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TECHNICAL REPORT 3391

TRAJECTORY EQUATIONS FOR A SIX-DEGREE-OF-FREEDOM MISSILE USING A FIXED-PLANE COORDINATE SYSTEM

B. BARNETT

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Technical Report 3391

TRAJECTORY EQUATIONS FOR A SIX-DEGREE-OF-FREEDOM MISSILE USING A FIXED-PLANE COORDINATE SYSTEM

by

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June 1966

Distribution of this document is unlimited

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ABSTRACT

This report derives from elementary principles the general equations of motion for a missile utilizing a fixed-plane coordinate system, i.e., a coordinate system with one axis constrained to lie in a given plane.

Included in the derivation are explicit expressions for introducing wind and an alternate set of equations to be used when singularity conditions are approached. Means are provided for automatically converting to the alternate set of equations so that uninterrupted trajectory simulation can proceed under all conditions. A complete discussion of initial conditions is included.

The general equations can be used for flat or spherical, rotating, or non-rotating earth cases.

INTRODUCTION

This report is part of a continuing program to give Picatinny Arsenal a complete capability in the flight simulation of all types of projectiles and missiles, whether ballistic or rocket-boosted, guided or unguided.

Contained herein is a rederivation, from elementary principles, and an elaboration of the fixed plane coordinate system described in Reference 1, "Trajectory Equations for a Six-Degree-of-Freedom Missile." In Reference 1, a list of only the basic equations unique to the fixed plane coordinate system is presented. This report completes this list and includes, for the first time, a derivation for initial conditions, as well as an alternate set of equations to be used when singularity conditions are approached. Means are provided for converting to this set during flight simulation. Finally, explicit equations for introducing wind into the equations of motion are derived.

This alternate set of fixed-plane trajectory equations was required to supplement (and, in many cases, replace) the existing equations utilizing the missile-fixed coordinate system as derived in Reference 1. To be explicit, the previous equations produced satisfactory trajectories for low-spin projectiles, but were inadequate for the simulation of spin-stabilized shells. The most obvious differences were in the large deflections accompanying most trajectories for high-spin projectiles, these being three or four times as large as had been anticipated. Discussion of this matter with personnel from the Naval Weapons Laboratory, Dahlgren, Virginia, indicated the existence of a narrow band of permissible integration step sizes (incremental time steps) whereby both truncation and round-off errors are of acceptable magnitudes. It is possible that when forces and moments are referred to a missile-fixed coordinate system (as was the case in Reference 1), and high-spin rates are to be accounted for, this band of acceptable time increments becomes even more narrow or perhaps nonexistent. Although this is not known with certainty, it provides the impetus for studying the fixed-plane coordinate system described in this report. In particular, this coordinate system has one axis constrained to lie in a given plane and, consequently, does not rotate with the missile. To provide a working simulation and to determine whether this new coordinate system alleviates the conditions producing unsatisfactory trajectories, the equations derived in this report were programmed by the Digital Applications Unit for the IBM 709 computer, Reference 5 contains a description of the corresponding computer program.

The equations again consider both flat and spherical, rotating and non-rotating earth cases, and again make use of the Euler transformations (as opposed to direction cosines) to express vector quantities in various coordinate systems.

As before, guidance factors, motion of the earth along its orbit, launcher effects, and asymmetric missiles are not included in the derivations. These represent potential areas of extension of the present equations of motion.

NOTATION

Unit vectors in each of the three orthogonal directions are represented by $\bar{i}_i$, $\bar{j}_i$, $\bar{k}_i$ where subscript i denotes the coordinate system under consideration. Components of vectors in each of these directions will have two subscripts. The first subscript (X, Y, or Z) denotes a component along the $\bar{i}$, $\bar{j}$, $\bar{k}$ axis, respectively; the second subscript (I, E, etc.) denotes the referencing coordinate system. Thus V_{XE} is the component of the vector $\vec{V}$ along the $\bar{i}$ axis of the E-coordinate system.

Arrows over vector quantities denote vectors of arbitrary magnitude, bars over vector quantities denote vectors of unit magnitude.

Subscripted vectors other than the subscripts mentioned above are enclosed in parentheses, i.e. $(\vec{\omega}_T)$.

Finally $\frac{d_i(\ )}{dt}$ denotes time differentiation of vectors relative to the ith coordinate system, while a dot over a given variable denotes scalar differentiation with respect to time.

SYMBOLS

Vector Quantities

$$\left.\begin{array}{l} \bar{i}_I, \bar{j}_I, \bar{k}_I \\ \bar{i}_E, \bar{j}_E, \bar{k}_E \\ \bar{i}_W, \bar{j}_W, \bar{k}_W \\ \bar{i}_H, \bar{j}_H, \bar{k}_H \\ \bar{i}_V, \bar{j}_V, \bar{k}_V \\ \bar{i}_M, \bar{j}_M, \bar{k}_M \end{array}\right\} \text{coordinate systems}$$

$\vec{\omega}$	Angular velocity of (H) relative to (I)
$\vec{\Omega}$	Angular velocity of (E) relative to (I), i.e., angular velocity of the earth about its axis
$\vec{\omega}'$	Angular velocity of (H) relative to (E)
$\vec{\Omega}'$	Angular velocity of (M) relative to (H)
$(\vec{\omega}_T)$	Angular velocity of (M) relative to (I)
$\vec{\omega}''$	Angular velocity of (V) relative to (E)
$\vec{R}$	Vector from center of earth to the current missile CG position
$\vec{V}$	Time rate of change of $\vec{R}$, i.e., missile velocity
$(\vec{V}_W)$	Vector describing wind
$\Sigma\vec{F}$	Summation of forces acting on missile
$\Sigma\vec{L}$	Summation of moments acting on missile
$\vec{J}$	Angular momentum of missile
$\vec{\gamma}$	Vector giving direction of gravitational force exerted on missile
$(\vec{V}_r)$	Velocity of missile relative to the air
$\vec{r}$	Thrust malalignment vector
$\vec{T}$	Thrust vector

SYMBOLS (Cont)

Scalar Quantities

m	mass of rocket
I_X	longitudinal moment of inertia
I_Y	transverse moment of inertia
θ, ψ, Θ, Ψ	Eulerian angles
Φ, Φ'	Angles relating (H) to (V) coordinates
ρ	Density of air
d	Diameter of missile
k_{DA}, k_N, k_F	Aerodynamic coefficients
k_H, $k_{\phi\rho}$, k_ϕ	Aerodynamic coefficients
r_C	Distance from nose of missile to CG
λ_N	Distance from nose of missile to normal center of pressure
λ_M	Distance from nose of missile to magnus center of pressure
g	Gravitational acceleration
R_{earth}	Radius of earth
g_o	Gravitational acceleration at sea level
h	Altitude of rocket
δ_T, δ_A	Thrust malalignment angles
k	Portion of thrust devoted to produce jet torque
α, β	Angles defining missile position
$\lvert V_W \rvert$	Magnitude of wind velocity
ϕ	Angle defining direction of wind
A, B, G, H	Angles defining initial conditions

SYMBOLS (Cont)

. Denotes time differentiation

' Denotes quantities referred to (V) coordinates

$\times$ Denotes the vector cross product

PROCEDURE

Several coordinate systems are utilized in deriving the equations of motion. They are tabulated here for later convenience.

$\bar{i}_I, \bar{j}_I, \bar{k}_I$ Inertial coordinate system; origin (O) at the center of the earth, $\bar{k}_I$ axis coincident with positive spin axis of the earth, $\bar{i}_I$ axis coincident with 0° longitude, and $\bar{j}_I$ axis so directed as to form a right-handed coordinate system.

$\bar{i}_E, \bar{j}_E, \bar{k}_E$ Earth-fixed coordinate system; identical position as $\bar{i}_I, \bar{j}_I, \bar{k}_I$ at time equal zero, but coordinates are to be fixed to the earth.

$\bar{i}_W, \bar{j}_W, \bar{k}_W$ Azimuthal coordinate system; this coordinate system is used to introduce wind data into the equations of motion. The origin of these coordinates is again at O, the $\bar{i}_W$ axis is directed towards the missile CG, the $\bar{k}_W$ axis is directed towards the positive spin axis of the earth, and $\bar{j}_W$ is so directed as to produce a right-handed coordinate system.

$\bar{i}_H, \bar{j}_H, \bar{k}_H$ Fixed horizontal plane coordinate system; $\bar{i}_H$ axis along missile longitudinal axis is directed from the centroid towards the nose of the missile, $\bar{j}_H$ is constrained to lie in a horizontal plane, parallel to the $\bar{i}_E, \bar{j}_E$ plane, and $\bar{k}_H$ is directed as to produce a right-handed coordinate system.

$\bar{i}_V, \bar{j}_V, \bar{k}_V$ Fixed vertical plane coordinate system; the $\bar{i}_V$ axis is coincident with the $\bar{i}_H$ axis, the $\bar{j}_V$ axis is constrained to lie perpendicular to the $\bar{i}_E$ axis (i.e., remain in a specified vertical plane), and the $\bar{k}_V$ axis is so directed as to determine a right-handed coordinate system.

$\bar{i}_M, \bar{j}_M, \bar{k}_M$ Missile-fixed coordinate system; the $\bar{i}_M$ axis is coincident with missile longitudinal axis, the $\bar{j}_M$ and $\bar{k}_M$ axes are rigidly attached to the missile to form a right-handed coordinate system.

The procedure will be divided into several sections, as outlined in the Table of Contents, each describing a particular aspect of the equations of motion, with the last section combining all that precedes it and serving as a summary of the equations of motion. The first section will derive the general equations of motion, and will include the necessary transformations between coordinate systems.

The General Equations of Motion

As usual, Equations 1 and 2, the basis for Newtonian mechanics, provide the foundation for the equations of motion.[1]

$$\Sigma \vec{F} = m \frac{d^2_I \vec{R}}{dt^2} \tag{1}$$

$$\Sigma \vec{L} = \frac{d_I \vec{J}}{dt} \tag{2}$$

[1] Strictly speaking Equation 1 should read:

$$\Sigma \vec{F} = \frac{d}{dt}(m\vec{V}) = m\frac{d_I \vec{V}}{dt} + \vec{u}\dot{m} = m\frac{d^2_I \vec{R}}{dt^2} + \vec{u}\dot{m} \tag{1'}$$

The latter term, $\vec{u}\dot{m}$, involves the amount of mass being expelled from the rocket system, and the relative velocities of the rocket and the exist gases, ($\vec{u}$). This term is known as the jet reaction and is considered as part of the thrust, whose total makeup also includes considerations of nozzle design, operating temperatures, and pressures. Equation 1' uses $\vec{u}$ in place of the seemingly indicated $\vec{V}$ because the rocket system itself has not really been defined in this presentation. The interested reader is referred to References 2 and 3 for a more complete discussion.

Here $\Sigma\vec{F}$ and $\Sigma\vec{L}$ are summations of forces and moments acting on the missile, $\vec{R}$ is a vector from the center of the earth to the current CG (center of gravity) position of the missile, and $\vec{J}$ is the total angular momentum of the rocket.

Since Equations 1 and 2 are vector equations in three-dimensional space, three component or scalar equations are implicit to each. To determine explicit directions relative to the missile along which forces and moments can be conveniently summed, consider a coordinate system whose origin is at the missile CG and whose axes are directed as given previously by $\bar{i}_H$, $\bar{j}_H$, $\bar{k}_H$. Let $\vec{\omega}$ be the angular velocity of this coordinate system relative to inertial coordinates, where

$$\vec{\omega} = \omega_{XH}\bar{i}_H + \omega_{YH}\bar{j}_H + \omega_{ZH}\bar{k}_H \tag{3}$$

in terms of the (H) coordinates.

Utilizing the derivation presented in Appendix A, Equation 1 may be rewritten as follows:

$$\Sigma\vec{F} = m\frac{d^2_I\vec{R}}{dt} = m\frac{d_I}{dt}\left(\frac{d_I}{dt}\vec{R}\right) = m\frac{d_I}{dt}\left(\frac{d_H\vec{R}}{dt} + \vec{\omega}\times\vec{R}\right) \tag{4}$$

where all quantities in the right hand side of the equation are understood to be expressed in (H) coordinates. One may correctly surmise that the presence of the $\vec{\omega}\times$ term relates the motion of the coordinate system (H), to which forces and moments are referred, to an inertial coordinate system (I) as required by Newton's Laws of Motion.

Rather than differentiate Equation 4 directly, it is convenient to let

$$\frac{d_H\vec{R}}{dt} + \vec{\omega}\times\vec{R} = \vec{V} = V_{XH}\bar{i}_H + V_{YH}\bar{j}_H + V_{ZH}\bar{k}_H \tag{5}$$

so that Equation 4 becomes

$$\Sigma\vec{F} = m\frac{d_I\vec{V}}{dt} = m\left(\frac{d_H\vec{V}}{dt} + \vec{\omega}\times\vec{V}\right) \tag{6}$$

Since $\dot{V}$ is already expressed in (H) coordinates, (see Equation 5), one can write

$$\frac{d_H\vec{V}}{dt} = \dot{V}_{XH}\bar{i}_H + \dot{V}_{YH}\bar{j}_H + \dot{V}_{ZH}\bar{k}_H \tag{7}$$

Combining Equations 3, 5, 6, and 7, and performing the indicated operations, one then obtains the three component force equations, namely:

$$\begin{aligned}
\Sigma F_{XH} &= m[\dot{V}_{XH} + (\omega_{YH}V_{ZH} - \omega_{ZH}V_{YH})] \text{ in the } \bar{i}_H \text{ direction} \\
\Sigma F_{YH} &= m[\dot{V}_{YH} + (\omega_{ZH}V_{XH} - \omega_{XH}V_{ZH})] \text{ in the } \bar{j}_H \text{ direction} \\
\Sigma F_{ZH} &= m[\dot{V}_{ZH} + (\omega_{XH}V_{YH} - \omega_{YH}V_{XH})] \text{ in the } \bar{k}_H \text{ direction}
\end{aligned} \tag{8}$$

One can treat the moment equation (Equation 2) in a similar manner, obtaining first

$$\Sigma\vec{L} = \frac{d_I\vec{J}}{dt} = \frac{d_H\vec{J}}{dt} + \vec{\omega} \times \vec{J} \tag{9}$$

Denoting $(\vec{\omega}_T)$ as the total angular velocity of the missile relative to inertial coordinates, we can write the angular momentum in the following form:

$$\vec{J} = I_X(\omega_T)_{XH}\bar{i}_H + I_Y(\omega_T)_{YH}\bar{j}_H + I_Y(\omega_T)_{ZH}\bar{k}_H \tag{10}$$

where I_X, I_Y are the moments of inertia of the missile about the longitudinal and transverse axes, respectively. Note that the assumption of rotational symmetry $(I_Z = I_Y)$ is implicitly made in Equation 10.

Since $\vec{J}$ is already expressed in the (H) coordinate system, the time rate of $\vec{J}$ can be expressed as follows:[1]

$$\frac{d_H\dot{J}}{dt} = I_X(\dot{\omega}_T)_{XH}\bar{i}_H + I_Y(\dot{\omega}_T)_{YH}\bar{j}_H + I_Y(\dot{\omega}_T)_{ZH}\bar{k}_H \tag{11}$$

---

[1] Again, strictly speaking, terms of the following form should be included in the differentiation during the burning stages:

$$\dot{I}_X(\omega_T)_{XH},\ \dot{I}_Y(\omega_T)_{YH},\ \dot{I}_Y(\omega_T)_{ZH}$$

However, for a fairly stable rocket (no tumbling), $\dot{I}_Y(\omega_T)_{YH}$, and $\dot{I}_Y(\omega_T)_{ZH}$ can be neglected. The expression $\dot{I}_X(\omega_T)_{XH}$ can be written as follows:

$$\dot{I}_X(\omega_T)_{XH} = \dot{m}k^2(\omega_T)_{XH} + m(\omega_T)_{XH}\frac{d}{dt}(k^2) \tag{2'}$$

where k is the radius of gyration. The latter term can be neglected because usually the burning fuel has little effect on k^2. In addition, for low-spinning rockets, the former term also can be neglected. For high-spin rockets, however, $m\,k^2\,(\omega_T)_{XH}$ is combined with other terms which comprise the jet torque which acts to produce high-spin rates. The reader is referred to Reference 3 for further details on this matter.

Appropriately combining Equations 3, 9, 10, and 11 now results in the component equations for the moments:

$$\Sigma L_{XH} = I_X(\dot{\omega}_T)_{XH} + (\omega_{YH} I_Y(\omega_T)_{ZH} - \omega_{ZH} I_Y(\omega_T)_{YH})$$

$$\Sigma L_{YH} = I_Y(\dot{\omega}_T)_{YH} + (\omega_{ZH} I_X(\omega_T)_{XH} - \omega_{XH} I_Y(\omega_T)_{ZH}) \qquad (12)$$

$$\Sigma L_{ZH} = I_Y(\dot{\omega}_T)_{ZH} + (\omega_{XH} I_Y(\omega_T)_{YH} - \omega_{YH} I_X(\omega_T)_{XH})$$

Equations 8 and 12 comprise the basic six-degree equations of motion. These equations are of limited use, however, until one knows explicitly how the (H) coordinate system is moving with respect to inertial or earth-fixed coordinates. Further, one often has vector quantities expressed in earth coordinates (such as wind and gravitational attraction) which must be properly introduced into the (H) coordinates. Consequently, additional relationships between the (H) and (E) coordinates must be derived. In particular, one must know how the (H) coordinate system is oriented relative to the (E) coordinates, at all times. The orientation of (E) relative to (I) must also be known. Although other methods exist, as the Introduction indicates, the use of Euler angles appears to be the most straightforward for the present study. In this approach, one rotates a coordinate system, initially coincident with the (E) system, about selected axes so that after the rotations are performed this coordinate system will have the same orientation as the (H) system. One will then have the means for expressing vectors of one coordinate system in terms of the other.

The reader is reminded that, in performing these rotations, the $\bar{i}_H$ axis is to be coincident with the missile axis, while $\bar{j}_H$ remains parallel to the horizontal $\bar{i}_E - \bar{j}_E$ plane.

To accomplish this end, assume the existence of an arbitrary vector, extending from the origin of the (E) system, to represent the missile axis. First rotate the (E) system about the $\bar{k}_E$ axis by an angle ψ so that the $\bar{i}'$ axis ($\bar{i}_E$ rotated) coincides with the projection of the missile axis on the $\bar{i}_E - \bar{j}_E$ plane, as shown in Figure 1 (p 11).

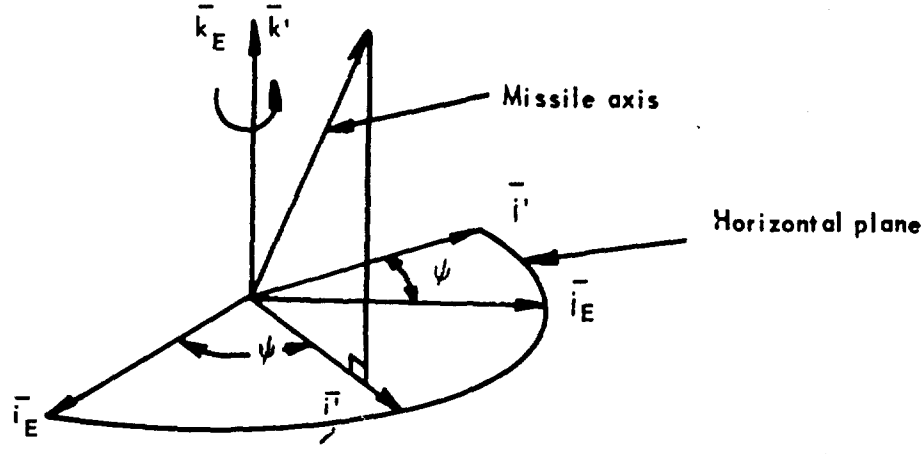


Figure 1

One may conveniently express this rotation in matrix form, as shown in Appendix B.

$$\begin{bmatrix} \bar{i}' \\ \bar{j}' \\ \bar{k}' \end{bmatrix} = \begin{bmatrix} \cos\psi & \sin\psi & 0 \\ -\sin\psi & \cos\psi & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \bar{i}_E \\ \bar{j}_E \\ \bar{k}_E \end{bmatrix} \tag{13}$$

Another rotation is yet required about the positive $\bar{j}'$ axis, ($\bar{j}_E$ rotated) so that $\bar{i}_H$ ($\bar{i}'$ rotated) is coincident with the missile axis. Denote the magnitude of this rotation by θ, leading to the matrix:

$$\begin{bmatrix} \bar{i}_H \\ \bar{j}_H \\ \bar{k}_H \end{bmatrix} = \begin{bmatrix} \cos\theta & 0 & -\sin\theta \\ 0 & 1 & 0 \\ \sin\theta & 0 & \cos\theta \end{bmatrix} \begin{bmatrix} \bar{i}' \\ \bar{j}' \\ \bar{k}' \end{bmatrix} \tag{14}$$

To avoid difficulties later on resulting from ambiguity in the determination of initial conditions and in other coordinate system transformations, let ψ be defined from $0° \leq \psi < 360°$ and $-90° \leq \theta \leq 90°$, for all time.[1] Obviously special attention must be given to the equations of motion when θ passes through $\pm\pi/2$ since sharp discontinuities in ψ will result. Each rotation matrix is a linear transformation; hence, one may obtain the (H) coordinates directly in terms of the (E) coordinates by combining both rotations. This is equivalent to the following matrix multiplication:

[1]As an example, Figure 1 would require θ to take on a negative value.

$$\begin{bmatrix} \overline{i_H} \\ \overline{j_H} \\ \overline{k_H} \end{bmatrix} = \begin{bmatrix} \cos\theta & 0 & -\sin\theta \\ 0 & 1 & 0 \\ \sin\theta & 0 & \cos\theta \end{bmatrix} \begin{bmatrix} \cos\psi & \sin\psi & 0 \\ -\sin\psi & \cos\psi & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \overline{i_E} \\ \overline{j_E} \\ \overline{k_E} \end{bmatrix} \tag{15}$$

producing

$$\begin{bmatrix} \overline{i_H} \\ \overline{j_H} \\ \overline{k_H} \end{bmatrix} = \begin{bmatrix} \cos\theta\cos\psi & \cos\theta\sin\psi & -\sin\theta \\ -\sin\psi & \cos\psi & 0 \\ \sin\theta\cos\psi & \sin\theta\sin\psi & \cos\theta \end{bmatrix} \begin{bmatrix} \overline{i_E} \\ \overline{j_E} \\ \overline{k_E} \end{bmatrix} \tag{16}$$

Note from Figure 1 (p 11) that $\overline{j_H}$ ($\overline{j}'$ rotated) remains in a horizontal plane, or is perpendicular to $\overline{k_E}$. This is reaffirmed in Equation 16 where the component of $\overline{j_H}$ on $\overline{k_E}$ is seen to be zero.

One may obtain the inverse of Equation 16 to obtain (E) in terms of (H) as follows:

$$\begin{bmatrix} \overline{i_E} \\ \overline{j_E} \\ \overline{k_E} \end{bmatrix} \begin{bmatrix} C\theta\, C\psi & -S\psi & S\theta\, C\psi \\ C\theta\, S\psi & C\psi & S\theta\, S\psi \\ -S\theta & 0 & C\theta \end{bmatrix} \begin{bmatrix} \overline{i_H} \\ \overline{j_H} \\ \overline{k_H} \end{bmatrix} \tag{17}$$

For brevity cos A and sin A have been replaced by CA and SA, respectively. It should be noted that Equations 16 and 17 actually represent three scalar equations. To illustrate this, let us assume the velocity components of the missile expressed in the (H) system are known and we wish to express the velocity in the (E) system. We have

$$\vec{V} = V_{XH}\overline{i_H} + V_{YH}\overline{j_H} + V_{ZH}\overline{k_H} = V_{XE}\overline{i_E} + V_{YE}\overline{j_E} + V_{ZE}\overline{k_E} \tag{18}$$

First, obtain $\overline{i_H}$, $\overline{j_H}$, $\overline{k_H}$ in terms of $\overline{i_E}$, $\overline{j_E}$, $\overline{k_E}$, which can be done by using Equation 16. Thus,

$$\begin{aligned} \overline{i_H} &= C\theta C\psi \overline{i_E} + C\theta S\psi \overline{j_E} - S\theta \overline{k_E} \\ \overline{j_H} &= -S\psi \overline{i_E} + C\psi \overline{j_E} + 0\,\overline{k_E} \\ \overline{k_H} &= S\theta C\psi \overline{i_E} + S\theta S\psi \overline{j_E} + C\theta \overline{k_E} \end{aligned} \tag{19}$$

It then remains to substitute these equations into Equation 18 and equate the coefficients of $\bar{i}_E$, $\bar{j}_E$, $\bar{k}_E$ to V_{XE}, V_{YE}, V_{ZE}, respectively. Performing these operations will yield:

$$
\begin{aligned}
V_{XE} &= V_{XH}C\theta C\psi - V_{YH}S\psi + V_{ZH}S\theta C\psi \\
V_{YE} &= V_{XH}C\theta S\psi + V_{YH}C\psi + V_{ZH}S\theta S\psi \\
V_{ZE} &= -V_{XH}S\theta + V_{ZH}C\theta
\end{aligned}
\tag{20}
$$

This is the required transformation.

Although much has been accomplished with Equations 13 through 20, one must go further to obtain the rates of change of the angles ψ and θ as the missile position varies in time.

Let $\vec{\omega}$ be the angular velocity of (H) relative to (E) where, in (H) coordinates,

$$\vec{\omega} = \omega_{XH}\bar{i}_H + \omega_{YH}\bar{j}_H + \omega_{ZH}\bar{k}_H \tag{21}$$

Appendix B shows $\vec{\omega}$ can also be written as

$$\vec{\omega} = \dot{\psi}\bar{k}_E + \dot{\theta}\bar{j}' \tag{22}$$

Using the matrices already developed, one proceeds by expressing both $\bar{k}_E$ and $\bar{j}'$ of Equation 22 in terms of the (H) coordinates and equating the resulting coefficients of $\bar{i}_H$, $\bar{j}_H$, $\bar{k}_H$ to the coefficients of $\bar{i}_H$, $\bar{j}_H$, $\bar{k}_H$ in Equation 21. This will produce

$$
\begin{aligned}
\omega_{XH} &= -\dot{\psi}S\theta \\
\omega_{YH} &= \dot{\theta} \\
\omega_{ZH} &= \dot{\psi}C\theta
\end{aligned}
\tag{23}
$$

The rates of change of the Euler angles become simply

$$\dot{\psi} = -\frac{\omega_{XH}}{S\theta} = \frac{\omega_{ZH}}{C\theta} \tag{24}$$

$$\dot{\theta} = \omega_{YH}$$

One may note that several angular velocities have been introduced into the equations of motion, namely, $\vec{\omega}$, $\vec{\omega}$, and $(\vec{\omega}_T)$. In component form, this amounts to nine unknowns, a completely overwhelming assignment for the three moment equations. Fortunately, not all these quantities are independent. It can be shown (see Ref 4, for example) that if a coordinate system (H) is rotating with angular velocity $\vec{\omega}$ relative to a coordinate system (E), and (E) is rotating with angular velocity $\vec{\Omega}$ relative to a coordinate system (I), then the angular velocity of (H) relative to (I), $\vec{\omega}$, is simply the vector sum of the individual angular velocities. Thus,

$$\vec{\omega} = \vec{\omega} + \vec{\Omega} \tag{26}$$

This can be generalized to more than three coordinate systems. For example, continuing to use the definitions given in the Table of Symbols, we are also at liberty to write

$$(\vec{\omega}_T) = \vec{\Omega} + \vec{\omega} = \vec{\Omega} + \vec{\omega} + \vec{\Omega} \tag{27}$$

It is now necessary to obtain explicit expressions for $\vec{\Omega}$, $(\vec{\omega}_T)$, and $(\dot{\vec{\omega}}_T)$, each of which is necessary to the solution of the general equations of motion.

To obtain components of $\vec{\Omega}$ in the (H) system, one need only convert $\overline{k_E}$ into the (H) coordinates, since the $\overline{k_E}$ axis was assigned to be coincident with the spin axis of the earth. Therefore

$$\vec{\Omega} = \Omega\overline{k_E} = -\Omega S\theta\overline{i_H} + \Omega C\theta\overline{k_H} \tag{28}$$

The total angular velocity, $(\vec{\omega}_T)$, of the missile is introduced by first making use of the (M) coordinate system. Since the $\overline{i_M}$ and $\overline{i_H}$ axes are to remain coincident at all times, the angular velocity $\vec{\Omega}$ of (M) relative to (H) is simply;

$$\vec{\Omega} = \Omega_{XM}\overline{i_M} = \Omega_{XH}\overline{i_H} \tag{29}$$

By the use of Equation 27, the total angular velocity can now be written in component form as follows:

$$\begin{aligned} (\omega_T)_{XH} &= \Omega_{XH} + \omega_{XH} - \Omega S\theta \\ (\omega_T)_{YH} &= \omega_{YH} \\ (\omega_T)_{ZH} &= \omega_{ZH} + \Omega C\theta \end{aligned} \tag{30}$$

One may still sense the existence of too many unknowns, namely, $\vec{\Omega}$, and the three components of $\vec{\omega}$. Fortunately Equation 23 gives a relation between two components of $\vec{\omega}$. Eliminating $\dot{\psi}$ produces

$$\omega_{XH} = -\omega_{ZH} \tan \theta \tag{31}$$

Finally, to obtain explicit expressions for $(\dot{\vec{\omega}}_T)$ as given in the basic moment equation, Equation 31 is substituted into Equation 30 and the latter is differentiated with respect to time. This leads to:

$$\begin{aligned}
(\dot{\omega}_T)_{XH} &= \dot{\Omega}_{XH} - \dot{\omega}_{ZH} \tan \theta - \omega_{ZH} \sec^2 \theta \omega_{YH} - \Omega C\theta \omega_{YH} \\
(\dot{\omega}_T)_{YH} &= \dot{\omega}_{YH} \\
(\dot{\omega}_T)_{ZH} &= \dot{\omega}_{ZH} - \Omega S\theta \omega_{YH}
\end{aligned} \tag{32}$$

This completes the discussion. The equations needed to obtain the actual trajectory are summarized beginning on page 51. The next section considers the forces and moments which constitute the "left" side of the equations of motion (Equations 8 and 12).

Forces and Moments

This section considers the forces and moments acting on the missile. Excluding guidance, these forces and moments can be divided into three categories as follows:

1. Aerodynamic
2. Gravitational
3. Jet

Since the forces and moments are essentially as presented in Reference 1, only a brief account of each will be given here.

Aerodynamic Forces

Three aerodynamic forces are considered. A brief table giving magnitude and direction of each is given on page 16.

Force	Magnitude[1]	Direction
Axial drag	$\rho d^2 (V_r)^2 k_{DA}$	Along missile axis
Normal force	$\rho d^2 (V_r)^2 k_N$	Perpendicular to missile axis in plane of yaw
Magnus force	$\rho d^3 (\omega_T)_{XH} (V_r) k_F$	Perpendicular to plane of yaw[2]

[1] (V_r) is the magnitude of the missile velocity relative to the air.

[2] The yaw plane is the plane that contains both the missile axis and resultant velocity vector.

These forces are illustrated in Figure 2.

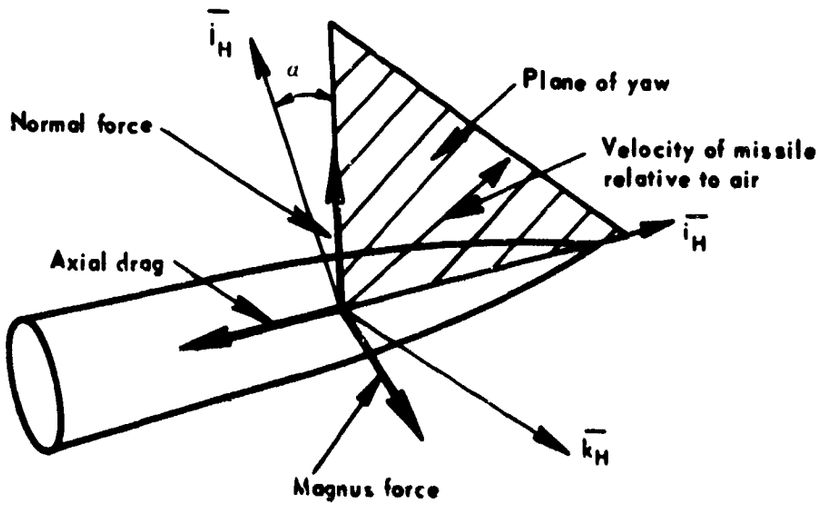


Figure 2

The axial drag acts along the negative $\bar{i}_H$ axis, which by definition is directed from the CG towards the nose of the missile. Explicitly,

$$(\text{Axial drag})_{XH} = -\rho d^2 (V_r)^2 k_{DA} \tag{33}$$

The normal force components act opposite to the directions of $(V_r)_{YH}$ and $(V_r)_{ZH}$. Schematically we have

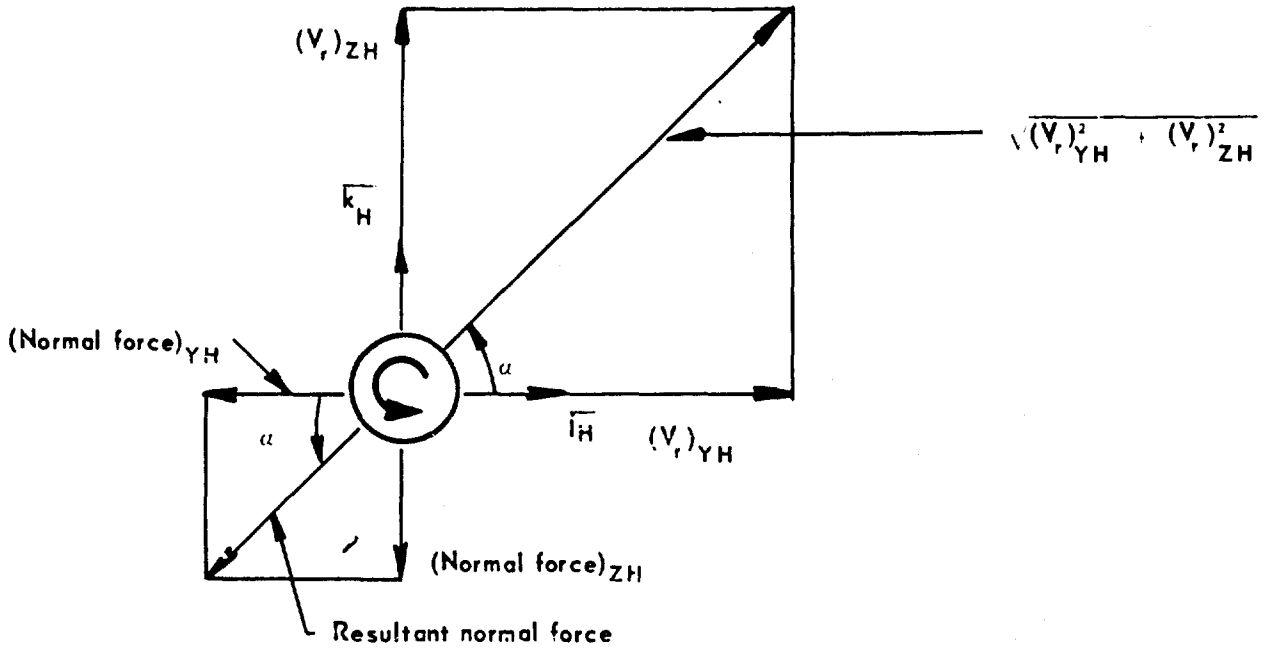


Figure 3

Knowing the scalar magnitude of the resultant normal force and using the geometry of Figure 3, one can deduce the component of this force acting along $\overline{i_H}$, namely

$$(\text{Normal force})_{YH} = -\rho d^2 (V_r)^2 k_N \cos\alpha = \frac{-\rho d^2 (V_r)^2 k_N (V_r)_{YH}}{\sqrt{(V_r)^2_{YH} + (V_r)^2_{ZH}}} \quad (34)$$

Similarly, in the $\overline{k_H}$ direction

$$(\text{Normal Force})_{ZH} = \frac{-\rho d^2 (V_r)^2 k_N (V_r)_{ZH}}{\sqrt{(V_r)^2_{YH} + (V_r)^2_{ZH}}} \quad (35)$$

The Magnus force acts in a direction perpendicular to the plane of yaw, as indicated in Figure 4. It should be noted that, for reversed spin, the Magnus force will act in the opposite direction.

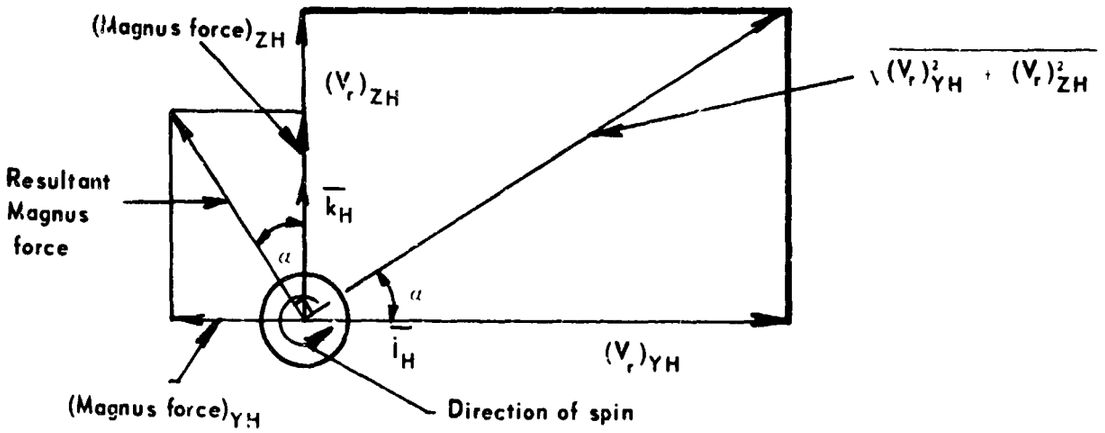


Figure 4

Using the scalar magnitude and the geometry of Figure 4 above, it is readily ascertained that in the $\overline{j}_H$ direction

$$(\text{Magnus force})_{YH} = -\rho d^3(\omega_T)_{XH}(V_r)k_F \sin\alpha = \frac{-\rho d^3(\omega_T)_{XH}(V_r)k_F(V_r)_{ZH}}{\sqrt{(V_r)^2_{YH} + (V_r)^2_{ZH}}} \quad (36)$$

Similarly, in the $\overline{k}_H$ direction

$$(\text{Magnus force})_{ZH} = \frac{\rho d^3(\omega_T)_{XH}(V_r)k_F(V_r)_{YH}}{\sqrt{(V_r)^2_{YH} + (V_r)^2_{ZH}}} \quad (37)$$

Aerodynamic Moments

Proceeding in a similar vein as for the aerodynamic forces, the following table gives the magnitude and direction of each of the aerodynamic moments considered:

Moment	Magnitude	Direction
Restoring	$\rho d^2 (V_r)^2 k_N (r_C - \lambda_N)$	Perpendicular to the plane of yaw
Magnus	$\rho d^3 (\omega_T)_{XH} (V_r) k_F (r_C - \lambda_M)$	In plane of yaw
Yaw damping	$\rho d^4 (V_r) \sqrt{(\omega_T)^2_{YH} + (\omega_T)^2_{ZH}}\, k_H$	Perpendicular to missile axis
Roll damping[1]	$\rho d^4 (V_r) k_{\phi p} (\omega_T)_{XH}$	Along missile axis
Roll[2]	$\rho d^3 (V_r)^2 k_\phi$	Along missile axis

[1] $k_{\phi p}$ is used for spin-stabilized missiles, while for fin-stabilized missiles $k_{\phi p}$ is to be changed to k_A.

[2] The roll moment is used only for missiles with canted fins.

In obtaining the components of the above moments, one may use the relation "distance × force = moment," applied in a manner consistent with the geometry of the system. When expressed in vector form the restoring moment equation becomes

$$(\text{Restoring moment}) = (r_C - \lambda_N)\overline{i}_H \times [(\text{normal force})_{YH}\overline{j}_H + (\text{normal force})_{ZH}\overline{k}_H] \quad (38)$$

This produces, in component form

$$(\text{Restoring moment})_{YH} = \frac{(r_C - \lambda_N)\, \rho d^2 (V_r)^2 k_N (V_r)_{ZH}}{\sqrt{(V_r)^2_{YH} + (V_r)^2_{ZH}}} \quad (39)$$

in the $\overline{j}_H$ direction, and

$$(\text{Restoring moment})_{ZH} = \frac{-(r_C - \lambda_N)\, \rho d^2 (V_r)^2 k_N (V_r)_{YH}}{\sqrt{(V_r)^2_{YH} + (V_r)^2_{ZH}}} \quad (40)$$

in the $\overline{k}_H$ direction. Note that both magnitude and direction are consistent with the above table.

Doing likewise for the Magnus moment produces along the $\overline{j}_H$ axis

$$(\text{Magnus moment})_{YH} = \frac{-(r_C - \lambda_M)\rho d^3 (\omega_T)_{XH} (V_r) k_F (V_r)_{YH}}{\sqrt{(V_r)^2_{YH} + (V_r)^2_{ZH}}} \quad (41)$$

while along the $\overline{k}_H$ axis

$$(\text{Magnus moment})_{ZH} = \frac{-(r_C - \lambda_M)\rho d^3(\omega_T)_{XH}(V_r)k_F(V_r)_{ZH}}{\sqrt{(V_r)^2_{YH} + (V_r)^2_{ZH}}} \qquad (42)$$

The yaw damping moment acts to diminish the yaw of the missile; hence, this moment acts opposite to the lateral angular velocity of the missile. The components of this moment are readily ascertained to be

$$(\text{Yaw damping})_{YH} = -\rho d^4(V_r)\sqrt{(\omega_T)^2_{YH} + (\omega_T)^2_{ZH}}\, k_H \cos\delta_Y \qquad (43)$$

$$(\text{Yaw damping})_{ZH} = -\rho d^4(V_r)\sqrt{(\omega_T)^2_{YH} + (\omega_T)^2_{ZH}}\, k_H \sin\delta_Y \qquad (44)$$

where δ_Y is shown in Figure 5 below.

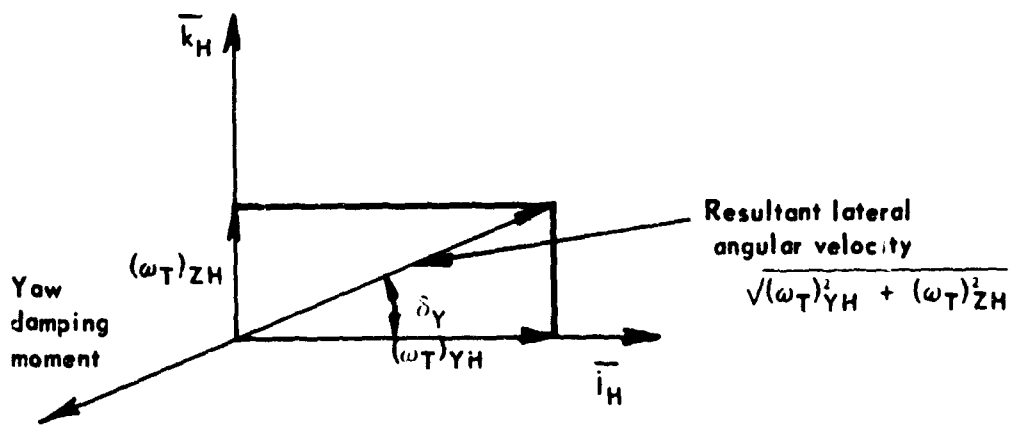


Figure 5

Writing $\cos\delta_Y$ in terms of $(\omega_T)_{YH}$ and $(\omega_T)_{ZH}$

$$\cos\delta_Y = \frac{(\omega_T)_{YH}}{\sqrt{(\omega_T)^2_{YH} + (\omega_T)^2_{ZH}}} \qquad (45)$$

and substituting into Equation 43 leads to the component along $\overline{i}_H$

$$(\text{Yaw damping})_{YH} = -\rho d^4(V_r)(\omega_T)_{YH}k_H \qquad (46)$$

similarly, for the component along $\overline{k}_H$,

$$(\text{Yaw damping})_{ZH} = -\rho d^4(V_r)(\omega_T)_{ZH}k_H \qquad (47)$$

The roll damping moment acts to reduce the spin; hence, it is introduced into the equations of motion with a minus sign prefixed.

The direction of the roll moment depends upon the orientation of the canted fins. If the cant produces positive spin (i.e., a clockwise rotation of the missile looking from the rear of the rocket), then this term is introduced with a plus sign. Like the roll damping moment, the direction is along the longitudinal axis of the missile.

It should be noted that, since none of the aerodynamic coefficients (k_{DA}, k_N, k_F, etc.) are assumed to be linear in nature, they are not to be taken as slopes to be multiplied by angle of attack. Rather, these coefficients are point values obtained directly from experiment as functions both of Mach number and angle of attack.

Table 1 (p 22) summarizes in component form the aerodynamic forces and moments considered thus far.

Gravitational Force

In evaluating this force, distinction must be made between spherical and flat earth cases. In the latter case, the gravitational force acts in a constant direction (independent of the missile position) while in the former case, the gravitational attraction is directed towards the earth's center from the missile CG. In both cases, specific expressions can be derived; the simpler case will be treated first.

Flat Earth Case

In the absence of the earth's rotation,[1] both the $\overline{k}_I$ and $\overline{k}_E$ axes are taken as extending in positive vertical direction. The origins of the (I) and the (E) coordinate systems are both located on the surface of the "flat" earth. Hence, $mg\dot{y}$ ($\dot{y}$ being a unit vector representing the direction of the gravitational force acting on the missile) becomes:

$$mg\dot{y} \quad -mg\overline{k}_I \quad -mg\overline{k}_E \tag{48}$$

[1] One should have little desire to refine the model to include the earth's rotation and yet allow the assumption of a plane to represent the earth's geometry.

TABLE 1

Summary of Aerodynamic Forces and Moments

Force or Moment	$\overline{i_H}$	$\overline{j_H}$	$\overline{k_H}$
Axial drag	$-\rho d^2 (V_r)^2 k_{DA}$	0	0
Normal force	0	$\dfrac{-\rho d^2 (V_r)^2 k_N (V_r)_{YH}}{\sqrt{(V_r)^2_{YH} + (V_r)^2_{ZH}}}$	$\dfrac{-\rho d^2 (V_r)^2 k_N (V_r)_{ZH}}{\sqrt{(V_r)^2_{YH} + (V_r)^2_{ZH}}}$
Magnus force	0	$\dfrac{-\rho d^3 (\omega_T)_{XH} (V_r) k_F (V_r)_{ZH}}{\sqrt{(V_r)^2_{YH} + (V_r)^2_{ZH}}}$	$\dfrac{\rho d^3 (\omega_T)_{XH} (V_r) k_F (V_r)_{YH}}{\sqrt{(V_r)^2_{YH} + (V_r)^2_{ZH}}}$
Restoring moment	0	$\dfrac{(r_C - \lambda_N) \rho d^2 (V_r)^2 k_N (V_r)_{ZH}}{\sqrt{(V_r)^2_{YH} + (V_r)^2_{ZH}}}$	$\dfrac{-(r_C - \lambda_N) \rho d^2 (V_r)^2 (V_r)_{YH} k_N}{\sqrt{(V_r)^2_{YH} + (V_r)^2_{ZH}}}$
Magnus moment	0	$\dfrac{-(r_C - \lambda_M) \rho d^3 (\omega_T)_{XH} (V_r) k_F (V_r)_{YH}}{\sqrt{(V_r)^2_{YH} + (V_r)^2_{ZH}}}$	$\dfrac{-(r_C - \lambda_M) \rho d^3 (\omega_T)_{XH} (V_r) k_F (V_r)_{ZH}}{\sqrt{(V_r)^2_{YH} + (V_r)^2_{ZH}}}$
Yaw damping moment	0	$-\rho d^4 (V_r) (\omega_T)_{YH} k_H$	$-\rho d^4 (V_r) (\omega_T)_{ZH} k_H$
Roll damping moment	$\rho d^4 (V_r) k_{\phi\rho} (\omega_T)_{XH}$	0	0
Roll moment	$\rho d^3 (V_r)^2 k_\phi$	0	0

Replacing $\overline{k}_E$ by its representation in (H) coordinates, similar to Equation 28, one obtains

$$mg\vec{y} = mgS\theta\overline{i}_H - mgC\theta\overline{k}_H \tag{49}$$

in a form suitable for substitution into the general equations of motion.

Spherical Earth Case

For this case, the direction of the gravitational force depends upon the missile position. In particular, let the current missile position be represented by a vector $\vec{R}$, where

$$\vec{R} = R_{XE}\overline{i}_E + R_{YE}\overline{j}_E + R_{ZE}\overline{k}_E \tag{50}$$

as shown in Figure 6.

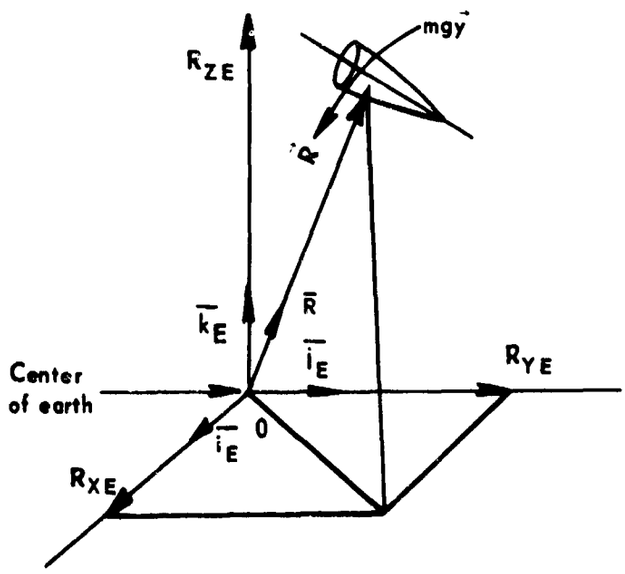


Figure 6

Dividing $\dot{R}$ by its magnitude produces the unit vector

$$\overline{R} = \frac{\vec{R}}{|\dot{R}|} = \frac{R_{XE}\overline{i}_E + R_{YE}\overline{j}_E + R_{ZE}\overline{k}_E}{\sqrt{R_{XE}^2 + R_{YE}^2 + R_{ZE}^2}} \tag{51}$$

Clearly,

$$mg\dot{y} = -mg\overline{R} \tag{52}$$

Combining Equations 51 and 52 and obtaining $\overline{i_E}$, $\overline{j_E}$, $\overline{k_E}$ in terms of the (H) coordinates, there results:

$$(\text{Gravity force})_{XH} = \frac{-mg}{\sqrt{R_{XE}^2 + R_{YE}^2 + R_{ZE}^2}}[R_{XE}C\theta C\psi + R_{YE}C\theta S\psi - R_{ZE}S\theta]$$

$$(\text{Gravity force})_{YH} = \frac{-mg}{\sqrt{R_{XE}^2 + R_{YE}^2 + R_{ZE}^2}}[-R_{XE}S\psi + R_{YE}C\psi] \quad (53)$$

$$(\text{Gravity force})_{ZH} = \frac{-mg}{\sqrt{R_{XE}^2 + R_{YE}^2 + R_{ZE}^2}}[R_{XE}S\theta C\psi + R_{YE}S\theta S\psi + R_{ZE}C\theta]$$

One should note that the gravitational acceleration, g, is a function of altitude. In both cases, one may write

$$g = g_o \left[\frac{R_{earth}}{h}\right]^2 \quad (54)$$

where

g_o = acceleration of gravity at sea level,

R_{earth} = radius of the earth,

$$h \begin{cases} = \text{current altitude of missile} \\ = (R_{ZE} \text{ for the flat earth case}) \\ = \sqrt{R_{XE}^2 + R_{YE}^2 + R_{ZE}^2} \text{ for the spherical earth case.} \end{cases}$$

Since the gravitational force acts at the CG of the missile, there are no moments associated with this force. Table 2 (p 25) summarizes the results of this derivation in component form.

Jet Forces

A single jet force is considered, the thrust, which imparts forward motion to the missile. Ideally, this force should act along the missile's longitudinal axis; however, due to imperfections in the rocket design, the actual nature of propellants and other factors, there may be a component of thrust perpendicular to the longitudinal axis. Since the thrust vector is defined at a given time relative to the missile body, the thrust components will first be specified in the (M) coordinate system (i.e., the coordinate system that is rigidly attached to the missile).

TABLE 2

Summary of Gravitational Force

Case	$\bar{i}_H$	$\bar{j}_H$	$\bar{k}_H$
Flat earth	$mg \sin\theta$	0	$-mg \cos\theta$
Spherical earth	$\frac{-mg}{h}\begin{pmatrix} R_{XE}C\theta C\psi \\ +R_{YE}C\theta S\psi \\ -R_{ZE}S\theta \end{pmatrix}$	$\frac{-mg}{h}\begin{pmatrix} -R_{XE}S\psi \\ +R_{YE}C\psi \end{pmatrix}$	$\frac{-mg}{h}\begin{pmatrix} R_{XE}S\theta C\psi \\ +R_{YE}S\theta S\psi \\ +R_{ZE}C\theta \end{pmatrix}$

$$h = \begin{cases} R_{ZE} & \text{Flat earth case} \\ \sqrt{R_{XE}^2 + R_{YE}^2 + R_{ZE}^2} & \text{Spherical earth case} \end{cases}$$

$$g = g_o \left[\frac{R_{earth}}{h} \right]^2$$

Using the symbols presented in Figure 7, one can easily ascertain these components to be as in Equation 55.

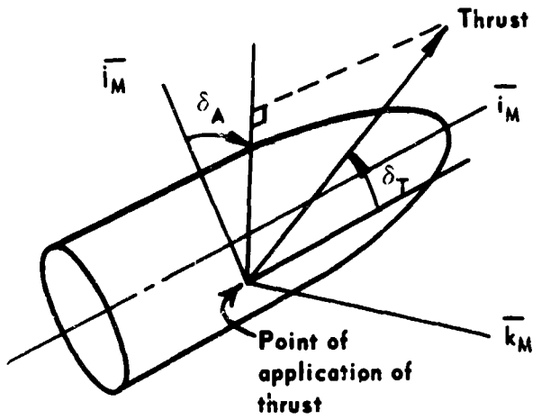


Figure 7

$$
\begin{aligned}
(\text{Thrust})_{XM} &= T_{XM} = T \cos \delta_T \\
(\text{Thrust})_{YM} &= T_{YM} = T \sin \delta_T \cos \delta_A \qquad (55) \\
(\text{Thrust})_{ZM} &= T_{ZM} = T \sin \delta_T \sin \delta_A
\end{aligned}
$$

It is necessary, of course, to determine the components of thrust in the (H) coordinate system along whose axes the forces are summed.

Figure 8 shows a typical relationship between the (H) and (M) coordinates.

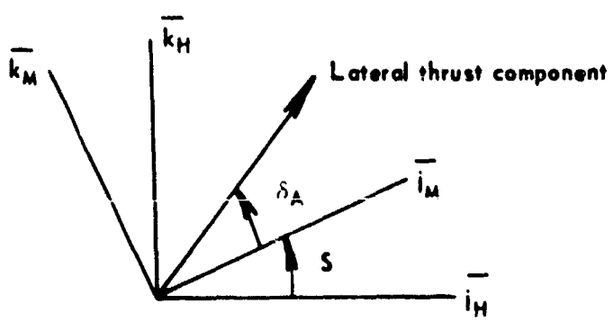


Figure 8

Here S is defined as the angle between the $\overline{j}_M$ and $\overline{j}_H$ axes. It is readily seen that

$$\begin{bmatrix} \overline{i}_M \\ \overline{j}_M \\ \overline{k}_M \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos S & \sin S \\ 0 & -\sin S & \cos S \end{bmatrix} \begin{bmatrix} \overline{i}_H \\ \overline{j}_H \\ \overline{k}_H \end{bmatrix} \tag{56}$$

and, from Equation 56,

$$\begin{aligned} \overline{i}_M &= \overline{i}_H \\ \overline{j}_M &= \cos S\, \overline{j}_H + \sin S\, \overline{k}_H \\ \overline{k}_M &= -\sin S\, \overline{j}_H + \cos S\, \overline{k}_H \end{aligned} \tag{57}$$

Combining Equations 55 and 57 produces the required result:

$$\begin{aligned} T_{XH} &= T \cos \delta_T \\ T_{YH} &= T \sin \delta_T \cos (\delta_A + S) \\ T_{ZH} &= T \sin \delta_T \sin (\delta_A + S) \end{aligned} \tag{58}$$

Jet Moments

Two jet moments exist. One unintentionally arises from the fact that the line of thrust may not pass through the centroid of the missile. The other is an intentional jet torque which causes the rocket to spin about its axis of revolution. To be explicit about the former, one may define a vector $\vec{r}$ from the CG of the rocket to the point of application of the thrust at the nozzle exit. As before, this vector is defined in the missile coordinate system, i.e.,

$$\vec{r} = r_{XM}\overline{i}_M + r_{YM}\overline{j}_M + r_{ZM}\overline{k}_M$$

The corresponding jet moment is now given by

$$(\text{Thrust moment}) = \vec{r} \times (\text{Thrust})$$

which in component form becomes:

$$
\begin{aligned}
(\text{Thrust moment})_{XM} &= r_{YM}T_{ZM} - r_{ZM}T_{YM} \\
(\text{Thrust moment})_{YM} &= r_{ZM}T_{XM} - r_{XM}T_{ZM} \\
(\text{Thrust moment})_{ZM} &= r_{XM}T_{YM} - r_{YM}T_{XM}
\end{aligned}
\tag{59}
$$

Converting these components into the (H) coordinates and substituting Equation 55 finally produces

$$
\begin{aligned}
(\text{Thrust moment})_{XH} &= T \sin \delta_T [r_{YM} \sin \delta_A - r_{ZM} \cos \delta_A] \\
(\text{Thrust moment})_{YH} &= T [r_{ZM} \cos \delta_T - r_{XM} \sin \delta_T \sin \delta_A] \cos S \\
&\quad - T [r_{XM} \sin \delta_T \cos \delta_A - r_{YM} \cos \delta_T] \sin S \\
(\text{Thrust moment})_{ZH} &= T [r_{ZM} \cos \delta_T - r_{XM} \sin \delta_T \sin \delta_A] \sin S \\
&\quad + T [r_{XM} \sin \delta_T \cos \delta_A - r_{YM} \cos \delta_T] \cos S
\end{aligned}
\tag{60}
$$

The final moment, the jet torque, will be assumed proportional to the thrust. If this moment produces positive spin, this term is to be introduced into the equations of motion with a plus sign prefixed. Mathematically,

$$
(\text{Jet torque}) = k\, T\, \overline{i_H} \tag{61}
$$

This concludes the discussion of forces and moments.

One might note that $\dot{V}_r$, the velocity of the missile relative to the air, has not been defined in the presence of winds. In the next section of this report explicit expressions will be derived for this vector in the (H) coordinate system.

Introduction of Winds

This section derives expressions that can be used to introduce wind into the equations of motion and thus determine ($\dot{V}_r$). For the spherical earth case, it will be assumed that wind data exists at preselected points on the earth's surface. For a flat earth, it is assumed that wind is presented as a function of range and altitude.

In both cases, if one knows the velocity of the missile (V_M) and the and the air velocity (V_A), each relative to the earth's surface, then the

velocity of the missile relative to the air ($\dot{V}_r$) is given by

$$(\dot{V}_r) \quad (\dot{V}_{M/A}) \quad \dot{V}_M - \dot{V}_A \tag{62}$$

For the flat non-rotating earth, wind is likely to be given relative to the ground; therefore, use of the (E) coordinate system is appropriate and Equation 62 may be written as

$$(\dot{V}_r) \quad (V_{XE} - (V_W)_{XE})\overline{i}_E + (V_{YE} - (V_W)_{YE})\overline{j}_E + (V_{ZE} - (V_W)_{ZE})\overline{k}_E \tag{63}$$

where ($\dot{V}_W$) is the actual wind vector.

It is an easy matter to convert Equation 63 into the (H) coordinates. Doing so produces

$$(V_r)_{XH} \quad [V_{XE} - (V_W)_{XE}]C\theta C\psi + [V_{YE} - (V_W)_{YE}]C\theta S\psi - [V_{ZE} - (V_W)_{ZE}]S\theta$$

$$(V_r)_{YH} \quad -[V_{XE} - (V_W)_{XE}]S\psi + [V_{YE} - (V_W)_{YE}]C\psi \tag{64}$$

$$(V_r)_{ZH} \quad [V_{XE} - (V_W)_{XE}]S\theta C\psi + [V_{YE} - (V_W)_{YE}]S\theta S\psi + [V_{ZE} - (V_W)_{ZE}]C\theta$$

Of course, $(V_W)_{XE}$, $(V_W)_{YE}$, and $(V_W)_{ZE}$ can be given as functions of range.

The spherical earth case is a bit more complicated because of the geometry. Also, one must account for the motion of the air induced by the earth's rotation in addition to wind (i.e., disturbance of the air within this rotating air mass).

Recalling that Equation 5

$$\dot{V} \quad \frac{d_H\dot{R}}{dt} \quad \dot{\omega} \cdot \dot{R} \quad V_{XH}\overline{i}_H + V_{YH}\overline{j}_H + V_{ZH}\overline{k}_H \tag{5}$$

is the velocity of the missile relative to inertial coordinates (by definition of $\dot{\omega}$), one must also express the motion of the air induced by the earth's rotation relative to inertial coordinates. By so doing one can obtain the velocity of the missile relative to the rotating air mass. This will be the equivalent of Equation 62 referred to inertial coordinates.

The air mass rotates as a whole with the earth; this is shown physically by noticing that the rotation of the air mass is not apparent to one on the earth's surface. The velocity of this rotating air mass at the current missile position (specified by the vector $\dot{R}$) is simply

$$\dot{V}_{\text{Rot. air mass}} \quad (\dot{\Omega} \cdot \dot{R}) \tag{65}$$

The velocity of the missile relative to the rotating air mass therefore becomes

$$\vec{V}_{M/\text{Rot. air mass}} = \vec{V} - (\vec{\Omega} \times \vec{R}) \tag{66}$$

One must now account for the wind itself. For the spherical earth case, winds may be conveniently introduced into the equations of motion by use of the azimuthal coordinate system (W). This coordinate system is described on page 6 and is illustrated in Figure 9 (p30), where the origin of (W) has been translated to the earth's surface at 0' for visual purposes. The wind can be considered to be a function of the height $\hat{h}$, of the missile CG above the earth's surface, (see Figure 9). One method of designating the wind in the (W) coordinates is to specify wind magnitude and wind direction measured from North at given altitudes. It will be assumed in this report that the wind vector, $\vec{V}_W(\hat{h})$, has no vertical component. Thus at a given altitude $\vec{V}_W(\hat{h})$ should be expressible only in the $\overline{j_W}-\overline{k_W}$ plane, as presented in Equation 67, using the geometry of Figure 9.

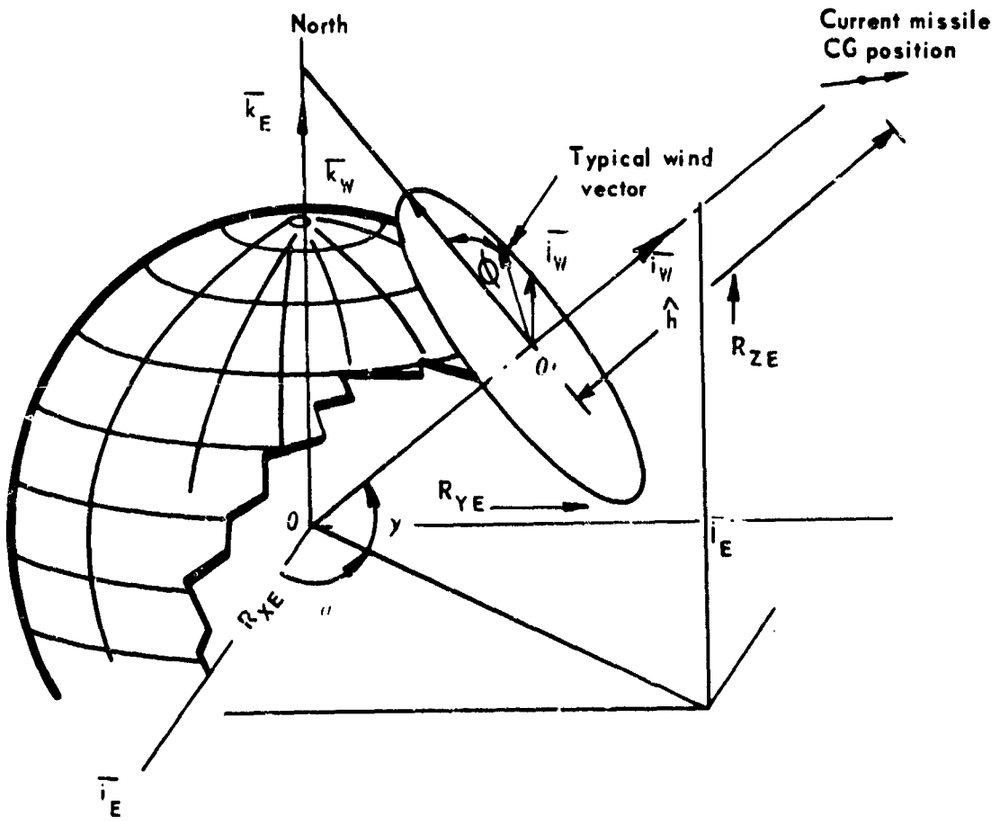


Figure 9

$$(\vec{V}_W) = |V_W(\hat{h})| \sin\phi\, \bar{j}_W + |V_W(\hat{h})| \cos\phi\, \bar{k}_W = (V_W)_{YW}\bar{j}_W + (V_W)_{ZW}\bar{k}_W \quad (67)$$

The presence of $\hat{h}$ emphasizes the dependence of wind on altitude. The point 0' on the earth's surface is defined by α and γ where

$$\alpha = \tan^{-1}\left\{\frac{R_{YE}}{R_{XE}}\right\}$$

$$\gamma = \tan^{-1}\left\{\frac{R_{ZE}}{\sqrt{R^2_{XE} + R^2_{YE}}}\right\} \quad (68)$$

where if

$$R_{XE} > 0 \quad \begin{array}{ll} R_{YE} > 0 & 0 \leq \alpha < 90^\circ \\ R_{YE} < 0 & 270^\circ \leq \alpha < 360^\circ \end{array}$$

$$R_{XE} < 0 \quad \begin{array}{ll} R_{YE} > 0 & 90^\circ \leq \alpha < 180^\circ \\ R_{YE} < 0 & 180^\circ \leq \alpha < 270^\circ \end{array}$$

and

$$-90^\circ \leq \gamma \leq 90^\circ$$

It remains to obtain the wind components in the (H) coordinate system and add the results to Equation 66. This is done by first obtaining $\bar{j}_W$ and $\bar{k}_W$ in terms of (E) and then (by the now overworked Equation 17) obtaining (E) in terms of (H).

Rotating first about $\bar{k}_E$ through the angle α and then rotating about $\bar{j}'$ through the angle β ($\beta = -\gamma$ in Fig 9), one can see that the resultant coordinate system will be coincident with (W).

Performing the indicated operations results in

$$\begin{bmatrix} \bar{i}_W \\ \bar{j}_W \\ \bar{k}_W \end{bmatrix} = \begin{bmatrix} C\beta C\alpha & C\beta S\alpha & -S\beta \\ -S\alpha & C\alpha & 0 \\ S\beta C\alpha & S\beta S\alpha & C\beta \end{bmatrix} \begin{bmatrix} \bar{i}_E \\ \bar{j}_E \\ \bar{k}_E \end{bmatrix} \quad (69)$$

Combining Equation 69 with Equation 17 completes the transformation. Thus,

$$\begin{bmatrix} \bar{i}_W \\ \bar{j}_W \\ \bar{k}_W \end{bmatrix} = \begin{bmatrix} C\beta C\alpha & C\beta S\alpha & -S\beta \\ -S\alpha & C\alpha & 0 \\ S\beta C\alpha & S\beta S\alpha & C\beta \end{bmatrix} \begin{bmatrix} C\theta C\psi & -S\psi & S\theta C\psi \\ C\theta S\psi & C\psi & S\theta S\psi \\ -S\theta & 0 & C\theta \end{bmatrix} \begin{bmatrix} \bar{i}_H \\ \bar{j}_H \\ \bar{k}_H \end{bmatrix}$$

leading to

$$\begin{bmatrix} \bar{i}_W \\ \bar{j}_W \\ \bar{k}_W \end{bmatrix} = \begin{bmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{bmatrix} \begin{bmatrix} \bar{i}_H \\ \bar{j}_H \\ \bar{k}_H \end{bmatrix} \tag{70}$$

where

$$A_{11} = C\beta C\alpha C\theta C\psi + C\beta S\alpha C\theta S\psi + S\beta S\theta$$

$$A_{12} = -C\beta C\alpha S\psi + C\beta S\alpha C\psi$$

$$A_{13} = C\beta C\alpha S\theta C\psi + C\beta S\alpha S\theta S\psi - S\beta C\theta$$

$$A_{21} = -S\alpha C\theta C\psi + C\alpha C\theta S\psi$$

$$A_{22} = S\alpha S\psi + C\alpha C\psi$$

$$A_{23} = -S\alpha S\theta C\psi + C\alpha S\theta S\psi$$

$$A_{31} = S\beta C\alpha C\theta C\psi + S\beta S\alpha C\theta S\psi - C\beta S\theta$$

$$A_{32} = -S\beta C\alpha S\psi + S\beta S\alpha C\psi$$

$$A_{33} = S\beta C\alpha S\theta C\psi + S\beta S\alpha S\theta S\psi + C\beta C\theta$$

Clearing up the remaining threads, we have

$$(V_W) = (V_W)_{YW}\bar{j}_W + (V_W)_{ZW}\bar{k}_W = (V_W)_{XH}\bar{i}_H + (V_W)_{YH}\bar{j}_H + (V_W)_{ZH}\bar{k}_H \tag{71}$$

where the usual algebraic manipulation results in the necessary equation

$$\begin{aligned} (V_W)_{XH} &= (V_W)_{YW} A_{21} + (V_W)_{ZW} A_{31} \\ (V_W)_{YH} &= (V_W)_{YW} A_{22} + (V_W)_{ZW} A_{32} \\ (V_W)_{ZH} &= (V_W)_{YW} A_{23} + (V_W)_{ZW} A_{33} \end{aligned} \tag{72}$$

Finally, the sought after vector is

$$(\vec{V}_r) = \vec{V} - (\vec{\Omega} \times \vec{R}) - (\vec{V}_W) \tag{73}$$

Note again that $\vec{V}_W$ prescribes the wind velocity relative to the rotating air mass. Thus, for a missile moving on or above the surface of the earth with a velocity of $\vec{\Omega} \times \vec{R}$ (stationary orbit or fixed surface point), in the absence of winds would cause the relative velocity, $\vec{V}_r$ to be zero. Writing Equation 73 in component form yields

$$\begin{aligned}
(V_r)_{XH} &= V_{XH} - \Omega C\theta R_{YH} - (V_W)_{XH} \\
(V_r)_{YH} &= V_{YH} - \Omega C\theta R_{XH} + \Omega S\theta R_{ZH} - (V_W)_{YH} \\
(V_r)_{ZH} &= V_{ZH} + \Omega S\theta R_{YH} - (V_W)_{ZH}
\end{aligned} \tag{74}$$

where, of course, the magnitude of $(\vec{V}_r)$ is

$$|\vec{V}_r| = \sqrt{(V_r)^2_{XH} + (V_r)^2_{YH} + (V_r)^2_{ZH}}$$

This completes the analysis of winds.

Initial Conditions

Initial conditions for the spherical earth case will be derived first. Initial conditions for the flat earth are immediately obtainable upon specializing certain of the parameters.

In order to start the trajectory, a complete set of initial conditions must be provided. These are tabulated below for convenience.

R_{XE}	V_{XH}	$\psi(0)$	Ω_{XH}
R_{YE}	V_{YH}	$\theta(0)$	ω_{YH}
R_{ZE}	V_{ZH}		ω_{ZH}

Those conditions that depend upon a coordinate system for representation can be given in terms of other coordinate systems, if desired, as long as known transformation equations exist between such coordinate systems.

Of the conditions stated above, the missile orientation expressed in terms of $\psi(0)$ and $\theta(0)$ is not usually known, per se. But the following measurable quantities are (or at least should be) known:

Longitude of the launch point	A°
Latitude of the launch point	B°
Azimuthal heading of the missile	G° (This will be defined more explicitly later in this report.)
Angle of declination of the missile from the local vertical	H°

It is possible, as will be shown, to express $\psi(0)$ and $\theta(0)$ as a function of the variables A°, B°, G°, and H°. To do this, one rotates a coordinate system initially ($t = 0$) coincident with (E) through the above measured angles. Upon completion of the rotations, the resultant coordinate system is identical in orientation with the (H) coordinate system at launch. The (H) system is also defined by simply specifying $\psi(0)$ and $\theta(0)$. Each of the resultant rotations can be expressed in matrix form, one matrix containing the four known variables and the other matrix containing the two unknown variables. One is then at liberty to equate corresponding terms of the two matrices, since each actually represents the same coordinate system. A particular notational scheme is used to distinguish the several rotations performed. Attached to each unit vector is a series of primes, the number of which denotes the number of rotations already performed.

The purpose of the first two rotations is to locate the $\bar{i}$ axis so that, when extended, it passes through the CG of the missile at time $t = 0$. This is done by first rotating about $\bar{k}_E$ through the angle A° so that the $\bar{i}'$ axis lies on the same longitude as the missile CG and, secondly, by rotating about $\bar{j}'$ through the angle B°, the latitude of the missile position. These rotations are used in the same order to obtain the position of the azimuthal coordinates; hence, one may use Equation 68 with α and β replaced by A and B, respectively. Thus

$$\begin{bmatrix} \bar{i}'' \\ \bar{j}'' \\ \bar{k}'' \end{bmatrix} \quad \begin{bmatrix} CACB & SACB & -SB \\ -SA & CA & 0 \\ CASB & SASB & CB \end{bmatrix} \quad \begin{bmatrix} \bar{i}_E \\ \bar{j}_E \\ \bar{k}_E \end{bmatrix} \tag{75}$$

The purpose of the next two rotations is to orient the $\bar{i}''$ axis so that it becomes coincident with the missile axis.

In particular, the third rotation is about $\bar{i}''$ through the angle G°, so that after this rotation is performed $-\bar{k}'''$ coincides with the projection of

the missile axis on the $\overline{j'''} - \overline{k'''}$ plane. In matrix form, we have

$$\begin{bmatrix} \overline{i'''} \\ \overline{j'''} \\ \overline{k'''} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & CG & SG \\ 0 & -SG & CG \end{bmatrix} \begin{bmatrix} \overline{i''} \\ \overline{j''} \\ \overline{k''} \end{bmatrix} \tag{76}$$

The fourth rotation is about $\overline{j'''}$ through the angle H° so that $\overline{i''''}$ is finally coincident with the missile axis. Upon examining the geometry of these rotations, H° is seen to be the angle of declination the missile makes with the local vertical $\overline{i''}$.

Mathematically,

$$\begin{bmatrix} \overline{i''''} \\ \overline{j''''} \\ \overline{k''''} \end{bmatrix} = \begin{bmatrix} CH & 0 & -SH \\ 0 & 1 & 0 \\ SH & 0 & CH \end{bmatrix} \begin{bmatrix} \overline{i'''} \\ \overline{j'''} \\ \overline{k'''} \end{bmatrix} \tag{77}$$

Combining all rotations performed results in

$$\begin{bmatrix} \overline{i''''} \\ \overline{j''''} \\ \overline{k''''} \end{bmatrix} = \begin{bmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \\ b_{31} & b_{32} & b_{33} \end{bmatrix} \begin{bmatrix} \overline{i_E} \\ \overline{j_E} \\ \overline{k_E} \end{bmatrix} \tag{78}$$

where

b_{11} CHCBCA - SHSGSA - SHCGSBCA

b_{12} CHCBSA + SHSGCA - SHCGSBSA

b_{13} -CHSB - SHCGCB

b_{21} -CGSA + SGSBCA

b_{22} CGCA + SGSBSA

b_{23} SGCB

b_{31} SHCBCA + CHSGSA + CHCGSBCA

b_{32} SHCBSA - CHSGCA + CHCGSBSA

b_{33} -SHSB + CHCGCB

Although one may feel that a sufficient number of rotations have already been performed to obtain $\overline{i''''}$ coincident with the missile axis, there is

yet no assurance that $\overline{j''''}$ will lie in a horizontal plane, as required. This situation is remedied by performing a fifth rotation about $\overline{i''''}$ through an angle L such that the resultant position of $\overline{j''''}$ lies in the horizontal plane. In matrix form

$$\begin{bmatrix} \overline{i_H} \\ \overline{j_H} \\ \overline{k_H} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & CL & SL \\ 0 & -SL & CL \end{bmatrix} \begin{bmatrix} \overline{i''''} \\ \overline{j''''} \\ \overline{k''''} \end{bmatrix} \tag{79}$$

The magnitude of L is as yet unknown. Substituting Equation 78 into Equation 79 produces the matrix

$$\begin{bmatrix} \overline{i_H} \\ \overline{j_H} \\ \overline{k_H} \end{bmatrix} \begin{bmatrix} b_{11} & b_{12} & b_{13} \\ b_{21}CL + b_{31}SL & b_{22}CL + b_{32}SL & b_{23}CL + b_{33}SL \\ -b_{21}SL + b_{31}CL & -b_{22}SL + b_{32}CL & -b_{23}SL + b_{33}CL \end{bmatrix} \begin{bmatrix} \overline{i_E} \\ \overline{j_E} \\ \overline{k_E} \end{bmatrix} \tag{80}$$

which enables one to convert from (H) to (E) coordinates. Equation 16

$$\begin{bmatrix} \overline{i_H} \\ \overline{j_H} \\ \overline{k_H} \end{bmatrix} \begin{bmatrix} C\theta C\psi & C\theta S\psi & -S\theta \\ -S\psi & C\psi & 0 \\ S\theta C\psi & S\theta S\psi & C\theta \end{bmatrix} \begin{bmatrix} \overline{i_E} \\ \overline{j_E} \\ \overline{k_E} \end{bmatrix} \tag{16}$$

also relates the (H) and (E) coordinates. Since these sets of equations are equivalent at the start of the trajectory, one simply compares corresponding terms, from which one obtains L as well as $\psi(0)$ and $\theta(0)$.

In particular,

$$\sin\theta(0) \quad -b_{13} \tag{81}$$

$$\sin\psi(0) \quad \frac{b_{12}}{\cos\theta(0)}$$

$$\cos\psi(0) \quad \frac{b_{11}}{\cos\theta(0)} \tag{82}$$

and by using

$$b_{23} \cos L + b_{33} \sin L = 0$$
$$b_{22} \cos L + b_{32} \sin L = C\psi \tag{83}$$

one obtains for L

$$\text{Cos } L = \frac{-b_{33}\, C\psi}{b_{23}\, b_{32} - b_{33}\, b_{22}}$$
$$\text{Sin } L = \frac{b_{23}\, C\psi}{b_{22}\, b_{32} - b_{33}\, b_{22}} \tag{84}$$

Although L need not be explicitly used in determining $\psi(0)$, $\theta(0)$, L completely describes the missile orientation at launch.

Since θ was restricted between $\pm 90°$, the quadrants of each angle are uniquely determined in the preceding equations.

The initial linear and angular velocities may be given in terms of either the (E) or the (H) coordinates, although $\vec{\omega}$ is usually most easily expressed in the (H) coordinate system. Perhaps it would be best at this point to illustrate the theory by a brief example.

Let us suppose that it is desired to launch a rocket as pictured in Figure 10 (p 38). We make the following assignments:

A = 30° (Longitude)

B = −40° (Latitude. Note the negative rotation about $\overline{j'}$ for the Northern Hemisphere; hence, the minus sign).

G = 10° (190° − 180° so that the negative $\overline{k'''}$ axis will coincide with the line 0' P).

H = 35° (Angle of declination from the local vertical).

$\psi(0)$ and $\theta(0)$ can then be determined by use of the equations up to and including 84.

Let us now introduce a 5° yaw angle in a plane containing the local vertical, measured towards the north, as pictured in Figure 11 (p 39). From the geometry of Figure 11 (b), one can obtain the components of the missile velocity in the (E) system and use the transformation Equation 17 to obtain V_{XH}, V_{YH}, and V_{ZH}. Ω is simply the spin of the missile as

Diagram Illustrating Initial Conditions (I)

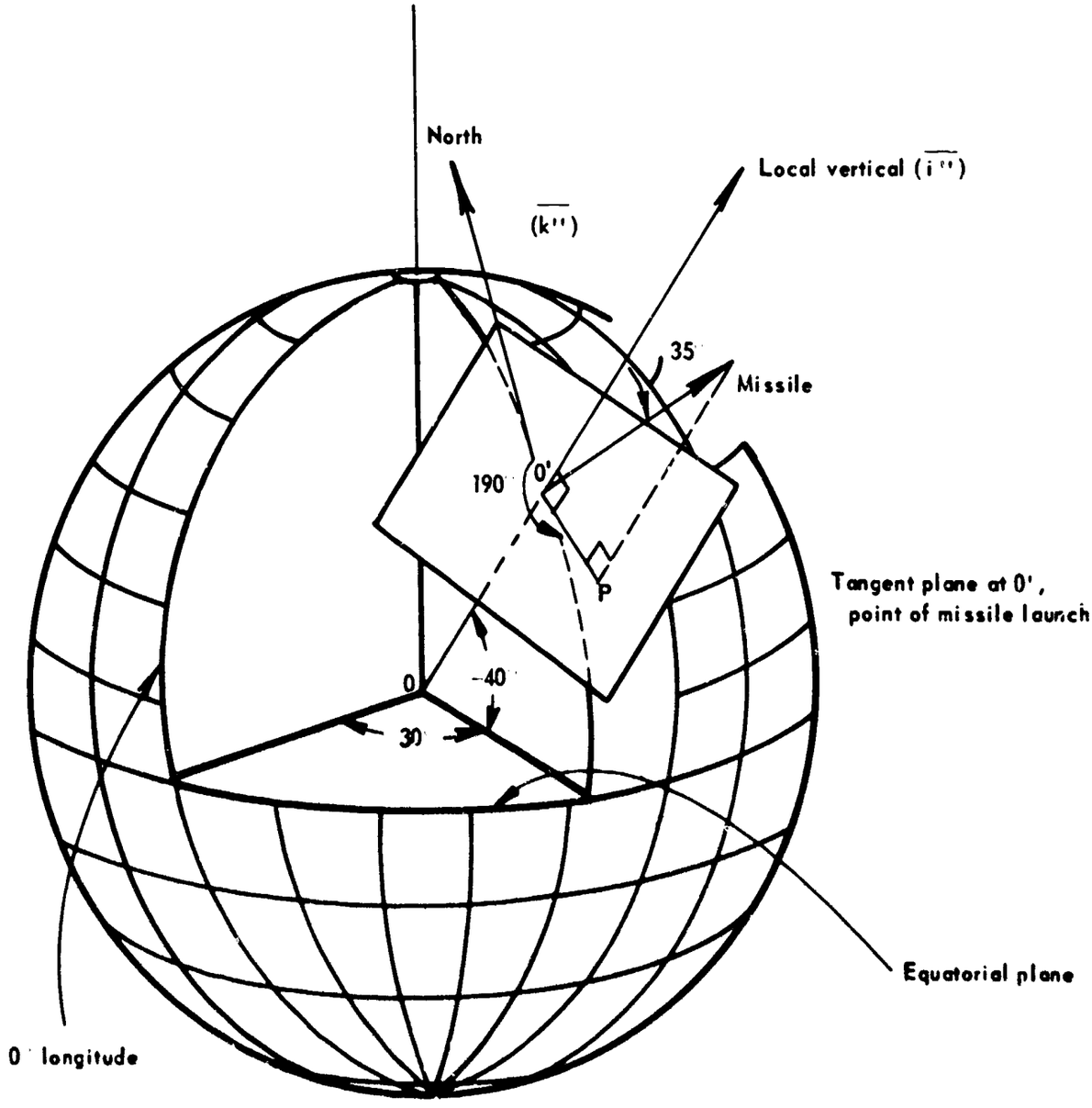


Figure 10

Diagram Illustrating Initial Conditions (II)

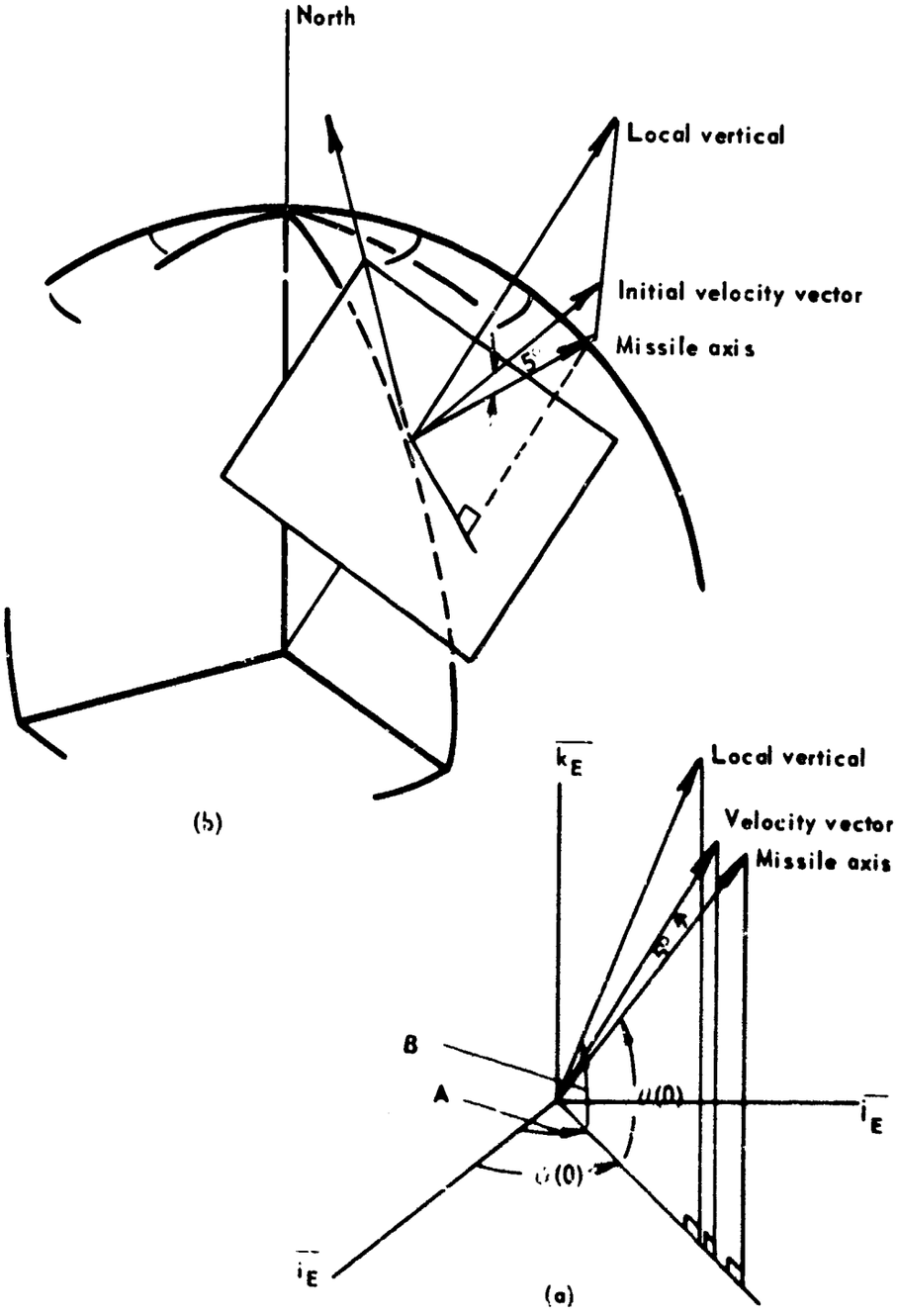


Figure 11

observed in the (H) coordinate system. ω_{YH} and ω_{ZH} are the lateral angular velocities of the missile as observed from the surface of the earth. Sketching the rotations in sequence before numerical values are assigned to these quantities can be helpful.

We will now specialize the results so that they are appropriate to the flat earth case. As has been mentioned previously, the vertical is to be the $\overline{k_E}$ axis, while the $\overline{i_E}$ axis will be taken as downrange. One convenient way of ascertaining initial conditions is to first obtain the $\overline{i''}$ axis coincident with the $\overline{k_E}$ axis; then, with G° and H° known, compute $\psi(0)$ and $\theta(0)$. Referring to Figure 10 (p 38), we replace the 30° with 0° and the −40° with −90° (or 270°). The $\overline{i''}$ axis thus achieves its first desired orientation. Mathematically,

$$A = 0°$$

$$B = 270°$$

Our matrix Equation 78 for arbitrary angles G and H now becomes

$$\begin{bmatrix} \overline{i''''} \\ \overline{j''''} \\ \overline{k''''} \end{bmatrix} = \begin{bmatrix} SHCG & SHSG & CH \\ -SG & CG & 0 \\ -CHCG & -CHSG & SH \end{bmatrix} \begin{bmatrix} \overline{i_E} \\ \overline{j_E} \\ \overline{k_E} \end{bmatrix} \tag{85}$$

When the appropriate quantities are substituted into Equation 83, L becomes zero. Using these results to obtain $\psi(0)$ and $\theta(0)$ for the flat earth case gives the following:

$$\begin{aligned} \psi(0) &= G \\ \theta(0) &= H - 90 \\ L &= 0° \end{aligned} \tag{86}$$

for

$$A = 0°$$

$$B = 270°$$

The other quantities $V_{XH}, \ldots, \omega_{ZH}$ are, as before, treated analogously, although R_{XE}, R_{YE}, and R_{ZE} may (if appropriate) be assigned values of zero.

This concludes the section on initial conditions. More information will be provided on this subject when the alternate set of equations, as derived in the next section, is used.

Singularity Conditions and an Alternate Set of Equations

This section presents an alternate set of equations to be used when singularity conditions are approached and, equally important, the means to convert to this set during flight simulation.

As indicated under the General Equations of Motion, special attention must be given to the equations of motion when θ approaches $+\pi/2$. As had been anticipated, the term $\omega_{ZH}\tan\theta$ in the equations of motion approaches infinity as θ approaches 90°. Furthermore, $\dot{\psi}$ becomes indeterminate when this singular condition is reached. Unfortunately, $\theta = 90°$ occurs whenever the missile axis becomes vertical, the occurrence of which cannot be ignored. To avoid this condition, the (H) coordinate system will be replaced by a new fixed-plane system labelled the (V) coordinates. As outlined in the beginning of the procedure, $\overline{j_V}$ is constrained to lie perpendicular to the $\overline{i_E}$ axis (i.e., to lie in a specified fixed vertical plane) while $\overline{i_V}$ is to be coincident with the missile axis. This change necessitates redefining the Euler angles, and deriving equations of transformation between the two coordinate systems.

To satisfy the conditions imposed upon the (V) coordinate system, the rotations indicated below are made.

1. A rotation about $\overline{i_E}$ of magnitude ψ.

2. A rotation about $\overline{j''}$ of magnitude Θ, so that $\overline{i''} = \overline{i_V}$ becomes the missile axis.

To avoid ambiguity, the following angle restrictions were observed:

$$0 \leq \psi \leq 360°$$

$$0 \leq \Theta \leq 180$$

In matrix form

$$\begin{bmatrix} \overline{i_V} \\ \overline{j_V} \\ \overline{k_V} \end{bmatrix} = \begin{bmatrix} C\Theta & 0 & -S\Theta \\ 0 & 1 & 0 \\ S\Theta & 0 & C\Theta \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & C\psi & S\psi \\ 0 & -S\psi & C\psi \end{bmatrix} \begin{bmatrix} \overline{i_E} \\ \overline{j_E} \\ \overline{k_E} \end{bmatrix} \qquad (87)$$

After multiplying, we have

$$\begin{bmatrix} \overline{i_V} \\ \overline{j_V} \\ \overline{k_V} \end{bmatrix} = \begin{bmatrix} C\Theta & S\Theta S\Psi & -S\Theta C\Psi \\ 0 & C\Psi & S\Psi \\ S\Theta & -C\Theta S\Psi & C\Theta C\Psi \end{bmatrix} \begin{bmatrix} \overline{i_E} \\ \overline{j_E} \\ \overline{k_E} \end{bmatrix} \tag{88}$$

Note that, as required, $\overline{j_V}$ has no component along $\overline{i_E}$.

Rather than introduce new notation, a prime over a given "omega" will denote that the particular angular velocity that was previously referred to in (H) coordinates will now refer to the (V) coordinates.

Continuing in a manner analogous to the horizontal coordinate system, we can write

$$\vec{\omega}' = \dot{\Psi}\,\overline{i_E} + \dot{\Theta}\,\overline{j'} = \dot{\Psi}\,[C\Theta\,\overline{i_V} + S\Theta\,\overline{k_V}] + \dot{\Theta}\,\overline{j_V} \tag{89}$$

or in component form

$$\begin{aligned} \omega'_{XV} &= \dot{\Psi}C\Theta \\ \omega'_{YV} &= \dot{\Theta} \\ \omega'_{ZV} &= \dot{\Psi}S\Theta \end{aligned} \tag{90}$$

and for later purposes

$$\omega'_{XV} = \frac{\omega'_{ZV}}{\tan\Theta} \tag{91}$$

Note that, when $\Theta = 0°$, a singular condition is again present; however, investigating the geometry of the rotations for Ψ and Θ will show that this occurs when the missile axis is horizontal rather than vertical, a situation for which the previous set of equations is applicable.

The angular velocity of the earth in the new coordinates becomes

$$\vec{\Omega} = \Omega\overline{k_E} = [-\Omega S\Theta C\Psi]\overline{i_V} + \Omega S\Psi\overline{j_V} + \Omega C\Theta C\Psi\overline{k_V} \tag{92}$$

Letting $\vec{\Omega}'$ be the angular velocity of the (M) coordinates relative to (V) coordinates, we can write

$$(\vec{\omega}'_T) = \vec{\Omega}' + \dot{\vec{\omega}}' = \vec{\Omega}' + \vec{\omega}' + \dot{\vec{\Omega}} \tag{93}$$

or, in component form (after using Equations 91 and 92),

$$\begin{aligned}
(\omega'_T)_{XV} &= \Omega'_{XV} + \frac{\omega'_{ZV}}{\tan\Theta} - \Omega S\Theta C\Psi \\
(\omega'_T)_{YV} &= \omega'_{YV} + \Omega S\Psi \\
(\omega'_T)_{ZV} &= \omega'_{ZV} + \Omega C\Theta C\Psi
\end{aligned} \tag{94}$$

Differentiating Equation 94 yields

$$\begin{aligned}
(\dot{\omega}'_T)_{XV} &= \dot{\Omega}'_{XV} - \frac{\omega'_{ZV}}{\sin^2\Theta}\dot{\Theta} + \frac{\dot{\omega}'_{ZV}}{\tan\Theta} - \Omega\,[C\Theta C\Psi\dot{\Theta} - S\Theta S\Psi\dot{\Psi}] \\
(\dot{\omega}'_T)_{YV} &= \dot{\omega}'_{YV} + \Omega\, C\Psi\dot{\Psi} \\
(\dot{\omega}'_T)_{ZV} &= \dot{\omega}'_{ZV} - \Omega\,[S\Theta C\Psi\dot{\Theta} + C\Theta S\Psi\dot{\Psi}]
\end{aligned} \tag{95}$$

One may note that Equations 8 and 12, the basic component equations of motion (as well as most forces and moments), were derived without first specifying the orientation of the (H) coordinate system; hence, they are equally valid in the (V) coordinates. Therefore, using the "omegas" and the "omega-dots" derived above, along with the new transformation equations, produces an alternate set of equations valid for vertical orientations of the missile.

Since both sets of equations have their own singularity conditions, it it necessary not only to have both sets of equations available but it should also be possible to convert from one system to the other as the need arises. It is the purpose of the next few equations to establish this conversion.

By definition the $\bar{i}$ axes of the (H) and (V) coordinates are coincident; hence, if an additional rotation is made about the $\overline{i_V}$ axis so that $\overline{j_V}$ rotated lies in a horizontal plane (as determined by $\overline{i_E}$ and $\overline{j_E}$), then the (V) coordinate system will become coincident with the (H) coordinates.

Let us denote this additional rotation by Φ, whose magnitude is yet unknown. In matrix form

$$\begin{bmatrix} \overline{i_H} \\ \overline{j_H} \\ \overline{k_H} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & C\Phi & S\Phi \\ 0 & -S\Phi & C\Phi \end{bmatrix} \begin{bmatrix} \overline{i_V} \\ \overline{j_V} \\ \overline{k_V} \end{bmatrix} \tag{96}$$

Combining the inverses of Equations 88 and 96 eventually produces

$$\begin{bmatrix} \overline{i_E} \\ \overline{j_E} \\ \overline{k_E} \end{bmatrix} = \begin{bmatrix} C\Theta & S\Theta S\Phi & S\Theta C\Phi \\ S\Theta S\Psi & C\Psi C\Phi - C\Theta S\Psi S\Phi & -C\Psi S\Phi - C\Theta S\Psi C\Phi \\ -S\Theta C\Psi & S\Psi C\Phi + C\Theta C\Psi S\Phi & -S\Psi S\Phi + C\Theta C\Psi C\Phi \end{bmatrix} \begin{bmatrix} \overline{i_H} \\ \overline{j_H} \\ \overline{k_H} \end{bmatrix} \tag{97}$$

Comparing this with Equation 17 yields explicit expressions for obtaining Ψ, Θ, and Φ in terms of ψ and θ, as follows:

$$\text{Cos}\ \Theta = \cos\theta \cos\psi \tag{98a}$$

$$\text{Sin}\ \Psi = \frac{\cos\theta \sin\psi}{\sin\Theta}; \quad \cos\Psi = \frac{\sin\theta}{\sin\Theta} \tag{98b}$$

and

$$\text{Sin}\ \Phi = \frac{-\sin\psi}{\sin\Theta} \quad ; \quad \cos\Phi = \frac{\sin\theta\cos\psi}{\sin\Theta} \tag{98c}$$

Again no ambiguity results from the restrictions made earlier on Θ (i.e., the quadrants of Θ, Ψ, and Φ are uniquely determined).

Knowing Ψ, Θ, and Φ and, of course, ψ, θ, $\dot{\psi}$, and $\dot{\theta}$, we are now in a position to determine the angular rates of change, $\dot{\Psi}$ and $\dot{\Theta}$.

This is accomplished by differentiating Equations 98a and 98b, producing

$$\dot{\Theta} \quad \frac{S\theta C\psi\dot{\theta} + C\theta S\psi\dot{\psi}}{S\Theta} \tag{99}$$

and

$$\dot{\Psi} \quad \frac{S\Theta[-S\theta S\psi\dot{\theta} + C\theta C\psi\dot{\psi}] - C\theta S\psi C\Theta\dot{\Theta}}{C\Psi \sin^2\Theta} \tag{100}$$

If Ψ happens to be either 90° or 270°, one can differentiate the second equation (98b), obtaining instead of Equation 100

$$\dot{\Psi} = \frac{-S\Theta C\theta\dot{\theta} + S\theta C\Theta\dot{\Theta}}{S\Psi \sin^2\Theta} \tag{101}$$

to eliminate the singularity.

Once these derivatives are known it follows, from Equation 90, that

$$\begin{aligned} \omega'_{ZV} &= \dot{\Psi} \sin \Theta \\ \omega'_{YV} &= \dot{\Theta} \end{aligned} \tag{102}$$

To obtain Ω'_{XV}, one need only equate the two expressions for $(\omega_T)_X$, obtaining

$$\Omega'_{XV} + \frac{\omega'_{ZV}}{\tan \Theta} - \Omega S\Theta C\Psi = \Omega_{XH} - \omega_{ZH} \tan \theta - \Omega S\theta \tag{103}$$

or, since $S\Theta C\Psi = S\theta$,

$$\Omega'_{XV} = \Omega_{XH} - \omega_{ZH} \tan \theta - \frac{\omega'_{ZV}}{\tan \Theta} \tag{104}$$

To complete the transformation, Equation 96 again comes into use. We have for the invariant vectors (force, moment, and velocity)

$$\begin{aligned} F'_{XV} &= F_{XH} & L'_{XV} &= L_{XH} \\ F'_{YV} &= F_{YH} C\Phi - F_{ZH} S\Phi & L'_{YV} &= L_{YH} C\Phi - L_{ZH} S\Phi \\ F'_{ZV} &= F_{YH} S\Phi + F_{ZH} C\Phi & L'_{ZV} &= L_{YH} S\Phi + L_{ZH} C\Phi \end{aligned} \tag{105}$$

$$\begin{aligned} V'_{XV} &= V_{XH} \\ V'_{YV} &= V_{YH} C\Phi - V_{ZH} S\Phi \\ V'_{ZV} &= V_{YH} S\Phi + V_{ZH} C\Phi \end{aligned}$$

Other equations required for using the fixed vertical plane are

$$V_{XE} = V'_{XV} C\Theta + V'_{ZV} S\Theta$$

$$(\text{Gravity force})_{XV} = mg\, S\Theta C\Psi \text{ (flat earth)}$$

$$(\text{Gravity force})_{XV} = \frac{-mg}{\sqrt{R^2_{XE} + R^2_{YE} + R^2_{ZE}}} \left[R_{XE} C\Theta + R_{YE} S\Theta S\Psi - R_{ZE} S\Theta C\Psi\right] \text{ (Spherical earth)} \tag{106}$$

with similar expressions (tabulated in the Summary of Equations, page 51) for the $\bar{j}$ and $\bar{k}$ directions.

Concerning winds, we use the two matrices shown below

$$\begin{bmatrix} \overline{i_W} \\ \overline{j_W} \\ \overline{k_W} \end{bmatrix} = \begin{bmatrix} C\beta C\alpha & C\beta S\alpha & -S\beta \\ -S\alpha & C\alpha & 0 \\ S\beta C\alpha & S\beta S\alpha & C\beta \end{bmatrix} \begin{bmatrix} C\Theta & 0 & S\Theta \\ S\Theta S\Psi & C\Psi & -C\Theta S\Psi \\ -S\Theta C\Psi & S\Psi & C\Theta C\Psi \end{bmatrix} \begin{bmatrix} \overline{i_V} \\ \overline{j_V} \\ \overline{k_V} \end{bmatrix} \tag{107}$$

which produces, in a manner analogous to a previous result,

$$(V'_W)_{XV} = (V_W)_{YW} A'_{21} + (V_W)_{ZW} A'_{31}$$

where now

$$\begin{aligned} A'_{21} &= -S\alpha C\Theta + C\alpha S\Theta S\Psi \\ A'_{31} &= S\beta C\alpha C\Theta + S\beta S\alpha S\Theta S\Psi - C\beta S\Theta C\Psi \end{aligned} \tag{108}$$

This result and the other two components are tabulated in the Summary of Equations (p 51). Needless to say, $(\dot{V}_r)$ is determined from Equation 72 written in terms of the (V) coordinates. Wind for the flat earth case is analogous to that of the (H) coordinates. The resulting equations are also tabulated in the Summary of Equations section.

Before leaving this section, three derivations are yet required. These involve expressing the thrust and thrust moments, specifying new initial conditions in the (V) coordinates, and converting from (V) back to (H). We begin with the thrust equations.

Since Φ is defined as the angle between the $\overline{j_V}$ and $\overline{j_H}$ axis, Figure 8 (p 26) may be modified as follows:

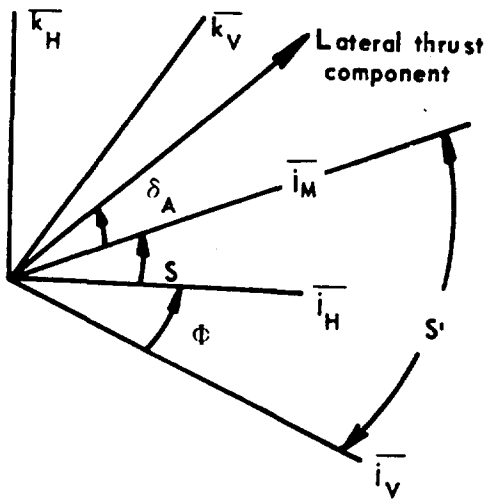


Figure 12

from which it is clear that the thrust components expressed in the new coordinates are given by Equation 58 with S replaced by S'. Thus,

$$T_{XV} = T \cos \delta_T$$

$$T_{YV} = T \sin \delta_T \cos (\delta_A + S') \qquad (109)$$

$$T_{ZV} = T \sin \delta_T \sin (\delta_A + S')$$

At changeover (i.e., when converting from (H) to (V) coordinates), angle S' is defined by

$$S' = \Phi + S \qquad (110)$$

after which the following is used

$$S' = \int_{t\ \text{changeover}}^{t} \Omega' \, dt + S' \Big|_{t\ \text{changeover}}$$

Use of these equations implies that there is no reorientation of the (M) coordinates; hence, all quantities previously referred to these coordinates (such as δ_A, δ_T, r_{XM}, r_{YM}, r_{ZM}) remain constant when changing from (H) to (V) coordinates.

One similarly replaces S by S' in Equation 62 to obtain the thrust moments in the (V) coordinates.

This completes the transformation equations pertinent to the thrust terms.

Initial conditions must be provided in the (V) coordinates, because of the possibility of "vertical launch." Actually, much that has already been derived in the section on initial conditions can be used here. The only real change is that angle L has to be redefined relative to the (V) coordinates. Angles A, B, G, and H are defined as before. Let L' denote this new angle. It is determined by noting that $\overline{j_V}$ is to have no component along $\overline{i_E}$, in place of $\overline{k_E}$. This is effected by equating Equation 80 (with H's replaced by V's) with Equation 88 (instead of with Equation 16 as was done before). Mathematically, we have

$$\cos \Theta(0) = b_{11}$$

$$S\Psi = \frac{b_{12}}{S\Theta} \qquad C\Psi = -\frac{b_{13}}{S\Theta} \tag{112}$$

We have for L'

$$CL' = -\frac{b_{31}\, C\Psi}{b_{21}\, b_{32} - b_{31}\, b_{22}}$$

$$SL' = \frac{b_{21}\, C\Psi}{b_{21}\, b_{32} - b_{31}\, b_{22}} \tag{113}$$

where $b_{11}, \ldots, b_{33}$ are defined as before.

For the flat earth case, again setting A = 0°, B = 270°, we have

$$\cos \Theta(0) = SHCG$$

$$\cos \Psi(0) = -\frac{CH}{S\Theta} \tag{114}$$

$$\sin \Psi(0) = -\frac{SG}{S\Theta}$$

Finally, it is necessary to determine expressions for converting from (V) to (H) coordinates. To obtain these expressions, one can proceed in a manner similar to previous derivations.

An angle Φ' is defined such that when a rotation of this magnitude is performed about $\overline{i_V}$, the $\overline{j_V}$ axis will then lie in the horizontal plane (i.e., parallel to the equatorial plane). This rotation expressed in matrix form will yield Equation 96, with Φ replaced by Φ'.

Postmultiplying the inverse of Equation 88 by the inverse of Equation 96 produces

$$\begin{bmatrix} \overline{i_E} \\ \overline{j_E} \\ \overline{k_E} \end{bmatrix} = \begin{bmatrix} C\Theta & S\Theta S\Phi' & S\Theta C\Phi' \\ S\Theta S\Psi & C\Psi C\Phi' - C\Theta S\Psi S\Phi' & -C\Psi S\Phi' - C\Theta S\Psi C\Phi' \\ -S\Theta C\Psi & S\Psi C\Phi' + C\Theta C\Psi S\Phi' & -S\Phi' S\Psi + C\Theta C\Phi' C\Psi \end{bmatrix} \begin{bmatrix} \overline{i_H} \\ \overline{j_H} \\ \overline{k_H} \end{bmatrix} \tag{115}$$

Comparing Equation 115 with Equation 17 below

$$\begin{bmatrix} \overline{i_E} \\ \overline{j_E} \\ \overline{k_E} \end{bmatrix} = \begin{bmatrix} C\theta C\psi & -S\psi & S\theta C\psi \\ C\theta S\psi & C\psi & S\theta S\psi \\ -S\theta & 0 & C\theta \end{bmatrix} \begin{bmatrix} \overline{i_H} \\ \overline{j_H} \\ \overline{k_H} \end{bmatrix} \tag{17}$$

results in the following transformations:

$$\begin{aligned} S\theta &= S\Theta C\Psi \\ C\psi &= \frac{C\Theta}{C\theta} & S\psi &= \frac{S\Theta S\Psi}{C\theta} \\ S\Phi' &= \frac{-S\psi}{S\Theta} & C\Phi' &= \frac{S\theta C\psi}{S\Theta} \end{aligned} \tag{116}$$

No ambiguity results, since $-\frac{\pi}{2} \le \theta \le \frac{\pi}{2}$.

To obtain the angular velocities, one differentiates Equation 116, obtaining

$$\dot{\theta} = \frac{C\Theta C\Psi\dot{\Theta} - S\Theta S\Psi\dot{\Psi}}{C\theta} \tag{117}$$

$$\dot{\psi} = \frac{C\theta[C\Theta S\Psi\dot{\Theta} + S\Theta C\Psi\dot{\Psi}] + S\Theta S\Psi S\theta\dot{\theta}}{C\psi \cos^2\theta} \tag{118}$$

If ψ is 90° or 270°, one can use

$$\dot{\psi} = \frac{C\theta S\Theta\dot{\Theta} - C\Theta S\theta\dot{\theta}}{S\psi \cos^2\theta} \tag{119}$$

Equation 104 completes the transformation of the angular velocities.

Solving for Ω_{XH} leads to

$$\Omega_{XH} = \Omega'_{XV} + \omega_{ZH} \tan \theta + \frac{\omega'_{ZV}}{\tan \Theta} \tag{120}$$

where both ω_{YH} and ω_{ZH} are defined by Equation 23.

When Equation 96 is used, the forces, moments, and velocities can be expressed in (H) coordinates, given their representations in (V) coordinates. These auxiliary equations are tabulated in the following section of this report.

This, therefore, completes the transformation from the (V) to the (H) system. A complete tabulation is presented in the following section.

Summary of Equations

The equations of motion as derived in the preceding five sections are summarized below. In the writing of the equations, the primes have been dropped when working in the (V) system. No ambiguity should result.

Force Equations (H-Coordinates)

$$-\rho d^2 (V_r)^2 k_{DA} - G_{XH} + T \cos \delta_T = m[\dot{V}_{XH} + \omega_{YH} V_{ZH} - \{\omega_{ZH} + \Omega C\theta\} V_{YH}]$$

$$-\rho d^2 (V_r)^2 k_N \mathbf{V}^{-1} (V_r)_{YH} - \rho d^3 (\omega_T)_{XH} (V_r) k_F \mathbf{V}^{-1} (V_r)_{ZH} - G_{YH} + T \sin \delta_T \cos (\delta_A + S)$$

$$= m [\dot{V}_{YH} + \{\omega_{ZH} + \Omega C\theta\} V_{XH} - \{-\omega_{ZH} \tan \theta - \Omega S\theta\} V_{ZH}]$$

$$-\rho d^2 (V_r)^2 k_N \mathbf{V}^{-1} (V_r)_{ZH} + \rho d^3 (\omega_T)_{XH} (V_r) k_F \mathbf{V}^{-1} (V_r)_{YH} - G_{ZH} + T \sin \delta_T \sin (\delta_A + S)$$

$$= m [\dot{V}_{ZH} + \{-\omega_{ZH} \tan \theta - \Omega S\theta\} V_{YH} - \omega_{YH} V_{XH}]$$

Moment Equations (H-Coordinates)

$$\rho d^4 (V_r) k\phi p (\omega_T)_{XH} + \rho d^3 (V_r)^2 k\phi + T \sin \delta_T [r_{YM} \sin \delta_A - r_{ZM} \cos \delta_A] + KT$$

$$= I_X [\dot{\Omega}_{XH} - \dot{\omega}_{ZH} \tan \theta - \omega_{ZH} \sec^2 \theta \omega_{YH} - \Omega C\theta \omega_{YH}]$$

$$\rho d^3 (V_r)^2 k_N \mathbf{V}^{-1} (V_r)_{ZH} (r_C - \lambda_N) - \rho d^3 (\omega_T)_{XH} (V_r) k_F \mathbf{V}^{-1} (V_r)_{YH} (r_C - \lambda_M) - \rho d^4 (V_r) k_H \omega_{YH}$$

$$+ T[r_{ZM} C\delta_T - r_{XM} S\delta_T S\delta_A] CS - T[r_{XM} S\delta_T C\delta_A - r_{YM} C\delta_T] SS$$

$$= I_Y \dot{\omega}_{YH} + [\{\omega_{ZH} + \Omega C\theta\} I_X \{\Omega_{XH} - \omega_{ZH} \tan \theta - \Omega S\theta\} - \{-\omega_{ZH} \tan \theta - \Omega S\theta\} I_Y \{\omega_{ZH} + \Omega C\theta\}]$$

$$-\rho d^2 (V_r)^2 k_N \mathbf{V}^{-1} (V_r)_{YH} (r_C - \lambda_N) - \rho d^3 (\omega_T)_{XH} (V_r) k_F \mathbf{V}^{-1} (V_r)_{ZH} (r_C - \lambda_M) - \rho d^4 (V_r) k_H \{\omega_{ZH} + \Omega C\theta\}$$

$$+ T[r_{ZM} C\delta_T - r_{XM} S\delta_T S\delta_A] SS + T[r_{XM} S\delta_T C\delta_A - r_{YM} C\delta_T] CS = I_Y [\dot{\omega}_{ZH} - \Omega S\theta \omega_{YH}]$$

$$+ \{-\omega_{ZH} \tan \theta - \Omega S\theta\} I_Y \omega_{YH} - \omega_{YH} I_X \{\Omega_{XH} - \omega_{ZH} \tan \theta - \Omega S\theta\}]$$

where

$$V^{-1} = \frac{1}{\sqrt{(V_r)^2_{YH} + (V_r)^2_{ZH}}} ;$$

$$(\omega_T)_{XH} = \Omega_{XH} - \omega_{ZH} \tan\theta - \Omega S\theta$$

$$(V_r) = \sqrt{(V_r)^2_{XH} + (V_r)^2_{YH} + (V_r)^2_{ZH}}$$

G, The Gravitational Attraction, Is

1. For The Spherical Earth Case

$$G_{XH} = \frac{mg}{\sqrt{R^2_{XE} + R^2_{YE} + R^2_{ZE}}} \left[R_{XE}\, C\theta\, C\psi + R_{YE}\, C\theta\, S\psi - R_{ZE}\, S\theta \right]$$

$$G_{YH} = \frac{mg}{\sqrt{R^2_{XE} + R^2_{YE} + R^2_{ZE}}} \left[-R_{XE}\, S\psi + R_{YE}\, C\psi \right]$$

$$G_{ZH} = \frac{mg}{\sqrt{R^2_{XE} + R^2_{YE} + R^2_{ZE}}} \left[R_{XE}\, S\theta\, C\psi + R_{YE}\, S\theta\, S\psi + R_{ZE}\, C\theta \right]$$

II. For The Flat Earth Case

$$G_{XH} = -mg\, S\theta$$

$$G_{YH} = 0$$

$$G_{ZH} = +mg\, C\theta$$

where

$$g = g_o \left[\frac{R_{earth}}{h}\right]^2 \qquad h = \begin{cases} \sqrt{R_{XE}^2 + R_{YE}^2 + R_{ZE}^2} & \text{(Spherical Earth)} \\ R_{ZE} & \text{(Flat Earth)} \end{cases}$$

Equations For Wind (H-Coordinates)

Spherical Earth

$$(V_r)_{XH} = V_{XH} - \Omega\, C\theta\, R_{YH} - (V_W)_{XH}$$

$$(V_r)_{YH} = V_{YH} - \Omega\, [C\theta\, R_{XH} - S\theta\, R_{ZH}] - (V_W)_{YH}$$

$$(V_r)_{ZH} = V_{ZH} + \Omega\, S\theta\, R_{YH} - (V_W)_{ZH}$$

$$(V_W)_{XH} = (V_W)_{YW}\, a_{21} + (V_W)_{ZW}\, a_{31}$$

$$(V_W)_{YH} = (V_W)_{YW}\, a_{22} + (V_W)_{ZW}\, a_{32}$$

$$(V_W)_{ZH} = (V_W)_{YW}\, a_{23} + (V_W)_{ZW}\, a_{33}$$

$$a_{21} = -S\alpha\, C\theta\, C\psi + C\alpha\, C\theta\, S\psi$$

$$a_{31} = S\beta\, C\alpha\, C\theta\, C\psi + S\beta\, S\alpha\, C\theta\, S\psi - C\beta\, S\theta$$

$$a_{22} = S\alpha\, S\psi + C\alpha\, C\psi$$

$$a_{32} = -S\beta\, C\alpha\, S\psi + S\beta\, S\alpha\, C\psi$$

$$a_{23} = -S\alpha\, S\theta\, C\psi + C\alpha\, S\theta\, S\psi$$

$$a_{33} = S\beta\, C\alpha\, S\theta\, C\psi + S\beta\, S\alpha\, S\theta\, S\psi + C\beta\, C\theta$$

$$(V_W)_{YW} = |V_W|\, S\phi$$

$$(V_W)_{ZW} = |V_W|\, C\phi$$

$|V_W|$, ϕ given as function of altitude

$$R_{XH} = R_{XE}\, C\theta\, C\psi + R_{YE}\, C\theta\, S\psi - R_{ZE}\, S\theta$$

$$R_{YH} = -R_{XE}\, S\psi + R_{YE}\, C\psi$$

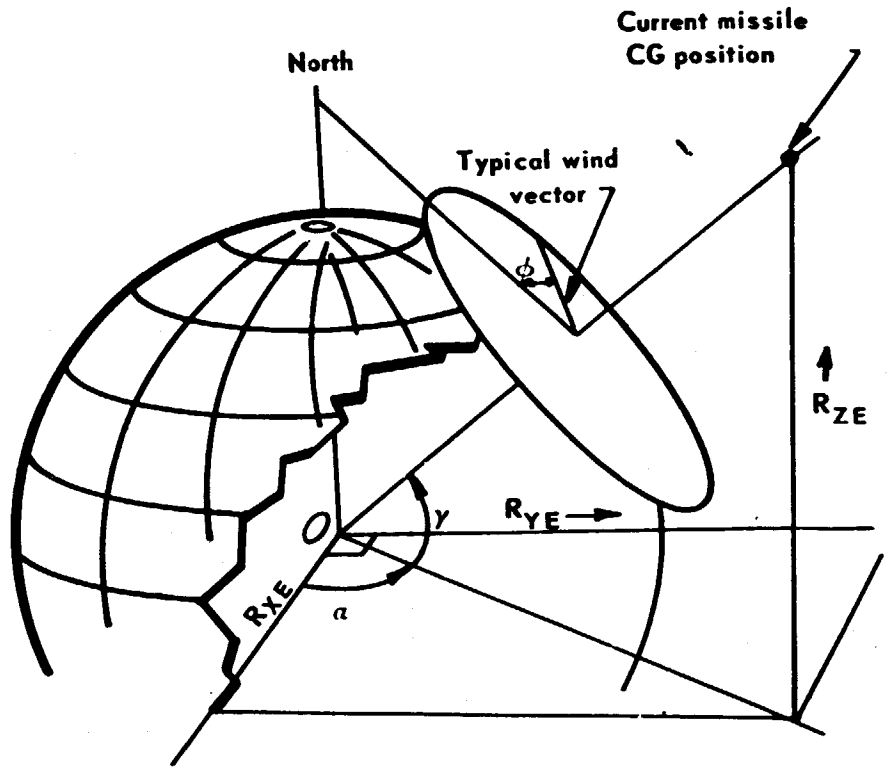
$$R_{ZH} = R_{XE}\, S\theta\, C\psi + R_{YE}\, S\theta\, S\psi + R_{ZE}\, C\theta$$

$$\alpha = \tan^{-1}\left\{\frac{R_{YE}}{R_{XE}}\right\}$$

$$\gamma = \tan^{-1}\left\{\frac{R_{ZE}}{\sqrt{R_{XE}^2 + R_{YE}^2}}\right\}$$

$$\beta = -\gamma$$

To determine appropriate wind profile



where if

$$R_{XE} > 0 \left\{ \begin{array}{ll} R_{YE} > 0 & 0 \leq \alpha < \pi/2 \\ R_{YE} < 0 & 3\pi/2 \leq \alpha < 2\pi \end{array} \right.$$

$$R_{XE} < 0 \left\{ \begin{array}{ll} R_{YE} > 0 & \pi/2 \leq \alpha < \pi \\ R_{YE} < 0 & \pi \leq \alpha < 3\pi/2 \end{array} \right.$$

and

$$-\pi/2 \leq \gamma \leq \pi/2$$

Equations For Wind – Flat Earth

$$(V_r)_{XH} = [V_{XE} - (V_W)_{XE}]\, C\theta\, C\psi + [V_{YE} - (V_W)_{YE}]\, C\theta\, S\psi - [V_{ZE} - (V_W)_{ZE}]\, S\theta$$

$$(V_r)_{YH} = -[V_{XE} - (V_W)_{XE}]\, S\psi + [V_{YE} - (V_W)_{YE}]\, C\psi$$

$$(V_r)_{ZH} = [V_{XE} - (V_W)_{XE}]\, S\theta\, C\psi + [V_{YE} - (V_W)_{YE}]\, S\theta\, S\psi + [V_{ZE} - (V_W)_{ZE}]\, C\theta$$

Equations For Euler Angles

$$\dot{\theta} = \omega_{YH}$$

$$\dot{\psi} = \omega_{ZH} \sec\theta$$

$$\dot{S} = \Omega_{XH}$$

Equations of Transformation

$$V_{XE} = V_{XH}\, C\theta\, C\psi - V_{YH}\, S\psi + V_{ZH}\, S\theta\, C\psi$$

$$V_{YE} = V_{XH}\, C\theta\, S\psi + V_{YH}\, C\psi + V_{ZH}\, S\theta\, S\psi$$

$$V_{ZE} = -\, V_{XH}\, S\theta + V_{ZH}\, C\theta$$

Equations For Initial Conditions

$$S\theta(0) = -\, b_{13}$$

$$S\psi(0) = \frac{b_{12}}{C\theta(0)} \qquad C\psi(0) = \frac{b_{11}}{C\theta(0)}$$

$$CL(0) = -\, \frac{b_{33}\, C\psi}{\Delta} \qquad SL(0) = \frac{b_{23}\, C\psi}{\Delta}$$

where

$$\Delta = b_{23}\, b_{32} - b_{33}\, b_{22}$$

$$b_{12} = CH\, CB\, SA + SH\, SG\, CA - SH\, CG\, SB\, SA$$

$$b_{31} = SH\, CB\, CA + CH\, SG\, SA + CH\, CG\, SB\, CA$$

$$b_{11} = CH\, CB\, CA - SH\, SG\, SA - SH\, CG\, SB\, CA$$

$$b_{21} = -\, CG\, SA + SG\, SB\, CA$$

$$b_{22} = CG\, CA + SG\, SB\, SA$$

$$b_{32} = SH\, CB\, SA - CH\, SG\, CA + CH\, CG\, SB\, SA$$

A, B, G, H given initially

$$b_{13} = -CHSB - SHCGCB$$

$$b_{23} = SGCB$$

$$b_{33} = -SHSB + CHCGCB$$

For Flat Earth

$$\psi(0) = G; \qquad \theta(0) = H - 90^\circ$$

$$A = L(0) = 0 \qquad B = 270^\circ$$

Step Equations

$$R_{XE} = \int_T^t V_{XE}\, dt + R_{XE}|_T$$

$$R_{YE} = \int_T^t V_{YE}\, dt + R_{YE}|_T$$

$$R_{ZE} = \int_T^t V_{ZE}\, dt + R_{ZE}|_T$$

$$\theta = \int_T^t \dot{\theta}\, dt + \theta|_T$$

$$\psi = \int_T^t \dot{\psi}\, dt + \psi|_T$$

$$S = \int_T^t \Omega_{XH}\, dt + S|_T$$

where

$T = 0$ if trajectory starts in H-Coordinates, $T = T_{changeover}$ otherwise.

In all equations

$$-\frac{\pi}{2} \leq \theta \leq \frac{\pi}{2}$$

$$0 \leq \psi < 2\pi$$

Equations of Transformation (H) → (V)

$$C\Theta = C\theta C\psi \qquad 0 \leq \Theta \leq \pi$$

$$S\Psi = \frac{C\theta S\psi}{S\Theta}\ ; \qquad C\Psi = \frac{S\theta}{S\Theta}$$

$$S\Phi = -\frac{S\psi}{S\Theta}\ ; \qquad C\Phi = \frac{S\theta C\psi}{S\Theta}$$

$$\dot{\Theta} = \frac{S\theta C\psi\dot{\theta} + C\theta S\psi\,\dot{\psi}}{S\Theta}$$

$$\dot{\Psi} = \frac{S\Theta\,[-S\theta S\psi\dot{\theta} + C\theta C\psi\dot{\psi}] - C\theta S\psi C\Theta\dot{\Theta}}{C\Psi \sin^2\Theta}$$

If $\Psi = \frac{\pi}{2}$ or $\frac{3\pi}{2}$, use the following

$$\dot{\Psi} = -\frac{S\Theta C\theta\,\dot{\theta} + S\theta C\Theta\,\dot{\Theta}}{S\Psi \sin^2\Theta}$$

$$\omega_{ZV} = \dot{\Psi}\,S\Theta$$

$$\omega_{YV} = \dot{\Theta}$$

$$\Omega_{XV} = \Omega_{XH} - \omega_{ZH}\tan\theta - \omega_{ZV}\cot\Theta$$

$$F_{XV} = F_{XH}$$

$$F_{YV} = F_{YH}\,C\Phi - F_{ZH}\,S\Phi$$

$$F_{ZV} = F_{YH}\,S\Phi + F_{ZH}\,C\Phi$$

$$L_{XV} = L_{XH}$$

$$L_{YV} = L_{YH}\,C\Phi - L_{ZH}\,S\Phi$$

$$L_{ZV} = L_{YH}\,S\Phi + L_{ZH}\,C\Phi$$

$$V_{XV} = V_{XH}$$

$$V_{YV} = V_{YH}\,C\Phi - V_{ZH}\,S\Phi$$

$$V_{ZV} = V_{YH}\,S\Phi + V_{ZH}\,C\Phi$$

δ_T, δ_A, r_{XM}, r_{YM}, r_{ZM} remain the same

$$T_{XV} = T_{XH}$$

$$T_{YV} = T_{YH}\,C\Phi - T_{ZH}\,S\Phi$$

$$T_{ZV} = T_{YH}\,S\Phi + T_{ZH}\,C\Phi$$

$$S' = \Phi + S$$

Force Equations (V-Coordinates)

$$-\rho d^2 (V_r)^2 k_{DA} - G_{XV} + T \cos \delta_T = m[\dot{V}_{XV} + \{\omega_{YV} + \Omega S\Psi\} V_{ZV} - \{\omega_{ZV} \cot \Theta - \Omega S\Theta C\Psi\} V_{YV}$$

$$-\rho d^2 (V_r)^2 k_N V^{-1} (V_r)_{YV} - \rho d^3 (\omega_T)_{XV} (V_r) k_F V^{-1} (V_r)_{ZV} - G_{YV} + T \sin \delta_T \cos (\delta_A + S')$$

$$= m [\dot{V}_{YV} + \{\omega_{ZV} + \Omega C\Theta C\Psi\} V_{XV} - \{\omega_{ZV} \cot \Theta - \Omega S\Theta C\Psi\} V_{ZV}]$$

$$-\rho d^2 (V_r)^2 k_N V^{-1} (V_r)_{ZV} + \rho d^3 (\omega_T)_{XV} (V_r) k_F V^{-1} (V_r)_{YV} - G_{ZV} + T \sin \delta_T \sin (\delta_A + S')$$

$$= m [\dot{V}_{ZV} + \{\omega_{ZV} \cot \Theta - \Omega S\Theta C\Psi\} V_{YV} - \{\omega_{YV} + \Omega S\Psi\} V_{XV}]$$

Moment Equations (V-Coordinates)

$$\rho d^4 (V_r) k\phi\rho (\omega_T)_{XV} + \rho d^3 (V_r)^2 k\phi + TS\delta_T [r_{YM} S\delta_A - r_{ZM} C\delta_A] + KT$$

$$= I_X [\dot{\Omega}'_{XV} + \dot{\omega}_{ZV} \cot \Theta - \omega_{ZV} CSC^2\Theta \, \omega_{YV} - \Omega \{C\Theta C\Psi \omega_{YV} - S\Psi \omega_{ZV}\}]$$

$$\rho d^2 (V_r)^2 k_N V^{-1} (V_r)_{ZV} (r_C - \lambda_N) - \rho d^3 (\omega_T)_{XV} (V_r) k_F V^{-1} (V_r)_{YV} (r_C - \lambda_M) - \rho d^4 (V_r) k_H (\omega_T)_{YV}$$

$$+ T [r_{ZM} C\delta_T - r_{XM} S\delta_T S\delta_A] CS' - T] r_{XM} S\delta_T C\delta_A - r_{YM} C\delta_T] SS'$$

$$= I_Y [\dot{\omega}_{YV} + \Omega C\Psi \omega_{ZV} CSC\Theta] + \{\omega_{ZV} + \Omega C\Theta C\Psi\} I_X \{\Omega'_{XV} + \omega_{ZV} \cot \Theta - \Omega S\Theta C\Psi\}$$

$$- \{\omega_{ZV} \cot \Theta - \Omega S\Theta C\Psi\} I_Y \{\omega_{ZV} + \Omega C\Theta C\Psi\}]$$

$$-\rho d^2 (V_T)^2 k_N V^{-1} (V_r)_{YV} (r_C - \lambda_N) - \rho d^3 (\omega_T)_{XV} (V_r) k_F V^{-1} (V_r)_{ZV} (r_C - \lambda_M) - \rho d^4 (V_r) k_H (\omega_T)_{ZV}$$

$$+ T[r_{ZM} C\delta_T - r_{XM} S\delta_T S\delta_A]SS' + T[r_{XM} S\delta_T C\delta_A - r_{YM} C\delta_T]CS'$$

$$= I_Y [\dot{\omega}_{ZV} - \Omega \{S\Theta C\Psi \omega_{YV} + S\Psi \cot\Theta\, \omega_{ZV}\}] + [\{\omega_{ZV} \cot\Theta - \Omega S\Theta C\Psi\} I_Y \{\omega_{YV} + \Omega S\Psi\}$$

$$- \{\omega_{YV} + \Omega S\Psi\} I_X \{\Omega'_{XV} + \omega_{ZV} \cot\Theta - \Omega S\Theta C\Psi\}]$$

where

$$(\omega_T)_{XV} = \Omega'_{XV} + \omega_{ZV} \cot\Theta - \Omega S\Theta C\Psi$$

G, The Gravitational Attraction, Is

1. For The Spherical Earth Case

$$G_{XV} = \frac{mg}{\sqrt{R_{XE}^2 + R_{YE}^2 + R_{ZE}^2}} \left[R_{XE} C\Theta + R_{YE} S\Theta S\Psi - R_{ZE} S\Theta C\Psi \right]$$

$$G_{YV} = \frac{mg}{\sqrt{R_{XE}^2 + R_{YE}^2 + R_{ZE}^2}} \left[R_{YE} C\Psi + R_{ZE} S\Psi \right]$$

$$G_{ZV} = \frac{mg}{\sqrt{R_{XE}^2 + R_{YE}^2 + R_{ZE}^2}} \left[R_{XE} S\Theta - R_{YE} C\Theta S\Psi + R_{ZE} C\Theta C\Psi \right]$$

II. For The Flat Earth Case

$$G_{XV} = -\,mg\,S\Theta\,C\Psi$$

$$G_{YV} = +\,mg\,S\Psi$$

$$G_{ZV} = +\,mg\,C\Theta\,C\Psi$$

where

$$h = \begin{cases} \sqrt{R^2_{XE} + R^2_{YE} + R^2_{ZE}} & \text{(Spherical Earth)} \\ R_{ZV} & \text{(Flat Earth)} \end{cases}$$

Equations For Wind (V-Coordinates)

Spherical Earth

$$(V_r)_{XV} = V_{XV} - [+\,\Omega\,S\Psi R_{ZV} - \Omega\,C\Theta\,C\Psi R_{YV}] - (V_W)_{XV}$$

$$(V_r)_{YV} = V_{YV} - [\Omega\,C\Theta\,C\Psi R_{XV} + \Omega\,S\Theta\,C\Psi R_{ZV}] - (V_W)_{YV}$$

$$(V_r)_{ZV} = V_{ZV} - [-\Omega\,S\Theta\,C\Psi R_{YV} - \Omega\,S\Psi R_{XV}] - (V_W)_{ZV}$$

$$(V_W)_{XV} = (V_W)_{YW}\,a'_{21} + (V_W)_{ZW}\,a'_{31}$$

$$(V_W)_{YV} = (V_W)_{YW}\,a'_{22} + (V_W)_{ZW}\,a'_{32}$$

$$(V_W)_{ZV} = (V_W)_{YW}\,a'_{23} + (V_W)_{ZW}\,a'_{33}$$

where

$$a'_{31} = S\beta\, C\alpha\, C\Theta + S\beta\, S\alpha\, S\Theta\, S\Psi - C\beta\, S\Theta\, C\Psi$$

$$a'_{22} = C\alpha\, C\Psi$$

$$a'_{32} = S\beta\, S\alpha\, C\Psi + C\beta\, S\Psi$$

$$a'_{21} = -S\alpha\, C\Theta + C\alpha\, S\Theta\, S\Psi$$

$$a'_{23} = -S\alpha\, S\Theta - C\alpha\, C\Theta\, S\Psi$$

$$a'_{33} = S\beta\, C\alpha\, S\Theta - S\beta\, S\alpha\, C\Theta\, S\Psi + C\beta\, C\Theta\, C\Psi$$

$(V_W)_{YV}$, $(V_W)_{ZV}$, α, β given as before

$$(V_r) = \sqrt{(V_r)^2_{XV} + (V_r)^2_{YV} + (V_r)^2_{ZV}}$$

$$R_{XV} = R_{XE}\, C\Theta + R_{YE}\, S\Theta\, S\Psi - R_{ZE}\, S\Theta\, C\Psi$$

$$R_{YV} = R_{YE}\, C\Psi + R_{ZE}\, S\Psi$$

$$R_{ZV} = R_{XE}\, S\Theta - R_{YE}\, C\Theta\, S\Psi + R_{ZE}\, C\Theta\, C\Psi$$

Flat Earth

$$(V_r)_{XV} = [V_{XE} - (V_W)_{XE}]\, C\Theta + [V_{YE} - (V_W)_{YE}]\, S\Theta\, S\Psi - [V_{ZE} - (V_W)_{ZE}]\, S\Theta\, C\Psi$$

$$(V_r)_{YV} = [V_{YE} - (V_W)_{YE}]\,C\Psi + [V_{ZE} - (V_W)_{ZE}]\,S\Psi$$

$$(V_r)_{ZV} = [V_{XE} - (V_W)_{XE}]\,S\Theta - [V_{YE} - (V_W)_{YE}]\,C\Theta\,S\Psi + [V_{ZE} - (V_W)_{ZE}]\,C\Theta\,C\Psi$$

Equations For Euler Angles

$$\dot{\Psi} = \omega_{ZV}\,CSC\Theta$$

$$\dot{\Theta} = \omega_{YV}$$

$$\dot{S}' = \Omega'_{XV}$$

Equations of Transformation

$$V_{XE} = V_{XV}\,C\Theta + V_{ZV}\,S\Theta$$

$$V_{YE} = V_{XV}\,S\Theta\,S\Psi + V_{YV}\,C\Psi - V_{ZV}\,C\Theta\,S\Psi$$

$$V_{ZE} = -\,V_{XV}\,S\Theta\,C\Psi + V_{YV}\,S\Psi + V_{ZV}\,C\Theta\,C\Psi$$

Equations For Initial Conditions

$$C\Theta(0) = b_{11}$$

$$S\Psi(0) = \frac{b_{12}}{S\Theta} \qquad C\Psi(0) = -\frac{b_{13}}{S\Theta}$$

$$CL'(0) = -\frac{b_{31}\,C\Psi}{\Delta'} \qquad SL'(0) = \frac{b_{21}\,C\Psi}{\Delta'}$$

where

$$\Delta' = b_{21}\, b_{32} - b_{31}\, b_{22}$$

$$b_{11} = CH\,CB\,CA - SH\,SG\,SA - SH\,CG\,SB\,CA$$

$b_{21}, b_{13}, b_{31}, b_{32}, b_{22}$, A, B, G, H given as before.

For Flat Earth

Set $A = 0^\circ$, $B = 270^\circ$

Step Equations

$$R_{XE} = \int_T^t V_{XE}\, dt + R_{XE}|_T$$

$$R_{YE} = \int_T^t V_{YE}\, dt + R_{YE}|_T$$

$$R_{ZE} = \int_T^t V_{ZE}\, dt + R_{ZE}|_T$$

$$\Theta = \int_T^t \dot{\Theta}\, dt + \Theta|_T$$

$$\Psi = \int_T^t \dot{\Psi}\, dt + \Psi|_T$$

$$S' = \int_T^t \Omega'_{XV}\, dt + S'|_T$$

where

$T = 0$ if trajectory starts in V-Coordinates, $T = T_{changeover}$ otherwise.

Equations of Transformation (V) → (H)

$$S\theta = S\Theta\, C\Psi \qquad -\frac{\pi}{2} \le \theta \le \frac{\pi}{2}$$

$$S\psi = \frac{S\Theta\, S\Psi}{C\theta} \qquad C\psi = \frac{C\Theta}{C\theta}$$

$$S\Phi' = -\frac{S\psi}{S\Theta} \qquad C\Phi' = \frac{S\theta\, C\psi}{S\Theta}$$

$$\dot{\theta} = \frac{C\Theta\, C\Psi\dot{\Theta} - S\Theta\, S\Psi\, \dot{\Psi}}{C\theta}$$

$$\dot{\psi} = \frac{C\theta\,[C\Theta\, S\Psi\dot{\Theta} + S\Theta\, C\Psi\dot{\Psi}] + S\Theta\, S\Psi\, S\theta\,\dot{\theta}}{C\psi\, \cos^2\theta}$$

if $\psi = \frac{\pi}{2}, \frac{3\pi}{2}$ use the following

$$\dot{\psi} = \frac{C\theta\, S\Theta\dot{\Theta} - C\Theta\, S\theta\,\dot{\theta}}{S\psi\, \cos^2\theta}$$

$$\omega_{ZH} = \dot{\psi}\, C\theta$$

$$\omega_{YH} = \dot{\theta}$$

$$\Omega_{XH} = \Omega'_{XV} + \omega_{ZH} \tan\theta + \omega_{ZV} \cot\Theta$$

$$F_{XH} = F_{XV}$$

$$F_{YH} = F_{YV}\, C\Phi' + F_{ZV}\, S\Phi'$$

$$F_{ZH} = -F_{YV}\, S\Phi' + F_{ZV}\, C\Phi'$$

$$L_{XH} = L_{XV}$$

$$L_{YH} = L_{YV}\, C\Phi' + L_{ZV}\, S\Phi'$$

$$L_{ZH} = -L_{YV}\, S\Phi' + L_{ZV}\, C\Phi'$$

$$V_{XH} = V_{XV}$$

$$V_{YH} = V_{YV}\, C\Phi' + V_{ZV}\, S\Phi'$$

$$V_{ZH} = -V_{YV}\, S\Phi' + V_{ZV}\, C\Phi'$$

$\delta_T, \delta_A, r_{XM}, r_{YM}, r_{ZM}$ remain the same

$$T_{XH} = T_{XV}$$

$$T_{YH} = T_{YV}\, C\Phi' + T_{ZV}\, S\Phi'$$

$$T_{ZH} = -T_{YV}\, S\Phi' + T_{ZH}\, C\Phi'$$

$$S = S' - \Phi'$$

In all equations

$$0 \le \Theta \le \pi \qquad 0 \le \Psi \le 2\pi$$

RESULTS AND DISCUSSION

The final equations resulting from the derivation are tabulated in the preceding section. There are, however, many aspects of the equations that have not been discussed. It is the purpose of this section to consider briefly some of these aspects and limitations and to mention what is involved in incorporating certain refinements in the equations as presently derived.

One might begin by noting that a typical rocket trajectory consists of several flight phases. Explicitly, these phases, typical for a two-stage rocket, may be tabulated as follows:

1. Acceleration of booster and main stage
2. Coasting of booster and main stage
3. Separation of booster and main stage
4. Coasting of main stage
5. Acceleration of main stage
6. Free flight of main stage.

These phases are in direct correspondence with those used in the six-degree-of-freedom digital computer trajectory program at Picatinny Arsenal. The user is thus able to run any or all of these phases. Phase 3 is effected by imparting a short thrust to the main stage, simulating the actual booster separation.

Since the aerodynamic coefficients, centers of gravity, pressure, etc. are not necessarily identical for all phases, several sets of such data are often required to simulate a complete trajectory. The equations in the preceding section of this report are used for all phases.

To numerically solve the differential equations of motion, a modification of the third order Runge-Kutta technique is used (Runge-Kutta-Gill). This numerical scheme appears to be adequate for both low- and high-spin projectiles, although inclusion of thrust malalignments remains to be investigated. As is indicated in the Introduction, however, it appeared that this numerical scheme was not satisfactory when high angular rates of change were implicit in the equations themselves. This condition pertains to high-spin projectiles for which the fixed-plane coordinate system is rigidly

attached to the missile (as was originally done). This is an advantage of the present coordinate system.

Other aspects of these equations, such as the use of thrust modification factors, the atmospheric model (ARDC Atmosphere of 1959), etc., will be discussed in a forthcoming report.

Some of the refinements of the equations are discussed below.

1. Introduction of an ellipsoidal earth model. This modification has far-reaching effects on the equations. For example, the direction and magnitude of the gravitational attraction (consistent with the earth model) would vary with missile position. Additional coordinates would have to be specified on the surface of the ellipsoid to properly introduce wind data. In addition, an iterative scheme is necessary to compute the altitude of the missile if it is naturally defined as the shortest distance from the missile CG to the earth model. Initial conditions would also be modified accordingly. For increased ranges and accuracy requirements of trajectory simulation, the ellipsoidal earth refinement may well become necessary.

2. Treatment of asymmetrical missiles. Asymmetrical missiles can arise from two sources. The more severe situation is present if the external missile configuration does not possess rotational symmetry. For this case, the transverse moments of inertia could vary about the fixed-plane coordinates as the missile rotates. Also, specific data relating the resultant point of application of the aerodynamic forces would have to be provided for accurate results. A second asymmetrical condition can occur for externally symmetrical missiles when the missile CG is offset by a prescribed amount from the longitudinal axis of the missile. This obviously modifies the lever arms in the moment equations and may also introduce time-dependent moments and products of inertia.

3. Other refinements could include guidance and launcher effects, both of which can play prominent roles in trajectory analysis.

In addition to the limitations listed earlier, two assumptions were implicit in the derivation. The first is that the motion of the earth about the sun was neglected; for most sub-orbital trajectories this phenomenon can be ignored. The second assumption is that the rate of change of inertias was neglected (i.e., $\dot{I}$ terms during the thrusting periods). For missiles

possessing excessively high burning rates with large angular velocity components, the magnitude of these terms should be investigated.

A final word about the equations is that there is no estimate of the dispersion of the missile. This requires the computation of several trajectories, each for a slightly different initial condition, with appropriate statistical combinations of the various ranges and deflections from a fixed standard.

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APPENDIX A

Rotating coordinate systems

It is the purpose of this Appendix to express the rate of change of a vector relative to fixed coordinates, in terms of rotating coordinates. Briefly, this analysis answers the following question: "Given a particle whose motion is known, how would the particle's motion appear to someone situated on a coordinate system that is itself in rotation?"

Obviously, one must know the motion of the rotating coordinates relative to the fixed coordinates. To be explicit, let (I) denote fixed coordinates and (H) a rotating coordinate system, whose origins are both coincident. Further, let $\vec{\omega}$ be the angular velocity of (H) relative to (I). This motion might be as depicted in Figure A-1, where the three dotted discs represent the paths of each axis of the (H) system for the given constant $\vec{\omega}$.

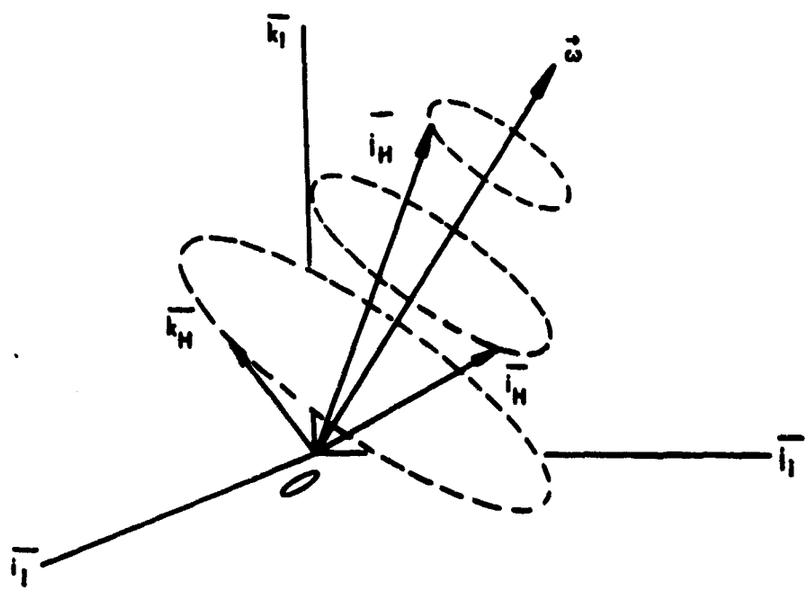

Fig A-1

Let $\vec{R}$ be an arbitrary vector. Clearly, $\vec{R}$ has a representation in the (H) system as follows:

$$\vec{R} = R_{XH}\,\bar{i}_H + R_{YH}\,\bar{j}_H + R_{ZH}\,\bar{k}_H \tag{A-1}$$

To determine the rate of change of $\vec{R}$ relative to the (I) coordinates in terms of the (H) coordinates one must account not only for the changing magnitude of $\vec{R}$, but also for the variation of the unit vectors $\bar{i}_H$, $\bar{j}_H$, $\bar{k}_H$ (which help represent $\vec{R}$) relative to (I). Mathematically:

$$\frac{d_I\vec{R}}{dt} = \dot{R}_{XH}\,\bar{i}_H + \dot{R}_{YH}\,\bar{j}_H + \dot{R}_{ZH}\,\bar{k}_H + R_{XH}\frac{d_I\bar{i}_H}{dt} + R_{YH}\frac{d_I\bar{j}_H}{dt} + R_{ZH}\frac{d_I\bar{k}_H}{dt} \tag{A-2}$$

The first three terms of Equation A-2 define simply how $\vec{R}$ itself is changing independent of any moving coordinates. The latter three terms describe the motion of (H) with respect to (I). It is noteworthy that the I subscript denotes differentiation with respect to the (I) coordinates. To elaborate further, if I were replaced by H then Equation A-2 would read

$$\frac{d_H\vec{R}}{dt} = \dot{R}_{XH}\,\bar{i}_H + \dot{R}_{YH}\,\bar{j}_H + \dot{R}_{ZH}\,\bar{k}_H \tag{A-3}$$

since, for purposes of the differentiation, $\bar{i}_H$, $\bar{j}_H$, $\bar{k}_H$ would be fixed.

The task remains of obtaining expressions for $\frac{d_I\bar{i}_H}{dt}, \ldots, \frac{d_I\bar{k}_H}{dt}$. To do this, let us "isolate" the $\bar{i}$ vector as presented in Figure A-1, and define some new quantities as given in Figure A-2. Further, let

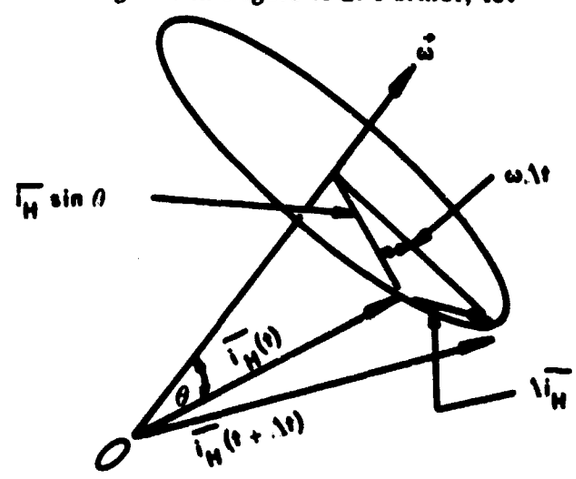

Fig A-2

$\bar{i}_H\,(t + \Delta t)$ denotes the position of $\bar{i}_H$ at time $t + \Delta t$. Clearly, by the definition of a derivative,

$$\frac{d_I\,\bar{i}_H}{dt} = \lim_{\Delta t \to 0} \frac{\bar{i}_H\,(t + \Delta t) - \bar{i}_H\,(t)}{\Delta t} \tag{A-4}$$

and from the geometry of Figure A-2

$$\bar{i}_H\,(t + \Delta t) - \bar{i}_H\,(t) = \Delta\,\bar{i}_H \approx \bar{i}_H\,\sin\theta\,[\omega\Delta t] \tag{A-5}$$

we may write

$$\left|\frac{d_I\,\bar{i}_H}{dt}\right| \approx \left|\frac{\bar{i}_H\,\sin\theta\,\omega\Delta t}{\Delta t}\right|$$

and in the limit

$$\left|\frac{d_I\,\bar{i}_H}{dt}\right| = \lim_{\Delta t \to 0} \left|\frac{\bar{i}_H\,\sin\theta\,\omega\Delta t}{\Delta t}\right| = \left|\bar{i}_H\,\sin\theta\,\omega\right| \tag{A-6}$$

Equation A-6, however, is very reminiscent of the cross-product, namely

$$\left|\frac{d_I\,\bar{i}_H}{dt}\right| = \left|\vec{\omega} \times \bar{i}_H\right| \tag{A-7}$$

and, if one examines the directions, one can indeed state

$$\frac{d_I\,\bar{i}_H}{dt} = \vec{\omega} \times \bar{i}_H \tag{A-8}$$

Since $\bar{i}_H$ is typically representative of each coordinate axis, we also have

$$\begin{aligned} \frac{d_I\,\bar{j}_H}{dt} &= \vec{\omega} \times \bar{j}_H \\ \frac{d_I\,\bar{k}_H}{dt} &= \vec{\omega} \times \bar{k}_H \end{aligned} \tag{A-9}$$

It is natural to substitute Equations A-8 and A-9 into Equation A-2. This produces

$$\frac{d_I \vec{R}}{dt} = \dot{R}_{XH}\,\bar{i}_H + \dot{R}_{YH}\,\bar{j}_H + \dot{R}_{ZH}\,\bar{k}_H + R_{XH}\,(\vec{\omega} \times \bar{i}_H)$$

$$+ R_{YH}\,(\vec{\omega} \times \bar{j}_H) + R_{ZH}\,(\vec{\omega} \times \bar{k}_H) \qquad \text{(A-10)}$$

which one may write as

$$\frac{d_I \vec{R}}{dt} = \frac{d_H \vec{R}}{dt} + \vec{\omega} \times R_{XH}\,\bar{i}_H + \vec{\omega} \times R_{YH}\,\bar{j}_H + \vec{\omega} \times R_{ZH}\,\bar{k}_H \qquad \text{(A-11)}$$

or finally

$$\frac{d_I \vec{R}}{dt} = \frac{d_H \vec{R}}{dt} + \vec{\omega} \times \vec{R} \qquad \text{(A-12)}$$

which, in the text proper, is the basis for Equation 4.

APPENDIX B

Matrix representation of rotations

Assume the geometry of Figure B-1, where a rotation of magnitude ψ has been performed about the $\bar{k}_E$ axis, producing new vectors $\bar{i}'$ and $\bar{j}'$.

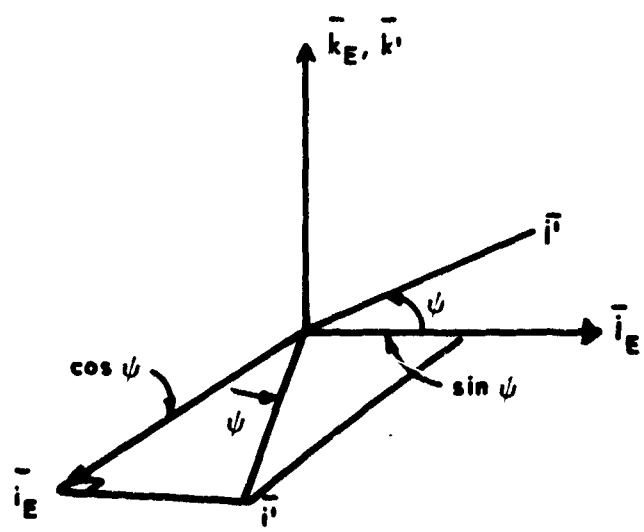

Fig B-1

Clearly the projection of $\bar{i}'$ on the $\bar{i}_E$ axis is $\bar{i}' \cos\psi = \cos\psi$ (all vectors shown are of unit magnitude). Continuing in like manner for all possible combinations, one may form the following table:

	$\bar{i}_E$	$\bar{j}_E$	$\bar{k}_E$
$\bar{i}'$	$\cos\psi$	$\sin\psi$	0
$\bar{j}'$	$-\sin\psi$	$\cos\psi$	0
$\bar{k}'$	0	0	1

Here, for example, $-\sin\psi$ is to be interpreted as being the projection of $\bar{j}'$ on the $\bar{i}_E$ axis.

If we now choose to represent $\bar{i}'$ in terms of its components along $\bar{i}_E$, $\bar{j}_E$, and $\bar{k}_E$, we write

$$\bar{i}' = \cos\psi\, \bar{i}_E + \sin\psi\, \bar{j}_E + 0\, \bar{k}_E \qquad \text{(B-1)}$$

Though matrix multiplication, we can write not only $\bar{i}'$, but also $\bar{j}'$ and $\bar{k}'$, not separately as in Equation B-1, but together in one matrix similar to the given table. Thus:

$$\begin{bmatrix} \bar{i}' \\ \bar{j}' \\ \bar{k}' \end{bmatrix} = \begin{bmatrix} \cos\psi & \sin\psi & 0 \\ -\sin\psi & \cos\psi & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \bar{i}_E \\ \bar{j}_E \\ \bar{k}_E \end{bmatrix} \qquad \text{(B-2)}$$

or Equation 13 of the text.

It might be mentioned that this development is not intended to be rigorous; however, it should enable one to obtain these rotation matrices without difficulty.

Finally, let us complicate the above and now assume that $\bar{i}'$ and $\bar{j}'$ are rotating about $\bar{k}_E$ with some angular velocity $\vec{\omega}_1$. Clearly $\vec{\omega}_1$ is directed along the $\bar{k}_E$ axis, and further, in time Δt we have the relation

$$|\vec{\omega}_1| \Delta t = \frac{\Delta\psi}{\Delta t} \Delta t \qquad \text{(B-3)}$$

or, in the limit, as Δt approaches zero

$$|\vec{\omega}_1| = \frac{d\psi}{dt} \qquad \text{(B-4)}$$

Finally, combining both magnitude and direction, we can write

$$\vec{\omega}_1 = \dot{\psi}\bar{k}_E \qquad \text{(B-5)}$$

We can do likewise for a rotation about $\bar{j}'$, obtaining, for example,

$$\vec{\omega}_2 = \dot{\theta}\bar{j}' \qquad \text{(B-6)}$$

where θ is an angle defined analogously in the $\bar{i}' - \bar{k}'$ plane as ψ was defined in the $\bar{i}_E - \bar{j}_E$ plane.

Since angular velocities may be added, we can combine Equations B-5 and B-6 and write

$$\vec{\omega} = \vec{\omega}_1 + \vec{\omega}_2 = \dot{\psi}k_E + \dot{\theta}j' \qquad \text{(B-7)}$$

which, in the text, is Equation 22.

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4 DESCRIPTIVE NOTES *(Type of report and inclusive dates)*

5 AUTHOR(S) *(Last name, first name, initial)*
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13 ABSTRACT
This report derives from elementary principles the general equations of motion for a missile utilizing a fixed-plane coordinate system, i.e., a coordinate system with one axis constrained to lie in a given plane.

Included in the derivation are explicit expressions for introducing wind and an alternate set of equations to be used when singularity conditions are approached. Means are provided for automatically converting to the alternate set of equations so that uninterrupted trajectory simulation can proceed under all conditions. A complete discussion of intial conditions is included.

The general equations can be used for flat or spherical, rotating, or nonrotating earth cases.

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Fixed plane coordinate systems						
General equations of motion						
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