ASYMPTOTICALLY OPTIMAL STATISTICS IN SOME MODELS WITH INCREASING FAILURE RATE AVERAGES

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ABSTRACT

Let \( F \) and \( G \) be defined by \( F(t) = H(yt) \) and \( G(t) = H(\theta t) \) where \( H \) is unknown and \( H(0) = 0 \). For testing the equality of the means of \( F \) and \( G \) in the two-sample problem; it is shown that the Savage (The Annals of Mathematical Statistics (1956) pp 590-615) statistic maximizes the minimum power over IFRA(or IFR) distributions asymptotically. Asymptotic uniqueness holds only in a class of rank tests. The results are extended to censored samples, the problem of estimating the ratio of the means, and the \( k \)-sample problem.
1. **INTRODUCTION AND SUMMARY.** Birnbaum, Esary and Marshall (1966) have shown that the class $\mathcal{F}$ of distributions with increasing failure rate averages (IFRA) characterizes the concept of wear-out in the sense that $\mathcal{F}$ is the smallest class that contains the exponential distributions and is closed under the formation of coherent systems.

In this note, statistical inference for models in which the distributions are unknown and IFRA will be considered. Let $F$ and $G$ be defined by

$$F(t) = H(t/\theta) \quad \text{and} \quad G(t) = H(t/Y)$$

where $H$ is an unknown IFRA distribution with $H(0) = 0$. Then, for the two-sample problem where one tests the equality of the means of $F$ and $G$, it is shown that the Savage (1956) statistic maximizes the minimum power over IFRA distributions asymptotically. This asymptotic minimax solution is extended to censored samples and it turns out that the Gastwirth (1965) modified version of the Savage statistic is asymptotically minimax for this case. Asymptotic uniqueness of these minimax solutions holds only in a class of rank tests. The results are extended to obtain an estimate of the ratio of the means that minimizes the maximum asymptotic variance over IFRA distributions.

Finally, the results are shown to hold also for distributions with increasing failure rates (IFR), and extensions to the $k$-sample problem are given.

2. **THE TWO-SAMPLE LIFE-TESTING PROBLEM.** $X_1, \ldots, X_m$ and $Y_1, \ldots, Y_n$ are independent random samples from populations with life distributions $F$ and $G$. $N = m + n$, $F(t) = H(t/\theta_N)$, $G(t) = H(t/Y_N)$, $H$ has the density $h$ and is IFRA, i.e., $H(0) = 0$ and for each $t > 0$,

$$\frac{d}{dt} \left\{ - \log \left[ 1 - H(t) \right]/t \right\} = \frac{1}{t^2} \ln \left[ 1 - H(t) \right] + \frac{h(t)}{t[1-H(t)]} \geq 0$$

$r_1, \ldots, r_m$ denote the ranks of the $x$'s.
in the combined sample. The level $\alpha$ Savage (1956) test $\psi_N$ of $H_0: \Delta_N = (\theta_N/\gamma_N) = 1$ against $\Delta_N > 1$ rejects for large values of the statistic

\begin{equation}
S_N = \frac{1}{m} \sum_{i=1}^{m} - \ln \left(1 - \frac{r_i}{N+1}\right)
\end{equation}

It is assumed throughout that

\begin{equation}
0 < \lim_{N \to \infty} (m/N) = \lambda < 1
\end{equation}

Let $0 < \alpha \leq \infty$ and consider sequences of alternatives $\{\Delta_N\}$ satisfying

\begin{equation}
\lim_{N \to \infty} N^{1/2} (\Delta_N - 1) = \alpha
\end{equation}

Then the asymptotic power function $\beta(c;\alpha, H)$ of a test $\psi_N$ is defined as the limit of the power for such alternatives, i.e.

\begin{equation}
\beta(c;\alpha, H) = \lim_{N \to \infty} \inf \beta_N(\psi_N|H)
\end{equation}

where $\beta_N(\psi_N|H) = E(\psi_N|F_N,G_N)$ denotes the power of $\psi_N$ when $F_N(t) = H(t/\theta_N)$, $G_N(t) = H(t/\gamma_N)$ and $\Delta_N = \theta_N/\gamma_N$ satisfies (2.4).

Let $\Phi$ be the standard normal distribution function. Then the results of Chernoff and Savage (1958), Fatou's Lemma, and a few computations yield

**Lemma 2.1.** Suppose $H$ has a density $h$ and that $H(0) = 0$, then the asymptotic power function of the level $\alpha$ Savage test $\psi_N$ is given by

\begin{equation}
\beta(c;\psi, H) = \Phi \left( \Phi^{-1}(\alpha) + c \left[\lambda(1-\lambda)\right]^{1/2} \int_0^\infty \frac{t h(t)}{1-H(t)} \, dH(t) \right)
\end{equation}

The next result shows that $\psi$ and the exponential distribution $K_\theta(x) = 1 - \exp(-x/\theta)$ is a saddle point for the asymptotic power function $\beta(c;\psi, H)$. In other words, $\psi$ is worst for the exponential distribution, but is better than all other tests for this distribution.
Theorem 2.1. For all \( 0 \leq c < \infty \) and all \( \sigma > 0 \),

\[
(2.7) \quad \sup_{\Phi} \beta(c;\Phi,K_0) = \beta(c;\Phi,K_0) = \inf_{H} \beta(c;\Phi,H)
\]

where \( H \) ranges over the class of IFRA distributions with a density, and \( \Phi_N \) ranges over the class of all level \( \alpha \) tests.

Proof. The left hand equality was proved by Capon (1961) by essentially comparing \( \Psi_n \) with the Neyman-Pearson test for \( K_0 \). To prove the right equality, note that (2.1) yields

\[
(2.8) \quad \frac{th(t)}{1-H(t)} \geq - \ln [1-H(t)]
\]

thus

\[
\int_{0}^{\infty} \frac{th(t)}{1-H(t)} dH(t) \geq \int_{0}^{\infty} - \ln [1-H(t)] dH(t) = 1
\]

The equality signs hold if and only if \( H \) has a constant failure rate average, i.e., if and only if \( H \) is exponential, thus

Corollary 2.1. If \( H \) is IFRA, has a density, and is not exponential, then

\[
(2.9) \quad \beta(c;\Phi,K_0) < \beta(c;\Phi,H)
\]

The minimax property of the Savage statistic now follows at once from Theorem 2.1.

Theorem 2.2. The level \( \alpha \) Savage test \( \Psi_n \) is asymptotically minimax over the class \( \Omega \) of all IFRA distributions with a density, i.e., if \( H \) ranges over \( \Omega \), then

\[
(2.10) \quad \inf_{H} \beta(c;\Phi,H) \geq \inf_{H} \beta(c;\Phi,H)
\]

for all level \( \alpha \) tests \( \Phi_N \).
Remarks

(i) \( H \) is said to have increasing failure rate (IFR) \([1]\) if \( H(0) = 0 \) and \( h(t)/[1-H(t)] \) is nondecreasing in \( t > 0 \). The class of IFR distributions contains the class of exponential distributions and is contained in the class of IFRA distributions. It follows that Theorem 2.1, Corollary 2.1 and Theorem 2.2 holds also for this class.

(ii) The results of this section are stronger than the minimax results of [6] in the sense that no conditions such as bounds on the Kolmogorov distances or variances of the distributions are needed.

(iii) The \( \lim \inf \) in (2.5) can be replaced by a limit if one assumes conditions as in Lemma 3 of Hodges and Lehmann (1963). The results hold if \( \lim \inf \) is replaced by \( \lim \sup \) or partially replaced by \( \lim \sup \) as in [6].

(iv) An asymptotically equivalent form of the Savage statistic is

\[
\sum_{i=1}^{m} J_0(r_i), \text{ where}
\]

\[
J_0(k) = \sum_{j=N-k+1}^{N} \frac{1}{j}
\]

(v) The results in this section hold if one, instead of considering level \( \alpha \) tests, considers \( \varphi_N \) with asymptotic level \( \alpha \), i.e. tests for which

\[
E(\varphi_N|\theta = \gamma) \to \alpha \text{ as } N \to \infty.
\]

(vi) The one-sided alternative \( \Delta > 1 \) can be replaced by the two-sided alternative \( \Delta \neq 1 \).
(vii) For the k-sample problem with model $F_i(x) = H(x/[1+\theta c_i]); i=1,\ldots, k$; the Puri (1964) extension of the Savage statistic is asymptotically minimax for testing $H_0^k: \theta = 0$ against $\theta > 0$ (or $\theta \neq 0$).

3. EFFICIENCY OF THE BEST TEST FOR EXPONENTIAL MODELS. When $H$ equals an exponential distribution $K(t) = 1 - \exp(-t/\alpha)$, then the uniformly most powerful level $\alpha$ test $\varphi_{N \alpha}$ of $\theta = \gamma$ against $\theta > \gamma$ rejects when

$$T = \frac{1}{m} \sum_{i=1}^{m} X_i / \frac{1}{n} \sum_{i=1}^{n} Y_i > F_{2m,2n}(\alpha)$$

where $F_{2m,2n}(\alpha)$ is obtained from the tables of the $F$ distribution. In this section the performance of $T$ is investigated when the assumption of exponentiality is violated and $H$ is an IFRA distribution.

Upon writing

$$\sqrt{N} (T - \Delta) = \sqrt{N} (\overline{X} - \Delta \overline{Y}) / \overline{Y},$$

it is clear that $\sqrt{N} (T - \Delta)$ has an asymptotic normal distribution with mean zero and variance

$$\sigma^2(T) = \Delta^2 \sigma^2(H) / \lambda(1-\lambda) \mu^2(H)$$

where

$$\mu(H) = \int_0^\infty t dH(t) \text{ and } \sigma^2(H) = \int_0^\infty t^2 dH(t) - \mu^2(H).$$

When $H$ is exponential, then $\sigma^2(H) = \mu^2(H)$. It follows that when $H$ is such that $\sigma^2(H) \neq \mu^2(H)$, then $\varphi_{N \alpha}$ does not have level $\alpha$ asymptotically, in fact

$$E(\varphi_{N \alpha} | \theta = \gamma) \to \Phi(\psi^{-1}(\alpha) \mu(H)/\sigma(H)) \text{ as } N \to \infty$$

Thus when $\alpha < \frac{1}{2}$ and $\mu(H) > \sigma(H)$, then the asymptotic level of $\varphi_{N \alpha}$ is less
than \(\alpha\). Barlow, Marshall and Proschan (1963) have essentially shown that for IFRA distributions, \(\mu(H) \geq \sigma(H)\). The asymptotic power function of \(\varphi_N^*\) is

\[
\beta(c;\varphi^*, H) = \Phi\left(\frac{\delta^{-1}(\alpha) + c[\lambda(1-\lambda)]^{1/2}}{\mu(H)/\sigma(H)}\right)
\]

\(\varphi_N^*\) can easily be modified to have asymptotic level \(\alpha\) by dividing \(\sqrt{N} (T-1)\) by a consistent estimate of

\[
r(H) = \sigma(H)/\mu(H); \text{ e.g.}
\]

\[
\hat{r}(H) = \hat{\sigma}(H)/\hat{\mu}(H) \text{ with}
\]

\[
\hat{\mu}(H) = \frac{1}{N} (\sum x_i + \sum y_i) \quad \text{and}
\]

\[
\hat{\sigma}(H) = \frac{1}{N} (\sum x_i^2 + \sum y_i^2 - \hat{\mu}^2(H)).
\]

For this test, \(\hat{\varphi}_N\), one has

\[
\beta(c;\hat{\varphi}, H) = \Phi\left(\frac{\delta^{-1}(\alpha) + c[\lambda(1-\lambda)]^{1/2}}{\mu(H)/\sigma(H)}\right)
\]

Since \(\mu(H) \geq \sigma(H)\) when \(H\) is IFRA, \(\mu(k_o) = \sigma(k_o)\), and since \(\beta(c;\hat{\varphi}, k_o) = \beta(c,\hat{\varphi}, k_o)\), then (2.7) and remark (i) of Section 2 yields

**Theorem 3.1.** For all \(0 \leq c \leq \infty\) and all \(\sigma > \sigma\),

\[
\sup_{\varphi} \beta(c;\varphi, k_o) = \beta(c;\hat{\varphi}, k_o) = \inf_H \beta(c,\hat{\varphi}, H)
\]

where \(H\) ranges over the class of IFRA distributions and \(\varphi_N\) ranges over the class of all tests with asymptotic level \(\alpha\).

Thus \(\hat{\varphi}_N\) is asymptotically minimax in the sense of Theorem 2.2 for the class of IFRA distributions and the class of tests with asymptotic level \(\alpha\). To see that this is not true for \(\varphi_N^*\), let \(H\) be an IFRA distribution with \(\mu(H) > \sigma(H)\), then for each \(\alpha < \frac{1}{2}\),

\[
\beta(c;\varphi^*, H) < \beta(c;\hat{\varphi}, k_o) \quad \text{for} \quad 0 \leq c < \sigma(H)/\mu(H).
\]
Let Pitman asymptotic efficiency be as defined in [10]. It follows from (2.6) and (3.6) that the Pitman efficiency of the Savage test $\hat{\psi}_N$ to the modified classical test $\hat{\varphi}_N$ is

$$e(\hat{\psi}, \hat{\varphi}) = \frac{\sigma^2(H) \left[ \int_0^\infty xq(x)dH(x) \right]^2}{\mu^2(H)}$$

(3.9)

where $q(x) = h(x)/[1-H(x)]$ is the failure rate of $H$.

The Weibull distribution is defined by

$$\hat{H}(x) = 1 - e^{-ax^b}; a, b > 0; x \geq 0$$

(3.10)

If $\mu_k$ denotes the $k$th moment about zero, then

$$\mu_k = a^b \Gamma \left( \frac{k}{b} + 1 \right), \quad q(x) = abx^{b-1},$$

(3.11)

and

$$\int_0^\infty xq(x)d\hat{H}(x) = ab \mu_b = b$$

Thus for the Weibull distribution

$$e_b(\hat{\psi}, \hat{\varphi}) = \frac{b^2 \left[ \Gamma \left( \frac{2}{b} + 1 \right) - \Gamma \left( \frac{1}{b} + 1 \right) \right]}{\Gamma^2 \left( \frac{1}{b} + 1 \right)}$$

(3.12)

For $b = 1$, the Weibull distribution coincides with exponential distribution and $e_1(\hat{\psi}, \hat{\varphi}) = 1$. For $b = 2$, one has the linear failure rate $q(x) = 2ax$ and (3.12) becomes

$$e_2(\hat{\psi}, \hat{\varphi}) = \frac{16}{\pi} - 4 \approx 1.093$$

(3.13)
Using L'Hôpital's rule, one finds that

\[ \lim_{b \to \infty} e_b(\psi, \hat{\phi}) = \infty \]

When \( b < 1 \), the failure rate is decreasing. Stirling's approximation shows that if \( k = (1/b) \) is large, then (3.12) is approximately \( 2^{2k+1} k^{-2} e^{-k} - k^{-2} \), and that

\[ \lim_{b \to 0} e_b(\psi, \hat{\phi}) = \infty \]

If \( k \) is an integer, then

\[ e_b(\psi, \hat{\phi}) = \frac{(2k)!}{k^2 (k!)^2} - \frac{1}{k^2} \]

For \( k = 2 \) and \( 3 \) (3.16) becomes 1.25 and \( (19/9) = 2.11 \) respectively.

It is easy to show ([15] and [9]) that \( \phi_N \) is asymptotically most powerful and locally most powerful for the Weibull distribution. Thus

\[ e_b(\psi, \phi) \geq 1 \quad \text{for all} \quad b > 0 \]

and for all test \( \phi_N \) for which this efficiency is computable. In particular (3.17) holds for \( \phi_N \). Thus the Savage test \( \psi_N \) is uniformly more efficient than the adjusted classical test \( \hat{\phi}_N \) for the Weibull distribution. Moreover, the Savage test is much better when the failure rate parameter \( b \) is large or close to zero. It is conjectured that the Savage statistic is uniformly more efficient than \( \phi_N \) for all distributions with monotone failure rates.
4. CENSORED SAMPLES. Fix $M \leq N$ and wait until a total of $M$ X's and Y's have failed. Let $m' \leq m$ be the number of X's observed, then the ranks $r_1, \ldots, r_m'$ can be computed from the data. The Gastwirth (1965) modified Savage statistic is

$$S_M = -\frac{1}{M} \sum_{i=1}^{m'} \ln \left( 1 - \frac{r_i}{N+1} \right) + m' + (m-m') \ln \left( 1 - \frac{M}{N+1} \right)$$

It is assumed that

$$0 < \lim_{N \to \infty} \frac{M}{N} = p < 1$$

The asymptotic power function of the level $\alpha$ test $\hat{\psi}_M$ that rejects for large values of $S_M$ can be computed using [9] and [8]. One gets

$$\beta(c;\hat{\psi}_p,H) = \psi \left( \psi^{-1}(\alpha) + c[\lambda(1-\lambda)/p]^{1/2} \int_0^H t h(t) \frac{dH(t)}{1-H(t)} \right)$$

From (2.8), it follows that when $H$ ranges over the class of IFRA distributions, then

$$\inf_H \beta(c;\hat{\psi}_p,H) = \beta(c;\hat{\psi}_p,H_0)$$

$$= \psi \left( \psi^{-1}(\alpha) + c[\lambda(1-\lambda)/p]^{1/2} [p+\ln(1-p)(1-p)] \right)$$

Since Hájek (1962) and Gastwirth (1965) has shown that

$$\beta(c;\hat{\psi}_p,K_0) \geq \beta(c;\hat{\psi}_p,K_0)$$

for all level $\alpha$ tests $\hat{\psi}_N$, then the results of Section 2 holds for $\hat{\psi}_M$.

5. ASYMPTOTIC UNIQUENESS. Stein (1956) and Hájek (1962) has shown that one can obtain asymptotically optimal statistics by estimating the underlying distribution. Although these statistics are impractical, they show that one can not hope for asymptotic uniqueness in the class of all tests with asymptotic level $\alpha$.
Consider the class of one-sided level \( \alpha \) rank tests \( \mathcal{F} \) [5] based on statistics of the form

\[
T_M = T_M(J_N) = \frac{1}{m} \sum_{i=1}^{m} J_{N \{N+1\}}(r_i)
\]

where there exist a function \( J \) which is continuous except for possibly a finite number of jump discontinuities and which satisfies

\[
\int_{0}^{\infty} J^2(u) du < \infty \quad \text{and} \quad \lim_{N \to \infty} \int_{0}^{\infty} [J_N(u) - J(u)]^2 du = 0
\]

and the conditions of Comment 3.8 of Hájek (1962). Let \( \mathcal{F}' \) be the class of IFRA distributions \( H \) with a density \( h \) which has the Radon - Nykodym derivative \( h' \) with respect to Lebesgue measure and satisfies

\[
\int_{0}^{\infty} [x^2 h'(x)/h(x)]^2 dH(x) < \infty
\]

For these classes one has

\textbf{Theorem 5.2.} The Savage - Gastwirth test \( T_p \) is asymptotically uniquely minimax for \( \mathcal{F} \) and \( \mathcal{F}' \), i.e., if \( \varphi_o = \varphi_o(J_N) \in \mathcal{F} \), if \( H \) ranges over \( \mathcal{F}' \), and if

\[
\inf_H \beta(c; \varphi_o, H) \geq \inf_H \beta(c; \varphi, H)
\]

for all \( \varphi \in \mathcal{F} \), then there exists constants \( a_N \) and \( b_N \) such that

\[
\sqrt{N} [S_M - (a_N T_M(J_N) + b_N)] \to 0
\]

in probability as \( N \to \infty \) provided (2.3), (4.2) and (2.4) hold with \( c < \infty \).

\textbf{Proof.} (2.7) and (5.4) show that \( \beta(c; \varphi_o, H) = \beta(c; \varphi_p, H) \). Thus \( \varphi_o \) is asymptotically optimal for \( K_o \). From Hájek (1962), it follows that the correlation coefficient satisfies

\[
\rho_N(S_M, T_M \mid K_o; \Delta = 1) \to 1 \quad \text{as} \quad N \to \infty
\]
This implies that for regression coefficients $a_N$ and $b_N$,

$$E(N[S_M - (a_N T_M + b_M)]^2 \mid K_o; \lambda = 1) \rightarrow 0$$

Since $S_M$ and $T_M$ are distribution free, (5.7) holds not only for $K_o$, but for general $H$. The result now follows from the continuity arguments of LeCam and Hájek (e.g., [9]).

### 6. ESTIMATION

Barlow and Prochan (1966) have shown that the estimates of the mean that are optimal for exponential models are not robust for IFR distributions. Here an asymptotically robust estimate of the ratio $\mu_1/\mu_2$ of the means of $X$ and $Y$ is constructed using the methods of Hodges and Lehmann (1963). Write $x = (x_1, \ldots, x_m), y = (y_1, \ldots, y_n), ax = (ax_1, \ldots, ax_n)$ etc., and let

$$s(x, y) = S_M$$

be the Savage - Gastwirth statistic (4.1). $\mu_1/\mu_2 = \theta_\mu(H)/\gamma_\mu(H) = \theta/\gamma = \Delta$, so one estimates $\Delta$.

Note that $\sqrt{N} s(X, \Delta Y)$ asymptotically tends to be normally distributed about the point $0$ [9]. Let

$$\Delta^* = \sup \{\Delta: s(x, \Delta y) \geq 0\} \quad \text{and} \quad (6.2)$$

$$\Delta^{**} = \inf \{\Delta: s(x, \Delta y) \leq 0\}$$

and define the estimate $\hat{\Delta}$ of $\Delta$ by

$$\hat{\Delta} = \hat{\Delta}(x, y) = \frac{1}{2}(\Delta^* + \Delta^{**}) \quad \text{(6.3)}$$

Since $s(ax, ay) = s(x, y)$ by the invariance properties of ranks, then

$$\hat{\Delta}(ax, ay) = \hat{\Delta}(x, y) \quad \text{for all } a > 0 \ , \ i.e., \ \hat{\Delta} \ \text{is scale invariant.} \quad (6.4)$$
Moreover, using this, the definition (6.2), and noting that \( s(x, \Delta y) \) is decreasing in \( \Delta \), one gets

(6.5) \( \hat{\Delta} (ax, by) = (a/b) \Delta (x, y) \),

(6.6) \( P_\Delta (\Delta/\Delta \leq t) = P_1(\Delta \leq t) \),

(6.7) \( \Delta^* \leq \Delta^{**} \),

(6.8) \( P(\Delta^* < t) = P(s(x, ty) < 0) \),

(6.9) \( P(\Delta^{**} < t) = P(s(x, ty) \leq 0) \),

(6.10) \( P(s(x, ty) < 0) \leq P(\Delta \leq t) \leq P(s(x, ty) \leq 0) \), and

**Lemma 6.1.** If \( H \) satisfies (5.3) and \( H(0) = 0 \), then

\[
\lim_{N \to \infty} P_\Delta (\sqrt{N} [((\Delta/\Delta) - 1) \leq t]) = \mathcal{S} \left( t^{\lambda(1-\lambda)/\rho} \int_0^{H^{-1}(p)} \frac{xh(x)}{1-H(x)} \, dx \right),
\]

**Proof.** (6.6) shows that one can let \( \Delta = 1 \). From (6.10) it follows that

\[
\lim_{N \to \infty} P_1 \left( N^\frac{1}{2}(\Delta - 1) \leq t \right) = \lim_{N \to \infty} P_1(\Delta \leq 1 + tN^{-\frac{1}{2}})
\]

\[
= \lim_{N \to \infty} P_1(s(X, (1 + tN^{-\frac{1}{2}})Y) \leq 0)
\]

\[
= \lim_{N \to \infty} P_\Delta (s(X, Y) \leq 0)
\]

where \( \Delta_N = 1/(1 + tN^{-\frac{1}{2}}) \). Since \( N^\frac{1}{2}(\Delta_N - 1) \to t \) as \( N \to \infty \), the result follows from (4.3).

**Lemma 6.1** shows that the asymptotic variance of \( \sqrt{N} [((\Delta/\Delta) - 1) \) is

(6.11) \[ V(\Delta, H) = 1/\lambda(1-\lambda)/\rho \int_0^{H^{-1}(p)} \frac{th(t)}{1-H(t)} \, dt \].

Moreover, (4.4) shows that the maximum asymptotic variance over IFRA distributions is
Let $\mathcal{F}'$ be as in Section 5, then the results of the previous sections yield

**Theorem 6.1.** $\tilde{\Delta}$ is asymptotically minimax over $\mathcal{F}'$ and the class $\mathcal{E}$ of scale invariant estimates that are asymptotically normal; i.e., if $V(\Delta,H)$ denotes the asymptotic variance of the estimate $\tilde{\Delta} \in \mathcal{E}$, then

$$
(6.13) \quad \sup_H \{V(\Delta,H): H \in \mathcal{F}'\} \leq \sup_{\tilde{\Delta}} \{V(\tilde{\Delta},H): \tilde{\Delta} \in \mathcal{E}\}
$$

$V(\Delta,H)$ also satisfies the saddle-point inequality

$$
(6.14) \quad \sup_H V(\Delta,H) = V(\Delta,K_0) = \inf_{\tilde{\Delta}} V(\tilde{\Delta},K_0)
$$

where $H$ ranges over $\mathcal{F}'$ and $\tilde{\Delta}$ over $\mathcal{E}$.

A different approach to the problem of obtaining asymptotic minimax estimates is given by Huber (1963).
REFERENCES


Let $F$ and $G$ be defined by $F(t) = H(Yt)$ and $G(t) = H(0t)$ where $H$ is unknown and $H(0) = 0$. For testing the equality of the means of $F$ and $G$ in the two-sample problem, it is shown that the Savage (The Annals of Mathematical Statistics 1956 pp 590-615) statistic maximizes the minimum power over IFRA (or IFR) distributions asymptotically. Asymptotic uniqueness holds only in a class of rank tests. The results are extended to censored samples, the problem of estimating the ratio of the means, and the $k$-sample problem.