TECHNICAL NOTE R-138

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# TRANSIENT CHAMBER PRESSURE AND THRUST IN SOLID ROCKET MOTORS

by Hans H. Seidel



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*RESEARCH LABORATORIES*

*BROWN ENGINEERING COMPANY, INC.*

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# TRANSIENT CHAMBER PRESSURE AND THRUST IN SOLID ROCKET MOTORS

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ABSTRACT

An analysis of solid rocket motors was made in order to find the chamber pressure and thrust during the transient periods of buildup and decay. The results are differential equations and solutions of closed form.


Approved

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Director of Research

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## LIST OF SYMBOLS

| | |
|---|---|
| $A$ | Cross-sectional area, surface area |
| $a$ | Coefficient |
| $C$ | Constant |
| $c_p$ | Specific heat at constant pressure |
| $d$ | Constant |
| $F$ | Thrust |
| $m$ | Mass of propellant |
| $n$ | Combustion index |
| $p$ | Pressure |
| $p_o$ | Ignition pressure |
| $R$ | Gas constant |
| $r$ | Burning rate |
| $T$ | Temperature |
| $t$ | Time |
| $V$ | Volume |
| $v$ | Velocity |
| $\alpha_d$ | Nozzle angle in divergent part |
| $\gamma$ | Ratio of specific heats |
| $\lambda$ | Correction factor |
| $\rho$ | Density |
| $\phi$ | Independent variable |

LIST OF SYMBOLS (Continued)

| | |
|---|---|
| $\psi$ | Discharge function |
| $\dot{\omega}$ | Mass flow rate |

Subscripts

| | |
|---|---|
| a | Refers to ambient condition |
| c | Location at chamber |
| e | Location at exit or separation |
| f | Refers to propellant |
| t | Location at throat |
| o | Initial condition |

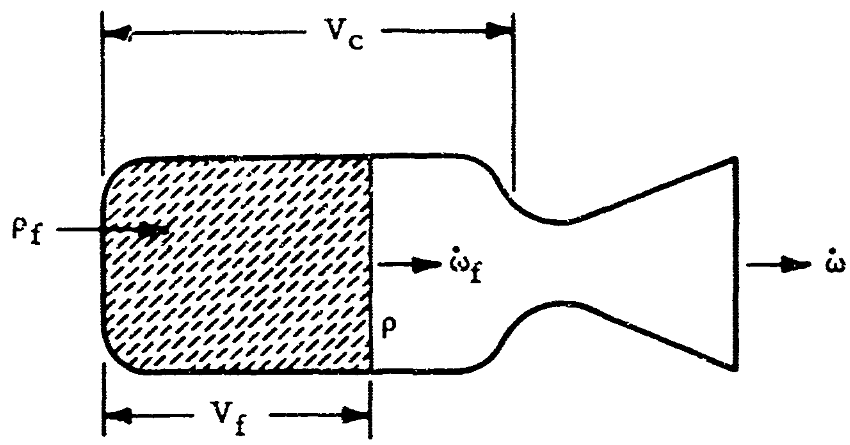
# INTRODUCTION

In determining the total impulse and performance values of solid propellant rocket motors usually the thrust buildup period, which is the time from ignition up to a certain rated thrust, and the ..rust decay period, which is the time from an average effective thrust down to zero, are not included. The total impulse can be accurately calculated by integration of the thrust over the total operating time. For calculating the thrust during these transient phases of buildup and decay, the chamber pressure as a function of time has to be evaluated first. During these periods quasi-steady flow through the nozzle is assumed. General equations will be derived which are differential equations applicable for cases when several parameters are variables, as area of flame front changing with time, combustion index changing over certain pressure ranges, average ratio of specific heats changing with expansion ratio, etc. Then, assumptions are made where those parameters are constant and solutions of closed form are derived.

## CHAMBER PRESSURE BUILDUP

### General Derivations

Consider a general case of burning fuel in a solid propellant rocket motor where the combustion gases escape through the nozzle.




By applying the law of conservation of mass in the chamber the following equation can be used

$$\frac{d\rho V}{dt} = \dot{\omega}_f - \dot{\omega} \tag{1}$$

where $\dot{\omega}_f$ is the rate of fuel consumption and $\dot{\omega}$ is the discharge rate through the nozzle.

The burning rate of a solid propellant increases strongly with an increase in pressure under which the propellant burns. The burning rate can be expressed as

$$r = a p_c^{\,n} \tag{2}$$

where a and n are coefficients which depend on the propellant composition, on temperature and on operating pressure. Introducing the surface area of the flame front $A_f$ and the density of the propellant $\rho_f$ the rate of fuel consumption can be written

$$\dot{\omega}_f = A_f \rho_f a p_c^{\,n} \quad . \tag{3}$$

The discharge rate through the nozzle, in this case through the throat region, can be expressed by the continuity equation

$$\dot{\omega} = \rho_t v_t A_t \tag{4}$$

where

$\rho$ - density of the gas,

A - cross-sectional area, and

v - velocity at the throat.

The energy equation is

$$\frac{v_t^2}{2} = c_p(T_c - T_t) = c_p T_c\left(1 - \frac{T_t}{T_c}\right) \tag{5}$$

where the velocity in the chamber is assumed to be negligible.

Substituting the specific heat at constant pressure

$$c_p = \frac{\gamma}{\gamma - 1} R \tag{6}$$

and introducing the isentropic expansion

$$\frac{T_t}{T_c} = \left(\frac{p_t}{p_c}\right)^{\frac{\gamma-1}{\gamma}} \tag{7}$$

the velocity at the throat becomes

$$v_t = \left\{\frac{2\gamma}{\gamma - 1} R\, T_c\left[1 - \left(\frac{p_t}{p_c}\right)^{\frac{\gamma-1}{\gamma}}\right]\right\}^{\frac{1}{2}} \quad . \tag{8}$$

The mass flow rate in Equation 4 can be changed with

$$\frac{\rho_t}{\rho_c} = \left(\frac{p_t}{p_c}\right)^{\frac{1}{\gamma}} \tag{9}$$

to

$$\dot{\omega} = \rho_c A_t (2R\, T_c)^{\frac{1}{2}} \left(\frac{p_t}{p_c}\right)^{\frac{1}{\gamma}} \left\{\frac{\gamma}{\gamma - 1}\left[1 - \left(\frac{p_t}{p_c}\right)^{\frac{\gamma-1}{\gamma}}\right]\right\}^{\frac{1}{2}} \quad . \tag{10}$$

Substitution of the density of the gas in the chamber with the aid of the perfect gas law

$$\rho_c = \frac{p_c}{R\, T_c} \tag{11}$$

results in

$$\dot{\omega} = p_c A_t \left(\frac{2}{R\, T_c}\right)^{\frac{1}{2}} \left(\frac{p_t}{p_c}\right)^{\frac{1}{\gamma}} \left\{\frac{\gamma}{\gamma - 1}\left[1 - \left(\frac{p_t}{p_c}\right)^{\frac{\gamma-1}{\gamma}}\right]\right\}^{\frac{1}{2}} \quad . \tag{12}$$

For convenience the right hand term in Equation 12 will be called a function $\psi$,

$$\psi = \left(\frac{p_t}{p_c}\right)^{\frac{1}{\gamma}} \left\{\frac{\gamma}{\gamma - 1}\left[1 - \left(\frac{p_t}{p_c}\right)^{\frac{\gamma-1}{\gamma}}\right]\right\}^{\frac{1}{2}} \tag{13}$$

which is really the discharge function, depending only on the ratio of specific heats and the pressure ratio between chamber and throat. This discharge function has a maximum value at the critical pressure ratio when sonic velocity exists in the throat. In order to obtain the maximum of the discharge function, i. e., finding the critical pressure ratio, the derivative of $\psi$ with respect to the pressure ratio has to be taken and set equal to zero.

$$\psi = \left(\frac{\gamma}{\gamma - 1}\right)^{\frac{1}{2}} \left[\left(\frac{p}{p_c}\right)^{\frac{2}{\gamma}} - \left(\frac{p}{p_c}\right)^{\frac{\gamma+1}{\gamma}}\right]^{\frac{1}{2}}$$

$$\frac{d\psi}{d(p/p_c)} = \left(\frac{\gamma}{\gamma - 1}\right)^{\frac{1}{2}} \cdot \frac{1}{2}\left[\left(\frac{p}{p_c}\right)^{\frac{2}{\gamma}} - \left(\frac{p}{p_c}\right)^{\frac{\gamma+1}{\gamma}}\right]^{-\frac{1}{2}} \left[\frac{2}{\gamma}\left(\frac{p}{p_c}\right)^{\frac{2-\gamma}{\gamma}} - \frac{\gamma + 1}{\gamma}\left(\frac{p}{p_c}\right)^{\frac{1}{\gamma}}\right] . \quad (14)$$

Setting $d\psi/d(p/p_c) = 0$ yields the critical pressure ratio

$$\left(\frac{\gamma}{\gamma - 1}\right)^{\frac{1}{2}} \frac{(2/\gamma)(p/p_c)^{\frac{2-\gamma}{\gamma}} - [(\gamma + 1)/\gamma](p/p_c)^{\frac{1}{\gamma}}}{2\left[(p/p_c)^{\frac{2}{\gamma}} - (p/p_c)^{\frac{\gamma+1}{\gamma}}\right]^{\frac{1}{2}}} = 0 \quad . \quad (15)$$

Equation 15 will be zero, if

$$\frac{2}{\gamma}\left(\frac{p}{p_c}\right)^{\frac{2-\gamma}{\gamma}} - \frac{\gamma + 1}{\gamma}\left(\frac{p}{p_c}\right)^{\frac{1}{\gamma}} = 0 \quad (16)$$

or

$$\frac{p}{p_c} = \left(\frac{2}{\gamma + 1}\right)^{\frac{\gamma}{\gamma-1}} \quad . \quad (17)$$

This is the critical pressure ratio at which sonic velocity just exists in the throat and the discharge function has reached a maximum at

$$\psi_{max} = \left(\frac{\gamma}{\gamma - 1}\right)^{\frac{1}{2}} \left[\left(\frac{2}{\gamma + 1}\right)^{\frac{2}{\gamma-1}} - \left(\frac{2}{\gamma + 1}\right)^{\frac{\gamma+1}{\gamma-1}}\right]^{\frac{1}{2}} \quad (18)$$

or

$$\psi_{max} = \left(\frac{\gamma}{\gamma + 1}\right)^{\frac{1}{2}} \left(\frac{2}{\gamma + 1}\right)^{\frac{1}{\gamma - 1}} \quad . \tag{19}$$

The result shows that the discharge function depends only on the ratio of the specific heats in the case of critical conditions.

For the subcritical case, the discharge function remains

$$\psi = \left(\frac{p_a}{p_c}\right)^{\frac{1}{\gamma}} \left\{\frac{\gamma}{\gamma - 1}\left[1 - \left(\frac{p_a}{p_c}\right)^{\frac{\gamma - 1}{\gamma}}\right]\right\}^{\frac{1}{2}} \quad . \tag{20}$$

With Equations 3, 12, and 13, Equation 1 can be written

$$\frac{d\rho V}{dt} = A_f \rho_f a p_c^{\,n} - p_c A_t \psi \left(\frac{2}{RT_c}\right)^{\frac{1}{2}} \quad . \tag{21}$$

The derivative of the gas mass in the free volume of the chamber can be expressed as

$$\frac{d\rho V}{dt} = V \frac{d\rho}{dt} + \rho \frac{dV}{dt} \tag{22}$$

where V is the instantaneous volume in the combustion chamber and $\rho$ is the instantaneous density of the gas. The free volume is a function of time due to combustion and consumption of the fuel

$$V = V_c - \frac{m_0}{\rho_f} + \int_0^t A_f a p_c^{\,n} \, dt \tag{23}$$

where

$V_c$ - the volume in the chamber at burnout,

$m_0$ - the initial mass of the propellant, and

$\int_0^t A_f a p_c^n dt$ - the volume of the consumed propellant or the created volume by combustion of propellant during the time after ignition.

Since the chamber volume and the initial mass of the propellant are constant, the derivative of the instantaneous volume with respect to time will be

$$\frac{dV}{dt} = A_f a p_c^n \quad . \tag{24}$$

Substituting into Equation 22 yields

$$\frac{d\rho V}{dt} = \left(V_c - \frac{m_0}{\rho_f} + \int_0^t A_f a p_c^n dt\right)\frac{d\rho}{dt} + \rho_c A_f a p_c^n \quad . \tag{25}$$

The experiment shows that the isobaric combustion temperature changes only slightly with combustion pressure, i.e., the temperature change is negligible.

Applying the equation of state and taking the derivative of the density with respect to time gives

$$\frac{d\rho_c}{dt} = \frac{1}{RT_c}\frac{dp_c}{dt} \quad . \tag{26}$$

With the above relation the expression in Equation 25 becomes

$$\frac{d\rho V}{dt} = \left(V_c - \frac{m_0}{\rho_f} + \int_0^t A_f a p_c^n dt\right)\frac{1}{RT_c}\frac{dp_c}{dt} + \rho_c A_f a p_c^n \quad . \tag{27}$$

The law of conservation of mass described in Equation 21 can be written with the derived expressions as

$$\left(V_c - \frac{m_0}{\rho_f} + \int_0^t A_f a p_c{}^n dt\right) \frac{1}{RT_c} \frac{dp_c}{dt} + \rho_c A_f a p_c{}^n = A_f \rho_f a p_c{}^n - p_c A_t \psi \left(\frac{2}{RT_c}\right)^{\frac{1}{2}} \tag{28}$$

or

$$\left(V_c - \frac{m_0}{\rho_f} + \int_0^t A_f a p_c{}^n dt\right) = RT_c A_f \rho_f a p_c{}^n \left(1 - \frac{\rho_c}{\rho_f}\right) - p_c A_t \psi \left(2RT_c\right)^{\frac{1}{2}} \quad . \tag{29}$$

In this relation the density in the chamber $\rho_c$ is less than one or two percent of the propellant density $\rho_f$, thus it can be neglected.

$$\left(V_c - \frac{m_0}{\rho_f} + \int_0^t A_f a p_c{}^n dt\right) \frac{dp_c}{dt} = RT_c A_f \rho_f a p_c{}^n - p_c A_t \psi \left(2RT_c\right)^{\frac{1}{2}} \quad . \tag{30}$$

The above differential equation describes the development of the chamber pressure as a function of time. The variables are

| | | |
|---|---|---|
| $p_c$ | - | chamber pressure, |
| $A_f$ | - | surface area of the flame front, and |
| a, n | - | coefficients vary with operating pressure and propellant temperature. |

As mentioned previously the chamber temperature can be set approximately constant as can the gas constant or molecular weight of the gas and the ratio of the specific heats.

Equation 30 must be solved numerically on a digital computer. Depending on the design condition of a solid propellant rocket motor, different cases have to be distinguished for determining the chamber pressure buildup.

### Closed Nozzle by Diaphragms

If the nozzle is closed for a moment during ignition by a membrane or diaphragm and no efflux through the nozzle takes place, the following relation has to be applied

$$\left(V_c - \frac{m_0}{\rho_f} + \int_0^{t_1} A_f a p_c^n dt\right) \frac{dp_c}{dt} = RT_c A_f \rho_f a p_c^n \quad . \tag{31}$$

This equation can be used both for the igniter propellant and for the main propellant; only the parameters for the propellant and the boundary conditions have to be changed.

At a certain pressure the membrane bursts and the efflux begins. Quasisteady flow will be assumed, i.e., the discharge rate is determined at any given moment by the equations for steady flow. Here, this equation is valid

$$\left(V_c - \frac{m_0}{\rho_f} + \int_0^{t_2} A_f a p_c^n dt\right) \frac{dp_c}{dt} = RT_c A_f \rho_f a p_c^n - p_c A_t \psi (2RT_c)^{\frac{1}{2}} \quad . \tag{32}$$

Usually, when the diaphragms burst, the chamber pressure greatly exceeds the critical pressure, i.e.,

$$\frac{p_a}{p_c} \leqq \left(\frac{2}{\gamma + 1}\right)^{\frac{\gamma}{\gamma - 1}} \tag{33}$$

so the discharge function, which is constant, can be introduced.

Equations 32 and 19 yield

$$\psi_{max} = \left(\frac{2}{\gamma + 1}\right)^{\frac{1}{\gamma - 1}} \left(\frac{\gamma}{\gamma + 1}\right)^{\frac{1}{2}}$$

$$\left(V_c - \frac{m_0}{\rho_f} + \int_0^{t_2} A_f a p_c^{\,n} dt\right)\frac{dp_c}{dt} = RT_c A_f \rho_f a p_c^{\,n} - (\gamma)^{\frac{1}{2}} \left(\frac{2}{\gamma + 1}\right)^{\frac{\gamma+1}{2(\gamma-1)}} p_c A_t (RT_c)^{\frac{1}{2}} \quad . \tag{34}$$

## Open End Nozzle

The nozzle is open and efflux through the nozzle takes place from beginning of ignition. Equation 20 gives the discharge function until the critical pressure ratio is reached

$$\psi = \left(\frac{p_a}{p_c}\right)^{\frac{1}{\gamma}} \left\{\frac{\gamma}{\gamma - 1}\left[1 - \left(\frac{p_a}{p_c}\right)^{\frac{\gamma-1}{\gamma}}\right]\right\}^{\frac{1}{2}} \quad .$$

Then the differential equation will be for $p_a/p_c \geqq [2/(\gamma+1)]^{\frac{\gamma}{\gamma-1}}$

$$\left(V_c - \frac{m_0}{\rho_f} + \int_0^{t_1} A_f a p_c^{\,n} dt\right)\frac{dp_c}{dt} = RT_c A_f \rho_f a p_c^{\,n} - p_c A_t \left(\frac{p_a}{p_c}\right)^{\frac{1}{\gamma}} \left\{\frac{2\gamma}{\gamma - 1} RT_c\left[1 - \left(\frac{p_a}{p_c}\right)^{\frac{\gamma-1}{\gamma}}\right]\right\}^{\frac{1}{2}} \tag{35}$$

and for the supercritical phase

$$\frac{p_a}{p_c} \leqq \left(\frac{2}{\gamma + 1}\right)^{\frac{\gamma}{\gamma-1}}$$

$$\left(V_c - \frac{m_0}{\rho_f} + \int_0^{t_2} A_f a p_c^{\,n} dt\right)\frac{dp_c}{dt} = RT_c A_f \rho_f a p_c^{\,n} - (\gamma)^{\frac{1}{2}} \left(\frac{2}{\gamma + 1}\right)^{\frac{\gamma+1}{2(\gamma-1)}} p_c A_t (RT_c)^{\frac{1}{2}} \quad . \tag{36}$$

The time required to build up the chamber pressure is the sum of time $t_1$ during which the pressure increases from $p_a$ to $p_a/[2/(\gamma+1)]^{\frac{\gamma}{\gamma-1}}$ and the time $t_2$ during which the chamber pressure varies from $p_a/[2/(\gamma+1)]^{\frac{\gamma}{\gamma-1}}$ to $p_c$. The efflux during the first interval of time is subcritical and $\psi$ depends on $p_a/p$; the efflux during the second interval is supercritical and $\psi$ is independent of $p_a/p$.

## Approximate Solution of the Differential Equations

An integration of Equation 30 can be accomplished on the basis of assumptions which will follow later

$$\left(V_c - \frac{m_0}{\rho_f} + \int_0^t A_f a p_c^n \, dt\right) \frac{dp_c}{dt} = RT_c A_f \rho_f a p_c^n - p_c A_t \psi (2RT_c)^{\frac{1}{2}} \quad .$$

A new independent variable will be introduced for the relative consumed propellant mass

$$\phi = \frac{\rho_f}{m_0} \int_0^t A_f a p_c^n \, dt \quad . \tag{37}$$

Equation 30 becomes with the above

$$\left[V_c - \frac{m_0}{\rho_f}(1 - \phi)\right] \frac{dp_c}{dt} = RT_c A_f \rho_f a p_c^n - p_c A_t \psi (2RT_c)^{\frac{1}{2}} \quad . \tag{38}$$

The derivative of the chamber pressure can be changed to

$$\frac{dp_c}{dt} = \frac{dp_c}{dt} \frac{d\phi}{d\phi} = \left(\frac{dp_c}{d\phi}\right)\left(\frac{d\phi}{dt}\right) \tag{39}$$

and with Equation 37

$$\frac{d\phi}{dt} = \frac{\rho_f}{m_0} A_f a p_c^n \tag{40}$$

to

$$\frac{dp_c}{dt} = \frac{\rho_f}{m_0} A_f a p_c^n \frac{dp_c}{d\phi} \quad . \tag{41}$$

Substitution into Equation 38 yields

$$\left[V_c - \frac{m_0}{\rho_f}(1 - \phi)\right] \frac{\rho_f}{m_0} A_f a p_c^n \frac{dp_c}{d\phi} = RT_c A_f \rho_f a p_c^n - p_c A_t \psi (2RT_c)^{\frac{1}{2}} \tag{42}$$

or simplified

$$\left(\frac{V_c \rho_f}{m_0} - 1 + \phi\right) \frac{dp_c}{d\phi} = RT_c \rho_f - \frac{A_t \psi}{A_f a} p_c^{1-n} (2RT_c)^{\frac{1}{2}} \quad .$$

Separation of the variable leads to

$$\frac{dp_c}{RT_c \rho_f - (A_t \psi / A_f a) p_c^{1-n} (2RT_c)^{\frac{1}{2}}} = \frac{d\phi}{[(V_c \rho_f / m_0) - 1 + \phi]} \quad . \tag{43}$$

The boundary conditions are

at $t = 0$, $\quad \phi = 0$

$\quad p_c = p_o$

where $p_o$ is the ignition pressure, and

at $t = t$, $\quad \phi = \phi$

$\quad p_c = p_c \quad .$

The integration can be carried out

$$\int_{p_o}^{p_c} \frac{dp_c}{RT_c \rho_f - (A_t \psi / A_f a) p_c^{1-n} (2RT_c)^{\frac{1}{2}}} = \int_0^{\phi} \frac{d\phi}{[(V_c \rho_f / m_0) - 1 + \phi]} \quad . \tag{44}$$

The right hand side integral can be solved very easily

$$\int_0^{\phi} \frac{d\phi}{[(V_o \rho_f/m_0) - 1 + \phi]} = \ln\left(\frac{V_c \rho_f}{m_0} - 1 + \phi\right)\Big|_0^{\phi}$$

$$I_1 = \ln\left[\frac{(V_c \rho_f/m_0) - 1 + \phi}{(V_c \rho_f/m_0) - 1}\right]$$

$$I_1 = \ln\left[1 + \frac{\phi}{(V_c \rho_f/m_0) - 1}\right] \quad . \tag{45}$$

The expression for the independent variable can be introduced again

$$I_1 = \ln\left[1 + \frac{(\rho_f/m_0)\int_0^t A_f a p_c^n \, dt}{(V_c \rho_f/m_0) - 1}\right] \tag{46}$$

or simplified

$$I_1 = \ln\left[1 + \frac{\int_0^t A_f a p_c^n \, dt}{V_c - (m_0/\rho_f)}\right] \quad . \tag{47}$$

If it is assumed that the area of the flame front is constant, which is used in many motors, and an average burning rate is defined between the ignition pressure $p_o$ and the steady state pressure $p_c$,

$$r_m = a\left(\frac{p_o + p_c}{2}\right)^n \tag{48}$$

the integral becomes

$$I_1 = \ln\left\{1 + \frac{A_f a[(p_o + p_c)/2]^n \, t}{V_c - (m_0/\rho_f)}\right\} \quad . \tag{49}$$

For the solution of the left hand side integral in Equation 44 some assumptions have to be made such as constant area of flame front and constant discharge function in the case of supercritical condition, which is true when the ignition pressure $p_o$ is above the critical pressure. The combustion index of many propellant types are in the neighborhood of n = 2/3, so it will be used for the integration.

$$\int_{p_o}^{p_c} \frac{dp_c}{RT_c \rho_f - (A_t \psi / A_f a)\, p_c^{1-n}\, (2RT_c)^{1/2}} = \int_{p_o}^{p_c} \frac{dp_c}{C - d\, p_c^{1/3}} = I_2 \qquad (50)$$

where

$$C = RT_c \rho_f$$

and

$$d = \frac{A_t \psi}{A_f a} (2RT_c)^{\frac{1}{2}}$$

which, by dividing out the integral, becomes

$$I_2 = \int_{p_o}^{p_c} \left[ -\frac{1}{d} p_c^{-1/3} - \frac{C}{d^2} p_c^{-2/3} + \frac{(C^2/d^2) p_c^{-2/3}}{C - d\, p_c^{1/3}} \right] dp_c \quad . \qquad (51)$$

Integration of the terms results in

$$I_2 = \left[ -\frac{3}{2d} p_c^{2/3} - \frac{3C}{d^2} p_c^{1/3} - \frac{3C^2}{d^3} \ln (C - d\, p_c^{1/3}) \right] \Bigg|_{p_o}^{p_c} \qquad (52)$$

and setting the limits yields

$$I_2 = \frac{3}{2d}\left(p_o^{2/3} - p_c^{2/3}\right) + \frac{3C}{d^2}\left(p_o^{1/3} - p_c^{1/3}\right) - \frac{3C^2}{d^3}\,\ell n\,\frac{C - d\,p_c^{1/3}}{C - d\,p_o^{1/3}}$$

or

$$I_2 = \frac{3C^2}{d^2}\left[\frac{1}{C}\left(p_o^{1/3} - p_c^{1/3}\right) + \frac{d}{2C^2}\left(p_o^{2/3} - p_c^{2/3}\right) - \frac{1}{d}\,\ell n\,\frac{C - d\,p_c^{1/3}}{C - d\,p_o^{1/3}}\right] \quad . \tag{53}$$

The whole expression of the solved integral $I_2$ is given with replacement of the constants

$$I_2 = \frac{3\rho_f^2\, a^2\, A_f^2\, RT_c}{2\psi^2\, A_t^2}\left(\frac{1}{\rho_f RT_c}\left(p_o^{1/3} - p_c^{1/3}\right) + \frac{A_t\psi\,(2RT_c)^{1/2}}{2A_f a\,\rho_f^2 R^2 T_c^2}\left(p_o^{2/3} - p_c^{2/3}\right) - \frac{A_f a}{A_t\psi(2RT_c)^{1/2}}\,\ell n\left\{\frac{\rho_f RT_c - [A_t\psi(2RT_c)^{1/2}/A_f a]\, p_c^{1/3}}{\rho_f RT_c - [A_t\psi(2RT_c)^{1/2}/A_f a]\, p_o^{1/3}}\right\}\right) \tag{54}$$

The solution of Equation 44 permits the determination of the chamber pressure as a function of time which is

$$\frac{3\rho_f^2\, a^2\, A_f^2\, RT_c}{2\psi^2\, A_t^2}\left(\frac{1}{\rho_f RT_c}\left(p_o^{1/3} - p_c^{1/3}\right) + \frac{A_t\psi(2RT_c)^{1/2}}{2A_f a\,\rho_f^2\, R^2\, T_c^2}\left(p_o^{2/3} - p_c^{2/3}\right) - \frac{A_f a}{A_t\psi(2RT_c)^{1/2}}\,\ell n\left\{\frac{\rho_f RT_c - [A_t\psi(2RT_c)^{1/2}/A_f a]\, p_c^{1/3}}{\rho_f RT_c - [A_t\psi(2RT_c)^{1/2}/A_f a]\, p_o^{1/3}}\right\}\right) \tag{55}$$

$$= \ell n\left[1 + \frac{A_f a[(p_o + p_c)/2]^{2/3}\, t}{V_c - (m_0/\rho_f)}\right] \quad .$$

If the area of the flame front is a variable, i.e., a function of time and the combustion index different from $n = 2/3$ especially in certain pressure ranges, Equation 30 has to be integrated numerically for obtaining an accurate chamber pressure curve.

## CHAMBER PRESSURE DECAY

### Decay During Supercritical Phase

After burnout the combustion chamber can be considered as a gas-filled vessel, in which $p_c$, $\rho_c$ and $T_c$ exist as initial conditions. It is assumed that the discharge of the gas through the nozzle is approximately quasistationary, i.e., the discharge rate is determined at any given moment by the equations for steady flow.

The law of conservation of mass applied to the gaseous mass contained in the combustion chamber can be introduced as

$$\frac{d\rho V}{dt} = - \dot{\omega} \tag{56}$$

where $\dot{\omega}$ is the discharge flow rate out of the chamber.

Since the volume of the chamber does not vary with time, Equation 56 becomes

$$V_c \frac{d\rho}{dt} = - \dot{\omega} \quad . \tag{57}$$

Using the derived relation for the discharge rate in Equation 10 the expression will be

$$V_c \frac{d\rho}{dt} = - \rho A_t (2RT)^{\frac{1}{2}} \left(\frac{Pt}{p}\right)^{\frac{1}{\gamma}} \left\{\frac{\gamma}{\gamma - 1}\left[1 - \left(\frac{Pt}{p}\right)^{\frac{\gamma-1}{\gamma}}\right]\right\}^{\frac{1}{2}} \tag{58}$$

where the discharge function $\psi$ is introduced again by

$$\psi = \left(\frac{Pt}{p}\right)^{\frac{1}{\gamma}} \left\{\frac{\gamma}{\gamma - 1}\left[1 - \left(\frac{Pt}{p}\right)^{\frac{\gamma-1}{\gamma}}\right]\right\}^{\frac{1}{2}} \quad . \tag{59}$$

Dividing by the initial chamber condition yields

$$V_c \frac{d}{dt}\frac{\rho}{\rho_c} = -\frac{\rho}{\rho_c} A_t (2RT_c)^{\frac{1}{2}} \left(\frac{T}{T_c}\right)^{\frac{1}{2}} \psi \quad . \tag{60}$$

Following the adiabatic relation and arranging gives

$$\frac{d}{dt}\left(\frac{p}{p_c}\right)^{\frac{1}{\gamma}} = -\frac{A_t}{V_c}\left(\frac{p}{p_c}\right)^{\frac{\gamma+1}{2\gamma}} \psi (2RT_c)^{\frac{1}{2}} \quad . \tag{61}$$

The derivative can be written as

$$\begin{aligned} \frac{d}{dt}\left(\frac{p}{p_c}\right)^{\frac{1}{\gamma}} &= \frac{1}{\gamma}\left(\frac{p}{p_c}\right)^{\frac{1}{\gamma}-1} \frac{d}{dt}\left(\frac{p}{p_c}\right) \\ &= \frac{1}{\gamma}\left(\frac{p}{p_c}\right)^{\frac{1-\gamma}{\gamma}} \frac{d}{dt}\left(\frac{p}{p_c}\right) \quad . \end{aligned} \tag{62}$$

Substituting into Equation 61 yields

$$\frac{d}{dt}\left(\frac{p}{p_c}\right) = -\frac{A_t}{V_c}\gamma\left(\frac{p}{p_c}\right)^{\frac{3\gamma-1}{2\gamma}} \psi (2RT_c)^{\frac{1}{2}} \quad . \tag{63}$$

This first order differential equation can be solved by separation of the variables as follows

$$\frac{d(p/p_c)}{\psi (p/p_c)^{\frac{3\gamma-1}{2\gamma}}} = -\frac{A_t}{V_c}\gamma (2RT_c)^{\frac{1}{2}}\, dt \quad . \tag{64}$$

The boundary conditions are

$$t = 0 \rightarrow \frac{p}{p_c} = 1$$

$$t = t_1 \rightarrow \frac{p}{p_c} = \frac{p}{p_c} \quad .$$

Equation 64 integrated results in

$$\int_{1}^{p/p_c} \frac{d(p/p_c)}{\psi (p/p_c)^{\frac{3\gamma-1}{2\gamma}}} = - \frac{A_t}{V_c} \gamma (2RT_c)^{\frac{1}{2}} t_1 \quad . \qquad (65)$$

Since the discharge function $\psi$ depends on the pressure ratio, two cases have to be distinguished, the supercritical and the subcritical conditions. For the supercritical condition the pressure ratio is as developed previously

$$\frac{p_a}{p} \leqq \left(\frac{2}{\gamma + 1}\right)^{\frac{\gamma}{\gamma-1}}$$

and the discharge function has reached a maximum with sonic velocity in the throat

$$\psi_{max} = \left(\frac{\gamma}{\gamma + 1}\right)^{\frac{1}{2}} \left(\frac{2}{\gamma + 1}\right)^{\frac{1}{\gamma-1}} \quad .$$

Equation 65 becomes

$$\frac{A_t}{V_c} \gamma (2RT_c)^{\frac{1}{2}} t_1 = - \int_{1}^{p/p_c} \frac{(p/p_c)^{-\frac{3\gamma-1}{2\gamma}}}{[\gamma/(\gamma + 1)]^{\frac{1}{2}} [2/(\gamma + 1)]^{\frac{1}{\gamma-1}}} d\left(\frac{p}{p_c}\right) \quad . \qquad (66)$$

The integral will be calculated separately

$$\int_{1}^{p/p_c} \left(\frac{p}{p_c}\right)^{-\frac{3\gamma-1}{2\gamma}} d\left(\frac{p}{p_c}\right) = \frac{1}{-(3\gamma - 1)/2\gamma + 1} \left(\frac{p}{p_c}\right)^{-\frac{3\gamma-1}{2\gamma} + 1} \Bigg|_{1}^{\left(\frac{p}{p_c}\right)}$$

$$\int_{1}^{p/p_c} \left(\frac{p}{p_c}\right)^{-\frac{3\gamma-1}{2\gamma}} d\left(\frac{p}{p_c}\right) = -\frac{2\gamma}{\gamma-1}\left(\frac{p}{p_c}\right)^{-\frac{\gamma-1}{2\gamma}} \Bigg|_{1}^{\left(\frac{p}{p_c}\right)}$$

$$= -\frac{2\gamma}{\gamma-1}\left[\left(\frac{p}{p_c}\right)^{-\frac{\gamma-1}{2\gamma}} - 1\right] \quad . \tag{67}$$

With this result Equation 66 becomes

$$\frac{A_t}{V_c}\gamma(2RT_c)^{\frac{1}{2}} t_1 = \frac{1}{[\gamma/(\gamma+1)]^{\frac{1}{2}}[2/(\gamma+1)]^{\frac{1}{\gamma-1}}} \frac{2\gamma}{\gamma-1}\left[\left(\frac{p}{p_c}\right)^{-\frac{\gamma-1}{2\gamma}} - 1\right] \quad . \tag{68}$$

Rearranging the above equation and expressing the chamber pressure explicitly as a function of time gives

$$\frac{p}{p_c} = \left[1 + \frac{A_t}{V_c}\gamma(2RT_c)^{\frac{1}{2}} t_1 \frac{\gamma-1}{2\gamma}\left(\frac{\gamma}{\gamma+1}\right)^{\frac{1}{2}}\left(\frac{2}{\gamma+1}\right)^{\frac{1}{\gamma-1}}\right]^{-\frac{2\gamma}{\gamma-1}}$$

or simplified

$$\frac{p}{p_c} = \left[1 + \frac{A_t t_1}{V_c}\frac{\gamma-1}{2}\left(\frac{2}{\gamma+1}\right)^{\frac{\gamma+1}{2(\gamma-1)}}(\gamma RT_c)^{\frac{1}{2}}\right]^{-\frac{2\gamma}{\gamma-1}} \quad . \tag{69}$$

Equation 69 shows that the efflux at supercritical condition is independent of $p_a/p_c$.

## Decay During Subcritical Phase

For this phase, Equation 65 is also valid which is

$$\frac{A_t}{V_c}\gamma(2RT_c)^{\frac{1}{2}} t = -\int_{1}^{p/p_c} \frac{(p/p_c)^{-\frac{3\gamma-1}{2\gamma}}}{\psi} d\left(\frac{p}{p_c}\right) \quad . \tag{70}$$

With the subcritical discharge function

$$\psi = \left(\frac{\gamma}{\gamma - 1}\right)^{\frac{1}{2}} \left(\frac{p_a}{p}\right)^{\frac{1}{\gamma}} \left[1 - \left(\frac{p_a}{p}\right)^{\frac{\gamma-1}{\gamma}}\right]^{\frac{1}{2}} \quad . \tag{71}$$

Taking into account the initial condition

$$t = 0 \rightarrow \frac{p_a}{p} = \left(\frac{2}{\gamma + 1}\right)^{\frac{\gamma}{\gamma-1}} = \frac{p_a}{p}\,\frac{p_c}{p_c}$$

or

$$\frac{p}{p_c} = \frac{p_a}{p_c[2/(\gamma + 1)]^{\frac{\gamma}{\gamma-1}}}$$

and

$$t = t_2 \rightarrow \frac{p}{p_c} = \frac{p}{p_c}$$

the relation becomes

$$\frac{A_t}{V_c}\,\gamma(2RT_c)^{\frac{1}{2}}\,t_2 = -\int_{p_a/\,p_c[2/(\gamma+1)]^{\frac{\gamma}{\gamma-1}}}^{p/p_c} \frac{(p/p_c)^{-\frac{3\gamma-1}{2\gamma}}\,d(p/p_c)}{[\gamma/(\gamma - 1)]^{\frac{1}{2}}\,(p_a/p)^{\frac{1}{\gamma}}\,[1 - (p_a/p)^{\frac{\gamma-1}{\gamma}}]^{\frac{1}{2}}} \tag{72}$$

For better integrability some minor changes must be performed

$$\int \frac{(p/p_c)^{-\frac{3\gamma-1}{2\gamma}} \, d(p/p_c)}{[(p_a/p)(p_c/p_c)]^{\frac{1}{\gamma}} \{1 - [(p_a/p)(p_c/p_c)]^{\frac{\gamma-1}{\gamma}}\}^{\frac{1}{2}}}$$

$$= \left(\frac{p_a}{p_c}\right)^{-\frac{1}{\gamma}} \int \frac{(p/p_c)^{-\frac{3(\gamma-1)}{2\gamma}} \, d(p/p_c)}{[1 - (p_a/p_c)^{\frac{\gamma-1}{\gamma}} (p/p_c)^{-\frac{\gamma-1}{\gamma}}]^{\frac{1}{2}}} \tag{73}$$

Finally the equation is

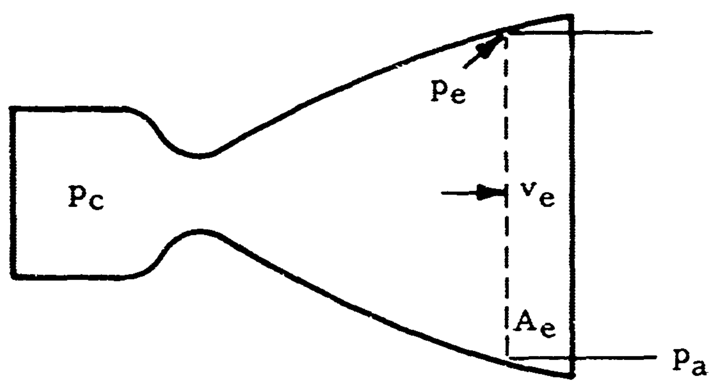
$$\frac{\gamma A_t t_2 \{[2\gamma/(\gamma - 1)] R T_c\}^{\frac{1}{2}}}{V_c} =$$

$$- \left(\frac{p_a}{p_c}\right)^{-\frac{1}{\gamma}} \int_{p_a/\,p_c[2/(\gamma+1)]^{\frac{\gamma}{\gamma-1}}}^{p/p_c} \frac{(p/p_c)^{-\frac{3(\gamma-1)}{2\gamma}} \, d(p/p_c)}{[1 - (p_a/p_c)^{\frac{\gamma-1}{\gamma}} (p/p_c)^{-\frac{\gamma-1}{\gamma}}]^{\frac{1}{2}}} . \tag{74}$$

The above integral can be calculated only by the numerical method.

## Thrust Buildup and Decay During Supercritical Phase

With the aid of the previously determined derivations the chamber pressure as a function of time can be calculated for the buildup and the decay, respectively. Applying the law of momentum on a rocket nozzle, the thrust is given by




$$F = \dot{\omega} v_e + A_e(p_e - p_a) \tag{75}$$

where $\dot{\omega}$ is the mass flow rate, $v_e$ is the exit velocity or the end velocity where separation of the gas from the wall occurs, $A_e$ is the area in the nozzle at separation and $p_e$ is the static pressure at this point.

In the above equation for the thrust the correction factor $\lambda$ will be assumed to be unity.

$$\lambda = \frac{1 + \cos \alpha_d}{2} \approx 1 \tag{76}$$

where $\alpha_d$ is the half nozzle angle for the divergent part of a conical nozzle.

The mass flow rate through the nozzle is given by Equation 12 which is

$$\dot{\omega} = p_c A_t \left(\frac{2}{RT_c}\right)^{\frac{1}{2}} \left(\frac{p_t}{p_c}\right)^{\frac{1}{\gamma}} \left\{\frac{\gamma}{\gamma - 1}\left[1 - \left(\frac{p_t}{p_c}\right)^{\frac{\gamma-1}{\gamma}}\right]\right\}^{\frac{1}{2}}$$

or

$$\dot{\omega} = p_c A_t \left(\frac{2}{RT_c}\right)^{\frac{1}{2}} \psi \tag{77}$$

where $\psi$ is the discharge function.

For the critical phase where

$$\frac{p_a}{p_c} \leqq \left(\frac{2}{\gamma + 1}\right)^{\frac{\gamma}{\gamma-1}}$$

the discharge function reaches a maximum and is a constant

$$\psi_{max} = \left(\frac{\gamma}{\gamma + 1}\right)^{\frac{1}{2}} \left(\frac{2}{\gamma + 1}\right)^{\frac{1}{\gamma-1}} \quad . \tag{78}$$

In this case the mass flow becomes

$$\dot{\omega} = p_c A_t \left(\frac{2}{RT_c}\right)^{\frac{1}{2}} \left(\frac{\gamma}{\gamma + 1}\right)^{\frac{1}{2}} \left(\frac{2}{\gamma + 1}\right)^{\frac{1}{\gamma-1}}$$

or

$$\dot{\omega} = p_c A_t \left(\frac{\gamma}{RT_c}\right)^{\frac{1}{2}} \left(\frac{2}{\gamma + 1}\right)^{\frac{\gamma+1}{2(\gamma-1)}} \quad . \tag{79}$$

From the energy equation

$$\frac{v_e^{\,2}}{2} = c_p(T_c - T_e) = c_p T_c\left(1 - \frac{T_e}{T_c}\right) \tag{80}$$

and the equation for isentropic expansion

$$\frac{T_e}{T_c} = \left(\frac{p_e}{p_c}\right)^{\frac{\gamma-1}{\gamma}} \quad , \tag{81}$$

the end velocity $v_e$ follows

$$v_e = \left\{\frac{2\gamma}{\gamma - 1} RT_c\left[1 - \left(\frac{p_e}{p_c}\right)^{\frac{\gamma-1}{\gamma}}\right]\right\}^{\frac{1}{2}} \quad . \tag{82}$$

For obtaining the area ratio where the separation occurs the continuity equation must be applied.

$$\rho_e v_e A_e = \rho_t v_t A_t \quad . \tag{83}$$

With the end velocity found in Equation 82 and the velocity of sound in the throat, the expression becomes

$$\begin{aligned} A_e\left\{\frac{2\gamma}{\gamma - 1} RT_c\left[1 - \left(\frac{p_e}{p_c}\right)^{\frac{\gamma-1}{\gamma}}\right]\right\}^{\frac{1}{2}} &= \frac{\rho_t}{\rho_e} (\gamma RT_t)^{\frac{1}{2}} A_t \\ &= \frac{\rho_t}{\rho_c}\frac{\rho_c}{\rho_e} (\gamma RT_c)^{\frac{1}{2}} \left(\frac{T_t}{T_c}\right)^{\frac{1}{2}} A_t \\ &= \left(\frac{2}{\gamma + 1}\right)^{\frac{1}{\gamma-1}} \left(\frac{p_c}{p_e}\right)^{\frac{1}{\gamma}} (\gamma RT_c)^{\frac{1}{2}} \left(\frac{2}{\gamma + 1}\right)^{\frac{1}{2}} A_t \\ &= (\gamma)^{\frac{1}{2}} (RT_c)^{\frac{1}{2}} \left(\frac{p_c}{p_e}\right)^{\frac{1}{\gamma}} \left(\frac{2}{\gamma + 1}\right)^{\frac{\gamma+1}{2(\gamma-1)}} A_t \quad . \end{aligned} \tag{84}$$

Finally, the area ratio is found by

$$\frac{A_e}{A_t} = \frac{(\gamma)^{\frac{1}{2}} [2/(\gamma + 1)]^{\frac{\gamma+1}{2(\gamma-1)}}}{(p_e/p_c)^{\frac{1}{\gamma}} \{[2\gamma/(\gamma - 1)][1 - (p_e/p_c)^{\frac{\gamma-1}{\gamma}}]\}^{\frac{1}{2}}} \quad . \tag{85}$$

The thrust is found by substituting into Equation 75:

$$F = p_c A_t (\gamma)^{\frac{1}{2}} \left(\frac{2}{\gamma + 1}\right)^{\frac{\gamma+1}{2(\gamma-1)}} \left\{\frac{2\gamma}{\gamma - 1}\left[1 - \left(\frac{p_e}{p_c}\right)^{\frac{\gamma-1}{\gamma}}\right]\right\}^{\frac{1}{2}}$$

$$+ \frac{A_t(\gamma)^{\frac{1}{2}}[2/(\gamma + 1)]^{\frac{\gamma+1}{2(\gamma-1)}}}{(p_e/p_c)^{\frac{1}{\gamma}}\{[2\gamma/(\gamma - 1)][1 - (p_e/p_c)^{\frac{\gamma-1}{\gamma}}]\}^{\frac{1}{2}}} (p_e - p_a) \quad . \qquad (86)$$

This equation is valid for

$$\frac{p_a}{p_c} \leq \left(\frac{2}{\gamma + 1}\right)^{\frac{\gamma}{\gamma-1}} \quad .$$

At the design point of the nozzle the gas is expanded down to the nozzle end where the exit pressure is equal to the ambient pressure $p_a$. If the chamber pressure is lowered, the gas is overexpanded and the exit pressure can be much lower than $p_a$; even separation from the wall occurs and the separation point travels upstream with decreasing chamber pressure. During overexpansion oblique shock waves exist in the nozzle.

From the standpoint of thrust, the nozzle may well be cut off at the separation station, since the internal and external pressures are nearly in balance beyond this point. The thrust of a rocket motor with separation in the nozzle is calculated on the assumption that the nozzle area ratio is not that corresponding to the actual exit but rather that of the separation station. Therefore, it is necessary to be able to predict the location of separation.

Experimental results on rocket motors which vary considerably from author to author will be used for finding the exit pressure where separation of the gas from the nozzle wall occurs. Summerfield recommends an average exit pressure ratio of $p_e/p_a = 0.4$ throughout all the expansion ratios, but this value is not realistic, especially when the separation exists in the neighborhood of the throat or when only sonic velocity is the maximum velocity due to the available pressure ratio. In Figure 1 the exit pressure can be found as a function of the chamber pressure ratio. With this exit pressure $p_e$ the thrust in Equation 86 is easily calculated.

Thrust Buildup and Decay During Subcritical Phase

In the case of subcritical pressure ratio

$$\frac{p_a}{p_c} \geqq \left(\frac{2}{\gamma + 1}\right)^{\frac{\gamma}{\gamma - 1}}$$

the pressure at the throat is equal to the ambient pressure

$$p_e = p_a \quad .$$

Using Equation 77 the mass flow rate becomes

$$\dot{\omega} = p_c A_t \left(\frac{2}{RT_c}\right)^{\frac{1}{2}} \left(\frac{p_a}{p_c}\right)^{\frac{1}{\gamma}} \left\{\frac{\gamma}{\gamma - 1}\left[1 - \left(\frac{p_a}{p_c}\right)^{\frac{\gamma - 1}{\gamma}}\right]\right\}^{\frac{1}{2}} \tag{87}$$

and with Equation 82 the end velocity is

$$v_e = \left\{\frac{2\gamma}{\gamma - 1} RT_c\left[1 - \left(\frac{p_a}{p_c}\right)^{\frac{\gamma - 1}{\gamma}}\right]\right\}^{\frac{1}{2}} \quad . \tag{88}$$

Finally, the thrust can be expressed as follows for the subcritical phase

$$F = \frac{2\gamma}{\gamma - 1} p_c A_t \left(\frac{p_a}{p_c}\right)^{\frac{1}{\gamma}} \left[1 - \left(\frac{p_a}{p_c}\right)^{\frac{\gamma - 1}{\gamma}}\right] \quad . \tag{89}$$

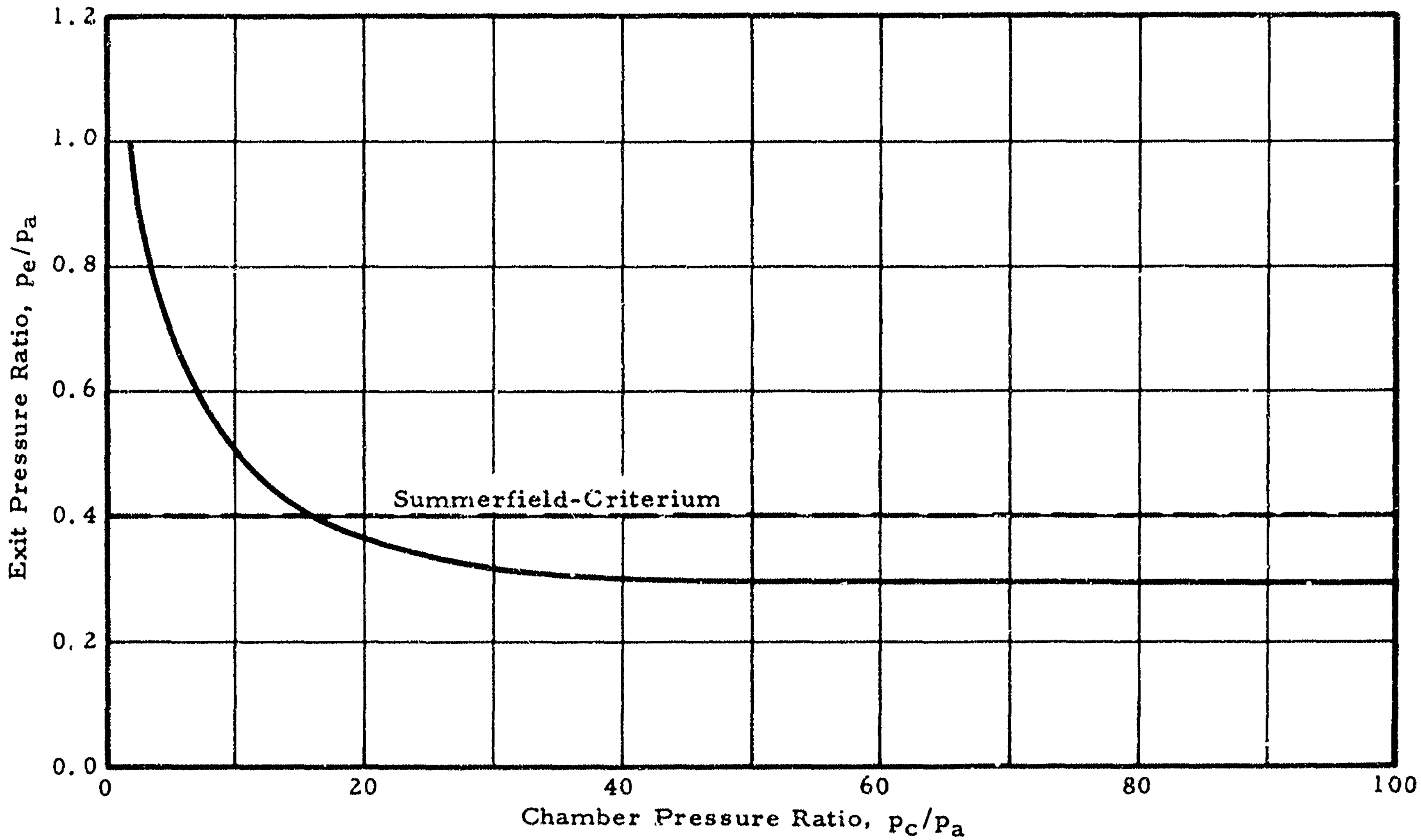



Figure 1. Exit Pressure at Separation as a Function of Chamber Pressure Ratio

# RESULTS

Expressions for the chamber pressure and thrust have been derived as a function of time for the supercritical and subcritical conditions during buildup and decay. Special attention was given to certain rocket motor design conditions, such as open end nozzles and closed nozzles by diaphragms during ignition. The results are differential equations which must be solved numerically for accurate purposes and in the case of variable parameters in the equations. In addition closed form solutions have been found based on assumptions such as constant propellant burning area and average burning rate.

# CONCLUSIONS

In many cases the total impulse during the thrust buildup and decay period is neglected by calculating average performance values. In order to evaluate the exact total impulse the transient conditions have to be taken into account. The instantaneous chamber pressure and thrust which are affected by the characteristics of the propellant and the design of the motor must be determined throughout the operating time of the rocket motor.

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