{p+1}\, v_o(t-2(p+1)\tau),$$

therefore,

$$i_o(t) = \frac{1}{R_o + Z_o}\left[v_o(t) + \sum_{k=0}^{p}(\Gamma_o^{k+1} - \Gamma_o^k)\,\Gamma_\ell^{k+1}\, v_o(t-2(k+1)\tau)\right]$$

$$-2\,\frac{Z_o}{(R_o + Z_o)(R_\ell + Z_o)}\sum_{k=0}^{p}(\Gamma_o\Gamma_\ell)^k\, v_\ell(t-(2k+1)\tau)$$

$$-\frac{1}{R_o + Z_o}(\Gamma_o\Gamma_\ell)^{p+1}\, v_o(t-2(p+1)\tau) + (\Gamma_o\Gamma_\ell)^{p+1}\, i_o(t-2(p+1)\tau).$$

Using

$$\Gamma_o^{k+1} - \Gamma_o^k = \Gamma_o^k(\Gamma_o - 1) = -2\,\frac{Z_o}{R_o + Z_o}\,\Gamma_o^k\,,$$

we obtain

$$i_o(t) = \frac{1}{R_o + Z_o}\left[v_o(t) - 2\,\frac{Z_o\Gamma_\ell}{R_o + Z_o}\sum_{k=0}^{p}(\Gamma_o\Gamma_\ell)^k\, v_o(t-2(k+1)\tau)\right]$$

$$-2\,\frac{Z_o}{(R_o + Z_o)(R_\ell + Z_o)}\sum_{k=0}^{p}(\Gamma_o\Gamma_\ell)^k\, v_\ell(t-(2k+1)\tau)$$

$$-\frac{1}{R_o + Z_o}(\Gamma_o\Gamma_\ell)^{p+1}\, v_o(t-2(p+1)\tau) + (\Gamma_o\Gamma_\ell)^{p+1}\, i_o(t-2(p+1)\tau).$$

Now, since $0 < R_o < \infty$ and/or $0 < R_\ell < \infty$, and hence $|\Gamma_o\Gamma_\ell| < 1$, we have, if $v_o$, $i_o$, $v_\ell$, $i_\ell$ are bounded on $(-\infty, T]$,

$$\lim_{p\to\infty}\frac{1}{R_o + Z_o}(\Gamma_o\Gamma_\ell)^{p+1}\, v_o(t-2(p+1)\tau) = 0,$$

$$\lim_{p\to\infty} (\Gamma_o\Gamma_\ell)^{p+1}\, i_o(t-2(p+1)\tau) = 0,$$

and ,

$$\lim_{p\to\infty} \sum_{k=0}^{p} (\Gamma_o\Gamma_\ell)^k\, v_o(t-2(k+1)\tau) = \sum_{k=0}^{\infty} (\Gamma_o\Gamma_\ell)^k\, v_o(t-2(k+1)\tau), \text{ etc.},$$

converge. Hence, using the symmetry referred to following Equations (3-11), we have derived the first two of Equations (3-8). Immediately following Equations (3-11) it was assumed that $\Gamma_o \neq 0$ and $\Gamma_\ell \neq 0$. We may, however, compare Equations (3-11) with the first two of Equations (3-8) and see that the latter are valid for all $\Gamma_o$, $\Gamma_\ell$ ( $|\Gamma_o| \leq 1$, $|\Gamma_\ell| \leq 1$, $|\Gamma_o\Gamma_\ell| < 1$ ), provided we define $(\Gamma_o\Gamma_\ell)^0 = 1$ if $\Gamma_o = 0$ or $\Gamma_\ell = 0$. We shall make this definition.

The remaining four of Equations (3-8) are easily derived in a similar manner.

## 5. The Lipschitz Condition

In this section we prove that the functional equations describing the electrical behavior at the ports of the networks considered in the last section satisfy a Lipschitz condition provided that the compact open topology on C is metrized with an appropriate metric. We consider the mapping $\hat{f}$, from $C_H( (-\infty,0], E^n )$ into $E^n$, defined as follows: For $i = 1, \ldots, n$, let $t_o^{(i)} > 0$, $s_o^{(i)} > 0$, $T^{(i)} > 0$, $S^{(i)} > 0$, and let $t_k^{(i)} = t_{k-1}^{(i)} + T^{(i)}$, $s_k^{(i)} = s_{k-1}^{(i)} + S^{(i)}$ for $k = 1, 2, \ldots$

Let A be a real n×n diagonal matrix,

$$A = \begin{bmatrix} a_1 & & 0 \\ & \ddots & \\ 0 & & a_n \end{bmatrix},$$

where $|a_i| < 1$ for $i = 1, \ldots, n$. Let $M_1, M_2, M_3$ denote real $n \times n$ matrices. Then, $\forall\, \bar{\varphi} \in C_H$,

$$F(\bar{\varphi}) \equiv M_1\bar{\varphi}(0) + M_2 \sum_{k=0}^{\infty} A^k\, \bar{\varphi}_{t_k} + M_3 \sum_{k=0}^{\infty} A^k\, \bar{\varphi}_{s_k}\,, \qquad (3\text{-}12)$$

where $\bar{\varphi}_{t_k}$ denotes the vector

$$( \varphi_1(-t_k^{(1)}) \,, \;.\;.\;.\;, \varphi_n(-t_k^{(n)}) \;)^t \quad ,$$

and similarly for $\bar{\varphi}_{s_k}$. If we consider any finite collection of two-port networks of the type considered in the last section, the system of functional equations describing the electrical behavior at their ports is of the form specified in Equation (3-12) above.

We first define the metric $\rho$ with which the compact open topology on $C$ will be metrized: Let $a = \max\{\, |a_i| : i = 1, \ldots, n \,\}$, and choose $b \ni 0 \le a < b < 1$. Let $t_o^* = 0$, $t_1^* > \max(t_o^{(i)}, s_o^{(i)}: i = 1,\ldots,n)$, and pick $T^* > \max(\, T^{(i)}, S^{(i)}: i = 1, \ldots, n)$. Let $t_k^* = t_{k-1}^* + T^*$ for $k = 2, 3, \ldots$. For $i = 1, \ldots, n$, define the integers[1] $N_i' = \left[T^*/T^{(i)}\right] + 1$, $N_i'' = \left[T^*/S^{(i)}\right] + 1$. Clearly, each interval $[-t_{k+1}^*, -t_k^*]$, for $k = 1, 2, \ldots$, contains at least one of the points $-t_k^{(i)}$ and at most $N_i'$ such points (for $i = 1, \ldots, n$); and, similarly, each such interval contains at least one of the points $-s_k^{(i)}$ and at most $N_i''$ such points. If $N_i''' = \left[(t_1^* - t_o^{(i)})/\, T^{(i)}\right] + 1$, and $N_i'''' = \left[(t_1^* - s_o^{(i)})/\, S^{(i)}\right] + 1$, then the interval $[-t_1^*, 0]$ contains $N_i'''$ of the points $-t_k^{(i)}$ and $N_i''''$ of the points $-s_k^{(i)}$ ,

---

[1] We use here the following notation: If $r$ is any real number then $[r]$ denotes the greatest integer $k$ such that $k \le r$

for $i = 1, \ldots, n$. Let $N = \max\{ N_i', N_i'', N_i''', N_i'''' : i=1, \ldots, n\}$; then, each interval $[-t^*_{k+1}, -t^*_k]$, for $k = 0, 1, 2, \ldots$, contains at least one of the points $-t_k^{(i)}$ and $-s_k^{(i)}$ and [illegible] N such points, for $i = 1, \ldots, n$. Let the compact open top[illegible]y on C be metrized with the metric $\rho$ defined by $\rho(\bar\varphi,\bar\Psi) = \sum_{k=0}^{\infty} m_k$, wh[illegible]e $m_k =$ $\min\{ b^k, \sup\{ \|\bar\varphi(t) - \bar\Psi(t)\| : -t^*_{k+1} \leq t \leq -t^*_k\} \}$, $\forall\ \bar\varphi, \bar\Psi \in C$.

We now prove two lemmas which will be used in our procedure for specifying the Lipschitz constant for $\bar f$.

<u>Lemma 1</u>. If $-t_k'$ denotes an arbitrary fixed point in one of the intervals $[-t^*_{k+1}, -t^*_k]$ and if $\bar g_k$ denotes the mapping from $C_H( (-\infty,0], E^n )$ to $E^n$ defined by $\bar g_k(\bar\varphi) = \bar\varphi(-t_k')$, $\forall\ \bar\varphi \in C_H$, then $\bar g_k$ satisfies a Lipschitz condition on $C_H$ with Lipschitz constant $L_k =$ $\max\{ 1, 2Hb^{-k} \}$.

<u>proof</u>: Let $\bar\varphi, \bar\Psi \in C_H$. If $\|\bar\varphi(-t_k') - \bar\Psi(-t_k')\| \leq b^k$, then $\| \bar\varphi(-t_k') - \bar\Psi(-t_k')\| \leq m_k \leq \rho(\bar\varphi,\bar\Psi) \leq L_k\ \rho(\bar\varphi,\bar\Psi)$. If $\|\bar\varphi(-t_k') - \bar\Psi(-t_k')\|$ $> b^k$ then $\rho(\bar\varphi,\bar\Psi) \geq m_k = b^k$. But then, $b^{-k}\rho(\bar\varphi,\bar\Psi) \geq 1 \Rightarrow 2Hb^{-k}\rho(\bar\varphi,\bar\Psi) \geq$ $2H$; or (since $\| \bar\varphi(-t_k') - \bar\Psi(-t_k')\| \leq 2H$), $\| \bar\varphi(-t_k') - \bar\Psi(-t_k')\| \leq$ $2Hb^{-k}\ \rho(\bar\varphi,\bar\Psi) \leq L\ \rho(\bar\varphi,\bar\Psi)$. Q.E.D.

<u>Lemma 2</u>. If, for $k = 0, 1, 2, \ldots, K$, $-t_k'$ denotes an arbitrary fixed point in the interval $[-t^*_{k+1}, -t^*_k]$, $\alpha_k$ denotes a real number, and $\bar g_k$ denotes a mapping of $C_H( (-\infty,0], E^n )$ into $E^n$ as described in Lemma 1, then if $L_K = \sum_{k=0}^{K} |\alpha_k| \cdot L_k$, where $L_k$ is the Lipschitz constant for each $\bar g_k$ (which is guaranteed to exist, by Lemma 1), then

$$\sum_{k=0}^{K} \alpha_k \|\bar\varphi(-t_k') - \bar\Psi(-t_k')\| \leq L_K\ \rho(\bar\varphi,\bar\Psi),\ \forall\ \bar\varphi,\bar\Psi \in C_H.$$

proof: $\sum_{k=0}^{K} \alpha_k \|\bar{\varphi}(-t_k') - \bar{\psi}(-t_k')\| \le \sum_{k=0}^{K} |\alpha_k| \cdot \|\bar{\varphi}(-t_k') - \bar{\psi}(-t_k')\| \le$

$\sum_{k=0}^{K} |\alpha_k| \cdot L_k\, \rho(\bar{\varphi},\bar{\psi}) = L_K\, \rho(\bar{\varphi},\bar{\psi}),\ \forall\ \bar{\varphi},\bar{\psi} \in C_H.$ Q.E.D.

It should be noted that the constants $L_k$ and $L_K$ of Lemmas 1 and 2 do not depend on the particular point $-t_k'$ specified in the interval $[-t_{k+1}^*, -t_k^*]$.

We now specify the Lipschitz constant L for the mapping $\bar{f}$; that is, we determine the constant L such that $\|\bar{f}(\bar{\varphi}) - \bar{f}(\bar{\psi})\| \le L\, \rho(\bar{\varphi},\bar{\psi})$, $\forall\ \bar{\varphi},\bar{\psi} \in C_H$: Pick the positive integer J such that $j \ge J \Rightarrow (a/b)^j < \frac{1}{2HNn}$ and $a^j < \frac{1}{Nn}$. Let $L_{J-1}$ denote the constant specified in Lemma 2 for the sequence of mappings $\{\bar{g}_k\}$ and constants $\alpha_k = a^k Nn$, $k = 0, \ldots, J-1$, and let $\ell_o$ denote the Lipschitz constant specified in Lemma 1 for $\bar{g}_o(\bar{\varphi}) \equiv \bar{\varphi}(o)$. Then, let $L = \|M_1\| \cdot \ell_o + (\|M_2\| + \|M_3\|)(1 + L_{J-1})$.

Theorem. If L is specified as above then $\forall\ \bar{\varphi},\bar{\psi} \in C_H$, $\|\bar{f}(\bar{\varphi}) - \bar{f}(\bar{\psi})\| \le L\, \rho(\bar{\varphi},\bar{\psi})$, where $\bar{f}$ is the mapping from $C_H(\ (-\infty,0],\ E^n\ )$ to $E^n$ defined by Equation (3-12).

proof: $\forall\ \bar{\varphi},\bar{\psi} \in C_H$:

$$\|\bar{f}(\bar{\varphi}) - \bar{f}(\bar{\psi})\| \le \|M_1\bar{\varphi}(0) - M_1\bar{\psi}(0)\| + \|M_2 \sum_{k=0}^{\infty} A^k \bar{\varphi}_{t_k} - M_2 \sum_{k=0}^{\infty} A^k \bar{\psi}_{t_k}\| +$$

$$\|M_3 \sum_{k=0}^{\infty} A^k \bar{\varphi}_{s_k} - M_3 \sum_{k=0}^{\infty} A^k \bar{\psi}_{s_k}\| \le \|M_1\| \cdot \|\bar{\varphi}(0) - \bar{\psi}(0)\| +$$

$$\|M_2\| \cdot \| \sum_{k=0}^{\infty} A^k (\bar{\varphi}_{t_k} - \bar{\psi}_{t_k})\| + \|M_3\| \cdot \| \sum_{k=0}^{\infty} A^k (\bar{\varphi}_{s_k} - \bar{\psi}_{s_k})\| \le \|M_1\| \cdot \ell_o \rho(\bar{\varphi},\bar{\psi}) +$$

$$\|M_2\|\cdot\|\sum_{k=0}^{\infty}A^k(\bar{\varphi}_{t_k}-\bar{\psi}_{t_k})\|+\|M_3\|\cdot\|\sum_{k=0}^{\infty}A^k(\bar{\varphi}_{s_k}-\bar{\psi}_{s_k})\|.$$

We now show that each of the terms $\|\sum_{k=0}^{\infty}A^k(\bar{\varphi}_{t_k}-\bar{\psi}_{t_k})\|, \|\sum_{k=0}^{\infty}A^k(\bar{\varphi}_{s_k}-\bar{\psi}_{s_k})\|$ is less than or equal to $(1+L_{j-1})\,\rho(\bar{\varphi},\bar{\psi})$, and then the theorem is proved.

$$\|\sum_{k=0}^{\infty}A^k(\bar{\varphi}_{t_k}-\bar{\psi}_{t_k})\| \le \sum_{k=0}^{\infty}\|A\|^k\cdot\|\bar{\varphi}_{t_k}-\bar{\psi}_{t_k}\| = \sum_{k=0}^{\infty}a^k\|\bar{\varphi}_{t_k}-\bar{\psi}_{t_k}\| \le$$

$$\sum_{k=0}^{\infty}a^k\left(\sum_{i=1}^{n}|\varphi_i(-t_k^{(i)})-\psi_i(-t_k^{(i)})|\right) \le \sum_{k=0}^{\infty}a^k\left(\sum_{i=1}^{n}\|\bar{\varphi}(-t_k^{(i)})-\bar{\psi}(-t_k^{(i)})\|\right)$$

$$=\sum_{i=1}^{n}\left(\sum_{k=0}^{\infty}a^k\|\bar{\varphi}(-t_k^{(i)})-\bar{\psi}(-t_k^{(i)})\|\right).$$

For each $i = 1, \dots, n$, $\exists$ a subsequence $\{-t_{k_j}^{(i)}\}$, $j = 0, 1, 2, \dots$, consisting of exactly one point $-t_k^{(i)}$ from each interval $[-t_{j+1}^*, -t_j^*)$ at which $\|\bar{\varphi}(-t_k^{(i)})-\bar{\psi}(-t_k^{(i)})\|$ is a maximum for all of the (at most $N$) points $-t_k^{(i)}$ in the interval. Obviously,

$$\sum_{k=0}^{\infty}a^k\|\bar{\varphi}(-t_k^{(i)})-\bar{\psi}(-t_k^{(i)})\| \le \sum_{j=0}^{\infty}a^jN\|\bar{\varphi}(-t_{k_j}^{(i)})-\bar{\psi}(-t_{k_j}^{(i)})\| \text{ for } i=1,\dots,n.$$ Thus

$$\|\sum_{k=0}^{\infty}A^k(\bar{\varphi}_{t_k}-\bar{\psi}_{t_k})\| \le \sum_{i=1}^{n}\left(\sum_{j=0}^{\infty}a^jN\|\bar{\varphi}(-t_{k_j}^{(i)})-\bar{\psi}(-t_{k_j}^{(i)})\|\right) =$$

$$\sum_{j=0}^{\infty}a^jN\left(\sum_{i=1}^{n}\|\bar{\varphi}(-t_{k_j}^{(i)})-\bar{\psi}(-t_{k_j}^{(i)})\|\right).$$

For each $j = 0, 1, 2, \ldots$, $\exists$ one point, of the n points $\{-t_{k_j}^{(i)}: i=1,\ldots,n\}$ in each interval $[-t_{k+1}^*, -t_k^*]$, at which $\|\bar{\varphi}(-t_{k_j}^{(i)}) - \bar{\psi}(-t_{k_j}^{(i)})\|$ is a maximum. Call it $-t_j'$. Then, $\sum_{i=1}^{n} \|\bar{\varphi}(-t_{k_j}^{(i)}) - \bar{\psi}(-t_{k_j}^{(i)})\| \leq$ $n\|\bar{\varphi}(-t_j') - \bar{\psi}(-t_j')\|$ and hence, $\|\sum_{k=0}^{\infty} A^k(\bar{\varphi}_{t_k} - \bar{\psi}_{t_k})\| \leq \sum_{j=0}^{\infty} a^j Nn\|\bar{\varphi}(-t_j') - \bar{\psi}(-t_j')\|$

$$= \sum_{j=0}^{J-1} a^j Nn\|\bar{\varphi}(-t_j') - \bar{\psi}(-t_j')\| + \sum_{j=J}^{\infty} a^j Nn\|\bar{\varphi}(-t_j') - \bar{\psi}(-t_j')\| \leq L_{J-1}\rho(\bar{\varphi},\bar{\psi})$$

$$+ \sum_{j=J}^{\infty} a^j Nn\|\bar{\varphi}(-t_j') - \bar{\psi}(-t_j')\|,$$ by Lemma 2. But, since $j \geq J \Rightarrow b^j > 2HNna^j$, $a^j Nn\|\bar{\varphi}(-t_j') - \bar{\psi}(-t_j')\| \leq a^j Nn \cdot 2H < b^j$ for all $j \geq J$. Therefore (since also $a^j Nn < 1$), $j \geq J \Rightarrow m_j = \min\{b^j, \sup(\|\bar{\varphi}(t) - \bar{\psi}(t)\|: -t_{j+1}^* \leq t \leq -t_j^*)\} \geq \min\{b^j, \sup(a^j Nn\|\bar{\varphi}(t) - \bar{\psi}(t)\|: -t_{j+1}^* \leq t \leq -t_j^*)\}$ $\geq \min\{a^j Nn\|\bar{\varphi}(-t_j') - \bar{\psi}(-t_j')\|, \sup(a^j Nn\|\bar{\varphi}(t) - \bar{\psi}(t)\|: -t_{j+1}^* \leq t \leq -t_j^*)\}$ $= a^j Nn\|\bar{\varphi}(-t_j') - \bar{\psi}(-t_j')\|$. Thus, $\rho(\bar{\varphi},\bar{\psi}) = \sum_{j=0}^{\infty} m_j \geq \sum_{j=J}^{\infty} m_j$

$$\geq \sum_{j=J}^{\infty} a^j Nn\|\bar{\varphi}(-t_j') - \bar{\psi}(-t_j')\|.$$

Thus,

$$\|\sum_{k=0}^{\infty} A^k(\bar{\varphi}_{t_k} - \bar{\psi}_{t_k})\| \leq L_{J-1}\,\rho(\bar{\varphi},\bar{\psi}) + \rho(\bar{\varphi},\bar{\psi}) = (L_{J-1} + 1)\,\rho(\bar{\varphi},\bar{\psi}).$$

The same technique exactly proves that

$$\|\sum_{k=0}^{\infty} A^k(\bar{\varphi}_{s_k} - \bar{\psi}_{s_k})\| \leq (L_{J-1} + 1)\rho(\bar{\varphi},\bar{\psi}).$$

Q.E.D.

## Chapter IV

## STATE EQUATIONS FOR A CLASS OF NONLINEAR DISTRIBUTED NETWORKS

In this chapter we define the class of nonlinear distributed networks for which our stability criteria apply. We also show that the behavior of any network in this class is determined by a system of functional-differential equations having the form

$$\dot{\bar{x}}(t) = \bar{f}(\bar{x}_t), \qquad t \geq 0,$$

where the state of the system at time $t \geq 0$ is represented by $\bar{x}_t$, a point in the space $C_H( (-\infty,0], E^n )$, and the function $\bar{f}$ is continuous and locally Lipschitzian. The form of the function $\bar{f}$ is also given, so that if one selects any network from the given class it will be evident how to construct the particular functional-differential equation which determines its behavior. We first consider the writing of state equations for lumped networks.

### 1. Lumped Networks and State Variables

Much has been written on the subject of writing state equations for lumped networks. See, for example, references [27] through [36]. Although both linear and nonlinear networks have been considered, we restrict our attention, for the moment, to lumped linear networks. Let the integer n denote the number of independent voltage and current sources in a given linear network. We may then consider the network to be a lumped linear n-port, containing no independent sources, with independent voltage and current sources connected at each port. For most such networks it is possible to designate a certain collection of the n-port's branch voltages and currents as the "state variables" of the network. These state variables have the property that the voltage across or the current through any branch of the network has a unique representation as a linear combination of the state variables and the independent source voltages and currents. Thus, the behavior of the network is completely determined if the behavior of the state variables and the independent sources is known.

Usually it is possible, and convenient, to select as state variables the voltages across capacitors and the currents through inductors in the n-port. We shall not dwell on the matter of when it is possible to select such a set of state variables to characterize a given linear network since this matter has received much attention in the literature [27, 28, 29, 30, 31, 32].

When it is possible to characterize a linear network having no mutually coupled inductors by a set of state variables as described above, we may write the system of linear differential equations

$$P\,\dot{\bar{x}}(t) = A\,\bar{x}(t) + B\,\bar{u}(t), \qquad t \geq 0, \qquad (4\text{-}1)$$

where: $\bar{x}(t)$ denotes the state vector, a vector whose components are the state variables for the network (the voltages across capacitors and the currents through inductors); $\bar{u}(t)$ denotes the vector whose components are the values of the independent sources; P denotes a diagonal matrix in which each diagonal element $p_{ii}$ is equal to the value of the reactive element (capacitor or inductor) corresponding to the i-th state variable.

We note that P is a positive definite symmetric matrix. Thus, the left-hand side of Equations (4-1) is equivalent to a vector whose components are equal to the values of the currents through capacitive branches and the voltages across inductive branches in the network. The right-hand side of Equations (4-1) is the expression of these same currents and voltages in terms of the state variables and independent sources. Thus, the rows of the matrices A and B are composed of the coefficients of the unique linear combinations of the state variables and the independent sources which are equal to the corresponding voltages and currents on the left-hand side.

In case mutual inductances are present in the n-port it may still be possible to choose a set of state variables as specified above and write Equations (4-1); however, P will no longer be a diagonal matrix. For all physically realizable n-ports the values of the coefficients of mutual inductance will be such that P will still be positive definite and symmetric.

In case the network possesses loops which contain only capacitors we also find that the matrix P may not be diagonal, but may be positive definite and symmetric. For example, for the network of Figure 4.1

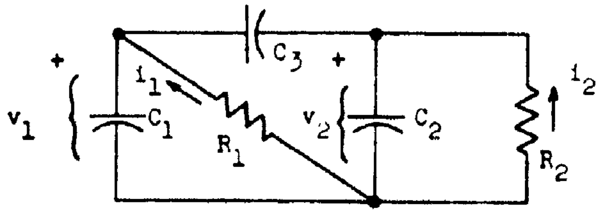


Figure 4.1. A network with a capacitive loop.

we may choose $\bar{x}(t) = (v_1(t),\ v_2(t)\ )^t$ and write

$$\begin{bmatrix} (C_1 + C_3) & -C_3 \\ -C_3 & (C_2 + C_3) \end{bmatrix} \dot{\bar{x}}(t) = \begin{bmatrix} -1/R_1 & 0 \\ 0 & -1/R_2 \end{bmatrix} \bar{x}(t).$$

The left-hand side of this equation is equivalent to the vector $(i_1(t), i_2(t)\ )^t$. Clearly, P is positive definite and symmetric. A similar remark can be made for networks having cut sets which contain only inductors.

In certain of the cases mentioned above, and sometimes when dependent sources are present in the network, it is convenient to choose as state variables linear combinations of certain branch voltages and currents. In any event, if some set of state variables may be chosen and Equations (4-1) written with a nonsingular P matrix then, upon multiplying both sides by $P^{-1}$, we obtain an equivalent set of equations in the form of Equations (4-1), with the new P matrix (the identity matrix) positive definite and symmetric.

In addition to Equations (4-1) we may also write

$$\bar{w}(t) = C\ \bar{x}(t) + D\ \bar{u}(t), \qquad (4\text{-}2)$$

where $\bar{w}(t)$ is a vector whose elements are the remaining port variables (those not included in $\bar{u}(t)$), and the matrices C and D are constructed in such a manner as to give the appropriate linear combinations of state variables and independent sources to represent these port variables.

If we consider a lumped linear multiport network containing no independent voltage and current sources, and assume that at each port one of the port variables (the port voltage and the current into the port) is specified independently, then it may be possible to write

Equations (4-1) and (4-2) as described above, where $\bar{u}(t)$ is the vector whose components are the independent port variables and $\bar{w}(t)$ is the vector containing the remaining port variables. Taking the Laplace transform of Equations (4-1) and (4-2) we easily obtain

$$\bar{W}(s) = [C(sP - A)^{-1} B + D] \bar{U}(s).$$

At $s = \infty$,

$$\bar{W}(s) = D\bar{U}(s).$$

Thus, if we consider the input variables to the multiport to be the components of the vector $\bar{u}$ and the output variables to be the components of the vector $\bar{w}$, the matrix D is the transmission matrix for the multiport when all of its capacitors are short circuited and all of its inductors are open circuited. If, for every pair of distinct ports there exists a zero of transmission at $s = \infty$, then D is a diagonal matrix.

## 2. A Class of Nonlinear Distributed Networks

Many nonlinear distributed networks may be represented as in Figure 4-2. This network consists of three main parts: One part is a lumped linear multiport which is connected to each of the other parts only at its ports. The second part consists of the collection of lossless transmission lines. This part is divided into three groups as explained below. Each end of each line is connected to one of the ports of the linear multiport. The remaining ports of the linear multiport are connected to the third part of the network, a nonlinear multiport.

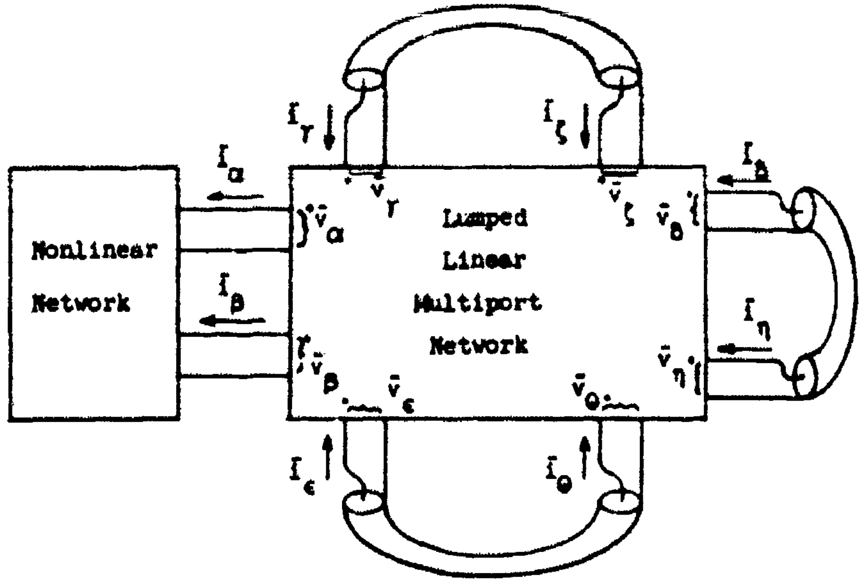


Figure 4.2. A typical nonlinear distributed network.

The nonlinear multiport is characterized as follows: We suppose that there are $n_\alpha$ ports for which the voltage is the independent variable, and $n_\beta$ ports for which the current is the independent variable, and let $\bar{v}_\alpha$ and $\bar{i}_\beta$ denote vectors whose components are the values of these port variables. Then, the remaining port variables are specified by a nonlinear function $\bar{U}$:

$$\begin{pmatrix} \bar{i}_\alpha \\ \bar{v}_\beta \end{pmatrix} = \bar{U} \left( \begin{pmatrix} \bar{v}_\alpha \\ \bar{i}_\beta \end{pmatrix} \right). \tag{4-3}$$

We assume that:

(A1) It is possible to specify vectors $\bar{u}$ and $\bar{w}$, $\bar{u}$ being a vector whose components consist of one port variable from each port of the lumped linear multiport, including the components of $\bar{i}_\alpha$ and $\bar{v}_\beta$, and $\bar{w}$ being a vector whose components consist of the remaining port variables, and a state vector $\bar{x}$ such that the resulting linear network may be characterized by Equations (4-1) and (4-2), with P a positive definite symmetric matrix, and such that there exists a zero of transmission at $s = \infty$ from each port of the lumped linear multiport to any other port at which a transmission line is connected, and vice versa.

For a given network, in order that the lumped linear multiport have the required transmission zeros as specified in condition (A1) above, it may be necessary that at certain ports a specific port variable be assumed independent. The choice of independent port variables divides the collection of transmission lines into three groups: One group contains all of the lines which have both ends connected to ports for which the current into the lumped linear multiport is chosen as the independent port variable. Another group contains all of the lines which have one end connected to a port for which the current into the multiport is the independent port variable, and the other end connected to a port for which the port voltage is the independent variable. The third group contains all of the remaining transmission lines, each of whose ends is connected to a port at which the port voltage is chosen as the independent port variable.

We let

$$\bar{u} = \begin{pmatrix} \bar{i}_\alpha \\ \bar{v}_\beta \\ \bar{i}_\gamma \\ \bar{i}_\delta \\ \bar{v}_\epsilon \\ \bar{i}_\zeta \\ \bar{v}_\eta \\ \bar{v}_\theta \end{pmatrix}, \text{ and } \quad \bar{w} = \begin{pmatrix} \bar{v}_\alpha \\ \bar{i}_\beta \\ \bar{v}_\gamma \\ \bar{v}_\delta \\ \bar{i}_\epsilon \\ \bar{v}_\zeta \\ \bar{i}_\eta \\ \bar{i}_\theta \end{pmatrix},$$

in Equations (4-1) and (4-2), where $\bar{i}_\gamma$ and $\bar{i}_\zeta$ denote $n_\gamma$ and $n_\zeta$ vectors ($n_\gamma = n_\zeta$) whose components are the independent port variables at those ports of the lumped linear multiport to which a transmission line of the first group is connected, etc. If $\bar{x}$ is an n-vector then P and A are $n \times n$ matrices, B is an $n \times (n_\alpha + n_\beta + \ldots + n_\theta)$ matrix, C is an $(n_\alpha + n_\beta + \ldots + n_\theta) \times n$ matrix, and D is an $(n_\alpha + n_\beta + \ldots + n_\theta) \times (n_\alpha + n_\beta + \ldots + n_\theta)$ matrix. We let B and C be partitioned in the following manner:

$$B = [B_I \,\vdots\, B_{II}],\quad B_I = [B_\alpha \,\vdots\, B_\beta],\quad B_{II} = [B_\gamma \,\vdots\, B_\delta \,\vdots\, B_\epsilon \,\vdots\, B_\zeta \,\vdots\, B_\eta \,\vdots\, B_\theta],$$

$$C = \begin{bmatrix} C_I \\ \hdashline C_{II} \end{bmatrix},\quad C_I = \begin{bmatrix} C_\alpha \\ \hdashline C_\beta \end{bmatrix},\quad C_{II} = \begin{bmatrix} C_\gamma \\ \hdashline C_\delta \\ \hdashline C_\epsilon \\ \hdashline C_\zeta \\ \hdashline C_\eta \\ \hdashline C_\theta \end{bmatrix},$$

where $B_I$ is an $n \times (n_\alpha + n_\beta)$ matrix, $B_{II}$ is an $n \times (n_\gamma + \ldots + n_\theta)$ matrix, $B_\gamma$ is an $n \times n_\gamma$ matrix, etc. Finally, we let $B_\gamma, B_\delta, \ldots, B_\theta$ be partitioned by columns as

$$B_\gamma = [B_{\gamma_1} \,\vdots\, B_{\gamma_2} \,\vdots\, \cdots \,\vdots\, B_{\gamma_{n_\gamma}}],\quad B_\delta = [B_{\delta_1} \,\vdots\, B_{\delta_2} \,\vdots\, \cdots \,\vdots\, B_{\delta_{n_\delta}}],\ \text{etc.},$$

and let $C_\gamma, C_\delta, \ldots, C_\theta$ be partitioned by rows as

$$C_\gamma = \begin{bmatrix} C_{\gamma_1} \\ \hdashline C_{\gamma_2} \\ \hdashline \vdots \\ \hdashline C_{\gamma_{n_\gamma}} \end{bmatrix},\qquad C_\delta = \begin{bmatrix} C_{\delta_1} \\ \hdashline C_{\delta_2} \\ \hdashline \vdots \\ \hdashline C_{\delta_{n_\delta}} \end{bmatrix},\ \text{etc.}$$

We also assume that the matrix D has the following form:

$$D = \begin{bmatrix} D_I & & & \bigcirc & \\ & D_\gamma & & & \\ & & D_\delta & & \\ \bigcirc & & & \ddots & \\ & & & & D_\theta \end{bmatrix},$$

where $D_I$ is an $(n_\alpha + n_\beta) \times (n_\alpha + n_\beta)$ matrix, $D_\gamma$ is an $n_\gamma \times n_\gamma$ matrix, etc. Condition (A1) above specifies that $D_\gamma$, $D_\delta$, ..., $D_\theta$ be diagonal matrices.

$$D_\gamma = \begin{bmatrix} d_{\gamma_1} & & & \bigcirc \\ & d_{\gamma_2} & & \\ & & \ddots & \\ \bigcirc & & & d_{\gamma_{n_\gamma}} \end{bmatrix}, \text{ etc.}$$

We also assume that:

(A2) Each of the diagonal elements of $D_\gamma$, $D_\delta$, ..., $D_\theta$ is a nonnegative real number; and for each transmission line, at least one of the two diagonal elements of the D matrix which correspond to the ports to which the line is connected to the lumped linear multiport, is a positive real number.

From Equations (4-1), (4-2), and (4-3), we obtain

$$\begin{pmatrix} \bar{i}_\alpha \\ \bar{v}_\beta \end{pmatrix} = \mathfrak{F}\left(C_I \bar{x} + D_I \begin{pmatrix} \bar{i}_\alpha \\ \bar{v}_\beta \end{pmatrix}\right). \qquad (4\text{-}4)$$

We assume that:

(A3) It is possible to solve Equation (4-4) for the vector $(\bar{i}_\alpha^t, \bar{v}_\beta^t)^t$ as an explicit function of $C_I \bar{x}$ in some neighborhood of $C_I \bar{x} = \bar{0}$. That is, in some neighborhood of $\bar{x} = \bar{0}$, $G_H = \{\bar{x} : \bar{x} \in E^n, \|\bar{x}\| < H\}$, we assume that there exists some function $\mathfrak{F}^*$ such that

$$\begin{pmatrix} \bar{i}_\alpha \\ \bar{v}_\beta \end{pmatrix} = \mathfrak{F}^*(C_I\bar{x}). \tag{4-5}$$

We further assume that:

(A4) Equation (4-5) satisfies a Lipschitz condition in $G_H$, and $\mathfrak{F}^*(\bar{0}) = \bar{0}$.

If conditions (A1) through (A4) above are satisfied for a given nonlinear distributed network then it is a member of the class of networks for which our stability theory applies. It is felt that most nonlinear distributed networks that one is likely to encounter will satisfy the above four conditions. If, however, a given network fails to satisfy one or more of these conditions, the following techniques might still be used to render it amenable to the application of our theory.

If D is not a block diagonal matrix with diagonal $D_\gamma$, $D_\delta$, . . , $D_\theta$ submatrices it may often be consistent with physical reality to consider the presence of small "stray" reactances at the ports of the lumped linear multiport. These reactances will have the effect of giving the necessary zeros of transmission at $s = \infty$. The addition of small stray reactances at those ports where the lumped linear and nonlinear multiports are connected will always allow condition (A3) to be satisfied; for, by adding enough strays, the matrix $D_I$ may be made to contain all zeros, and hence $\mathfrak{F}^* \equiv \mathfrak{F}$. Finally, if the function $\mathfrak{F}^*$ does not satisfy the required Lipschitz condition it might be satisfactory to approximate $\mathfrak{F}^*$ by some function which does--a polynomial, perhaps.

In our theory we consider only undriven nonlinear distributed networks; that is, the networks are assumed to contain no independent voltage and current sources. All of our networks have an equilibrium state $\bar{x}(t) = \bar{0}$, and it is the stability of this equilibrium which we study. If a network contains independent sources which are constant for all time (bias voltages, for example), and if it has an equilibrium state other than $\bar{0}$, it may still be possible to study the stability of this equilibrium by first finding an "equivalent" network with a corresponding equilibrium at $\bar{x} = \bar{0}$. For example, in Figure 4.3 we have shown such network; this network has three equilibrium points,

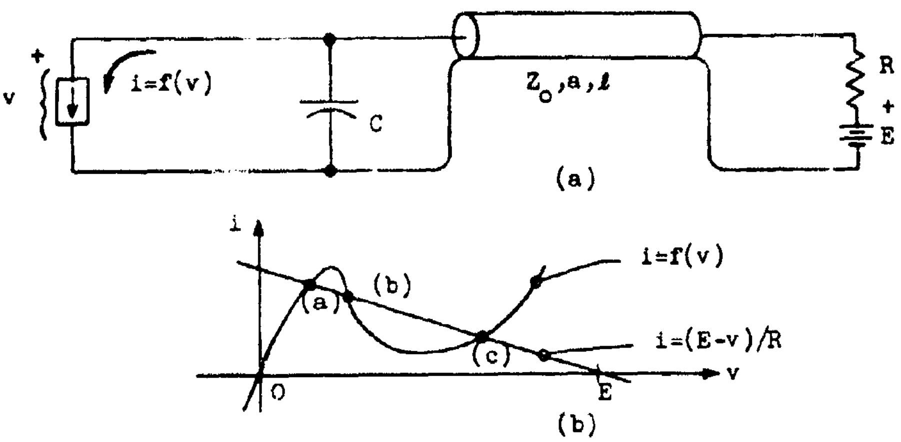


Figure 4.3. A nonlinear distributed network with a bias voltage.

labeled (a), (b), and (c). We may study the stability of any of these equilibria by considering the equivalent network, Figure 4.3a with $E = 0$ and nonlinear function f described by the curve of Figure 4.3b with the origin of the v-i coordinates shifted to the appropriate point (either (a), (b), or (c)).

We shall now derive the system of functional-differential equations which describes the behavior of any network in our class

3. A Functional-Differential Equation

For any nonlinear distributed network in the class described above we have, from Equations (4-2), for $j = 1, 2, \ldots, n_\gamma = n_\zeta$,

$$v_{\gamma_j}(t) = c_{\gamma_j}\bar{x}(t) + d_{\gamma_j} i_{\gamma_j}(t),$$

and

$$v_{\zeta_j}(t) = c_{\zeta_j}\bar{x}(t) + d_{\zeta_j} i_{\zeta_j}(t).$$

Thus, we can represent our network, as far as the behavior of the j-th transmission line connecting the $\gamma$ and $\zeta$ ports is concerned, by the network of Figure 4.4a. Hence, since $0 < d_{\gamma_j} < \infty$ and/or $0 < d_{\zeta_j} < \infty$, we have (using the functional equations for the transmission line two-ports)

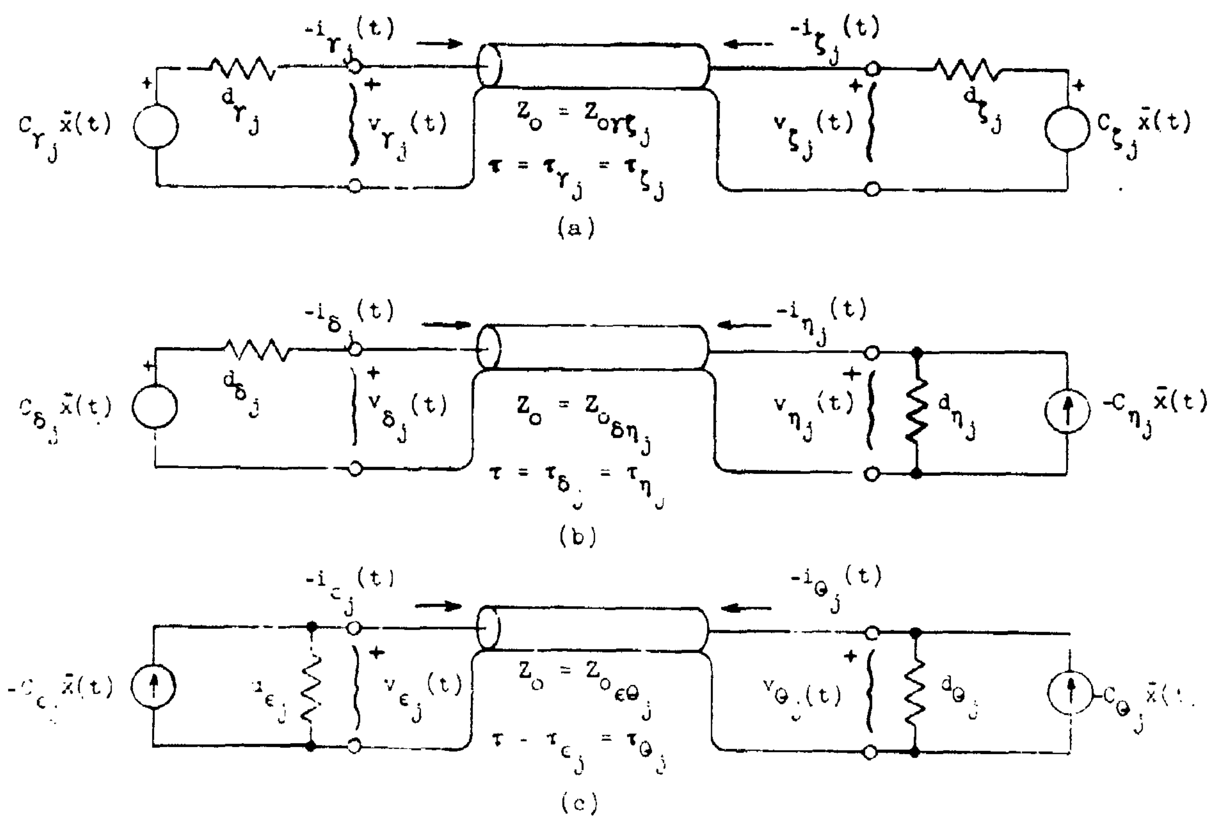


Figure 4.4. Equivalent networks connected to typical transmission lines.

$$i_{\Upsilon_j}(t) = -\lambda_{\Upsilon_j} C_{\Upsilon_j}\bar{x}(t) + 2\mu_{\Upsilon_j}\sum_{k=0}^{\infty}\rho_{\Upsilon_j}^k C_{\Upsilon_j}\bar{x}(t - 2(k+1)\tau_{\Upsilon_j})$$

$$+ 2\nu_{\Upsilon_j}\sum_{k=0}^{\infty}\rho_{\Upsilon_j}^k C_{\zeta_j}\bar{x}(t - (2k+1)\tau_{\Upsilon_j}),$$

$$i_{\zeta_j}(t) = -\lambda_{\zeta_j} C_{\zeta_j}\bar{x}(t) + 2\mu_{\zeta_j}\sum_{k=0}^{\infty}\rho_{\zeta_j}^k C_{\zeta_j}\bar{x}(t - 2(k+1)\tau_{\zeta_j})$$

$$+ 2\nu_{\zeta_j}\sum_{k=0}^{\infty}\rho_{\zeta_j}^k C_{\Upsilon_j}\bar{x}(t - (2k+1)\tau_{\zeta_j}),$$

where we have defined

$$\lambda_{\Upsilon_j} = \frac{1}{d_{\Upsilon_j} + Z_{o_{\Upsilon\zeta_j}}}, \qquad \lambda_{\zeta_j} = \frac{1}{d_{\zeta_j} + Z_{o_{\Upsilon\zeta_j}}}$$

$$\mu_{\Upsilon_j} = \frac{Z_{o_{\Upsilon\zeta_j}}\Gamma_{\zeta_j}}{(d_{\Upsilon_j} + Z_{o_{\Upsilon\zeta_j}})^2}, \qquad \mu_{\zeta_j} = \frac{Z_{o_{\Upsilon\zeta_j}}\Gamma_{\Upsilon_j}}{(d_{\zeta_j} + Z_{o_{\Upsilon\zeta_j}})^2},$$

$$\nu_{\Upsilon_j} = \nu_{\zeta_j} = \frac{Z_{o_{\Upsilon\zeta_j}}}{(d_{\Upsilon_j} + Z_{o_{\Upsilon\zeta_j}})(d_{\zeta_j} + Z_{o_{\Upsilon\zeta_j}})},$$

and

$$\rho_{\Upsilon_j} = \rho_{\zeta_j} = \Gamma_{\Upsilon_j}\Gamma_{\zeta_j} = \left(\frac{d_{\Upsilon_j} - Z_{o_{\Upsilon\zeta_j}}}{d_{\Upsilon_j} + Z_{o_{\Upsilon\zeta_j}}}\right)\left(\frac{d_{\zeta_j} - Z_{o_{\Upsilon\zeta_j}}}{d_{\zeta_j} + Z_{o_{\Upsilon\zeta_j}}}\right).$$

Similarly, for $j = 1, 2, \ldots, n_\delta = n_\eta$, we have

$$v_{\delta_j}(t) = C_{\delta_j}\bar{x}(t) + d_{\delta_j} i_{\delta_j}(t),$$

and

$$i_{\eta_j}(t) = C_{\eta_j}\bar{x}(t) + d_{\eta_j} v_{\eta_j}(t).$$

Thus, we can represent our network, as far as the behavior of the j-th transmission line connecting the $\delta$ and $\eta$ ports is concerned by the network of Figure 4.4b. Hence, since $0 < d_{\delta_j} < \infty$ and/or $0 < d_{\eta_j} < \infty$, we have

$$i_{\delta_j}(t) = -\lambda_{\delta_j} C_{\delta_j}\bar{x}(t) + 2\mu_{\delta_j} \sum_{k=0}^{\infty} \rho_{\delta_j}^k C_{\delta_j}\bar{x}(t - 2(k+1)\tau_{\delta_j})$$

$$+ 2\nu_{\delta_j} \sum_{k=0}^{\infty} \rho_{\delta_j}^k C_{\eta_j}\bar{x}(t - (2k+1)\tau_{\delta_j}),$$

$$v_{\eta_j}(t) = -\lambda_{\eta_j} C_{\eta_j}\bar{x}(t) + 2\mu_{\eta_j} \sum_{k=0}^{\infty} \rho_{\eta_j}^k C_{\eta_j}\bar{x}(t - 2(k+1)\tau_{\eta_j})$$

$$+ 2\nu_{\eta_j} \sum_{k=0}^{\infty} \rho_{\eta_j}^k C_{\delta_j}\bar{x}(t - (2k+1)\tau_{\eta_j})$$

where we have defined

$$\lambda_{\delta_j} = \frac{1}{d_{\delta_j} + Z_{o_{\delta\eta_j}}}, \qquad \lambda_{\eta_j} = \frac{1}{d_{\eta_j} + Y_{o_{\delta\eta_j}}},$$

$$\mu_{\delta_j} = \frac{Z_{o_{\delta\eta_j}} \Gamma_{\eta_j}}{(d_{\delta_j} + Z_{o_{\delta\eta_j}})^2}, \qquad \mu_{\eta_j} = \frac{-Y_{o_{\delta\eta_j}} \Gamma_{\delta_j}}{(d_{\eta_j} + Y_{o_{\delta\eta_j}})^2},$$

$$\nu_{\delta_j} = -\nu_{\eta_j} = \frac{-1}{(d_{\delta_j} + Z_{o_{\delta\eta_j}})(d_{\eta_j} + Y_{o_{\delta\eta_j}})},$$

and

$$\rho_{\delta_j} = \rho_{\eta_j} = \Gamma_{\delta_j}\Gamma_{\eta_j} = \left(\frac{d_{\delta_j} - Z_{o_{\delta\eta_j}}}{d_{\delta_j} + Z_{o_{\delta\eta_j}}}\right)\left(\frac{Y_{o_{\delta\eta_j}} - d_{\eta_j}}{Y_{o_{\delta\eta_j}} + d_{\eta_j}}\right).$$

Similarly, for $j = 1, 2, \ldots, n_\epsilon = n_\theta$, we have

$$i_{\epsilon_j}(t) = C_{\epsilon_j}\bar{x}(t) + d_{\epsilon_j}v_{\epsilon_j}(t),$$

and

$$i_{\theta_j}(t) = C_{\theta_j}\bar{x}(t) + d_{\theta_j}v_{\theta_j}(t).$$

Thus, we can represent our network, as far as the behavior of the j-th transmission line connecting the $\epsilon$ and $\theta$ ports is concerned by the network of Figure 4.4c. Hence, since $0 < d_{\epsilon_j} < \infty$ and/or $0 < d_{\theta_j} < \infty$, we have

$$v_{\epsilon_j}(t) = -\lambda_{\epsilon_j}C_{\epsilon_j}\bar{x}(t) + 2\mu_{\epsilon_j}\sum_{k=0}^{\infty}\rho_{\epsilon_j}^k C_{\epsilon_j}\bar{x}(t - 2(k+1)\tau_{\epsilon_j})$$

$$+ 2\nu_{\epsilon_j}\sum_{k=0}^{\infty}\rho_{\epsilon_j}^k C_{\theta_j}\bar{x}(t - (2k+1)\tau_{\epsilon_j}),$$

$$v_{\theta_j}(t) = -\lambda_{\theta_j}C_{\theta_j}\bar{x}(t) + 2\mu_{\theta_j}\sum_{k=0}^{\infty}\rho_{\theta_j}^k C_{\theta_j}\bar{x}(t - 2(k+1)\tau_{\theta_j})$$

$$+ 2\nu_{\theta_j}\sum_{k=0}^{\infty}\rho_{\theta_j}^k C_{\epsilon_j}\bar{x}(t - (2k+1)\tau_{\theta_j}),$$

where we have defined

$$\lambda_{\epsilon_j} = \frac{1}{d_{\epsilon_j} + Y_{o_{\epsilon\theta_j}}}, \qquad \lambda_{\theta_j} = \frac{1}{d_{\theta_j} + Y_{o_{\epsilon\theta_j}}},$$

$$\mu_{\epsilon_j} = \frac{-Y_{o_{\epsilon\theta_j}} \Gamma_{\theta_j}}{(d_{\epsilon_j} + Y_{o_{\epsilon\theta_j}})^2}, \qquad \mu_{\theta_j} = \frac{-Y_{o_{\epsilon\theta_j}} \Gamma_{\epsilon_j}}{(d_{\theta_j} + Y_{o_{\epsilon\theta_j}})^2},$$

$$\nu_{\epsilon_j} = \nu_{\theta_j} = \frac{-Y_{o_{\epsilon\theta_j}}}{(d_{\epsilon_j} + Y_{o_{\epsilon\theta_j}})(d_{\theta_j} + Y_{o_{\epsilon\theta_j}})},$$

and

$$\rho_{\epsilon_j} = \rho_{\theta_j} = \Gamma_{\epsilon_j} \Gamma_{\theta_j} = \left( \frac{Y_{o_{\epsilon\theta_j}} - d_{\epsilon_j}}{Y_{o_{\epsilon\theta_j}} + d_{\epsilon_j}} \right) \left( \frac{Y_{o_{\epsilon\theta_j}} - d_{\theta_j}}{Y_{o_{\epsilon\theta_j}} + d_{\theta_j}} \right).$$

Thus, if we let $\Omega = \{\gamma_1, \gamma_2, \ldots, \gamma_{n_\gamma}, \delta_1, \delta_2, \ldots, \delta_{n_\delta}, \epsilon_1, \ldots, \theta_{n_\theta}\}$, and define $\xi' = \xi'(\xi)$ on $\Omega$ as follows:

$$\xi' = \zeta_j \text{ for } \xi = \gamma_j \quad (j = 1, 2, \ldots, n_\gamma = n_\zeta),$$

$$\xi' = \eta_j \text{ for } \xi = \delta_j \quad (j = 1, 2, \ldots, n_\delta = n_\eta),$$

$$\xi' = \theta_j \text{ for } \xi = \epsilon_j \quad (j = 1, 2, \ldots, n_\epsilon = n_\theta),$$

$$\xi' = \gamma_j \text{ for } \xi = \zeta_j \quad (j = 1, 2, \ldots, n_\zeta = n_\gamma),$$

$$\xi' = \delta_j \text{ for } \xi = \eta_j \quad (j = 1, 2, \ldots, n_\eta = n_\delta),$$

$$\xi' = \epsilon_j \text{ for } \xi = \theta_j \quad (j = 1, 2, \ldots, n_\theta = n_\epsilon),$$

we have

$$[B_\gamma \mid B_\delta \mid \cdots \mid B_\theta] \begin{pmatrix} \bar{i}_\gamma \\ \bar{i}_\delta \\ \bar{v}_\epsilon \\ \bar{i}_\zeta \\ \bar{v}_\eta \\ \bar{v}_\theta \end{pmatrix} = \sum_{\xi\in\Omega} \Big[ -\lambda_\xi B_\xi C_\xi \bar{x}(t) +$$

$$2\sum_{k=0}^{\infty} \rho_\xi^k \mu_\xi B_\xi C_\xi \bar{x}(t - 2(k+1)\tau_\xi)$$

$$+ 2\sum_{k=0}^{\infty} \rho_\xi^k \nu_\xi B_\xi C_\xi \bar{x}(t - (2k+1)\tau_\xi)\Big].$$

Substituting the above equation and Equation (4-5) into Equation (4-1), we obtain

$$P\dot{\bar{x}}(t) = A\bar{x}(t) + B_I\, \mathcal{G}*(C_I\bar{x}(t)) - \sum_{\xi\in\Omega} Q_\xi \bar{x}(t)$$

$$+ 2\sum_{\xi\in\Omega}\sum_{k=0}^{\infty} \rho_\xi^k M_\xi \bar{x}(t - 2(k+1)\tau_\xi)$$

$$+ 2\sum_{\xi\in\Omega}\sum_{k=0}^{\infty} \rho_\xi^k N_\xi \bar{x}(t - (2k+1)\tau_\xi), \qquad (4\text{-}6)$$

where we have defined, for $\xi\in\Omega$,

$$Q_\xi = \lambda_\xi B_\xi C_\xi, \qquad M_\xi = \mu_\xi B_\xi C_\xi, \qquad N_\xi = \nu_\xi B_\xi C_\xi. .$$

Multiplying each side of Equation (4-6) by $P^{-1}$ (which exists since P is positive definite, by condition (A1)) yields a functional-differential equation of the desired form which describes the behavior of the network.

4. The Lipschitz Condition

It is easily shown that the right-hand side of the functional-differential equation which describes the behavior of any network in our class satisfies a Lipschitz condition on $C_H$: If we let $\bar{x}_t$ be a point in $C_H$, then by Lemma 1 of Section 3-3, the mapping $\bar{f}$ from $C_H$ to $E_n$ defined by $\bar{f}(\bar{x}_t) \equiv \bar{x}_t(0) = \bar{x}(t)$, satisfies a Lipschitz condition. Thus, so does the mapping $\bar{f}_A$ defined by $\bar{f}_A(\bar{x}_t) \equiv P^{-1}A\bar{x}(t)$, since if there exists L such that $\|\bar{\varphi}(0)-\bar{\psi}(0)\| \leq L\rho(\bar{\varphi},\bar{\psi})$ for every $\bar{\varphi}$, $\bar{\psi} \cdot C_H$, then $\|P^{-1}A\bar{\varphi}(0) - P^{-1}A\bar{\psi}(0)\| \leq \|P^{-1}A\|\cdot\|\bar{\varphi}(0)-\bar{\psi}(0)\| \leq (\|P^{-1}A\|\cdot L)\rho(\bar{\varphi},\bar{\psi})$. Also, the mapping $\bar{f}(\bar{x}_t) \equiv \bar{\mathcal{B}}^*(C_I\bar{x}(t))$ satisfies a Lipschitz condition, by condition (A4), and hence so does the mapping $\bar{f}_I(\bar{x}_t) \equiv P^{-1}B_I\bar{\mathcal{B}}^*(C_I\bar{x}(t))$. Finally for each $\xi$, the mapping $\bar{f}_\xi$ of $C_H$ into $E_n$ defined by

$$\bar{f}_\xi(\bar{x}_t) = -P^{-1}Q_\xi\bar{x}(t)$$

$$+ 2P^{-1}M_\xi \sum_{k=0}^{\infty}[\rho_\xi I_n]^k \,\bar{x}(t - 2(k+1)\tau_\xi)$$

$$+ 2P^{-1}N_\xi \sum_{k=0}^{\infty}[\rho_\xi I_n]^k \,\bar{x}(t - (2k+1)\tau_\xi)$$

has the form of Equation (3-12), which has been shown to satisfy a Lipschitz condition. Thus, a simple application of the triangle inequality shows that the mapping $\bar{f} \equiv \bar{f}_A + \bar{f}_I + \sum_{\xi\in\Omega} \bar{f}_\xi$, which is the right-hand side of our functional-differential equation, satisfies a Lipschitz condition.

## Chapter V

## STABILITY OF NONLINEAR DISTRIBUTED NETWORKS

It is obvious that $\bar{x} = \bar{0}$ is an equilibrium solution of the functional-differential equation which describes the behavior of any network in the class of nonlinear distributed networks defined in the previous chapter. In this chapter we state and prove several theorems concerning the stability and instability of this equilibrium solution. We use the same notation as in Chapter IV and consider our distributed networks to be characterized by Equation (4-6), with $\bar{x}_t \in C_H$.

### 1. A Lyapunov Functional

Before defining our Lyapunov functional, V, we shall prove several useful lemmas.

**Lemma 1.** If A is an $n \times m$ matrix and if $B = AA^t$, then $\|B\| = 0$ if and only if $A = 0$.

Proof: ("If") Proof of this part is trivial. ("Only if") Since B is symmetric there exists a nonsingular matrix P such that $P^{-1}BP = \Lambda$, where $\Lambda$ is a diagonal matrix whose diagonal elements are the eigenvalues of B. Since B is positive semidefinite all elements of the main diagonal are nonnegative; but since $\|B\| = 0$, all elements of the main diagonal are, in fact, zero. Thus, $\Lambda = 0$ and hence, $B = P\Lambda P^{-1} = 0$. If $B_{ij}$ represents the element of B in the i-th row and j-th column, and similarly for A,

$$B_{ij} = \sum_{k=1}^{m} A_{ik}A_{jk}.$$

In particular, for $i = 1, \ldots, n$, $B_{ii} = \sum_{k=1}^{m} A_{ik}^2$ and hence, for $k = 1, \ldots, m$, $A_{ik} = 0$. Thus, $A = 0$. Q.E.D.

__Lemma 2.__ If A, B, and $\rho$ are any real numbers such that $A > 0$ and $|\rho| < 1$, and $\bar{x}$ and $\bar{y}$ are arbitrary n-vectors, then the following inequality holds:

$$2B\rho^k\langle \bar{x},\bar{y}\rangle - A|\rho|^k\langle \bar{y},\bar{y}\rangle \le \frac{B^2}{A}|\rho|^k\langle \bar{x},\bar{x}\rangle, \qquad \text{for } k = 0, 1, 2, \ldots$$

__Proof:__ By the Schwarz inequality,

$$|\langle \bar{x},\bar{y}\rangle| \le \|\bar{x}\|\cdot\|\bar{y}\|, \quad \forall \ \bar{x}, \bar{y} \in E^n.$$

But for any $\bar{x}$, $\bar{y}$, $(\|\bar{x}\| - \|\bar{y}\|)^2 \ge 0$ and hence,

$$\|\bar{x}\|^2 + \|\bar{y}\|^2 \ge 2\|\bar{x}\|\cdot\|\bar{y}\|.$$

Therefore,

$$2|\langle \bar{x},\bar{y}\rangle| \le \|\bar{x}\|^2 + \|\bar{y}\|^2$$

and hence,

$$\pm 2\langle \bar{x},\bar{y}\rangle \le \langle \bar{x},\bar{x}\rangle + \langle \bar{y},\bar{y}\rangle$$

$$\Rightarrow \pm 2|\rho|^k\langle \bar{x},\bar{y}\rangle \le |\rho|^k\langle \bar{x},\bar{x}\rangle + |\rho|^k\langle \bar{y},\bar{y}\rangle$$

$$\Rightarrow \pm 2|\rho|^k\langle \bar{x},\bar{y}\rangle - |\rho|^k\langle \bar{y},\bar{y}\rangle \le |\rho|^k\langle \bar{x},\bar{x}\rangle$$

$$\Rightarrow 2\rho^k\langle \bar{x},\bar{y}\rangle - |\rho|^k\langle \bar{y},\bar{y}\rangle \le |\rho|^k\langle \bar{x},\bar{x}\rangle$$

$$\Rightarrow 2\rho^k\langle B\bar{x},A\bar{y}\rangle - |\rho|^k\langle A\bar{y},A\bar{y}\rangle \le |\rho|^k\langle B\bar{x},B\bar{x}\rangle$$

$$\Rightarrow 2AB\rho^k\langle \bar{x},\bar{y}\rangle - A^2|\rho|^k\langle \bar{y},\bar{y}\rangle \le B^2|\rho|^k\langle \bar{x},\bar{x}\rangle$$

$$\Rightarrow 2B\rho^k\langle \bar{x},\bar{y}\rangle - A|\rho|^k\langle \bar{y},\bar{y}\rangle \le \frac{B^2}{A}|\rho|^k\langle \bar{x},\bar{x}\rangle.$$

Q.E.D.

__Lemma 3.__ If A, B, and $\rho$ are any real numbers such that $A > 0$ and $|\rho| < 1$, and $\bar{x}$ and $\bar{y}$ are arbitrary n-vectors, then the following inequality holds:

$$2B\rho^k\langle \bar{x},\bar{y}\rangle + A|\rho|^k\langle \bar{y},\bar{y}\rangle \geq -\frac{1}{A}|\rho|^k\langle \bar{x},\bar{x}\rangle$$

for $k = 0, 1, 2, \ldots$

<u>Proof</u>: With only a few sign changes, the proof is identical to that of Lemma 2.

We now define a Lyapunov functional V on $C^* = \bigcup_{0<r<\infty} C_r$. For each $\xi\in\Omega$ let $a_\xi$ and $b_\xi$ denote fixed nonnegative real numbers, the values of which will be chosen later. For $H > 0$, define the functional V for every $\bar{\varphi}$ in $C_H( (-\infty,0], E^n )$ by

$$V(\bar{\varphi}) = \frac{1}{2}\bar{\varphi}^t(0)\, P\, \bar{\varphi}(0) + \sum_{\xi\in\Omega}\Bigg(a_\xi\sum_{k=0}^{\infty}|\rho_\xi|^k \int_{\sigma=-2(k+1)\tau_\xi}^{-2k\tau_\xi} \bar{\varphi}^t(\sigma)\bar{\varphi}(\sigma)d\sigma$$

$$+b_\xi\Bigg[\int_{\sigma=-\tau_\xi}^{0}\bar{\varphi}^t(\sigma)\bar{\varphi}(\sigma)\, d\sigma + \sum_{k=0}^{\infty}|\rho_\xi|^{k+1}\int_{\sigma=-(2(k+1)+1)\tau_\xi}^{-(2k+1)\tau_\xi}\bar{\varphi}^t(\sigma)\,\bar{\varphi}(\sigma)\, d\sigma\Bigg]\Bigg) \qquad (5\text{-}1)$$

Along a trajectory of our system we have

$$V(\bar{x}_t) = \frac{1}{2}\bar{x}^t(t)P\bar{x}(t) + \sum_{\xi\in\Omega}\Bigg(a_\xi\sum_{k=0}^{\infty}|\rho_\xi|^k\int_{\sigma=t-2(k+1)\tau_\xi}^{t-2k\tau_\xi}\bar{x}^t(\sigma)\,\bar{x}(\sigma)\, d\sigma$$

$$+b_\xi\Bigg[\int_{\sigma=t-\tau_\xi}^{t}\bar{x}^t(\sigma)\,\bar{x}(\sigma)\, d\sigma + \sum_{k=0}^{\infty}|\rho_\xi|^{k+1}\int_{\sigma=t-(2(k+1)+1)\tau_\xi}^{t-(2k+1)\tau_\xi}\bar{x}^t(\sigma)\,\bar{x}(\sigma)\, d\sigma\Bigg]\Bigg)$$

The derivative of this functional along a trajectory of the system has the value

$$\frac{d}{dt}V(\bar{x}_t) = \bar{x}^t(t)\, P\, \dot{\bar{x}}(t) + \sum_{\xi\in\Omega}\Bigg(a_\xi\sum_{k=0}^{\infty}|\rho_\xi|^k\big(\bar{x}^t(t-2k\tau_\xi)\,\bar{x}(t-2k\tau_\xi) -$$

$$\bar{x}^t(t-2(k+1)\tau_\xi)\,\bar{x}(t-2(k+1)\tau_\xi)] + b_\xi \Big[\bar{x}^t(t)\,\bar{x}(t)$$

$$- \bar{x}^t(t-\tau_\xi)\,\bar{x}(t-\tau_\xi) + \sum_{k=0}^{\infty} |\rho_\xi|^{k+1} \big(\bar{x}^t(t-(2k+1)\tau_\xi)\bar{x}(t-(2k+1)\tau_\xi)$$

$$- \bar{x}^t(t-(2(k+1)+1)\tau_\xi)\,\bar{x}(t-(2(k+1)+1)\tau_\xi)\big)\Big]\Big\}.$$

where $P\,\dot{\bar{x}}(t)$ may be replaced by the right hand side of Equation (4-6). Before making this substitution, however, we rewrite this equation as

$$\frac{d}{dt} V(\bar{x}_t) = \bar{x}^t(t)\,P\dot{\bar{x}}(t) + \sum_{\xi\in\Omega} a_\xi\,\bar{x}^t(t)\,\bar{x}(t)$$

$$+ \sum_{\xi\in\Omega} a_\xi \Big[\sum_{k=1}^{\infty} |\rho_\xi|^k\,\bar{x}^t(t-2k\tau_\xi)\,\bar{x}(t-2k\tau_\xi)$$

$$- \sum_{k=0}^{\infty} |\rho_\xi|^k\,\bar{x}^t(t-2(k+1)\tau_\xi)\,\bar{x}(t-2(k+1)\tau_\xi)\Big]$$

$$+ \sum_{\xi\in\Omega} b_\xi\,\bar{x}^t(t)\,\bar{x}(t) - \sum_{\xi\in\Omega} b_\xi\,\bar{x}^t(t-\tau_\xi)\,\bar{x}(t-\tau_\xi)$$

$$+ \sum_{\xi\in\Omega} b_\xi \Big[\sum_{k=0}^{\infty} |\rho_\xi|^{k+1}\,\bar{x}^t(t-(2k+1)\tau_\xi)\bar{x}(t-(2k+1)\tau_\xi)$$

$$- \sum_{k=1}^{\infty} |\rho_\xi|^k\,\bar{x}^t(t-(2k+1)\tau_\xi)\bar{x}(t-(2k+1)\tau_\xi)\Big]$$

$$= \bar{x}^t(t)P\,\dot{\bar{x}}(t) + \sum_{\xi\in\Omega} a_\xi\bar{x}^t(t)\bar{x}(t) + \sum_{\xi\in\Omega} b_\xi\,\bar{x}^t(t)\,\bar{x}(t)$$

$$+ \sum_{\xi\in\Omega} a_\xi(|\rho_\xi|-1) \sum_{k=0}^{\infty} |\rho_\xi|^k\,\bar{x}^t(t-2(k+1)\tau_\xi)\bar{x}(t-2(k+1)\tau_\xi)$$

$$+ \sum_{\xi\in\Omega} b_\xi(|\rho_\xi|-1) \sum_{k=0}^{\infty} |\rho_\xi|^k\,\bar{x}^t(t-(2k+1)\tau_\xi)\bar{x}(t-(2k+1)\tau_\xi).$$

Thus,

$$\frac{d}{dt} V(\bar{x}_t) = \bar{x}^t(t)\, A\bar{x}(t) + \bar{x}^t(t)\, B_I \mathfrak{F}^*(C_I\bar{x}(t)) - \bar{x}^t(t) \sum_{\xi\in\Omega} Q_\xi \bar{x}(t)$$

$$+ \bar{x}^t(t) \sum_{\xi\in\Omega} a_\xi I_n \bar{x}(t) + \bar{x}^t(t) \sum_{\xi\in\Omega} b_\xi I_n \bar{x}(t)$$

$$+ \sum_{\xi\in\Omega} \sum_{k=0}^{\infty} \Big[ 2\rho_\xi^k\, \bar{x}^t(t)\, M_\xi\, \bar{x}(t-2(k+1)\tau_\xi)$$

$$- a_\xi (1-|\rho_\xi|)\, |\rho_\xi|^k\, \bar{x}^t(t-2(k+1)\tau_\xi)\, \bar{x}(t-2(k+1)\tau_\xi) \Big]$$

$$+ \sum_{\xi\in\Omega} \sum_{k=0}^{\infty} \Big[ 2\rho_\xi^k\, \bar{x}^t(t)\, N_\xi\, \bar{x}(t-(2k+1)\tau_\xi)$$

$$- b_\xi (1-|\rho_\xi|)\, |\rho_\xi|^k\, \bar{x}^t(t-(2k+1)\tau_\xi)\bar{x}(t-(2k+1)\tau_\xi) \Big],$$

or,

$$\frac{d}{dt} V(\bar{x}_t) = \bar{x}^t(t) A\bar{x}(t) + \bar{x}^t(t)\, B_I \mathfrak{F}^*(C_I\bar{x}(t)) - \bar{x}^t(t) \sum_{\xi\in\Omega} Q_\xi \bar{x}(t)$$

$$+ \bar{x}^t(t) \sum_{\xi\in\Omega} a_\xi I_n \bar{x}(t) + \bar{x}^t(t) \sum_{\xi\in\Omega} b_\xi I_n \bar{x}(t)$$

$$+ \sum_{\xi\in\Omega} \sum_{k=0}^{\infty} \Big[ 2\rho_\xi^k \langle M_\xi^t\, \bar{x}(t),\, \bar{x}(t-2(k+1)\tau_\xi) \rangle$$

$$- a_\xi (1-|\rho_\xi|)\, |\rho_\xi|^k \langle \bar{x}(t-2(k+1)\tau_\xi),\, \bar{x}(t-2(k+1)\tau_\xi) \rangle \Big]$$

$$+ \sum_{\xi\in\Omega} \sum_{k=0}^{\infty} \Big[ 2\rho_\xi^k \langle N_\xi^t\, \bar{x}(t),\, \bar{x}(t-(2k+1)\tau_\xi) \rangle$$

$$- b_\xi (1-|\rho_\xi|)\, |\rho_\xi|^k \langle \bar{x}(t-(2k+1)\tau_\xi),\, \bar{x}(t-(2k+1)\tau_\xi) \rangle \Big].$$

In case $M_\xi = 0$ for some $\xi \in \Omega$, let us pick $a_\xi = 0$; and in case $N_\xi = 0$ for some $\xi \in \Omega$, let us pick $b_\xi = 0$. Otherwise, we shall require $a_\xi > 0$ and $b_\xi > 0$. That this is an appropriate choice of $a_\xi$ and $b_\xi$ for these cases can be seen by comparing Equation (4-6) with our Lyapunov functional V in Equation (5-1). If $M_\xi = 0$ then the terms involving $\bar{x}(t - 2(k+1)\tau_\xi)$ are not present in Equation (4-6). Similarly, if $N_\xi = 0$, the terms involving $\bar{x}(t - (2k+1)\tau_\xi)$ are not present in Equation (4-6). Thus, for such values of $\xi$ we have no need for the terms

$$a_\xi \sum_{k=0}^{\infty} |\rho_\xi|^k \int_{\sigma=-2(k+1)\tau_\xi}^{-2k\tau_\xi} \bar{\varphi}^t(\sigma)\, \bar{\varphi}(\sigma)\, d\sigma,$$

$$b_\xi \left[ \int_{\sigma=-\tau_\xi}^{0} \bar{q}^t(\sigma)\, \bar{q}(\sigma)\, d\sigma + \sum_{k=0}^{\infty} |\rho_\xi|^{k+1} \int_{\sigma=-(2(k+1)+1)\tau_\xi}^{-(2k+1)\tau_\xi} \bar{\varphi}^t(\sigma)\bar{\varphi}(\sigma)\, d\sigma \right],$$

in our Lyapunov functional. We also note, by Lemma 1, that $a_\xi = 0$ if and only if $\|M_\xi M_\xi^t\| = 0$; and $b_\xi = 0$ if and only if $\|N_\xi N_\xi^t\| = 0$. To avoid awkward notation we shall use the following conventions: the symbols

$$\sum_{\xi \in \Omega}{}' \qquad \text{and} \qquad \sum_{\xi \in \Omega}{}''$$

denote summations over only those $\xi \in \Omega$ for which $a_\xi \neq 0$ and $b_\xi \neq 0$, respectively. Thus,

$$\frac{d}{dt}V(\bar{x}_t) = \bar{x}^t(t)\, A\bar{x}(t) + \bar{x}^t(t)\, B_I D^*(C_I\bar{x}(t))$$

$$-\bar{x}^t(t) \sum_{\xi \in \Omega} Q_\xi \bar{x}(t) + \bar{x}^t(t) \sum_{\xi \in \Omega}{}' a_\xi I_n \bar{x}(t) + \bar{x}^t(t) \sum_{\xi \in \Omega}{}'' b_\xi I_n \bar{x}(t) +$$

$$\sum_{\xi\in\Omega}{}' \sum_{k=0}^{\infty}\Big[2\rho_\xi^k\langle M_\xi^t\bar{x}(t),\ \bar{x}(t-2(k+1)\tau_\xi)\rangle$$

$$-a_\xi(1-|\rho_\xi|)|\rho_\xi|^k\langle\bar{x}(t-2(k+1)\tau_\xi),\ \bar{x}(t-2(k+1)\tau_\xi)\rangle\Big]$$

$$+\sum_{\xi\in\Omega}{}'' \sum_{k=0}^{\infty}\Big[2\rho_\xi^k\langle N_\xi^t\bar{x}(t),\ \bar{x}(t-(2k+1)\tau_\xi)\rangle$$

$$-b_\xi(1-|\rho_\xi|)|\rho_\xi|^k\langle\bar{x}(t-(2k+1)\tau_\xi),\ \bar{x}(t-(2k+1)\tau_\xi)\rangle\Big].$$

But then, using Lemma 2, we have

$$\frac{d}{dt}V(\bar{x}_t)\le\bar{x}^t(t)A\bar{x}(t)+\bar{x}^t(t)B_I\mho^*(C_I\bar{x}(t))$$

$$-\bar{x}^t(t)\sum_{\xi\in\Omega}Q_\xi\bar{x}(t)+\bar{x}^t(t)\sum_{\xi\in\Omega}{}' a_\xi I_n\bar{x}(t)+\bar{x}^t(t)\sum_{\xi\in\Omega}{}'' b_\xi I_n\bar{x}(t)$$

$$+\sum_{\xi\in\Omega}{}' \sum_{k=0}^{\infty}\frac{1}{a_\xi(1-|\rho_\xi|)}|\rho_\xi|^k\langle M_\xi^t\bar{x}(t),\ M_\xi^t\bar{x}(t)\rangle$$

$$+\sum_{\xi\in\Omega}{}'' \sum_{k=0}^{\infty}\frac{1}{b_\xi(1-|\rho_\xi|)}|\rho_\xi|^k\langle N_\xi^t\bar{x}(t),\ N_\xi^t\bar{x}(t)\rangle\ .$$

Since $$\sum_{k=0}^{\infty}|\rho_\xi|^k=1/(1-|\rho_\xi|),$$

we have

$$\frac{d}{dt}V(\bar{x}_t)\le\bar{x}^t(t)A\bar{x}(t)+\bar{x}^t(t)B_I\mho^*(C_I\bar{x}(t))$$

$$-\bar{x}^t(t)\sum_{\xi\in\Omega}Q_\xi\bar{x}(t)\quad+$$

$$\bar{x}^t(t)\sum_{\xi\in\Omega}{}^{'}\,[a_\xi I_n + \frac{1}{a_\xi(1 - |\rho_\xi|)^2} M_\xi M_\xi^t]\,\bar{x}(t)$$

$$+ \bar{x}^t(t)\sum_{\xi\in\Omega}{}^{''}\,[b_\xi I_n + \frac{1}{b_\xi(1 - |\rho_\xi|)^2} N_\xi N_\xi^t]\,\bar{x}(t).$$

Thus,

$$\frac{d}{dt} V(\bar{x}_t) \leq 0$$

for all $\bar{x}_t \in C_H$ if

$$\left|-\bar{x}^t A\bar{x} - \bar{x}^t B_I \bar{g}^*(C_I\bar{x}) + \bar{x}^t \sum_{\xi\in\Omega} Q_\xi \bar{x}\right| \geq \bar{x}^t \,(\sum_{\xi\in\Omega}{}^{'}\,[a_\xi I_n$$

$$+ \frac{1}{a_\xi(1 - |\rho_\xi|)^2} M_\xi M_\xi^t] + \sum_{\xi\in\Omega}{}^{''}\,[b_\xi I_n + \frac{1}{b_\xi(1 - |\rho_\xi|)^2} N_\xi N_\xi^t])\,\bar{x},$$

for all $\bar{x} \in G_H = \{\bar{x} : \bar{x} \in E^n, \|\bar{x}\| < H\}$. We note, however, that

$$\bar{x}^t(\sum_{\xi\in\Omega}{}^{'}\,[a_\xi I_n + \frac{1}{a_\xi(1 - |\rho_\xi|)^2} M_\xi M_\xi^t] + \sum_{\xi\in\Omega}{}^{''}\,[b_\xi I_n + \frac{1}{b_\xi(1 - |\rho_\xi|)^2} N_\xi N_\xi^t])\,\bar{x}$$

$$\leq (\sum_{\xi\in\Omega}{}^{'}\,[a_\xi + \frac{1}{a_\xi(1-|\rho_\xi|)^2} \|M_\xi M_\xi^t\|] + \sum_{\xi\in\Omega}{}^{''}\,[b_\xi + \frac{1}{b_\xi(1-|\rho_\xi|)^2} \|N_\xi N_\xi^t\|]) \cdot \|\bar{x}\|^2;$$

hence,

$$\frac{d}{dt} V(\bar{x}_t) \leq 0$$

for all $\bar{x}_t \in C_H$ if

$$\left[-\bar{x}^t A\bar{x}^t - \bar{x}^t B_I B^*(C_I\bar{x}) + \bar{x}^t \sum_{\xi\in\Omega} Q_\xi \bar{x}\right] / \|\bar{x}\|^2$$

$$\geq \quad {\sum_{\xi\in\Omega}}' \left[a_\xi + \frac{1}{a_\xi(1-|\rho_\xi|)^2} \|M_\xi M_\xi^t\|\right]$$

$$+ {\sum_{\xi\in\Omega}}'' \left[b_\xi + \frac{1}{b_\xi(1-|\rho_\xi|)^2} \|N_\xi N_\xi^t\|\right],$$

for all $\bar{x} \in G_H$, $\bar{x} \neq 0$. In order to obtain conditions for ensuring that $\frac{d}{dt} V(\bar{x}_t) \leq 0$ that are in general as weak as possible for this method, we shall now choose the constants $a_\xi > 0$ and $b_\xi > 0$ such that

$$\left[a_\xi + \frac{1}{a_\xi(1 - |\rho_\xi|)^2} \|M_\xi M_\xi^t\|\right]$$

and

$$\left[b_\xi + \frac{1}{b_\xi(1 - |\rho_\xi|)^2} \|N_\xi N_\xi^t\|\right]$$

are minimized. The function $u(s) = s + k/s$, $k > 0$, has a minimum for $s > 0$ at $s = \sqrt{k}$; therefore, it is clear that we should choose

$$a_\xi = \frac{\sqrt{\|M_\xi M_\xi^t\|}}{(1 - |\rho_\xi|)}, \qquad \text{for } \xi \in \Omega,\ a_\xi \neq 0;$$

and

$$b_\xi = \frac{\sqrt{\|N_\xi N_\xi^t\|}}{(1 - |\rho_\xi|)} \qquad \text{for } \xi \in \Omega,\ b_\xi \neq 0.$$

Thus, with this choice of $a_\xi$ and $b_\xi$ we may write

$$\frac{d}{dt} V(\bar{x}_t) \le \bar{x}^t(t)[A - \sum_{\xi\in\Omega} Q_\xi + \sum_{\xi\in\Omega}' \frac{\sqrt{\|M_\xi M_\xi^t\|}}{(1 - |\rho_\xi|)} (I_n + \frac{1}{\|M_\xi M_\xi^t\|} M_\xi M_\xi^t)$$

$$+ \sum_{\xi\in\Omega}' \frac{\sqrt{\|N_\xi N_\xi^t\|}}{(1 - |\rho_\xi|)} (I_n + \frac{1}{\|N_\xi N_\xi^t\|} N_\xi N_\xi^t)] \bar{x}(t)$$

$$+ \bar{x}^t(t) B_I \mathcal{J}^*(C_I \bar{x}(t)).$$

Defining

$$\Lambda = \begin{bmatrix} \lambda_{\gamma_1} & & & \bigcirc \\ & \lambda_{\gamma_2} & & \\ & & \ddots & \\ \bigcirc & & & \lambda_{\theta_{n_\theta}} \end{bmatrix}, \tag{5-2}$$

we note that

$$\sum_{\xi\in\Omega} Q_\xi = \sum_{\xi\in\Omega} \lambda_\xi B_\xi C_\xi = \sum_{\xi\in\Omega} B_\xi \lambda_\xi C_\xi = [B_{\gamma_1} | B_{\gamma_2} | \cdots | B_{\theta_{n_\theta}}] \begin{bmatrix} \lambda_{\gamma_1} C_{\gamma_1} \\ \hline \lambda_{\gamma_2} C_{\gamma_2} \\ \hline \vdots \\ \hline \lambda_{\theta_{n_\theta}} C_{\theta_{n_\theta}} \end{bmatrix}$$

$$= [B_{\gamma_1} | B_{\gamma_2} | \cdots | B_{\theta_{n_\theta}}] \; \Lambda \begin{bmatrix} C_{\gamma_1} \\ \hline C_{\gamma_2} \\ \hline \vdots \\ \hline C_{\theta_{n_\theta}} \end{bmatrix} = B_{\Pi} \Lambda C_{\Pi} .$$

$\therefore$ ,

$$M_\xi M_\xi^t = \mu_\xi^2 B_\xi C_\xi C_\xi^t B_\xi^t = \mu_\xi^2 \|C_\xi\|^2 \, B_\xi B_\xi^t,$$

and,

$$N_\xi N_\xi^t = \nu_\xi^2 B_\xi C_{\xi'} C_{\xi'}^t B_\xi^t = \nu_\xi^2 \|C_{\xi'}\|^2 \, B_\xi B_\xi^t.$$

Therefore,

$$\sum_{\xi\in\Omega}{}' \frac{\sqrt{\|M_\xi M_\xi^t\|}}{(1 - |\rho_\xi|)} I_n + \sum_{\xi\in\Omega}{}'' \frac{\sqrt{\|N_\xi N_\xi^t\|}}{(1 - |\rho_\xi|)} I_n$$

$$= \sum_{\xi\in\Omega} \frac{\sqrt{\|M_\xi M_\xi^t\|}}{(1 - |\rho_\xi|)} I_n + \sum_{\xi\in\Omega} \frac{\sqrt{\|N_\xi N_\xi^t\|}}{(1 - |\rho_\xi|)} I_n$$

$$= \sum_{\xi\in\Omega} \left( \frac{\sqrt{\|M_\xi M_\xi^t\|} + \sqrt{\|N_\xi N_\xi^t\|}}{(1 - |\rho_\xi|)} \right) I_n$$

$$= \sum_{\xi\in\Omega} \left( \frac{(|\mu_\xi| \cdot \|C_\xi\| + |\nu_\xi| \cdot \|C_{\xi'}\|) \sqrt{\|B_\xi B_\xi^t\|}}{(1 - |\rho_\xi|)} \right) I_n$$

$$= \sum_{\xi\in\Omega} \left( \frac{(|\mu_\xi| \cdot \|C_\xi\| + |\nu_\xi| \cdot \|C_{\xi'}\|) \cdot \|B_\xi\|}{(1 - |\rho_\xi|)} \right) I_n,$$

since it is easily shown[1] that $\|B_\xi B_\xi^t\| = \|B_\xi\|^2$.

---

[1] Let A be a column vector. Since, for any square matrix M, the eigenvalues of $M^2$ are the squares of the eigenvalues of M, and since for positive semi-definite M, $\|M\|$ = maximum eigenvalue of M, $\|A\,A^t\| = \sqrt{\|(A\,A^t)^2\|} = \sqrt{\|(A\,A^t)(A\,A^t)\|} = \sqrt{\|A(A^tA)A^t\|}$. Let $a^2 = \|A\|^2 = A^tA$; then $\|A\,A^t\| = \sqrt{a^2\|A\,A^t\|} = a\sqrt{\|A\,A^t\|}$ (clearly $a \geq 0$). Thus, $\|A\,A^t\|^2 = a^2\|A\,A^t\| \Rightarrow \|A\,A^t\| = a$, if $\|A\,A^t\| \neq 0$. But if $\|A\,A^t\| = 0$ then trace $A\,A^t = 0$ (since the trace equals the sum of the eigenvalues of $A\,A^t$); but, trace $A\,A^t = A^tA = a^2$. Hence, $\|A\,A^t\| = 0 \Rightarrow a = 0$. Thus, $\|A\,A^t\| = a$; or, $\|A\,A^t\| = \|A\|^2$. Q.E.D.

If we now define, for $\xi \in \omega$,

$$E_\xi = \begin{cases} \dfrac{1}{\|B_\xi\|} \cdot B_\xi, & \text{if } \|B_\xi\| \neq 0 \\ 0, & \text{if } \|B_\xi\| = 0 \end{cases}$$

and

$$E_\Omega = [E_{\eta_1} | E_{\eta_2} | \cdots | E_{\eta_{n_\omega}}],$$

then

$$\sum_{\xi \in \Omega}{}' \frac{\sqrt{\|M_\xi M_\xi^t\|}}{(1 - |\rho_\xi|)} \left( \frac{1}{\|M_\xi M_\xi^t\|} \; M_\xi M_\xi^t \right)$$

$$= \sum_{\xi \in \Omega}{}' \frac{|\mu_\xi| \cdot \|C_\xi\| \cdot \|B_\xi\|}{(1 - |\rho_\xi|)} \left( \frac{\mu_\xi^2 \; \|C_\xi\|^2}{\mu_\xi^2 \; \|C_\xi\|^2 \; \|B_\xi\|^2} \; B_\xi B_\xi^t \right)$$

$$= \sum_{\xi \in \Omega}{}' \frac{|\mu_\xi| \cdot \|C_\xi\| \cdot \|B_\xi\|}{(1 - |\rho_\xi|)} \; E_\xi E_\xi^t$$

$$= \sum_{\xi \in \Omega} \frac{|\mu_\xi| \cdot \|C_\xi\| \; \|B_\xi\|}{(1 - |\rho_\xi|)} \; E_\xi E_\xi^t.$$

and similarly,

$$\sum_{\xi \in \Omega}{}'' \frac{\sqrt{\|N_\xi N_\xi^t\|}}{(1 - |\rho_\xi|)} \left( \frac{1}{\|N_\xi N_\xi^t\|} \; N_\xi N_\xi^t \right)$$

$$= \sum_{\xi \in \Omega} \frac{|\nu_\xi| \cdot \|C_\xi\| \cdot \|B_\xi\|}{(1 - |\rho_\xi|)} \; E_\xi E_\xi^t \; .$$

Thus, if we define

$$\psi_\xi = \frac{(|\mu_\xi|\cdot\|C_\xi\| + |\nu_\xi|\cdot\|C_{\xi'}\|)\cdot\|B_\xi\|}{(1 - |\rho_\xi|)}, \qquad \text{for } \xi \in \Omega,$$

and

$$\Psi = \begin{bmatrix} \psi_{\gamma_1} & & & \bigcirc \\ & \psi_{\gamma_2} & & \\ & & \ddots & \\ \bigcirc & & & \psi_{\theta_{n_6}} \end{bmatrix}, \tag{5-3}$$

we have

$$\frac{d}{dt} V(\bar{x}_t) \leq \bar{x}^t(t)\,[A - B_\Pi \Lambda C_\Pi + E_\Pi \Psi E_\Pi^t + (\mathrm{tr}\,\Psi) I_n]\bar{x}(t)$$

$$+ \bar{x}^t(t)\, B_I \mho^*(C_I \bar{x}(t)). \tag{5-4}$$ [1]

We may also define a functional V on C* by

$$V(\bar{\varphi}) = \tfrac{1}{2}\bar{\varphi}^t(0) P \bar{\varphi}(0) - \sum_{\xi\in\Omega}\left( a_\xi \sum_{k=0}^{\infty} |\rho_\xi|^k \int_{\sigma=-2(k+1)\tau_\xi}^{-2k\tau_\xi} \bar{\varphi}^t(\sigma)\bar{\varphi}(\sigma)\, d\sigma \right.$$

$$\left. + b_\xi\left[\int_{\sigma=-\tau_\xi}^{0} \bar{\varphi}^t(\sigma)\bar{\varphi}(\sigma)\, d\sigma + \sum_{k=0}^{\infty}|\rho_\xi|^{k+1}\int_{\sigma=-(2(k+1)+1)\tau_\xi}^{-(2k+1)\tau_\xi} \bar{\varphi}^t(\sigma)\bar{\varphi}(\sigma)\, d\sigma\right]\right). \tag{5-5}$$

Equation (5-5) is identical to Equation (5-1) except that the first plus sign is changed to a minus sign. Proceeding in exactly the same manner as above, with the use of Lemma 3 instead of Lemma 2, one easily obtains:

$$\frac{d}{dt} V(\bar{x}_t) \geq \bar{x}^t(t)[A - B_\Pi \Lambda C_\Pi - E_\Pi \Psi E_\Pi^t - (\mathrm{tr}\,\Psi) I_n]\,\bar{x}(t)$$

$$+ \bar{x}^t(t)\, B_I \mho^*(C_I \bar{x}(t)). \tag{5-6}$$

[1] tr Ψ denotes the trace of the matrix Ψ.

One may easily show that for any $H > 0$ the functions $V$ of Equations (5-1) and (5-5) are continuous and bounded on $C_H$; this is done in Appendix C. It is also obvious that $V$ of Equation (5-1) satisfies $V(\bar{\varphi}) \geq 0$ for all $\bar{\varphi} \in C^*$.

We now state and prove the basic stability and instability theorems for nonlinear distributed networks.

2. <u>Stability and Instability Theorems</u>

The following theorems apply to any network in the class of nonlinear distributed networks defined in the previous chapter. That is, we assume that conditions (A1) through (A4) of Section 4-2 are satisfied by every network for which these theorems are to be used.

<u>Theorem 1</u>. If

$$\ell = \left(\tfrac{1}{2}\|P\| + \sum_{\xi\in\Omega}\left[\frac{2\tau_\xi(a_\xi + b_\xi|\rho_\xi|)}{(1 - |\rho_\xi|)} + b_\xi\tau_\xi\right]\right)\cdot H^2,$$

$K = (2\ell/\lambda_p)^{1/2}$, where $\lambda_p$ is the smallest eigenvalue of the matrix $P$, and if $\bar{x}^t[A - B_\Omega \Lambda C_\Omega + E_\Omega \Psi E_\Omega^t + (\mathrm{tr}\,\Psi)I_n]\bar{x} + \bar{x}^t B_I \bar{\sigma}^*(C_I\bar{x}) \leq 0$ for all $\bar{x}$ in a neighborhood of the origin $G_K = \{\bar{x} : \bar{x} \in E^n, \|\bar{x}\| < K\}$, then the solution $\bar{x} = \bar{0}$ of Equation (4-6) is stable. Furthermore, if $\bar{x}^t[A - B_\Omega \Lambda C_\Omega + E_\Omega \Psi E_\Omega^t + (\mathrm{tr}\,\Psi)I_n]\bar{x} + \bar{x}^t B_I \bar{\sigma}^*(C_I\bar{x}) < 0$ for all $\bar{x} \in G_K$, $\bar{x} \neq \bar{0}$, then the solution $\bar{x} = \bar{0}$ is asymptotically stable and every solution of Equation (4-6) with initial condition $\bar{x}_0$ in $C_H$ approaches zero as $t \to \infty$.

<u>Proof</u>. Let the continuous functional $V$ be defined on $C^*$ by Equation (5-1). Clearly, $V(\bar{0}) = 0$. For the given $\ell$, let $U_\ell$ denote that region of $C^*$

where $V(\bar{\varphi}) < \ell$. $\bar{\varphi} \in U_\ell \Rightarrow V(\bar{\varphi}) < \ell \Rightarrow \frac{1}{2}\lambda_p \|\bar{\varphi}(0)\|^2 \leq \frac{1}{2}\bar{\varphi}^t(0)P\,\bar{\varphi}(0)$ [1] $\leq V(\bar{\varphi}) < \ell$ $\Rightarrow \|\bar{\varphi}(0)\| < (2\ell/\lambda_p)^{1/2}$ ∴. Let $u(s)$ be defined on $[0,K)$ by $u(s) = \frac{1}{2}\lambda_p s^2$. Clearly, $u(s)$ is continuous and increasing for $0 \leq s < K$, and $u(0) = 0$. From Appendix C we find that $\bar{\varphi} \in C_H \Rightarrow V(\bar{\varphi}) < \ell$, and hence $C_H \subset U_\ell$. Clearly, $u(\|\bar{\varphi}(0)\|) = \frac{1}{2}\lambda_p\|\bar{\varphi}(0)\|^2 \leq V(\bar{\varphi})$, for all $\bar{\varphi} \in U_\ell$, as observed above. According to Equation (5-4), the condition that $\bar{x}^t[A - B_{II} \wedge C_{II} + E_{II} \Psi E_{II}^t + (\mathrm{tr}\,\Psi)I_n]\bar{x} + \bar{x}^t B_I \mho^*(C_I\bar{x}) \leq 0$ for all $\bar{x} \in G_K$ implies that $\dot{V}_{(4-6)}(\bar{\varphi}) \leq 0$ for all $\bar{\varphi} \in U_\ell$. Hence, by Theorem 2 of Section 2-2 the solution $\bar{x} = \bar{0}$ of Equation (4-6) is stable.

The condition that $\bar{x}^t[A - B_{II} \wedge C_{II} + E_{II} \Psi E_{II}^t + (\mathrm{tr}\,\Psi)I_n]\bar{x} + \bar{x}^t B_I \mho^*(C_I\bar{x}) < 0$ for all $\bar{x} \in G_K$, $\bar{x} \neq \bar{0}$, implies that M, the largest invariant set in R (the set of all points in $U_\ell$ where $\dot{V}_{(4-6)}(\bar{\varphi}) = 0$), contains only the point $\bar{\varphi} = \bar{0}$.[2] Thus by Theorem 1 of Section 2-2, every solution of Equation (4-6) with initial condition $\bar{x}_o$ in $U_\ell$ (in particular, all $\bar{x}_o \in C_H$) approaches zero as $t \to \infty$. The solution $\bar{x} = \bar{0}$ is therefore asymptotically stable. Q.E.D.

Theorem 2. If $\bar{x}^t[A - B_{II} \wedge C_{II} - E_{II} \Psi E_{II}^t - (\mathrm{tr}\,\Psi)I_n]\bar{x} + \bar{x}^t B_I \mho^*(C_I\bar{x}) > 0$ for all $\bar{x} \neq \bar{0}$ in some neighborhood of the origin, $G_\gamma = \{\bar{x} : \bar{x} \in E^n, \|\bar{x}\| \leq \gamma < H\}$, then the solution $\bar{x} = \bar{0}$ of Equation (4-6) is unstable.

Proof. Let the bounded continuous functional V be defined on $C_H$ by Equation (5-5). Let $U_\gamma$ denote that region of $C_\gamma$ where $\frac{1}{2}\,\bar{\varphi}^t(0)\,P\,\bar{\varphi}(0) >$

[1] See pp. 110-111 of reference [37].

[2] This is seen as follows: Clearly $\{\bar{0}\}$ is an invariant set in R. Now if $\bar{\varphi} \in R$ then $\bar{\varphi}(0) = \bar{0}$ (since $\bar{\varphi} \in U_\ell$, $\bar{\varphi}(0) \neq 0 \Rightarrow \dot{V}_{(4-6)}(\bar{\varphi}) < 0$). Suppose $\bar{\varphi} \in M$; then, according to the definition of an invariant set, ∃ a function $\bar{x}$, defined on $(-\infty,\infty)$, with $\bar{x}_t \in M$ ∀ t in $(-\infty,\infty)$ and $\bar{x}_o = \bar{\varphi}$. But then, for all $t \leq 0$, $\bar{x}_t \in M \subset R \to \bar{x}_t(0) = \bar{0}$. That is, $\bar{x}(t) \equiv \bar{0}$ for $-\infty < t \leq 0$. Hence, $\bar{\varphi} = \bar{0}$.

$$\sum_{\xi \in \mathcal{M}} \Bigg( a_\xi \sum_{k=0}^{\infty} |\rho_\xi|^k \int_{\sigma=-2(k+1)\tau_\xi}^{-2k\tau_\xi} \bar{\varphi}^t(\sigma)\, \bar{\varphi}(\sigma)\, d\sigma + b_\xi \Bigg[ \int_{\sigma=-\tau_\xi}^{0} \bar{\varphi}^t(\sigma)\, \bar{\varphi}(\sigma)\, d\sigma$$

$$\sum_{k=0}^{\infty} |\rho_\xi|^{k+1} \int_{\sigma=-(2(k+1)+1)\tau_\xi}^{-(2k+1)\tau_\xi} \bar{\varphi}^t(\sigma)\, \bar{\varphi}(\sigma)\, d\sigma \Bigg] \Bigg);$$

that is, $U_\gamma$ denotes that region of $C_\gamma$ where $V(\bar{\varphi}) > 0$. Now, since $\{z : z \in E^1, z > 0\}$ is open in $E^1$, and since V is a continuous mapping of $C_\gamma$ into $E^1$, $U_\gamma$ is open in $C_\gamma$. Thus, $\exists$ an open set U in C such that $V(\bar{\varphi}) > 0$ on $U_\gamma = U \cap C_\gamma$.[1] Clearly, that part of the boundary of U, $\partial U$, which is in $C_\gamma$ consists of the collection of all those points $\bar{\varphi}$ in $C_\gamma$ for which $V(\bar{\varphi}) = 0$. It is also clear that $\bar{0}$ belongs to the closure of $U_\gamma = U \cap C_\gamma$ since $\bar{0} \in C_\gamma$, and $V(\bar{0}) = 0 \Rightarrow \bar{0} \in \partial U_\gamma$. Let us define the function u(s) mapping the interval [0,H) into $E^1$ by $u(s) = \frac{1}{2}\lambda_P s^2$, where $\lambda_P$ is the largest eigenvalue of the matrix P. Clearly, u(s) is continuous and increasing for $0 \le s < H$ and, $u(0) = 0$. Furthermore, $u(\|\bar{\varphi}(0)\|) = \frac{1}{2}\lambda_P \|\bar{\varphi}(0)\|^2 \ge \frac{1}{2}\bar{\varphi}^t(0) P\, \bar{\varphi}(0)$. Hence, $V(\bar{\varphi}) \le u(\|\bar{\varphi}(0)\|)$ on $U_\gamma = U \cap C_\gamma$. According to Equation (5-6), the condition that $\bar{x}^t[A - B_{II} \Lambda C_{II} - E_{II} \Psi E_{II}^t - (\mathrm{tr}\, \Psi) I_n]\bar{x} + \bar{x}^t B_I\, \Phi^*(C_I x) > 0\ \forall\ \bar{x} \in G_\gamma,\ \bar{x} \ne \bar{0}$, implies that $\dot{V}^*_{(4-6)}(\bar{\varphi}) \ge 0$ on the closure of $U_\gamma = U \cap C_\gamma$, and that the set of $\bar{\varphi}$ in the closure of $U_\gamma$ such that $\dot{V}^*_{(4-6)}(\bar{\varphi}) = 0$ contains no invariant set of Equation (4-6) except $\bar{\varphi} = \bar{0}$. Thus, by Theorem 3 of Section 2-2, the solution $\bar{x} = \bar{0}$ of Equation (4-6) is unstable. Q. E. D.

Several corollaries to the above theorems may also be stated. One having a trivial proof is:

[1] The existence of the open set U follows from Theorem 14, page 51, of reference [22].

Corollary 1 (Complete Stability). If conditions (A3) and (A4) of Section 4-2 hold for all $H > 0$ and, if $\bar{x}^t[A - B_U \Lambda C_U + E_U \Psi E_U^t + (\mathrm{tr}\,\Psi)I_n]\bar{x} + \bar{x}^t B_I \emptyset^*(C_I \bar{x}) < 0$ for all $\bar{x} \in E^n$, $\bar{x} \neq \bar{0}$, then every solution of Equation (4-6) with bounded initial condition $\bar{x}_o \in C$ approaches zero as $t \to \infty$ and the system is asymptotically stable in the large (completely stable).

Proof. Let $\bar{x}_o$ be some bounded initial condition in C. Then, $\exists\, H > 0$ such that $\|\bar{x}_o(t)\| < H$ for all $t \in (-\infty, 0]$; that is $\bar{x}_o \in C_H$. We may use this value of H in Theorem 1. Q.E.D.

Before stating the remaining corollaries we define the concept of a critical point of a mapping f from $E^n$ to $E^1$, and state a well-known theorem and two lemmas.

If f is a differentiable mapping from $E^n$ to $E^1$ and if for $\bar{x}_o \in E^n$, $\partial f/\partial x_i\,(\bar{x}_o) = 0$ for $i = 1, \ldots, n$, then $\bar{x}_o$ is said to be a critical point of f. A theorem which is available in many references [38 p. 62, 39 p. 61] is the following:

Theorem. If f is a twice continuously differentiable function mapping a neighborhood $N \subset E^n$ of $\bar{x}_o$ into $E^1$, and if $\bar{x}_o \in N$ is a critical point of f then, if the quadratic form

$$\sum_{i=1}^{n} \sum_{j=1}^{n} \frac{\partial^2 f}{\partial x_i \partial x_j}(\bar{x}_o)\, \lambda_i \lambda_j$$

is positive definite, f has a strict relative minimum at $\bar{x}_o$; that is, there exists an open set $G \subset N$ containing $\bar{x}_o$ such that $\bar{x} \in G$, $\bar{x} \neq \bar{x}_o \Rightarrow f(\bar{x}) > f(\bar{x}_o)$.

The hypotheses of Theorems 1 and 2 of this section involve conditions on functions f of the form $f(\bar{x}) = \bar{x}^t Q \bar{x} + \bar{x}^t B_I \emptyset^*(C_I \bar{x})$, where Q is an

$n \times n$ matrix,[1] be nonnegative or nonpositive in some open set $G_K$ containing $\bar{x} = \bar{0}$. We may use the above theorem to give conditions which may be easier to verify. We first prove two simple lemmas:

Lemma 1. Let $f(\bar{x}) \equiv \bar{x}^t Q\bar{x} + \bar{x}^t B_I \mathfrak{F}^*(C_I\bar{x})$. If

$$\left[\frac{\partial^2 f}{\partial x_i \, \partial x_j}(\bar{0})\right]$$

denotes the matrix whose i-jth element is

$$\frac{\partial^2 f}{\partial x_i \, \partial x_j}(\bar{0}), \quad \text{then} \left[\frac{\partial^2 f}{\partial x_i \, \partial x_j}(\bar{0})\right] = Q + B_I \mathfrak{F}^{*\prime}(\bar{0}) C_I + Q^t + [B_I \mathfrak{F}^{*\prime}(\bar{0}) C_I]^t,$$

where $\mathfrak{F}^{*\prime}(\bar{0})$ denotes the Jacobian matrix of the mapping $\mathfrak{F}^*$, evaluated at $\bar{0}$.

Proof.

$$\frac{\partial f}{\partial x_j} = \frac{\partial \bar{x}^t}{\partial x_j} Q\bar{x} + \bar{x}^t Q \frac{\partial \bar{x}}{\partial x_j} + \frac{\partial \bar{x}^t}{\partial x_j} B_I \mathfrak{F}^*(C_I\bar{x}) + \bar{x}^t B_I \mathfrak{F}^{*\prime}(C_I\bar{x}) C_I \frac{\partial \bar{x}}{\partial x_j} .$$

also,

$$\frac{\partial^2 f}{\partial x_i \, \partial x_j}(\bar{0}) = \frac{\partial \bar{x}^t}{\partial x_j} Q \frac{\partial \bar{x}}{\partial x_i} + \frac{\partial \bar{x}^t}{\partial x_i} Q \frac{\partial \bar{x}}{\partial x_j} + \frac{\partial \bar{x}^t}{\partial x_j} B_I \mathfrak{F}^{*\prime}(\bar{0}) C_I \frac{\partial \bar{x}}{\partial x_i} + \frac{\partial \bar{x}^t}{\partial x_i} B_I \mathfrak{F}^{*\prime}(0) C_I \frac{\partial \bar{x}}{\partial x_j}$$

$$= Q_{ji} + Q_{ij} + [B_I \mathfrak{F}^{*\prime}(\bar{0}) C_I]_{ji} + [B_I \mathfrak{F}^{*\prime}(\bar{0}) C_I]_{ij}$$

$$= Q^t_{ij} + Q_{ij} + [B_I \mathfrak{F}^{*\prime}(\bar{0}) C_I]^t_{ij} + [B_I \mathfrak{F}^{*\prime}(\bar{0}) C_I]_{ij}.$$

Hence,

$$\left[\frac{\partial^2 f}{\partial x_i \, \partial x_j}(\bar{0})\right] = Q + B_I \mathfrak{F}^{*\prime}(\bar{0}) C_I + Q^t + [B_I \mathfrak{F}^{*\prime}(\bar{0}) C_I]^t. \qquad \text{Q.E.D.}$$

---

[1] $Q = A - B_{\Pi} \Lambda C_{\Pi} \pm E_{\mu} \Psi E_{\mu}' \pm (\mathrm{tr}\, \Psi) I_n$.

Lemma 2. The matrix $M + M^t$ is positive definite if and only if $M$ is positive definite.

Proof. ("If") Proof of this part is trivial.

("Only if") Let $\bar{x}^t(M + M^t)\bar{x} > 0$ . Then, $\bar{x}^t M\bar{x} + \bar{x}^t M^t \bar{x} > 0 \Rightarrow 2\bar{x}^t M\bar{x} > 0 \Rightarrow$ $\bar{x}^t M\bar{x} > 0$. Q.E.D.

Lemmas 1 and 2 prove that if $f(\bar{x}) = \bar{x}^t Q\bar{x} + \bar{x}^t B_I \mathfrak{F}^*(C_I\bar{x})$ then the matrix

$$\left[\frac{\partial^2 f}{\partial x_i \, \partial x_j}(\bar{0})\right]$$

is positive definite if and only if the matrix $Q + B_I \mathfrak{F}^{*\prime}(\bar{0})C_I$ is positive definite. We now state corollaries of Theorems 1 and 2:

Corollary 2 (Asymptotic Stability). If, for all $\bar{x}$ in some open set containing the origin, $G_K = \{\bar{x} : \bar{x} \in E^n, \|\bar{x}\| < K\}$, the function $\bar{x}^t B_I \mathfrak{F}^*(C_I\bar{x})$ mapping $G_K$ into $E^1$ has continuous second partial derivatives, and if the matrix $-A + B_{II}\Lambda C_{II} - E_{II}\Psi E_{II}^t - (\mathrm{tr}\,\Psi)I_n - B_I \mathfrak{F}^{*\prime}(\bar{0})C_I$, where $\mathfrak{F}^{*\prime}(\bar{0})$ denotes the Jacobian matrix of the mapping $\mathfrak{F}^*$ evaluated at the origin, is positive definite, then the solution $\bar{x} = \bar{0}$ of Equation (4-6) is asymptotically stable.

Proof. Letting $f(\bar{x}) = -\bar{x}^t[A - B_{II}\Lambda C_{II} + E_{II}\Psi E_{II}^t + (\mathrm{tr}\,\Psi)I_n]\bar{x} - \bar{x}^t B_I \mathfrak{F}^*(C_I\bar{x})$, we see that $f(\bar{0}) = 0$. The hypotheses of this corollary imply that the quadratic form

$$\sum_{i=1}^{n} \sum_{j=1}^{n} \frac{\partial^2 f}{\partial x_i \, \partial x_j}(\bar{0})\lambda_i\lambda_j$$

is positive definite. Clearly, (see first line of proof of Lemma 1) $\partial f/\partial x_i = 0$ for $i = 1, \ldots, n$, and thus, $\bar{x} = \bar{0}$ is a critical point of $f$. Hence, by the above theorem, there exists an open set $G_{K'} \subset G_K$, containing $\bar{x} = \bar{0}$, such that $\bar{x} \in G_K$, $\bar{x} \neq \bar{0} \Rightarrow f(\bar{x}) > 0 \Rightarrow -f(\bar{x}) < 0$. Thus, by Theorem 1, the solution $\bar{x} = \bar{0}$ of Equation (4-6) is asymptotically stable. Q.E.D.

<u>Corollary 3 (Instability)</u>. If, for all $\bar{x}$ in some open set containing the origin, $G_\gamma = \{\bar{x} : \bar{x} \in E^n, \|\bar{x}\| < \gamma\}$, the function $\bar{x}^t B_I \mathfrak{F}^*(C_I \bar{x})$ mapping $G_\gamma$ into $E^1$ has continuous second partial derivatives, and if the matrix $A - B_{II} \Lambda C_{II} - E_{II} \Psi E_{II}^t - (\mathrm{tr}\,\Psi) I_n + B_I \mathfrak{F}^{*\prime}(\bar{0}) C_I$, where $\mathfrak{F}^{*\prime}(\bar{0})$ denotes the Jacobian matrix of the mapping $\mathfrak{F}^*$ evaluated at the origin, is positive definite, then the solution $\bar{x} = \bar{0}$ of Equation (4-6) is unstable.

<u>Proof</u>. The proof proceeds in the same manner as that of Corollary 3. Q.E.D.

We now give several examples of the application of the above results.

5. <u>Example 1</u>

For our first example we consider the distributed network of Figure 1.2 which was examined from the viewpoint of linear network theory in Chapter I. The network is redrawn in Figure 5.1 to show explicitly that it is a member of the class of networks having the form of Figure 4.2. Note that we have replaced the resistor r which was characterized by the equation $i = Gv$ by a lumped memoryless nonlinear element characterized by the equation $i = f(v)$. We assume that the function f satisfies a Lipschitz condition in some neighborhood of $v = 0$, and that $f(0) = 0$. Thus, the resistor r of Chapter 1 is but a special case of the type of element we shall consider. For this network Equations (4-1) and (4-2) become:

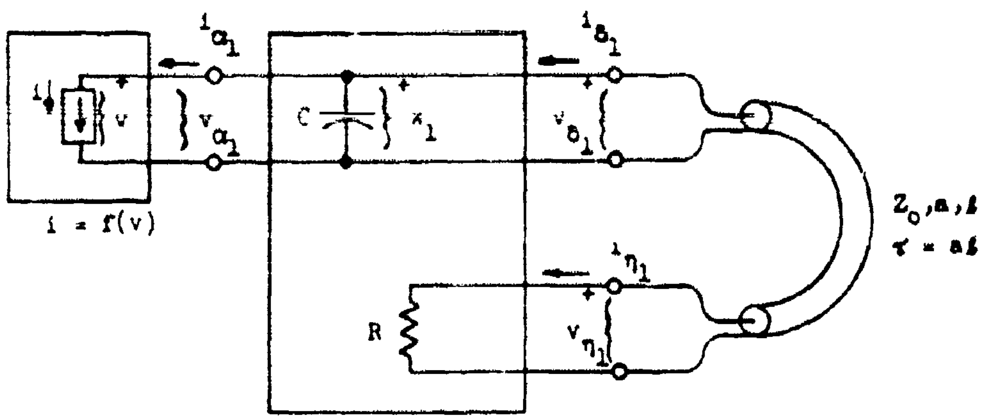


Figure 5.1. Network for Example 1.

$$[C]\dot{x}_1(t) = [0]x_1(t) + [-1 \quad 1 \quad 0] \begin{bmatrix} i_{\alpha_1}(t) \\ i_{\delta_1}(t) \\ v_{\eta_1}(t) \end{bmatrix} ,$$

$$\begin{bmatrix} v_{\alpha_1}(t) \\ v_{\delta_1}(t) \\ i_{\eta_1}(t) \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} x_1(t) + \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1/R \end{bmatrix} \begin{bmatrix} i_{\alpha_1}(t) \\ i_{\delta_1}(t) \\ v_{\eta_1}(t) \end{bmatrix} .$$

Thus, $P = [C]$, $A = [0]$, $B_I = [-1]$, $B_{II} = [1 \quad 0]$, $B_{\delta_1} = [1]$, $B_{\eta_1} = [0]$, $C_I = [1]$, $C_{II} = [1 \quad 0]^t$, $C_{\delta_1} = [1]$, $C_{\eta_1} = [0]$, $D_I = [0]$, $D_\delta = [0]$, $D_{\eta_1} = [1/R]$. Also, $B_I \mathcal{F}^*(C_I \bar{x}) = -f(x_1)$, $\Omega = [\delta_1, \eta_1]$, and

$$Q_{\delta_1} = \lambda_{\delta_1}[1]\,[1] = [\lambda_{\delta_1}], \qquad Q_{\eta_1} = \lambda_{\eta_1}\,[0]\,[0] = [0],$$

$$M_{\delta_1} = \mu_{\delta_1}[1]\,[1] = [\mu_{\delta_1}], \qquad M_{\eta_1} = \mu_{\eta_1}\,[0]\,[0] = [0],$$

$$N_{\delta_1} = \nu_{\delta_1}[1]\,[0] = [0], \qquad N_{\eta_1} = \nu_{\eta_1}\,[0]\,[1] = [0],$$

where,

$$\lambda_{\delta_1} = \frac{1}{0 + Z_o} = Z_o^{-1},$$

$$\mu_{\delta_1} = Z_o \frac{(R - Z_o)/(R + Z_o)}{(0 + Z_o)^2} = \frac{1}{Z_o}\left(\frac{R - Z_o}{R + Z_o}\right),$$

$$\rho_{\delta_1} = \Gamma_{\delta_1}\,\Gamma_{\eta_1} = -1\cdot\left(\frac{R - Z_o}{R + Z_o}\right) = -\frac{R - Z_o}{R + Z_o}.$$

The functional-differential equation which governs the behavior of this network is:

$$C\dot{x}_1(t) = -f(x_1(t)) - \frac{1}{Z_o}\,x_1(t) + 2\sum_{k=0}^{\infty} \rho_{\delta_1}^{k}\left[\frac{1}{Z_o}\left(\frac{R - Z_o}{R + Z_o}\right)\right] x_1(t - 2(k+1)\tau).$$

This is Equation (4-6) for our particular example. We now find:

$$B_{\mathcal{B}}\,\Lambda\,C_{\mathcal{B}} = [1 \quad 0]\begin{bmatrix}\lambda_{\delta_1} & 0 \\ 0 & \lambda_{\eta_1}\end{bmatrix}\begin{bmatrix}1 \\ 0\end{bmatrix} = \lambda_{\delta_1} = 1/Z_o,$$

$$E_{\mathcal{B}} = B_{\mathcal{B}} = [1 \quad 0],$$

$$\psi_{\delta_1} = Z_o^{-1}\,\frac{|R - Z_o|}{(R + Z_o) - |R - Z_o|}, \qquad \psi_{\eta_1} = 0,$$

and therefore

$$E_{\mathcal{B}}\,\Psi\,E'_{\mathcal{B}} = \text{[illegible]}\,\Psi\,\text{[illegible]} = \psi_{\delta_1}$$

Hence,

$$A - B_{\Omega} \Lambda C_{\Omega} + E_{\Omega} \Psi E_{\Omega}^{t} + (\mathrm{tr}\,\Psi) I_n = \left[ -\frac{1}{Z_o} \left( \frac{(R + Z_o) - 3|R - Z_o|}{(R + Z_o) - |R - Z_o|} \right) \right].$$

Thus, by Theorem 1 of Section 5-2, the solution $x_1 = 0$ is asymptotically stable if there exists $K > 0$ such that

$$x_1 \left( \frac{1}{Z_o} \left( \frac{(R + Z_o) - 3|R - Z_o|}{(R + Z_o) - |R - Z_o|} \right) x_1 + f(x_1) \right) > 0, \text{ when } |x_1| < K.$$

In a similar manner, we may apply Theorem 2 of Section 5-2 to obtain: The origin is unstable if there exists $\gamma > 0$ such that

$$x_1 \left( -\frac{1}{Z_o} \left( \frac{(R + Z_o) + |R - Z_o|}{(R + Z_o) - |R - Z_o|} \right) x_1 - f(x_1) \right) > 0, \text{ when } |x_1| < \gamma.$$

These criteria may be specified graphically as in Figure 5.2. If, in some neighborhood of the origin, the function f lies within the open

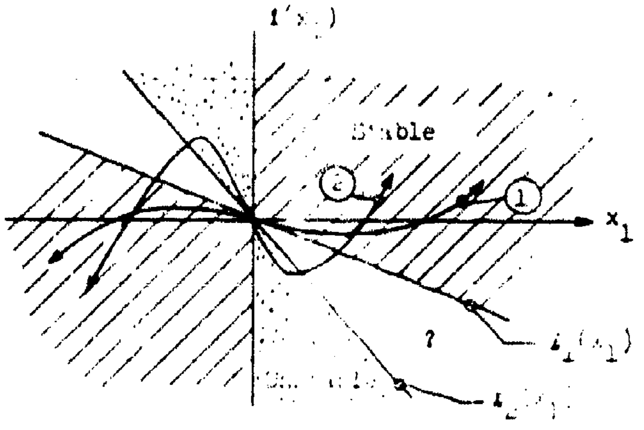


Figure 5.2. Stability criteria for Example

region labeled stable in Figure 5.2 (as curve 1 does), then the equilibrium solution $x_1 = 0$ is asymptotically stable. If, in some neighborhood of the origin, f lies within the open region labeled unstable (curve 2, for example), the solution $x_1 = 0$ is unstable. If the function f lies within the remaining region, the solution $x_1 = 0$ may or may not be stable. Also, according to Corollary 1 of Section 4-2, if for all $x_1$, $f(x_1)$ lies within the open region labeled stable, then the solution $x_1 = 0$ is completely stable. It was mentioned in Chapter I that results of this type would be obtained.

There are two cases to consider in determining the stability and instability regions of Figure 5.2. If $R \geq Z_o$, then the straight lines $\ell_1$ and $\ell_2$ are determined by the equations

$$\ell_1(x_1) = (\frac{R}{Z_o^2} - \frac{2}{Z_o})x_1,$$

$$\ell_2(x_1) = (-\frac{R}{Z_o^2})\, x_1.$$

In case $R \leq Z_o$, then

$$\ell_1(x_1) = (\frac{1}{R} - \frac{2}{Z_o})x_1,$$

$$\ell_2(x_1) = (-\frac{1}{R})x_1.$$

As mentioned in Chapter I, we note that as $Z_o \to R$ the lines $\ell_1$ and $\ell_2$ both approach the line $\ell(x_1) = (-1/R)x_1$. This results in stability and instability regions which approach those shown in Figure 1.3a. One would expect to obtain such regions for the lumped network of Figure 5.2, having no transmission line. Of course, when $Z_o = R$ the network of Figure 5.2 is equivalent to such a lumped network.

4. __Example 2__

In this example we consider a network which contains two memoryless nonlinear elements. The network, shown in Figure 5.3, consists of two lumped networks connected together by a lossless transmission line. The lumped networks are identical except that the nonlinear elements are characterized by (in general) different functions $f_1$ and $f_2$, and the values of $C_1$ and $C_2$ need not be equal. We assume that $f_1$ and $f_2$ satisfy Lipschitz conditions in some neighborhood of the origin, and that $f_1(0) = f_2(0) = 0$. For this network Equations (4-1) and (4-2) become:

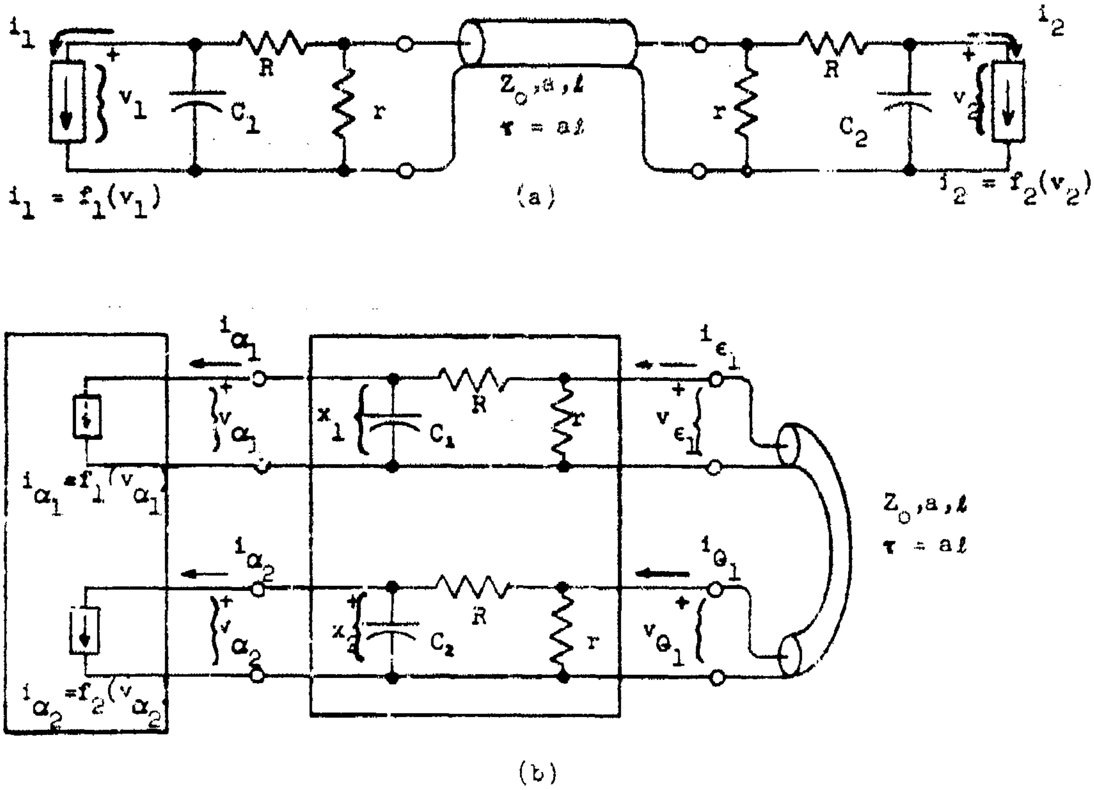


Figure 5.3. Network for Example 2.

$$\begin{bmatrix} C_1 & 0 \\ 0 & C_2 \end{bmatrix} \dot{\bar{x}}(t) = \begin{bmatrix} -1/R & 0 \\ 0 & -1/R \end{bmatrix} \bar{x}(t) + \begin{bmatrix} -1 & 0 & 1/R & 0 \\ 0 & -1 & 0 & 1/R \end{bmatrix} \begin{pmatrix} i_{\alpha_1}(t) \\ i_{\alpha_2}(t) \\ v_{\epsilon_1}(t) \\ v_{\Theta_1}(t) \end{pmatrix} ,$$

$$\begin{pmatrix} v_{\alpha_1}(t) \\ v_{\alpha_2}(t) \\ i_{\epsilon_1}(t) \\ i_{\Theta_1}(t) \end{pmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ -1/R & 0 \\ 0 & -1/R \end{bmatrix} \bar{x}(t) + \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{r}+\frac{1}{R} & 0 \\ 0 & 0 & 0 & \frac{1}{r}+\frac{1}{R} \end{bmatrix} \begin{pmatrix} i_{\alpha_1}(t) \\ i_{\alpha_2}(t) \\ v_{\epsilon_1}(t) \\ v_{\Theta_1}(t) \end{pmatrix} ,$$

where, $\bar{x} = (x_1, x_2)^t$. Thus,

$$P = \begin{bmatrix} C_1 & 0 \\ 0 & C_2 \end{bmatrix}, \quad A = \begin{bmatrix} -1/R & 0 \\ 0 & -1/R \end{bmatrix}, \quad B_I = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}, \quad B_{II} = \begin{bmatrix} 1/R & 0 \\ 0 & 1/R \end{bmatrix},$$

$$B_{\epsilon_1} = \begin{bmatrix} 1/R \\ 0 \end{bmatrix}, \quad B_{\Theta_1} = \begin{bmatrix} 0 \\ 1/R \end{bmatrix}, \quad C_I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad C_{II} = \begin{bmatrix} -1/R & 0 \\ 0 & -1/R \end{bmatrix},$$

$$C_{\epsilon_1} = [\ -1/R \quad 0\ ], \quad C_{\Theta_1} = [\ 0 \quad -1/R\ ], \quad D_I = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix},$$

$D_\epsilon = [\ \frac{1}{r} + \frac{1}{R}\ ]$, $D_\Theta = [\ \frac{1}{r} + \frac{1}{R}\ ]$. Also, $B_I\, \mathfrak{J}^*(C_I\, \bar{x}) =$

$-\begin{pmatrix} f_1(x_1) \\ f_2(x_2) \end{pmatrix}$, $\Omega = \{\ \epsilon_1, \Theta_1\ \}$, and (letting $G = 1/R$, $g = \frac{1}{r} + \frac{1}{R}$)

$$Q_{\epsilon_1} = \lambda_{\epsilon_1}\begin{bmatrix} G \\ 0 \end{bmatrix}[-G \quad 0] = \begin{bmatrix} -G^2\lambda_{\epsilon_1} & 0 \\ 0 & 0 \end{bmatrix}, \quad Q_{\Theta_1} = \lambda_{\Theta_1}\begin{bmatrix} 0 \\ G \end{bmatrix}[0 \quad -G] = \begin{bmatrix} 0 & 0 \\ 0 & -G^2\lambda_{\Theta_1} \end{bmatrix},$$

$$M_{\epsilon_1} = \mu_{\epsilon_1}\begin{bmatrix} G \\ 0 \end{bmatrix}[-G \quad 0] = \begin{bmatrix} -G^2\mu_{\epsilon_1} & 0 \\ 0 & 0 \end{bmatrix}, \quad M_{\Theta_1} = \mu_{\Theta_1}\begin{bmatrix} 0 \\ G \end{bmatrix}[0 \quad -G] = \begin{bmatrix} 0 & 0 \\ 0 & -G^2\mu_{\Theta_1} \end{bmatrix},$$

$$N_{\epsilon_1} = \nu_{\epsilon_1}\begin{bmatrix} G \\ 0 \end{bmatrix}[0 \quad -G] = \begin{bmatrix} 0 & -G^2\nu_{\epsilon_1} \\ 0 & 0 \end{bmatrix}, \quad N_{\Theta_1} = \nu_{\Theta_1}\begin{bmatrix} 0 \\ G \end{bmatrix}[-G \quad 0] = \begin{bmatrix} 0 & 0 \\ -G^2\nu_{\Theta_1} & 0 \end{bmatrix},$$

where,

$$\lambda_{\epsilon_1} = \frac{1}{g + Y_o}, \qquad \lambda_{\Theta_1} = \frac{1}{g + Y_o},$$

$$\mu_{\epsilon_1} = -Y_o \frac{(Y_o-g)/(Y_o+g)}{(g+Y_o)^2}, \qquad \mu_{\Theta_1} = -Y_o \frac{(Y_o-g)/(Y_o+g)}{(g+Y_o)^2},$$

$$\nu_{\epsilon_1} = \nu_{\Theta_1} = \frac{-Y_o}{(g + Y_o)(g + Y_o)}$$

$$\rho_{\epsilon_1} = \rho_{\Theta_1} = \frac{(Y_o-g)(Y_o-g)}{(Y_o+g)(Y_o+g)}$$

We now find:

$$B_{\square} \Lambda C_{\square} = \begin{bmatrix} G & 0 \\ 0 & G \end{bmatrix}\begin{bmatrix} (g+Y_o)^{-1} & 0 \\ 0 & (g+Y_o)^{-1} \end{bmatrix}\begin{bmatrix} -G & 0 \\ C & -G \end{bmatrix} = \begin{bmatrix} \frac{-G^2}{g+Y_o} & 0 \\ 0 & \frac{-G^2}{g+Y_o} \end{bmatrix},$$

$$E_{\Pi} = E_{\Pi}^{t} = I_2 ,$$

$$\Psi_{\epsilon_1} = \Psi_{\theta_1} = \frac{\left( Y_o \dfrac{|Y_o - g|}{(g+Y_o)^3} \cdot G + Y_o \dfrac{1}{(g+Y_o)^2} \cdot G \right) \cdot G}{1 - (Y_o-g)^2/(Y_o+g)^2}$$

$$= \frac{Y_o G^2}{(Y_o+g)} \left( \frac{|Y_o-g| + (Y_o+g)}{(Y_o+g)^2 - (Y_o-g)^2} \right)$$

$$= \frac{Y_o G^2}{(Y_o+g)} \left( \frac{|Y_o-g| + (Y_o+g)}{4Y_o g} \right) = \frac{G^2}{4g} \left( \frac{|Y_o-g| + Y_o+g}{Y_o+g} \right) ,$$

and therefore,

$$E_{\Pi} \Psi E_{\Pi}^{t} = \Psi ,$$

$$(\mathrm{tr}\,\Psi) I_n = 2\Psi .$$

Hence,

$$A - B_{\Pi} \Lambda C_{\Pi} + E_{\Pi} \Psi E_{\Pi}^{t} + (\mathrm{tr}\,\Psi) I_n = A + G^2 \Lambda + 3\Psi$$

$$= \begin{bmatrix} -G + \dfrac{G^2}{g+Y_o} + \dfrac{3}{4} \cdot \dfrac{G^2}{g} \left( \dfrac{|Y_o-g| + Y_o+g}{Y_o+g} \right) & 0 \\ 0 & -G + \dfrac{G^2}{g+Y_o} + \dfrac{3}{4} \cdot \dfrac{G^2}{g} \left( \dfrac{|Y_o-g| + Y_o+g}{Y_o+g} \right) \end{bmatrix}$$

$$= \begin{cases} \begin{bmatrix} -G + \dfrac{G^2}{g+Y_o} \left( \dfrac{2g+3Y_o}{2g} \right) & 0 \\ 0 & -G + \dfrac{G^2}{g+Y_o} \left( \dfrac{2g+3Y_o}{2g} \right) \end{bmatrix} , & \text{if } Y_o \geq g , \\[2em] \begin{bmatrix} -G + \dfrac{5}{2} \left( \dfrac{G^2}{g+Y_o} \right) & 0 \\ 0 & -G + \dfrac{5}{2} \left( \dfrac{G^2}{g+Y_o} \right) \end{bmatrix} , & \text{if } Y_o \leq g . \end{cases}$$

Thus, by Theorem 1 of Section 5-2, the solution $\bar{x} = \bar{0}$ is asymptotically stable if there exists some $K > 0$ such that:

When $Y_o \geq g$,

$$x_j \left(\left[G - \frac{G^2}{g+Y_o}\left(\frac{2g+3Y_o}{2g}\right)\right] x_j + f_j(x_j)\right) > 0 , \quad \text{for } |x_j|<K, \ j=1,2 .$$

when $Y_o \leq g$ ,

$$x_j \left(\left[G - \frac{5}{2}\left(\frac{G^2}{g+Y_o}\right)\right] x_j + f_j(x_j)\right) > 0 , \quad \text{for } |x_j|<K , \ j=1,2 .$$

Similarly,

$$A - B_{\Gamma} \Lambda C_{\Pi} - E_{\Pi} \Psi E_{\Pi}^{\vee} - (\mathrm{tr}\, \Psi)I_n = A + G^2\Lambda - 3\Psi$$

$$= \begin{bmatrix} -G + \frac{G^2}{g+Y_o} - \frac{3}{4} \cdot \frac{G^2}{g}\left(\frac{|Y_o-g| + Y_o+g}{Y_o+g}\right) & 0 \\ 0 & -G + \frac{G^2}{g+Y_o} - \frac{3}{4} \cdot \frac{G^2}{g}\left(\frac{|Y_o-g| + Y_o+g}{Y_o+g}\right) \end{bmatrix}$$

$$= \begin{cases} \begin{bmatrix} -G + \frac{G^2}{g+Y_o}\left(\frac{2g-3Y_o}{2g}\right) & 0 \\ 0 & -G + \frac{G^2}{g+Y_o}\left(\frac{2g-3Y_o}{2g}\right) \end{bmatrix} , & \text{if } Y_o \geq g , \\ \begin{bmatrix} -G - \frac{1}{2}\left(\frac{G^2}{g+Y_o}\right) & 0 \\ 0 & -G - \frac{1}{2}\left(\frac{G^2}{g+Y_o}\right) \end{bmatrix} , & \text{if } Y_o \leq g . \end{cases}$$

Thus, by Theorem 2 of Section 5-2, the solution $\bar{x} = \bar{0}$ is unstable if there exists some $\gamma > 0$ such that:

When $Y_o \geq g$,

$$x_j([G - \frac{G^2}{g+Y_o}\left(\frac{2g-3Y_o}{2g}\right)]\, x_j + f_j(x_j)) < 0 \quad \text{for } |x_j| < \gamma \;,\; j=1,2 \;.$$

When $Y_o \leq g$,

$$x_j([G + \frac{1}{2}\left(\frac{G^2}{g+Y_o}\right)]\, x_j + f_j(x_j)) < 0 \;,\; \text{for } |x_j| < \gamma \;,\; j=1,2 \;.$$

As in the first example, the above criteria may be specified graphically. We may draw a figure, identical to Figure 5.2, and require the curves of both $f_1$ and $f_2$ to lie within the open region labeled stable to ensure complete stability of the solution $\bar{x} = \bar{0}$. Similarly, if in some neighborhood of the origin the curves of both $f_1$ and $f_2$ lie within the region labeled unstable then the solution $\bar{x} = \bar{0}$ is unstable. The lines $\ell_1$ and $\ell_2$ are now determined by the following equations:

If $Y_o \geq g$, then

$$\ell_1(x_j) = -[G - \frac{G^2}{g+Y_o}(\frac{2g+3Y_o}{2g})]x_j \;,$$

$$\ell_2(x_j) = -[G - \frac{G^2}{g+Y_o}(\frac{2g-3Y_o}{2g})]x_j \;.$$

If $Y_o \leq g$, then

$$\ell_1(x_j) = -[G - \frac{5}{2}(\frac{G^2}{g+Y_o})]x_j \;,$$

$$\ell_2(x_j) = -[G + \frac{1}{2}(\frac{G^2}{g+Y_o})]x_j \;.$$

We note that as $r \to 0$ the lumped networks tend to become uncoupled from the transmission line, and hence from each other. Also,

$r \to 0 \Rightarrow g \to \infty$, and each of the above lines approaches the line $f(x_j) = -Gx_j$. Such a result is satisfying since when $r = 0$ our network becomes two simple networks of a type already considered for which, $f(x_j) > -Gx_j \Rightarrow$ complete stability and, $f(x_j) < -Gx_j \Rightarrow$ instability.

If, in this example, the two nonlinear elements were not independent of one another, it might be more convenient to use Corollaries 2 and 3 of Section 5-2 to determine asymptotic stability or instability of the solution $\bar{x} = \bar{0}$. Suppose the nonlinear multiport of Figure 5.3b is characterized by the equation

$$\begin{pmatrix} {}^{i}\alpha_1 \\ {}^{i}\alpha_2 \end{pmatrix} = \mathfrak{J}\left(\begin{pmatrix} {}^{v}\alpha_1 \\ {}^{v}\alpha_2 \end{pmatrix}\right) ,$$

where $\mathfrak{J}(\bar{0}) = \bar{0}$, and $\mathfrak{J}$ satisfies a Lipschitz condition in some neighborhood of the origin. Since $D_I = 0$ we have $\mathfrak{J}^* \equiv \mathfrak{J}$. It would, in general, be rather awkward to try to obtain graphical stability criteria as before. If, however, for all $\bar{x}$ in some open set containing the origin, $G_K = \{\bar{x} : \bar{x} \in E^n, \|\bar{x}\| < K\}$, the function $\bar{x}^t B_I \; \mathfrak{J}(C_I \bar{x})$ has continuous second partial derivatives and, if $\mathfrak{J}'(\bar{0})$ denotes the Jacobian matrix of the mapping evaluated at the origin, then, according to Corollaries 2 and 3 of Section 5-2: If the matrix

$$\begin{bmatrix} G - \dfrac{G^2}{g+Y_o} - \dfrac{3}{4} \cdot \dfrac{G^2}{g}\left(\dfrac{|Y_o - g| + Y_o + g}{Y_o + g}\right) & 0 \\ 0 & G - \dfrac{G^2}{g+Y_o} - \dfrac{3}{4} \cdot \dfrac{G^2}{g}\left(\dfrac{|Y_o - g| + Y_o + g}{Y_o + g}\right) \end{bmatrix} + \mathfrak{J}'(\bar{0})$$

is positive definite, the solution $\bar{x} = \bar{0}$ is asymptotically stable. If the matrix

$$\begin{bmatrix} -G + \dfrac{G^2}{g+Y_o} - \dfrac{3}{4} \cdot \dfrac{G^2}{g}\left(\dfrac{|Y_o-g| + Y_o+g}{Y_o+g}\right) & 0 \\ 0 & -G + \dfrac{G^2}{g+Y_o} - \dfrac{3}{4} \cdot \dfrac{G^2}{g}\left(\dfrac{|Y_o-g| + Y_o+g}{Y_o+g}\right) \end{bmatrix} = \mathbf{S}'(\bar{0})$$

is positive definite, the solution $\bar{x} = \bar{0}$ is unstable. Such criteria as these, even for a much larger network, should not be too difficult to verify, provided that adequate computing facilities are available.

5. Example 3.

In this example we consider a large network consisting of an arbitrary (finite) number of voltage controlled nonlinear resistors, having capacitance in parallel, connected together in an arbitrary manner by lossless transmission lines having lumped resistance at each end. The elements of the network are shown in Figure 5.4. We assume that all transmission lines in the entire distributed network have the same characteristic impedance $Z_o$. The parameters a and $\ell$, however, may be different for different lines. We let n denote the number of

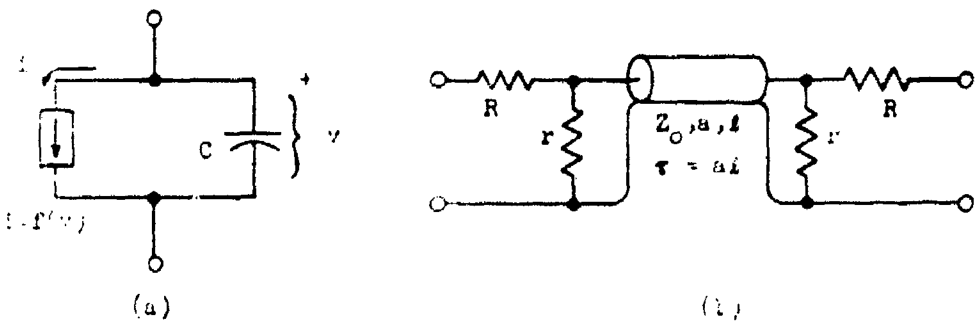


Figure 5.4. (a) Typical lumped network (b) Typical interconnecting line

lumped networks in our large distributed network, and use the subscript $j$, $j=1, \ldots, n$, to denote each particular lumped network. We denote by the positive integer $k_j$ the number of interconnecting lines which are connected to the $j$-th lumped network. In Figure 5.5 we show a typical lumped network and the lines connected to it.

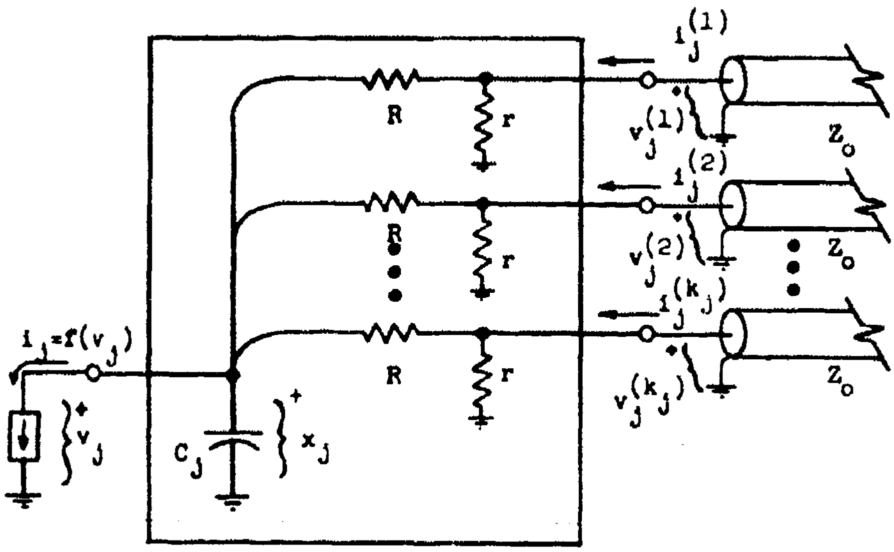


Figure 5.5. A typical subnetwork for Example 3.

The Kirchoff's voltage and current law equations for this subnetwork are:

$$[C_j]\dot{x}_j(t) = [-k_j/R]x_j(t) + [-1 \mid 1/R \;\cdots\cdots\; 1/R] \begin{pmatrix} i_j(t) \\ v_j^{(1)}(t) \\ \vdots \\ v_j^{(k_j)}(t) \end{pmatrix} ,$$

$$\begin{pmatrix} v_j(t) \\ i_j^{(1)}(t) \\ \vdots \\ i_j^{(k_j)}(t) \end{pmatrix} = \left[\begin{array}{c} 1 \\ \hline -1/R \\ \vdots \\ -1/R \end{array}\right] x_j(t) + \left[\begin{array}{c|ccc} 0 & 0 & \cdots & 0 \\ \hline 0 & (\frac{1}{R}+\frac{1}{r}) & & \bigcirc \\ \vdots & & \ddots & \\ 0 & \bigcirc & & (\frac{1}{R}+\frac{1}{r}) \end{array}\right] \begin{pmatrix} i_j(t) \\ v_j^{(1)}(t) \\ \vdots \\ v_j^{(k_j)}(t) \end{pmatrix}$$

For this example, the lumped linear multiport of Figure 4.2 consists of the collection of all lumped linear subnetworks of the type shown in Figure 5.5. Thus, for the entire network we have:

$$\begin{bmatrix} C_1 & & \bigcirc \\ & C_2 & \\ \bigcirc & & \ddots \; C_n \end{bmatrix} \dot{\bar{x}}(t) = \begin{bmatrix} -k_1/R & & \bigcirc \\ & -k_2/R & \\ \bigcirc & & \ddots \; -k_n/R \end{bmatrix} \bar{x}(t) +$$

$$\left[\begin{array}{ccc|cccc} -1 & & \bigcirc & 1/R \ldots 1/R & & & \bigcirc \\ & -1 & & & 1/R \ldots 1/R & & \\ \bigcirc & & \ddots \; -1 & \bigcirc & & \ddots & 1/R \ldots 1/R \end{array}\right] \bar{u}^*(t) \quad ,$$

$$\bar{w}^*(t) = \left[\begin{array}{ccc} 1 & & \bigcirc \\ \bigcirc & \ddots & \\ & & 1 \\ \hline -1/R & & \\ \vdots & & \\ -1/R & \bigcirc & \\ & -1/R & \\ & \vdots & \\ & -1/R & \\ & & -1/R \\ \bigcirc & & \vdots \\ & & -1/R \end{array}\right] \bar{x}(t) + \left[\begin{array}{c|ccc} \bigcirc & & \bigcirc & \\ \hline & (\frac{1}{R}+\frac{1}{r}) & & \\ & & & \bigcirc \\ \bigcirc & & \ddots & \\ & \bigcirc & & \\ & & & (\frac{1}{R}+\frac{1}{r}) \end{array}\right] \bar{u}^*(t), \tag{5-7}$$

where

$\bar{x} = (x_1, x_2, \ldots, x_n)^t$ ,

$\bar{u}^* = (i_1, i_2, \ldots, i_n, v_1^{(1)}, \ldots, v_1^{(k_1)}, v_2^{(1)}, \ldots, v_2^{(k_2)}, \ldots, v_n^{(1)}, \ldots, v_n^{(k_n)})^t$,

and

$\bar{w}^* = (v_1, v_2, \ldots, v_n, i_1^{(1)}, \ldots, i_1^{(k_1)}, i_2^{(1)}, \ldots, i_2^{(k_2)}, \ldots, i_n^{(1)}, \ldots, i_n^{(k_n)})^t$.

Equations (5-7) are not necessarily in the same form as Equations (4-1) and (4-2). This is because the elements of the vectors $\bar{u}^*$ and $\bar{w}^*$ may not be in the proper order as defined in Chapter IV. Let us investigate this matter: Since each transmission line in the distributed network is connected to a port of the lumped linear multiport for which the port voltage has been chosen as the independent variable, the vectors $\bar{u}$ and $\bar{w}$ of Equations (4-1) and (4-2) must, according to Chapter IV, be of the form

$$\bar{u} = \begin{pmatrix} \bar{I}_\alpha \\ \bar{v}_\epsilon \\ \bar{v}_\theta \end{pmatrix} \quad , \qquad \bar{w} = \begin{pmatrix} \bar{v}_\alpha \\ \bar{I}_\epsilon \\ \bar{I}_\theta \end{pmatrix} \quad .$$

It appears, therefore, that we need only relabel the port variables to give $\bar{u}^*$ and $\bar{w}^*$ the proper form. Things are not this simple, however, since this relabeling cannot always be accomplished. We see that the voltage variables in the vector $\bar{u}^*$ occur in groups such that all voltage variables associated with any lumped subnetwork (e.g., the j-th subnetwork of Figure 5.5) appear in adjacent locations in the vector $\bar{u}^*$. Hence, if these variables could be relabeled as required then at most one subnetwork would have some of its independent port voltage variables contained in both of the vectors $\bar{v}_c$ and $\bar{v}_\theta$. Consider now the following counterexample: suppose there are four lumped subnetworks and six interconnecting lines. Let the interconnections be made such that each subnetwork is connected to one end of three different lines, each of which is connected to each of the other three subnetworks, as indicated in Figure 5.6. If at most one

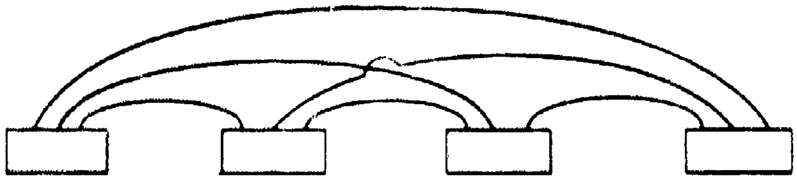

Figure 5.6. Network for Counterexample.

subnetwork has some of its independent port voltages in both of the vectors $\bar{v}_\epsilon$ and $\bar{v}_\theta$, then the port voltages associated with at least two other subnetworks are all members of either $\bar{v}_\epsilon$ or $\bar{v}_\theta$. However, since the port voltages associated with each end of any line must be labeled, one with an $\epsilon$ subscript, and the other with a $\theta$ subscript, it is clear that if any two subnetworks are considered, all six of their port voltage variables cannot be relabeled with the $\epsilon$ subscript. Similarly, they cannot all be relabeled with the $\theta$ subscript. Arriving at this contridiction proves that the proper relabeling cannot be accomplished for this network.

In order to put Equations (5-7) in the form of Equations (4-1) and (4-2) one must, in general, rearrange the port voltage variables in the vector $\bar{u}^*$, and similarly rearrange the elements of $\bar{w}^*$. If, for each line in the distributed network, one end is chosen arbitrarily and the corresponding port variables relabeled with an $\epsilon$ subscript, and if at all other ports to which a transmission line is connected the port variables are relabeled with a $\theta$ subscript, then the vector $\bar{u}^*$ may be put in the required form by simply rearranging several pairs of its elements. Performing the same operation on the vector $\bar{w}^*$ will put it in the required form. Let U denote a matrix which performs this rearrangement operation on the vectors $\bar{u}^*$ and $\bar{w}^*$. Then

$$\bar{u} = U\bar{u}^* , \quad \text{and} \quad \bar{w} = U\bar{w}^* . \tag{5-8}$$

We shall now briefly investigate the nature of the matrix U.

If one starts with the identity matrix and interchanges the i-th and the j-th columns, the resulting matrix $U_{ij}$, when premultiplying any vector, will give the original vector with the i-th and j-th elements interchanged. In the same manner, we may construct a matrix U which performs the operation of interchanging several pairs of elements, by interchanging the corresponding columns of the identity matrix. It is thus clear that U is nonsingular and, moreover, is its own inverse, since $UU\tilde{x} = I\tilde{x}$ for all $\tilde{x}$. We also note that if $\Lambda$ is a diagonal matrix, then $U\Lambda U = \Lambda$.

If in Equation (5-7) we define

$$P = \begin{bmatrix} C_1 & & \bigcirc \\ & C_2 & \\ \bigcirc & & \ddots \; C_n \end{bmatrix}, \quad A = \begin{bmatrix} -k_1/R & & \bigcirc \\ & -k_2/R & \\ \bigcirc & & \ddots \; -k_n/R \end{bmatrix},$$

$$B^* = [\, B_I^* \mid B_{II}^* \,] = \left[\begin{array}{ccc|ccc} -1 & & \bigcirc & 1/R \ldots 1/R & & \\ & -1 & & & 1/R \ldots 1/R & \bigcirc \\ \bigcirc & & \ddots \; -1 & \bigcirc & & \ddots \; 1/R \ldots 1/R \end{array}\right],$$

$$C^* = \begin{bmatrix} C_I^* \\ C_{II}^* \end{bmatrix} = \left[\begin{array}{ccc} 1 & & \bigcirc \\ \bigcirc & 1 & \\ & & \ddots \; 1 \\ \hline -1/R & & \\ \vdots & \bigcirc & \\ -1/R & & \\ & -1/R & \\ & \vdots & \\ & -1/R & \\ & & \ddots \\ & & -1/R \\ \bigcirc & & \vdots \\ & & -1/R \end{array}\right], \quad D^* = \begin{bmatrix} D_I^* & 0 \\ 0 & D_{II}^* \end{bmatrix} = \left[\begin{array}{c|ccc} \bigcirc & & \bigcirc & \\ \hline & (\frac{1}{R}+\frac{1}{r}) & & \\ & & \ddots & \bigcirc \\ \bigcirc & \bigcirc & & \\ & & & (\frac{1}{R}+\frac{1}{r}) \end{array}\right],$$

and if in Equation (5-8)

$$U = \begin{bmatrix} I & 0 \\ 0 & U_{II} \end{bmatrix},$$

then, since $\bar{u}^* = U\bar{u}$ and $\bar{w}^* = U\bar{w}$, Equations (5-7) become

$$P\dot{\bar{x}}(t) = A\bar{x}(t) + B^*U\bar{u}(t)$$

$$U\bar{w}(t) = C^*\bar{x}(t) + D^*U\bar{u}(t) \quad ,$$

and hence (noting $UD^*U = D^*$) ,

$$P\dot{\bar{x}}(t) = A\bar{x}(t) + B^*U\bar{u}(t)$$

$$\bar{w}(t) = UC^*\bar{x}(t) + D^*\bar{u}(t) \ .$$

These equations are in the form of Equations (4-1) and (4-2) if we define

$$B = B^*U = [\ B^*_I \ \vert\ B^*_{II}U_{II}\ ] \quad ,$$

$$C = UC^* = \begin{bmatrix} C^*_I \\ \hline U_{II}C^*_{II} \end{bmatrix} \ .$$

Now, letting $G = 1/R$ and $\varepsilon = \frac{1}{R} + \frac{1}{r}$,

$$\lambda_\xi = \lambda = \frac{1}{g+Y_o} \quad ,$$

$$\mu_\xi = \frac{-Y_o\left(\frac{Y_o-g}{Y_o+g}\right)}{(g+Y_o)^2} \quad ,$$

$$\nu_\xi = \frac{-Y_o}{(g+Y_o)^2} \ ,$$

$$\rho_\xi = \frac{(Y_o-g)^2}{(Y_o+g)^2} \ , \text{ for all } \xi \in \Omega \ .$$

Also, $\|B_\xi\| = \|C_\xi\| = 0$ for all $\xi \in \Omega$ and, hence

$$E = \begin{bmatrix} 1\dots1 & & & \bigcirc \\ & 1\dots1 & & \\ \bigcirc & & \ddots & \\ & & & 1\dots1 \end{bmatrix} .$$

Thus

$$B_{II} \Lambda C_{II} = B_{II}^* U_{II} \Lambda U_{II} C_{II}^* = B_{II}^* \Lambda C_{II}^*$$

$$= \begin{bmatrix} \frac{1}{R}\dots\frac{1}{R} & & & \bigcirc \\ & \frac{1}{R}\dots\frac{1}{R} & & \\ & & \ddots & \\ \bigcirc & & & \frac{1}{R}\dots\frac{1}{R} \end{bmatrix} \begin{bmatrix} \lambda & & & \bigcirc \\ & \lambda & & \\ \bigcirc & & \ddots & \\ & & & \lambda \end{bmatrix} \begin{bmatrix} -1/R & & & & & & & \\ \vdots & & & & \bigcirc & & & \\ -1/R & & & & & & & \\ & -1/R & & & & & & \\ & \vdots & & & & & & \\ & -1/R & & & & & & \\ & & \ddots & & & & & \\ & & & -1/R & & & & \\ & \bigcirc & & \vdots & & & & \\ & & & -1/R & & & & \end{bmatrix}$$

$$= \begin{bmatrix} \frac{1}{R}\dots\frac{1}{R} & & & \bigcirc \\ & \frac{1}{R}\dots\frac{1}{R} & & \\ & & \ddots & \\ \bigcirc & & & \frac{1}{R}\dots\frac{1}{R} \end{bmatrix} \begin{bmatrix} -\lambda/R & & & & \\ \vdots & & & \bigcirc & \\ -\lambda/R & & & & \\ & -\lambda/R & & & \\ & \vdots & & & \\ & -\lambda/R & & & \\ & & \ddots & & \\ & & & -\lambda/R & \\ & \bigcirc & & \vdots & \\ & & & -\lambda/R & \end{bmatrix}$$

$$= \begin{bmatrix} -k_1\lambda G^2 & & & \bigcirc \\ & -k_2\lambda G^2 & & \\ & & \ddots & \\ \bigcirc & & & -k_n\lambda G^2 \end{bmatrix} = -\lambda G^2 \begin{bmatrix} k_1 & & & \bigcirc \\ & k_2 & & \\ & & \ddots & \\ \bigcirc & & & k_n \end{bmatrix},$$

where $\lambda = (g+Y_o)^{-1}$

Similarly, since

$$\Psi_\xi = \frac{\left(Y_o \dfrac{|Y_o-g|}{(g+Y_o)^3} \cdot G + Y_o \dfrac{1}{(g+Y_o)^2} \cdot G\right) \cdot G}{1 - (Y_o-g)^2/(Y_o+g)^2}$$

$$= \frac{G^2}{4g}\left(\frac{|Y_o-g| + Y_o+g}{Y_o+g}\right), \quad \text{for all } \xi \in \Omega,$$

$$E_{\Pi} \Psi E_{\Pi}^t = \frac{G^2}{4g}\left(\frac{|Y_o-g| + Y_o+g}{Y_o+g}\right) \begin{bmatrix} k_1 & & & \bigcirc \\ & k_2 & & \\ & & \ddots & \\ \bigcirc & & & k_n \end{bmatrix}, \quad \text{and}$$

$$(\mathrm{tr}\, \Psi) I_n = \frac{G^2}{4g}\left(\frac{|Y_o-g| + Y_o + g}{Y_o+g}\right) \begin{bmatrix} 2N_\ell & & & \bigcirc \\ & 2N_\ell & & \\ & & \ddots & \\ \bigcirc & & & 2N_\ell \end{bmatrix},$$

where $N_\ell$ denotes the total number of lines in the entire distributed netowrk. Therefore

$$A - B_{\Pi}\Lambda C_{\Pi} + E_{\Pi}\Psi E_{\Pi}^{t} + (\mathrm{tr}\,\Psi)I_n = A + \lambda G^2 \begin{bmatrix} k_1 & & \bigcirc \\ & k_2 & \\ \bigcirc & & \ddots \; k_n \end{bmatrix}$$

$$+ \frac{G^2}{4g}\left(\frac{|Y_o-g| + Y_o+g}{Y_o+g}\right)\begin{bmatrix} k_1+2N_\ell & & \bigcirc \\ & k_2+2N_\ell & \\ \bigcirc & & \ddots \; k_n+2N_\ell \end{bmatrix}$$

$$= \begin{bmatrix} -k_1G + \frac{k_1G^2}{g+Y_o} + \frac{(k_1+2N_\ell)G^2}{4g}\left(\frac{|Y_o-g| + Y_o+g}{Y_o+g}\right) & & \bigcirc \\ & \ddots & \\ \bigcirc & & -k_nG + \frac{k_nG^2}{g+Y_o} + \frac{(k_n+2N_\ell)G^2}{4g}\left(\frac{|Y_o-g| + Y_o+g}{Y_o+g}\right) \end{bmatrix} .$$

For $j = 1, \ldots, n$, let

$$p_j = -k_jG + \frac{k_jG^2}{g+Y_o} + \frac{(k_j + 2N_\ell)G^2}{4g}\left(\frac{|Y_o-g| + Y_o+g}{Y_o+g}\right) .$$

Then, if $Y_o \geq g$,

$$p_j = -k_j\left[G - \frac{G^2}{g+Y_o}\left(1 + \frac{Y_o}{2g}\right)\right] + N_\ell\,\frac{G^2Y_o}{g(g+Y_o)} \qquad . \qquad (5\text{-}9)$$

If $Y_o \leq g$,

$$p_j = -k_j\left[G - \frac{3}{2}\left(\frac{G^2}{g+Y_o}\right)\right] + N_\ell\,\frac{G^2}{g+Y_o} \quad . \qquad (5\text{-}10)$$

As $r \to 0$ note that $g \to \infty$, and in Equations (5-9) and (5-10), $p_j \to -k_jG = -k_j/R$. Causing $r$ to become small has, of course, the effect of tending to uncouple the lumped circuits from one another. We also have,

$$A - B_I \Lambda C_{II} - E_{II} \Psi E_{II}^t - (\mathrm{tr}\,\Psi) I_n$$

$$= \begin{bmatrix} -k_1 G + \frac{k_1 G^2}{g+Y_o} - \frac{(k_1+2N_\ell)G^2}{4g}\left(\frac{|Y_o-g| + Y_o g}{Y_o+g}\right) & & \bigcirc \\ & \ddots & \\ \bigcirc & & -k_n G + \frac{k_n G^2}{g+Y_o} - \frac{(k_n+2N_\ell)G^2}{4g}\left(\frac{|Y_o-g| + Y_o+g}{Y_o+g}\right) \end{bmatrix} .$$

For $j = 1, \ldots, n$ , let

$$q_j = -k_j G + \frac{k_j G^2}{g+Y_o} - \frac{(k_j + 2N_\ell)G^2}{4g}\left(\frac{|Y_o-g| + Y_o+g}{Y_o+g}\right)$$

Then, if $Y_o \geq g$,

$$q_j = -k_j\left[G - \frac{G^2}{g+Y_o}\left(1 - \frac{Y_o}{2g}\right)\right] - N_\ell \frac{G^2 Y_o}{g(g+Y_o)} \quad . \qquad (5\text{-}11)$$

If $Y_o \leq g$ ,

$$q_j = -k_j\left[G - \frac{1}{2}\left(\frac{G^2}{g+Y_o}\right)\right] - N_\ell \frac{G^2}{g+Y_o} \quad . \qquad (5\text{-}12)$$

In Equations (5-11) and (5-12) it is also true that as $r \to 0$, $q_j \to -k_j/R$.

Since

$$B_I\ \mathcal{U}^*(C_I\bar{x}) = B_I^*\ \mathcal{U}^*(C_I^*\ \bar{x}) = -\begin{pmatrix} f_1(x_1) \\ \vdots \\ f_n(x_n) \end{pmatrix} \quad ,$$

the application of Theorems 1 and 2 of Section 5-2 yield the following stability criteria: The solution $\bar{x} = \bar{0}$ is asymptotically stable if there exists some $K > 0$ such that $x_j(-p_j x_j + f_j(x_j)) > 0$ , for

$|x_j| < K$, $j = 1, \ldots, n$, where $p_j$ is given by Equation (5-9) for $Y_o \geq g$, and by Equation (5-10) for $Y_o \leq g$. The solution $\bar{x} = \bar{0}$ is unstable if there exists some $\gamma > 0$ such that $x_j(-q_j x_j + f_j(x_j)) < 0$, for $|x_j| < \gamma$, $j = 1, \ldots, n$, where $q_j$ is given by Equation (5-11) for $Y_o \geq g$, and by Equation (5-12) for $Y_o \leq g$. As in the other examples we may specify these criteria graphically. For $j = 1, \ldots, n$, we may draw, as in Figure 5.7,

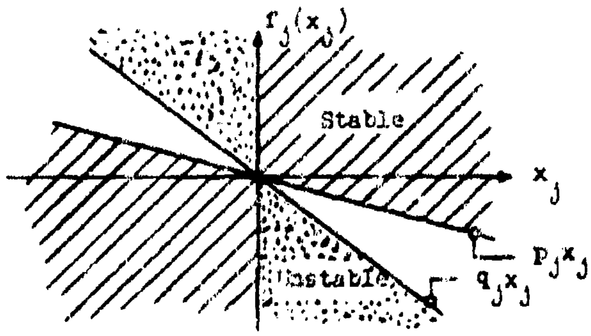


Figure 5.7. Stability criteria for Example 3.

regions in the $x_j, f_j(x_j)$ plane. If $f_j$ lies in the open region labeled stable for all $x_j \neq 0$ in some neighborhood of the origin, the solution $\bar{x} = \bar{0}$ is asymptotically stable. If for all $x_j \neq 0$, $f_j(x_j)$ lies in the open region labeled stable, then the solution $\bar{x} = \bar{0}$ is completely stable. Similarly, if $f_j$ lies in the open region labeled unstable for all $x_j \neq 0$ in some neighborhood of the origin, the solution $\bar{x} = \bar{0}$ is unstable.

We have noted that as $r \to 0$, each of the $p_j$, $q_j$ approach $-k_j/R$. This is exactly what we would hope to obtain since when

$r = 0$, the $n$ lumped networks are uncoupled and they are completely stable if $f_j(x_j) > -k_j G$. With our theory we have established criteria which ensure complete stability (and instability) even when the lumped networks are coupled. Note that the criteria are independent of $a$ and $\ell$ for every line, and are also independent of the value of $C_j$ for $j = 1, \ldots, n$.

## Chapter VI

## CONCLUSIONS

A stability theory for nonlinear distributed networks has been presented. In the development of this theory three major steps were taken. First, it was shown that the electrical behavior at the ports of certain two-port networks containing lossless transmission lines may be described by a system of functional equations. Next, a class of nonlinear distributed networks was defined and it was shown, using state variable techniques, that the behavior of any network in this class may be characterized by a system of functional-differential equations. Finally, a Lyapunov functional was presented and the stability theory for functional-differential equations was used to obtain several theorems and corollaries which specify sufficient conditions to ensure that the equilibrium state of a given network in the above class is stable, asymptotically stable, completely stable, or unstable.

It has been shown that the stability criteria that this theory may specify for any particular distributed network are independent of the length of the transmission lines and also independent of the phase velocity ( $= 1/a$) of the lines. Generally, the criteria are also independent of the values of the reactive elements contained in the network.

The stability criteria may not always be the best that one might hope to obtain. For example, in the paper by Brayton and Miranker [10], which is, at this time, the only other comparable theory known to the author, a stability criterion for the network of Example 1, Chapter V, is given.

They obtain: If

$$f' > -\frac{R}{Z_o^2 + R^2}\ , \tag{6-1}$$

then the solution $x_1 = 0$ is completely stable. By application of Corollary 1 to this example, we obtain: If $Z_o \geq R$ and

$$f' > \frac{1}{R} - \frac{2}{Z_o}\ , \tag{6-2}$$

or if $Z_o \leq R$ and

$$f' > \frac{R}{Z_o^2} - \frac{2}{Z_o}\ , \tag{6-3}$$

then the solution $x_1 = 0$ is completely stable. It is easily shown that if the ratio $R/Z_o$ is less than approximately 0.648 then $1/R - 2/Z_o > -R/(Z_o^2 + R^2)$; and hence the criterion of Equation (6-1) is less restrictive (in terms of admissible functions $f$) than the criterion of Equation (6-2). In case the ratio $R/Z_o$ is greater than approximately 1.54 then $R/Z_o^2 - 2/Z_o > -R/(Z_o^2 + R^2)$, and hence the criterion of Equation (6-1) is less restrictive than the criterion of Equation (6-3). If, however,

$$0.648 < R/Z_o < 1.54,$$

then the criteria of Equations (6-2) and (6-3) are less restrictive than the criterion of Equation (6-1). For example, if $Z_o = 10\ \Omega$ and $R = 9\ \Omega$, then our criteria imply complete stability if $f' > -0.089$; Brayton and Miranker's criterion implies complete stability if $f' > -0.0497$.

Several areas of future research along the lines of this work might prove fruitful. First of all, it would be significant to extend the class of distributed networks which may be described by functional-differential equations (and hence, to which our stability theory might apply) beyond the class which we have defined. One such extension might proceed along the lines of removing the restriction in condition (A1) concerning the zeros of transmission at $s = \infty$ of the matrix D. If this could be done, then networks containing transmission lines terminated at one end by only lumped memoryless nonlinear elements could be studied. We could study, for example, the network of Figure 6.1. The stability of the equilibrium state $v = 0$ for this network cannot be studied by our theory.

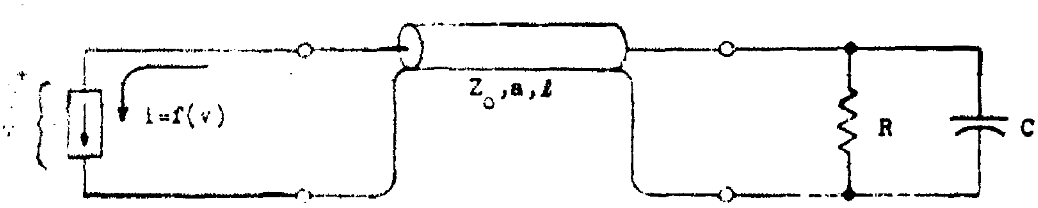


Figure 6.1. Another nonlinear distributed network.

Another useful extension of our work might be to consider networks containing types of distributed elements which are more general than lossless (LC) transmission lines. A theory for networks containing RC lines, for example, should be quite useful.

The Lyapunov functional of Equation ([illegible]-1) is probably not the only one which might be found that leads to a useful stability theory.

Of course, the conception of Lyapunov functionals is more of an art than a science. If, however, one is clever enough to find other Lyapunov functionals, it should be rather routine to develop new, and perhaps better, stability criteria for the class of distributed networks which we have defined.

Networks containing only lumped memoryless elements and transmission lines (i.e., no reactances) can certainly also have stable and unstable equilibria. A stability theory for such networks should be a valuable contribution.

Finally, having defined certain two-ports by functional equations, one is lead to conjecture about the perhaps academic problem of developing a theory of analysis of networks containing what might be called "functional elements". A functional one-port might be described by an equation of the type

$$i(t) = f(v_t),$$

where $i(t)$ denotes the value of the current through the element at the time $t$, and $v_t$ is a function (the voltage across the element) on some time interval for which $t$ is the right-hand end. Perhaps the theory of dynamical systems would be the proper setting for such a problem. Choosing the proper state space, and defining the order of complexity for such networks does not, at this time, seem to be a trivial problem.

## APPENDIX A

In this appendix we study some of the aspects of the problem of locating the zeros of the function

$$Y(s) \equiv \frac{1}{r} + sC + \frac{1}{Z_o}\left[\frac{(R + Z_o)e^{\tau s} - (R - Z_o)e^{-\tau s}}{(R + Z_o)e^{\tau s} + (R - Z_o)e^{-\tau s}}\right]$$

The equation $Y(s) = 0$ is the characteristic equation of the distributed network considered as an example in Chapter I. It is clear that $Y(s) = 0$ if and only if

$$(Z_o + sCrZ_o)[(R + Z_o)e^{2\tau s} + (R - Z_o)] = -r(R + Z_o)e^{2\tau s} + r(R - Z_o),$$

or

$$[(r + Z_o)(R + Z_o) + sCrZ_o(R + Z_o)]e^{2\tau s} = [(R - Z_o)(r - Z_o) - sCrZ_o(R - Z_o)].$$

Let us define $z = 2\tau s$, and let

$$\alpha = \frac{CrZ_o(R + Z_o)}{2\tau},$$

$$\beta = (r + Z_o)(R + Z_o),$$

$$\gamma = -\frac{CrZ_o(R - Z_o)}{2\tau},$$

$$\delta = (R - Z_o)(r - Z_o).$$

It follows that $Y(s) = 0$ if and only if $s = \zeta/2\tau$, where $\zeta$ is a root of the equation

$$(\alpha z + \beta)e^{z} = \gamma z + \delta. \qquad \text{(A-1)}$$

The location of the roots of Equation (A-1) has been studied extensively by E.M. Wright [5,6]. We shall adopt some of his techniques here. We first transform Equation (A-1) into a simpler equation by replacing $z$ by $z - \ln|\alpha/\gamma|$. Note that $\alpha/\gamma = -(R + Z_o)/(R - Z_o) = -1/\Gamma$, where $\Gamma$ is the reflection coefficient at the right-hand end of the transmission line in Figure 1.2. In case $\alpha/\gamma > 0$ (i.e., $\Gamma < 0$, which is the case for the example of Chapter I) we obtain

$$(z - A + B)e^{z} = z - A - B, \tag{A-2}$$

where

$$A = -\frac{\beta/\alpha + \delta/\gamma}{2} + \ln|\alpha/\gamma| = -\frac{2\tau}{Cr} + \ln|-1/\Gamma|,$$

$$B = \frac{\beta/\alpha - \delta/\gamma}{2} = \frac{2\tau}{CZ_o} .$$

Suppose $r = -1.78$, $R = 1.42$, $C = 1$, $Z_o = 2.19$, $\tau = \pi/6$. It then follows that $\ln|\alpha/\gamma| \approx 1.55$, $A \approx 2.14$, and $B \approx 0.48$. If we define the function $c_1(B)$ by $c_1(B) = \ln(B + 1 + \sqrt{B^2 + 2B}) + \sqrt{B^2 + 2B}$, then $c_1(0.48) \approx 2.03$. According to Wright [5], if $B > 0$ and $A > c_1(B)$ then Equation (A-2) has exactly two real roots. Furthermore, all of the roots of Equation (A-2) have real parts which are less than or equal to the value of the larger of these two real roots. It is clear that this statement also applies to the zeros of $Y(s)$ for the above parameter values.

We shall now show that the two real zeros of $Y(s)$ lie in the left half of the s-plane for the parameter values given above. This will then prove that $Y(s)$ has no zeros with positive real parts. We first rewrite $Y(s)$ as

$$Y(s) = \frac{1}{r} + sC + \frac{1}{Z_o}\left[\frac{e^{2\tau s} - \Gamma}{e^{2\tau s} + \Gamma}\right] .$$

Then, if we define $y_1(s) = -\frac{1}{r} - sC$ and $y_2(s) = \frac{1}{Z_o}\left[\frac{e^{2\tau s} - \Gamma}{e^{2\tau s} + \Gamma}\right]$, it follows that $Y(s) = 0$ if and only if $y_1(s) = y_2(s)$. On the real axis in the s-plane we have $\frac{d}{ds} y_2(s) = \frac{1}{Z_o} [(e^{2\tau s} + \Gamma)2\tau e^{2\tau s} - (e^{2\tau s} - \Gamma)2\tau e^{2\tau s}] / (e^{2\tau s} + \Gamma)^2 = \frac{4\Gamma\tau e^{2\tau s}}{Z_o(e^{2\tau s} + \Gamma)^2}$.

Thus, $\Gamma < 0 \Rightarrow \frac{d}{ds} y_2(s) < 0$ for all real s. Since we also have $y_2(-\infty) = -\frac{1}{Z_o}$, $y_2(0) = \frac{1}{Z_o}\left(\frac{1 - \Gamma}{1 + \Gamma}\right) = \frac{1}{R}$, and $y_2(\infty) = \frac{1}{Z_o}$, the function $y_2(s)$ behaves as shown in Figure A.1, for real s. The point $s = \sigma$ is the solution of the equation $e^{2\tau s} + \Gamma = 0$. The important thing to

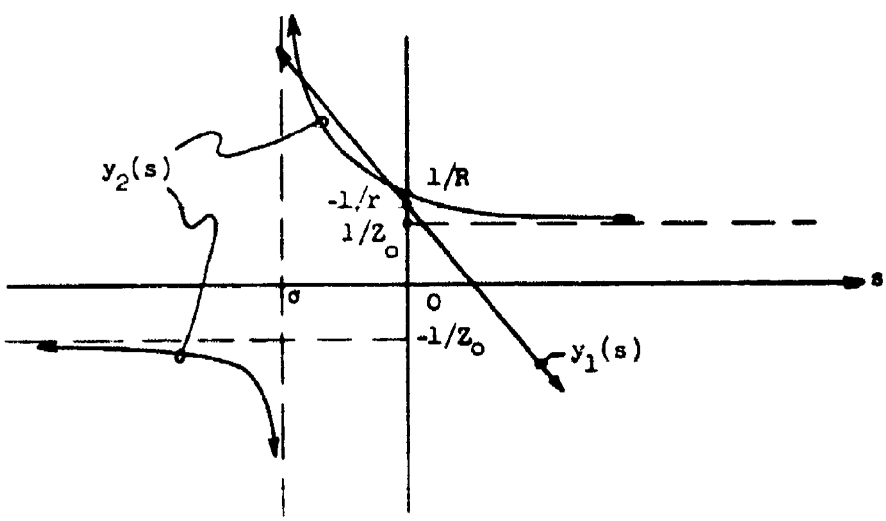


Figure A.1. The functions $y_1(s)$ and $y_2(s)$ for real s, when $\Gamma < 0$.

note here is that $y_2(s)$ is a strictly monotonic decreasing function and $\frac{d}{ds} y_2(s)$ is strictly monotonic increasing for $s > \sigma$; hence, the straight line $y_1 = -1/r - sC$ intersects this curve in, at most, two points. This agrees with Wright's results. In our case there are two intersections and they occur at approximately $s = -0.819$ and $s = -0.242$. That is, $Y(s)$ has exactly two real zeros and they are both negative. Hence, we have shown that for the parameter values: $r = -1.78$, $R = 1.42$, $C = 1$, $Z_o = 2.19$, and $\tau = \pi/6$, the zeros of $Y(s)$ all have negative real parts.

The main reason that Figure A.1 has been introduced is that it can provide a certain amount of insight into the stability problem for the linear distributed network of Figure 1.2. We shall not consider this topic in great depth, however the following remarks seem to be appropriate. First, we saw in Chapter I that an unstable network was made stable by simply increasing the value of the capacitor C. In Figure A.1 we see that varying the value of C simply causes the slope of the straight line representing $y_1(s)$ to vary. Hence, when $1/Z_o < -1/r < 1/R$, we can always make C small enough that $Y(s)$ will have two positive real zeros. Conversely, as we increase C we cause the real zeros to vanish and then reappear in the left half plane. Similarly, we can cause positive real zeros to exist for any value of $C > 0$, $1/Z_o < -1/r < 1/R$, by simply causing the point $\sigma$ to move close enough to the $s = 0$ axis. This corresponds to an increase in the value of $\tau$ ($\tau = a\ell$), which in turn is caused easily enough by increasing either the value of a ($a = \sqrt{L_t C_t}$) or $\ell$. We can also force the real zeros to occur in the left half of the s-plane by using a short enough transmission line (small $\ell$) or by making a small enough.

Let us now consider the location of the real zeros of $Y(s)$ when $-1/r > 1/R$ (the lumped network of Figure 1.1 is unstable if and only if r and R have values such that this inequality is satisfied, since $-1/r > 1/R \Leftrightarrow g < -G$). If $\Gamma < 0$, it is obvious, from Figure A.1, that $Y(s)$ will always have one positive real zero when $-1/r > 1/R$. If $\Gamma > 0$, then $\frac{d}{ds}\, y_2(s) > 0$ for all real s, and hence $y(s)$ behaves as shown in Figure A.2. Again, we see that $Y(s)$ will always have a positive real zero when $-1/r > 1/R$. Hence, if the lumped network of Figure 1.1 is unstable, then so is the distributed network of Figure 1.2.

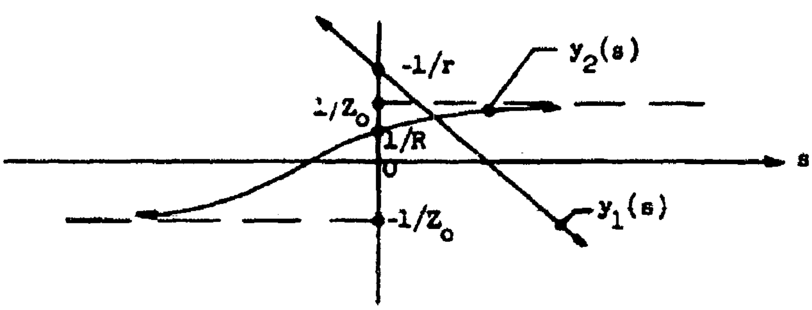


Figure A.2. The functions $y_1(s)$ and $y_2(s)$ for real s, when $\Gamma > 0$.

APPENDIX B

In this appendix we shall prove that the linear space $C =$ $C((-\infty,0], E^n)$ with the compact open topology is metrizable with metric $\rho$ defined as follows: Let $\{t_k\}$ be a sequence of real numbers with $0 = t_o < t_1 < \cdots < t_k < t_{k+1} < \cdots$ and such that $\lim_{k\to\infty} t_k = \infty$. For a fixed real number b, $0 < b < 1$, and for every $\bar{\varphi}, \bar{\psi} \in C$, let $\rho(\bar{\varphi},\bar{\psi}) = \sum_{k=o}^{\infty} m_k$, where $m_k = \min( b^k, \sup( \|\bar{\varphi}(t) - \bar{\psi}(t)\| :$ $-t_{k+1} \le t \le -t_k \} )$. Since $0 \le m_k \le b^k$ for all k, and since $0 < b < 1$, it is clear that $\sum_{k=o}^{\infty} m_k$ always converges and hence $\rho(\bar{\varphi},\bar{\psi})$ is well defined. We first verify that $\rho$ is, indeed, a metric on C; that is $\forall\, \bar{\varphi}, \bar{x}, \bar{\psi} \in C$, $\rho$ satisfies the three properties:

1) $\rho(\bar{\varphi},\bar{\psi}) \ge 0$, $\qquad \rho(\bar{\varphi},\bar{\psi}) = 0$ if and only if $\bar{\varphi} = \bar{\psi}$;

2) $\rho(\bar{\varphi},\bar{\psi}) = \rho(\bar{\psi},\bar{\varphi})$;

3) $\rho(\bar{\varphi},\bar{\psi}) \le \rho(\bar{\varphi},\bar{x}) + \rho(\bar{x},\bar{\psi})$.

Properties 1) and 2) are, of course, obvious. To prove property 3) we need the lemma:

<u>Lemma</u>: If A, B, C, D are nonnegative real numbers and if $A \le B + C$, then $\min( A,D ) \le \min( B,D ) + \min( C,D )$.

<u>proof</u>: If $D \le B$ and $D \le C$,

$\min( A,D ) \le D \le D + D = \min( B,D ) + \min( C,D )$.

If $D \le B$ and $D > C$,

$\min( A,D ) \le D \le D + C = \min( B,D ) + \min( C,D )$.

If $D > B$ and $D \le C$,

$\min( A,D ) \le D \le B + D = \min( B,D ) + \min( C,D )$.

If $D > B$ and $D > C$,

$$\min\{ A, D \} \leq A \leq B + C - \min\{ B, D \} + \min\{ C, D \} .$$

Having exhausted all possibilities, the lemma is proved. Q.E.D.

Now, for any k, if $\sigma$ is an arbitrary point in the interval $[-t_{k+1}, -t_k]$ then

$$\|\bar{\varphi}(\sigma) - \bar{\psi}(\sigma)\| = \|\bar{\varphi}(\sigma) - \bar{\chi}(\sigma) + \bar{\chi}(\sigma) - \bar{\psi}(\sigma)\| \leq \|\bar{\varphi}(\sigma) - \bar{\chi}(\sigma)\|$$
$$+ \|\bar{\chi}(\sigma) - \bar{\psi}(\sigma)\| \leq \sup\{ \|\bar{\varphi}(t) - \bar{\chi}(t)\| : -t_{k+1} \leq t \leq -t_k \} +$$
$$\sup\{ \|\bar{\chi}(t) - \bar{\psi}(t)\| : -t_{k+1} \leq t \leq -t_k \} .$$

But, since $\sigma$ is an arbitrary point in $[-t_{k+1}, -t_k]$, we therefore have

$$\sup\{ \|\bar{\varphi}(t) - \bar{\psi}(t)\| : -t_{k+1} \leq t \leq -t_k \} \leq \sup\{ \|\bar{\varphi}(t) - \bar{\chi}(t)\| :$$
$$-t_{k+1} \leq t \leq -t_k \} + \sup\{ \|\bar{\chi}(t) - \bar{\psi}(t)\| : -t_{k+1} \leq t \leq -t_k \}.$$

By the above lemma, we then have

$$\min\{ b^k, \sup\{ \|\bar{\varphi}(t) - \bar{\psi}(t)\| : -t_{k+1} \leq t \leq -t_k \} \} \leq \min\{ b^k, \sup\{ \|\bar{\varphi}(t)$$
$$- \bar{\chi}(t)\| : -t_{k+1} \leq t \leq -t_k \} \} + \min\{ b^k, \sup\{ \|\bar{\chi}(t) - \bar{\psi}(t)\| : -t_{k+1} \leq t$$
$$\leq -t_k \} \},$$ for all k. Thus, $\rho(\bar{\varphi},\bar{\psi}) \leq \rho(\bar{\varphi},\bar{\chi}) + \rho(\bar{\chi},\bar{\psi})$. Q.E.D.

Let $\mathfrak{I}_c$ denote the compact open topology for C. The definition of $\mathfrak{I}_c$ is given in the first footnote of Chapter II. Then $(C, \mathfrak{I}_c)$ denotes the topological space consisting of C with the compact open topology. If $\rho$ is the metric on C, defined above ( for any fixed b, $0 < b < 1$), let $\mathfrak{M}_\rho$ denote the metric topology on C. Then $(C, \mathfrak{M}_\rho)$ denotes the topological space consisting of C with the metric topology $\mathfrak{M}_\rho$. If $\bar{\varphi} \in C$, let $S(\bar{\varphi},\mu) \in \mathfrak{M}_\rho$ denote the open $\mu$-sphere about $\bar{\varphi}$, i.e., $S(\bar{\varphi},\mu) = \{ \bar{\psi} : \bar{\psi} \in C, \rho(\bar{\varphi},\bar{\psi}) < \mu \}$.

*Theorem.* The topological space $(C, \mathfrak{I}_c)$ is metrizable with the metric $\rho$ defined above proof: Let $\mathcal{B}_c$ denote the base for $\mathfrak{S}_c$ consisting of finite intersections of members of $\mathcal{Q}$ (see first footnote of Chapter II). Let $\mathcal{B}_\rho$ denote the base for $\mathfrak{M}_\rho$ consisting of all open $\mu$-spheres in $(C, \mathfrak{M}_\rho)$. We prove that $(C, \mathfrak{S}_c) = (C, \mathfrak{M}_\rho)$ by showing that if $\bar{\varphi}$ is a point in an element $B_c$ of $\mathcal{B}_c$, then there is an element $B_\rho$ of $\mathcal{B}_\rho$ containing the point $\bar{\varphi}$ and contained in $B_c$, and conversely.

Let $B_c$ be a member of $\mathcal{B}_c$. Thus, $B_c = \bigcap_{i=1}^{m} A(K_i, U_i)$, where $A(K_i, U_i)$ is the set of all $\bar{\varphi} \in C$ which map $K_i$, a compact subset of $(-\infty, 0]$, into $U_i$, an open subset of $E^n$. Let $\bar{\varphi} \in B_c$. We will show that there exists an open $\mu$-sphere about $\bar{\varphi}$, $S(\bar{\varphi}, \mu)$, which is contained in $B_c$. For $i = 1, \ldots, m$, $K_i$ is compact in $(-\infty, 0]$; therefore $\exists$ an integer $k^* \geq 1$ such that $-t_{k^*} < t$ for every $t \in K_i$. Also, $\bar{\varphi}[K_i]$ is compact in $E^n$. For every $\bar{y} \in \bar{\varphi}[K_i]$ $\exists$ $\delta_y$, $0 < \delta_y < b^{k^*}$, such that $\|\bar{y}' - \bar{y}\| < \delta_y \Rightarrow \bar{y}' \in U_i$. The family of $\delta_y/2$ neighborhoods $\{ N_{\delta_y/2}(\bar{y}) : \bar{y} \in \bar{\varphi}[K_i] \}$ is an open cover of $\bar{\varphi}[K_i]$ and, since $\bar{\varphi}[K_i]$ is compact, $\exists$ a finite subcover $\{ N_{\delta_{y_1}/2}(\bar{y}_1), \ldots, N_{\delta_{y_\ell}/2}(\bar{y}_\ell) \}$ of $\bar{\varphi}[K_i]$. Note that $N_{\delta_{y_j}}(\bar{y}_j) \subset U_i$ for $j = 1, \ldots, \ell$. Let $\delta_i = \min( \delta_{y_1}/2, \ldots, \delta_{y_\ell}/2 )$. Then, if $\bar{\psi} \in S(\bar{\varphi}, \delta_i)$ we have that, for $k \in \{ 0, \ldots, k^*-1 \}$ : $\min( b^{k^*}, \sup\{ \|\bar{\varphi}(t) - \bar{\psi}(t)\| : -t_{k+1} \leq t \leq -t_k \} )$

$$\leq \sum_{k=0}^{k^*-1} \min( b^{k^*}, \sup\{ \|\bar{\varphi}(t) - \bar{\psi}(t)\| : -t_{k+1} \leq t \leq -t_k \} ) \leq$$

$$\sum_{k=0}^{k^*-1} \min( b^k, \sup\{ \|\bar{\varphi}(t) - \bar{\psi}(t)\| : -t_{k+1} \leq t \leq -t_k \} ) = \sum_{k=0}^{k^*-1} m_k \leq$$

$\sum_{k=0}^{\infty} m_k = \rho(\bar{\varphi},\bar{\psi}) < \delta_i$. Therefore, $\forall\ t \in K_i$, $\min(\ b^{k^*}, \|\bar{\varphi}(t) - \bar{\psi}(t)\|\ ) <$ $\delta_i$. But $\delta_i < b^{k^*} \Rightarrow \|\bar{\varphi}(t) - \bar{\psi}(t)\| < \delta_i \quad \forall\ t \in K_i$. Thus, $\|\bar{\psi}(t) - \bar{y}_j\|$ $= \|\bar{\psi}(t) - \bar{\varphi}(t) + \bar{\varphi}(t) - \bar{y}_j\| \leq \|\bar{\psi}(t) - \bar{\varphi}(t)\| + \|\bar{\varphi}(t) - \bar{y}_j\| < \delta_i + \delta_{y_j}/2$ $\leq \delta_{y_j}/2 + \delta_{y_j}/2 = \delta_{y_j} \Rightarrow \bar{\psi}(t) \in N_{\delta_{y_j}}(\bar{y}_j) \subset U_i$. Now, let $\mu =$ $\min(\ \delta_1, \ldots, \delta_m\ )$; then, $S(\bar{\varphi},\mu)$ is the required $\mu$-sphere, since if $\bar{\psi} \in S(\bar{\varphi},\mu)$ and if $t \in K_i$ for any $i \in \{1, \ldots, m\}$ then $\bar{\psi}(t) \in U_i$, that is $\bar{\psi}$ maps $K_i$ into $U_i$, thus $\bar{\psi} \in B_c$.

Let $B_\rho$ be a member of $\mathcal{B}_\rho$. Thus, $B_\rho = S(\bar{\varphi}',\mu')$ for some $\bar{\varphi}' \in C$ and some $\mu' > 0$. Let $\bar{\varphi} \in B_\rho$. Since $S(\bar{\varphi}',\mu')$ is open in $(C,\mathfrak{m}_\rho)$, $\exists$ $\mu > 0$ such that $S(\bar{\varphi},\mu) \subset S(\bar{\varphi}',\mu')$; therefore, it is sufficient to show that $\exists$ a set of the form $\bigcap_{i=1}^{m} A(K_i, U_i)$ which contains $\bar{\varphi}$ and is contained in $S(\bar{\varphi},\mu)$. In case $\mu > \sum_{k=0}^{\infty} b^k$, we may simply let $\bigcap_{i=1}^{m} A(K_i, U_i) = A(K_1, U_1)$, where $K_1 = \{t: -1 \leq t \leq 0\ \}$, $U_1 = E^n$. $A(K_1,U_1) = C$ and hence contains $\bar{\varphi}$. Also, if $\bar{\psi} \in A(K_1, U_1)$ then $\rho(\bar{\varphi},\bar{\psi}) \leq \sum_{k=0}^{\infty} b^k < \mu$, hence $\bar{\psi} \in S(\bar{\varphi},\mu)$, $\Rightarrow A(K_1, U_1) \subset S(\bar{\varphi},\mu)$. If $\mu \leq \sum_{k=0}^{\infty} b^k$, then let the integer $k^* \geq 1$ be chosen such that $b^{k^*} \leq$ $\mu(1-b)/2$. Consider the closed interval $[-t_{k^*}, 0]$. Since $\bar{\varphi}$ is continuous on $[-t_{k^*}, 0]$, $\forall\ t \in [-t_{k^*}, 0]\ \exists\ \delta_t > 0$ such that if $-t_{k^*} \leq t' \leq 0$ and $|t' - t| \leq \delta_t$ then $\|\bar{\varphi}(t') - \bar{\varphi}(t)\| < \frac{\mu}{4k^*}$. Let $N_{\delta_t}(t)$ denote the open $\delta_t$ neighborhood of $t$. Clearly, $N_{\delta_t} \subset$ $\{\ t' : |t' - t| \leq \delta_t\ \}$. The family $\{\ N_{\delta_t}(t) : t \in [-t_{k^*}, 0]\ \}$ is an open cover of $[-t_{k^*}, 0]$ and since $[-t_{k^*}, 0]$ is compact, $\exists$ a finite subcover $\{ N_{\delta_{t_1}}(t_1), \ldots, N_{\delta_{t_m}}(t_m) \}$ of $[-t_{k^*}, 0]$. For $i = 1, \ldots, m$, let $K_i = \{ t: |t-t_i| \le \delta_{t_i}, t \in [-t_{k^*}, 0] \}$. Note that each $K_i$ is a compact subset of $(-\infty, 0]$; in fact, each $K_i$ is a compact subset of $[-t_{k^*}, 0]$, and clearly $\bigcup_{i=1}^{m} K_i = [-t_{k^*}, 0]$. Also, if $t \in K_i$ then $\|\bar{\varphi}(t) - \bar{\varphi}(t_i)\| < \frac{\mu}{4k^*}$. For $i = 1, \ldots, m$, let $U_i = \{ \bar{y}: \bar{y} \in E^n, \|\bar{y} - \bar{\varphi}(t_i)\| < \frac{\mu}{4k^*} \}$, and let $A(K_i, U_i)$ denote the set of all $\bar{\psi} \in C$ mapping $K_i$ into $U_i$. Then, $\bigcap_{i=1}^{m} A(K_i, U_i)$ is the required set: Clearly $\bar{\varphi} \in \bigcap_{i=1}^{m} A(K_i, U_i)$; and, if $\bar{\psi} \in \bigcap_{i=1}^{m} A(K_i, U_i)$ then $\|\bar{\varphi}(t) - \bar{\psi}(t)\| < 2\left(\frac{\mu}{4k^*}\right)$ for all $t$ in $[-t_{k^*}, 0]$, and hence $\rho(\bar{\varphi}, \bar{\psi}) =$

$$\sum_{k=0}^{\infty} m_k \le \sum_{k=k^*}^{\infty} m_k + \sum_{k=0}^{k^*-1} \sup\{ \|\bar{\varphi}(t) - \bar{\psi}(t)\|: -t_{k+1} \le t \le -t_k \} \le$$

$$\sum_{k=k^*}^{\infty} m_k + k^* \sup\{ \|\bar{\varphi}(t) - \bar{\psi}(t)\|: -t_{k^*} \le t \le 0 \} <$$

$$\sum_{k=k^*}^{\infty} m_k + k^* \cdot 2 \cdot \left(\frac{\mu}{4k^*}\right) \le \sum_{k=k^*}^{\infty} b^k + \mu/2 = \frac{b^{k^*}}{1-b} + \mu/2 \le \mu/2 + \mu/2 =$$

$\mu$, or $\rho(\bar{\varphi}, \bar{\psi}) < \mu$. Thus, $\bar{\psi} \in S(\bar{\varphi}, \mu)$, and hence $\bigcap_{i=1}^{m} A(K_i, U_i) \subset S(\bar{\varphi}, \mu)$. Q.E.D.

APPENDIX C

In this appendix we shall prove that the functionals V defined in Equations (5-1) and (5-5) are continuous on $C_H$ for any $H > 0$.

__Theorem__. For any $H > 0$, if V is defined on $C_H$ by either Equation (5-1) or Equation (5-5), then V is uniformly continuous on $C_H$.

__Proof__: Let $\epsilon > 0$ be given. Let $n_\Omega = n_\gamma + n_\delta + \dots + n_\theta$. Choose the positive integer $N_1$ so large that

$$a_\xi \frac{2H^2 \tau_\xi}{(1-|\rho_\xi|)} |\rho_\xi|^k < \frac{\epsilon}{4n_\Omega}, \forall \xi \in \Omega \text{ and } k \geq N_1 .$$

Choose the positive integer $N_2$ so large that

$$b_\xi \frac{2H^2 \tau_\xi}{(1-|\rho_\xi|)} |\rho_\xi|^{k+1} < \frac{\epsilon}{4n_\Omega}, \forall \xi \in \Omega \text{ and } k \geq N_2 .$$

Let $N = \max\{N_1, N_2\}$. Let $T = \max\{\tau_\xi : \xi \in \Omega\}$ and let

$$K_n = \sum_{\xi \in \Omega} \left[ \frac{2\tau_\xi(1-|\rho_\xi|^N)}{(1-|\rho_\xi|)} (a_\xi + b_\xi|\rho_\xi|) + b_\xi \tau_\xi \right]$$

and choose $\delta > 0$ such that $\bar{\Phi}, \bar{\Psi} \in C_H$, $\rho(\bar{\Phi}, \bar{\Psi}) < \delta$

$\Rightarrow \quad \sup\{ \|\bar{\Phi}(\sigma) - \bar{\Psi}(\sigma)\| : -(2N+1)T \leq \sigma \leq 0\} < \min\{\frac{\epsilon}{8HK_N}, \frac{\epsilon}{4H\|P\|}\}$ .

This is possible since convergence in the compact open topology on $(-\infty, 0]$ is equivalent to uniform convergence on every compact subset of $(-\infty, 0]$. For every $\bar{\Phi}, \bar{\Psi} \in C_H$ with $\rho(\bar{\Phi}, \bar{\Psi}) < \delta$ ,

$$
\begin{aligned}
[V(\bar\varphi) - V(\bar\Psi)] = \Bigg| & \frac{1}{2}\,\bar\varphi^t(0)\,P\bar\varphi(0) - \frac{1}{2}\,\bar\Psi^t(0)\,P\bar\Psi(0) \\
& \pm \sum_{\xi\in\Omega}\Bigg( a_\xi \sum_{k=0}^{\infty}|\rho_\xi|^k \int_{\sigma=-2(k+1)\tau_\xi}^{-2k\tau_\xi} \bar\varphi^t(\sigma)\bar\varphi(\sigma)d\sigma + b_\xi\Bigg[\int_{\sigma=-\tau_\xi}^{0}\bar\varphi^t(\sigma)\bar\varphi(\sigma)d\sigma \\
& + \sum_{k=0}^{\infty}|\rho_\xi|^{k+1}\int_{\sigma=-(2(k+1)+1)\tau_\xi}^{-2(k+1)\tau_\xi}\bar\varphi^t(\sigma)\bar\varphi(\sigma)d\sigma\Bigg]\Bigg) \\
& \mp \sum_{\xi\in\Omega}\Bigg( a_\xi \sum_{k=0}^{\infty}|\rho_\xi|^k \int_{\sigma=-2(k+1)\tau_\xi}^{-2k\tau_\xi} \bar\Psi^t(\sigma)\bar\Psi(\sigma)d\sigma + b_\xi\Bigg[\int_{\sigma=-\tau_\xi}^{0}\bar\Psi^t(\sigma)\bar\Psi(\sigma)d\sigma \\
& + \sum_{k=0}^{\infty}|\rho_\xi|^{k+1}\int_{\sigma=-(2(k+1)+1)\tau_\xi}^{-(2k+1)\tau_\xi}\bar\Psi^t(\sigma)\bar\Psi(\sigma)d\sigma\Bigg]\Bigg)\Bigg|
\end{aligned}
$$

$$
\begin{aligned}
\le \frac{1}{2}\,|\bar\varphi^t(0)\,P\,\bar\varphi(0) - \bar\Psi^t(0)\,P\,\bar\Psi(0)| + \Bigg| & \sum_{\xi\in\Omega}\Bigg( a_\xi\sum_{k=0}^{\infty}|\rho_\xi|^k\int_{\sigma=-2(k+1)\tau_\xi}^{-2k\tau_\xi}\Big[\bar\varphi^t(\sigma)\bar\varphi(\sigma) - \\
& \bar\Psi^t(\sigma)\bar\Psi(\sigma)\Big]d\sigma + b_\xi\int_{\sigma=-\tau_\xi}^{0}\Big[\bar\varphi^t(\sigma)\bar\varphi(\sigma) - \bar\Psi^t(\sigma)\bar\Psi(\sigma)\Big]d\sigma \\
& + b_\xi\sum_{k=0}^{\infty}|\rho_\xi|^{k+1}\int_{\sigma=-(2(k+1)+1)\tau_\xi}^{-(2k+1)\tau_\xi}\Big[\bar\varphi^t(\sigma)\bar\varphi(\sigma) - \bar\Psi^t(\sigma)\bar\Psi(\sigma)\Big]d\sigma\Bigg)\Bigg|
\end{aligned}
$$

$$
\begin{aligned}
\le \frac{1}{2}\,|\bar\varphi^t(0)\,P\,\bar\varphi(0) & - \bar\varphi^t(0)\,P\,\bar\Psi(0) + \bar\varphi^t(0)\,P\,\bar\Psi(0) - \bar\Psi^t(0)\,P\,\bar\Psi(0)| \\
& + \sum_{\xi\in\Omega}\Bigg[a_\xi\sum_{k=0}^{\infty}|\rho_\xi|^k\int_{\sigma=-2(k+1)\tau_\xi}^{-2k\tau_\xi}\Big|\,\|\bar\varphi(\sigma)\|^2 - \|\bar\Psi(\sigma)\|^2\,\Big|d\sigma \\
& + b_\xi\int_{\sigma=-\tau_\xi}^{0}\Big|\,\|\bar\varphi(\sigma)\|^2 - \|\bar\Psi(\sigma)\|^2\,\Big|d\sigma \\
& + b_\xi\sum_{k=0}^{\infty}|\rho_\xi|^{k+1}\int_{\sigma=-(2(k+1)+1)\tau_\xi}^{-(2k+1)\tau_\xi}\Big|\,\|\bar\varphi(\sigma)\|^2 - \|\bar\Psi(\sigma)\|^2\,\Big|d\sigma\Bigg]
\end{aligned}
$$

$$\leq^{1} \frac{1}{2}\left|\langle \bar\varphi(0), P(\bar\varphi(0) - \bar\Psi(0)\rangle\right| + \frac{1}{2}\left|\langle P^t(\bar\varphi(0) - \bar\Psi(0)), \bar\Psi(0)\rangle\right|$$

$$+ \sum_{\xi\in\Omega}\left[a_\xi \sum_{k=0}^{N-1} |\rho_\xi|^k \int_{\sigma=-2(k+1)\tau_\xi}^{-2k\tau_\xi} 2H\cdot\|\bar\varphi(\sigma) - \bar\Psi(\sigma)\|d\sigma\right.$$

$$\left.+ b_\xi \int_{\sigma=-\tau_\xi}^{0} 2H\cdot\|\bar\varphi(\sigma) - \bar\Psi(\sigma)\|d\sigma + b_\xi \sum_{k=0}^{N-1} |\rho_\xi|^{k+1} \int_{\sigma=-(2(k+1)+1)\tau_\xi}^{-(2k+1)\tau_\xi} 2H\cdot\|\bar\varphi(\sigma) - \bar\Psi(\sigma)\|d\sigma\right]$$

$$+ \sum_{\xi\in\Omega}\left[a_\xi \sum_{k=N}^{\infty} |\rho_\xi|^k \int_{\sigma=-2(k+1)\tau_\xi}^{-2k\tau_\xi} \left|\,\|\bar\varphi(\sigma)\|^2 - \|\bar\Psi(\sigma)\|^2\right| d\sigma\right.$$

$$\left.+ b_\xi \sum_{k=N}^{\infty} |\rho_\xi|^{k+1} \int_{\sigma=-(2(k+1)+1)\tau_\xi}^{-(2k+1)\tau_\xi} \left|\,\|\bar\varphi(\sigma)\|^2 - \|\bar\Psi(\sigma)\|^2\right| d\sigma\right]$$

$$\leq \frac{1}{2}\|\bar\varphi(0)\|\cdot\|P\|\cdot\|\bar\varphi(0) - \bar\Psi(0)\| + \frac{1}{2}\|P^t\|\cdot\|\bar\varphi(0) - \bar\Psi(0)\|\cdot\|\bar\Psi(0)\|$$

$$+ \sum_{\xi\in\Omega}\left[a_\xi \sum_{k=0}^{N-1} |\rho_\xi|^k\left(\frac{\epsilon}{4K_N}\right)2\tau_\xi + b_\xi\left(\frac{\epsilon}{4K_N}\right)\tau_\xi + b_\xi \sum_{k=0}^{N-1}|\rho_\xi|^{k+1}\left(\frac{\epsilon}{4K_N}\right)2\tau_\xi\right]$$

$$+ \sum_{\xi\in\Omega}\left[a_\xi \sum_{k=N}^{\infty}|\rho_\xi|^k (H^2)2\tau_\xi + b_\xi \sum_{k=N}^{\infty} |\rho_\xi|^{k+1} (H^2)2\tau_\xi\right]$$

$$\leq \frac{1}{2}\left(H\cdot\|P\|\cdot \frac{\epsilon}{4H\|P\|} + \|P^t\|\cdot \frac{\epsilon}{4H\|P\|}\cdot H\right)$$

$$+ \sum_{\xi\in\Omega}\left[a_\xi\left(\frac{1 - |\rho_\xi|^N}{1 - |\rho_\xi|}\right)\left(\frac{\epsilon}{4K_N}\right)2\tau_\xi + b_\xi\left(\frac{\epsilon}{4K_N}\right)\tau_\xi + b_\xi|\rho_\xi|\left(\frac{1 - |\rho_\xi|^N}{1 - |\rho_\xi|}\right)\left(\frac{\epsilon}{4K_N}\right)2\tau_\xi\right]$$

$$+ \sum_{\xi\in\Omega}\left[a_\xi \frac{2H^2\tau_\xi}{(1 - |\rho_\xi|)}|\rho_\xi|^N + b_\xi \frac{2H^2\tau_\xi}{(1 - |\rho_\xi|)}|\rho_\xi|^{N+1}\right]$$

---

[1] We use here the inequality $\left|\,\|\bar\varphi(\sigma)\|^2 - \|\bar\Psi(\sigma)\|^2\right| \leq 2H\cdot\|\bar\varphi(\sigma) - \bar\Psi(\sigma)\|$, which follows immediately from the inequalities $\left|\,\|\bar\varphi(\sigma)\| - \|\bar\Psi(\sigma)\|\,\right| \leq \|\bar\varphi(\sigma) - \bar\Psi(\sigma)\|$ and $\left|\,\|\bar\varphi(\sigma)\| + \|\bar\Psi(\sigma)\|\,\right| \leq 2H$.

$$\leq \frac{1}{2}\left(2\,\frac{\epsilon}{4}\right) + \sum_{\xi\in\Omega}\left[\frac{2\tau_\xi(1-|\rho_\xi|^N)}{(1-|\rho_\xi|)}(a_\xi + b_\xi|\rho_\xi|) + b_\xi\tau_\xi\right]\frac{\epsilon}{4K_N}$$

$$+ \sum_{\xi\in\Omega}\left(\frac{\epsilon}{4n_\Omega} + \frac{\epsilon}{4n_\Omega}\right) = \frac{\epsilon}{4} + \frac{\epsilon}{4} + \frac{\epsilon}{2} = \epsilon$$

That is, $\bar{\varphi}, \bar{\Psi} \in C_H$, $\rho(\bar{\varphi}, \bar{\Psi}) < \delta \Rightarrow |V(\bar{\varphi}) - V(\bar{\Psi})| \leq \epsilon$.

Thus, V is uniformly continuous on $C_H$. Q.E.D.

<u>Theorem.</u> **For any** $H > 0$, if V is **defined on** $C_H$ by either Equation (5-1) or Equation (5-5), then there exists **a real number** $\ell$ such that $|V(\bar{\varphi})| \leq \ell$ for every $\bar{\varphi} \in C_H$.

<u>proof</u>: Let $\ell = \left(\frac{1}{2}\|P\| + \sum_{\xi\in\Omega}\left[\frac{2\tau_\xi(a_\xi + b_\xi|\rho_\xi|)}{(1-|\rho_\xi|)} + b_\xi\tau_\xi\right]\right)\cdot H^2$ .

Then, if $\bar{\varphi} \in C_H$,

$$|V(\bar{\varphi})| \leq \frac{1}{2}\,\|P\|\cdot\|\bar{\varphi}(0)\|^2 + \sum_{\xi\in\Omega}\left[a_\xi \sum_{k=0}^{\infty}|\rho_\xi|^k \int_{\sigma=-2(k+1)\tau_\xi}^{-2k\tau_\xi}\|\bar{\varphi}(\sigma)\|^2 d\sigma\right.$$

$$\left. + b_\xi\int_{\sigma=-\tau_\xi}^{0}\|\bar{\varphi}(\sigma)\|^2 d\sigma + b_\xi\sum_{k=0}^{\infty}|\rho_\xi|^{k+1}\int_{\sigma=-(2(k+1)+1)\tau_\xi}^{-(2k+1)\tau_\xi}\|\bar{\varphi}(\sigma)\|^2 d\sigma\right]$$

$$\leq \frac{1}{2}\,\|P\|\cdot H^2 + \sum_{\xi\in\Omega}\left[a_\xi\sum_{k=0}^{\infty}|\rho_\xi|^k\,2\tau_\xi H^2 + b_\xi\tau_\xi H^2 + b_\xi\sum_{k=0}^{\infty}|\rho_\xi|^{k+1}2\tau_\xi H^2\right]$$

$$= \left(\frac{1}{2}\,\|P\| + \sum_{\xi\in\Omega}\left[\frac{2a_\xi\tau_\xi}{(1-|\rho_\xi|)} + b_\xi\tau_\xi + \frac{2b_\xi\tau_\xi|\rho_\xi|}{(1-|\rho_\xi|)}\right]\right)\cdot H^2$$

$= \ell$. Q.E.D.

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13. ABSTRACT


Electrical networks consisting of lumped linear and memoryless non-linear elements and an arbitrary number of lossless transmission lines are considered. It is shown that a large class of such networks may be described by a system of functional differential equations having the form

$$\dot{\overline{x}}(t) = \overline{f}(\overline{x}_t)$$

where the state of the system at time $t \geq 0$ is represented by $\overline{x}_t$, a point in the space $C_{\parallel}((-\infty, 0], E^n)$ of bounded continuous functions mapping the interval $(-\infty, 0]$ into $E^n$, with the compact open topology, and the function $\overline{f}$ mapping $C_{\parallel}((-\infty, 0], E^n)$ into $E^n$ is continuous and locally Lipschitizian. A Lyapunov functional is presented and used to obtain several theorems concerning the stability and instability of the equilibrium solution, $\overline{x} = \overline{0}$, of the network. Several examples of the theory are presented.


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| Non-linear | | | | | | |
| Stability | | | | | | |
| State variable | | | | | | |