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A CRITICAL SURVEY OF PERT/COST, WITH EMPHASIS ON
THE MONTE CARLO TECHNIQUE OF NETWORK CALCULATION

by

John L. Underwood

Corporal, United States Navy

Submitted in partial fulfillment of
the requirements for the degree of

MASTER OF SCIENCE
IN
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Graduate School of Business Administration
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Berkeley, California

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A CRITICAL SURVEY OF PERT/COST, WITH A CASE STUDY ON
THE MARKOV CARLO TECHNIQUE OF NETWORK CALCULATION

by

John L. Underwood

This work is accepted as fulfilling
the thesis requirements for the degree of
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ABSTRACT

The PERT/CPM system, now in current use, requires several simplifying assumptions in order to permit the use of analytic methods for determining predictions of critical path activities and project duration. As a result of these assumptions, the predictions may involve significant errors, and will always be optimistic. The Monte Carlo technique of network calculation does not require these assumptions and, hence, is capable of yielding more accurate predictions and providing more useful information. This technique is discussed in detail. A versatile reliability density model for activity times is introduced, and a resource allocation technique based on the probability that an activity will be on the critical path is developed. Finally, an application of PERT network theory to military operations planning is described.
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1. Introduction.

The PERT system was devised by a study group under the direction of the Special Projects Office, Bureau of Naval Weapons, Navy Department, in order to develop a methodology for providing the management of the Fleet Ballistic Missile (Polaris) program with an information reduction system for program monitoring and evaluation. By this system, management was to be continuously apprised of progress to date for the program as a whole, and be furnished valid predictions as to outlook toward accomplishing program objectives. [2]

The system, as developed, was a giant stride in management technology, and is credited with a major contribution toward the rather phenomenal success of the PERT development program. Not only did the PERT system accomplish the objectives already mentioned, but it allowed management to predict those activities in the development project whose completion time would have a direct effect on the overall project duration. These critical activities could then receive appropriate managerial attention.

The PERT system has since been employed extensively in industry, and has been made a standard procedure for monitoring research and development projects under the cognizance of the Federal government. Industry acceptance of the system has been, in general, enthusiastic and widespread.

The system has been recently extended to include a cost control feature, PERT/COST, which has now been designated the standardized system for monitoring federally sponsored R&D projects.

Another system, the Critical Path Method, has paralleled PERT in
development. It is similar in many respects to PERT, lacking PERT's
stochastic representation of time, but incorporating a correlation be-
tween time and cost, enabling management to schedule optimally. The
Department of Defense and NASA have incorporated the desirable features
of both systems into PERT/COST.

The success of PERT and PERT-type systems has stimulated the inter-

test of professionals in the fields of mathematics, management science,
and operations research. The professional journals have featured many
papers on the relative merits of the systems, proposals for refinement
and extension of the systems, consolidation and integration of the sys-
tems, and the like. It is the purpose of this paper to consolidate many
of these ideas, and to prepare some refinements and extensions to the
system, designed to extend its usefulness and increase the accuracy of
predictions about the projects, and to generate specific recommendations
for control of time and cost.

We shall first briefly describe the basic PERT system, critically
discuss certain features of the system, and tender proposals by which
its basic system may be improved. In particular, we shall discuss the
Monte Carlo method of calculation of the PERT network, and show how this
method corrects many of the deficiencies inherent in the analytic method.

Next we shall describe the standard PERT/COST system currently being
implemented by DoD and NASA. We shall follow this with our proposal for
an integrated project control, scheduling, and resource allocation tech-
nique.

Finally, we shall discuss an extension of the PERT network approach
to the problem of planning projects involving major future decisions,
such as military campaigns.
2. Conclusions and Recommendations

As a result of this study, it is concluded that:
(a) The assumptions employed in the currently standard analytic PERT network calculations may induce significant error in the results.
(b) The standard analytic technique for network calculation fails to provide a sufficiently definitive measure for activity criticality.
(c) A more flexible model for activity completion time, providing a range of variance to allow for varying uncertainty in time estimates, is needed.
(d) The Monte Carlo technique does not depend on the assumptions which induce the inaccuracy in the analytic technique, hence is capable of yielding results limited in accuracy only by the validity of the activities for activity time. In addition, this method yields the probability that a given activity will be on the critical path, and other useful information not obtained by the analytic technique. The Monte Carlo technique can easily provide the flexibility discussed in (c), above.
(e) The Monte Carlo technique requires increased computer running time, but the increase is not prohibitive, and is well justified by the improvement in quality of the results obtained.
(f) A resource allocation technique, based on the probability that an activity is on the critical path, is feasible, and provides a reasonable method of optimum resource allocation.
(g) The network representation of the complex interrelationships between activities in a project is a technique which may well be applied to the task of planning military operations.
It is recommended that the Monte Carlo technique for PERT network calculations be employed in lieu of the analytic technique. A method of utilizing this technique is discussed in detail in Appendix I.
1. The Basic PERT System

The basic PERT system is a means of defining the relationship between the various activities comprising a project, and for estimating the time required to complete each activity and the project as a whole. Use of the system enables realistic schedules to be imposed on the project, and assists in controlling the execution of the project.

By a project we shall mean a collection of tasks, or activities, each a necessary step toward the achievement of some final objective. Subsets, called paths, of these activities, are dependent sequentially, that is, any activity in the path, other than the beginning activity, may not be commenced until its predecessor has been completed. Each activity in the path, except the beginning activity, has a predecessor in the path. An example of a project in this sense may be as simple as the construction of a house, where the activities are tasks such as laying foundation, erecting subfloor, installing rough plumbing, erecting framing, installing rough wiring, etc. Or a project may be as complex as development of the Fleet Ballistic Missile system, with many hundreds of activities involved.

A network of directed arcs is a very convenient way of representing a project. The activities are represented by directed line segments, called arcs, terminating at nodes, called events. (See Figure 1.) The time required to complete an activity is associated with the length of the arc representing it. Events represent points in time. Several activities may terminate at one event. If so, the event is said to occur when the last of these terminating activities is completed. At that point in time, other activities, whose commencement is contingent upon
A Network Representation of a Project

Figure 1

The events may sequence. These activities are represented by arcs directed toward from the event. In the house construction example, the installation of the interior wallboard may commence only after the rough plumbing and rough wiring are completed.

Figure 2

Two independent activities incident to an event

Activities may be represented by a network with one beginning and one end event. Problems, on delay, activities may be drawn.
In the network in order to achieve this format. Such dummy activities may consume zero time, or may represent a predetermined amount of scheduled "dead" time. It is common practice for the network to be drawn with beginning event on the left, and with the end event, representing the achievement of the final objective, at the extreme right of the network. 

A path may be defined as an unbroken chain of events, with origin at the beginning event and terminus at the end event. The network consists of many such paths, overlapping, parallelizing, and crossing each other. 

Activities generally consume time. The exact amount of time, $t$, which will be consumed in completing any one activity is not generally known in advance. If one looked at the history of a completed project, one could discover the exact amount of time, $t$, required to complete any given activity, and could label each event with its actual time of occurrence. The longest path, with respect to time, from the beginning event to the end event could then be found. This path is called the critical path. The activities on this path are called critical activities. The length of the critical path, the sum, $\Sigma t$, of the completion times for all activities on the path, is the total elapsed time from the beginning to the end of the project, and is called the duration, $D$. It is apparent that a small change in $t$ for one of the activities on the critical path causes the same amount of change in $D$, but a small change in $t$ for an activity not on the critical path would cause no change in $D$. Hence it is important that project management be able to predict the critical path in order to predict $D$, and in order to allocate resources, set fix
schedules, most effectively.

Any path, other than the critical path, is shorter than the critical path. The difference in lengths of the critical path and any other path is called the slack, $S$, for the non-critical path. An activity or event may be located on several paths simultaneously. The smallest slack associated with any path which includes a given activity or event is the value of slack associated with that activity/event. The activities/events on the critical path have zero slack. Any single activity may be delayed an amount less than or equal to its slack without affecting the duration of the project.

The term, slack, may have a slightly different meaning if some authority has imposed a Planned Completion Date, PCD, on the end event of the project. If so, we define the Scheduled Duration, $D_s$, as the elapsed time from the beginning event to the PCD. Then slack for the critical path is $(D_s - D)$. Slack for any other path is $(D_s - $ path length$)$. With this definition, slack for any activity may be either positive, negative, or zero, but all activities on the critical path will have the same value and will be less than that for any other activity in the network.

The total system is a graph by which predictions of the critical path slack for any activity may be made with sufficient accuracy to enable management to operate more effectively.

**Stochastic Network Model.** In the planning stage, before a project is undertaken, the actual time, $t$, which will be required to complete an activity, is an unknown quantity. It is unlikely that this time could be predicted exactly. We shall call such uncertain quantities
random variables, and symbolize this with an overbar, as \( \bar{e} \). The corresponding unbarred symbol will represent a specific value of the variable. It is possible to set limits, within which \( \bar{e} \) is almost certain to fall. It is also possible to estimate the shape of the probability density function governing the random variable, \( e \). The probability density function for \( e \) is a curve whose ordinate in a region is a measure of the likelihood that \( e \) will fall in that region. To be more precise, the area under the curve in a region is the probability that \( e \) will occur in that region. In Figure 3 the ratio of the shaded area to the total area under the curve is the probability that \( e \) will fall between \( e_1 \) and \( e_2 \).

\[ f(e) \]

\[ e_1 \quad e_2 \]

**Figure 3**

A Probability Density Function

The actual probability density function for \( e \) is difficult, if not impossible, to obtain, and will be unique for any given activity performed in a given time period. Theoretically, if the activity were performed repeatedly, under the same conditions, with no learning taking place,
then a histogram showing the number of completions occurring in a time, say, plotted against t in discrete segments, would approximate the probability density function. Since this procedure is not possible, we must be satisfied with the best estimate we can formulate, using available knowledge of the nature of the activity.

In order to estimate the probability density function of tv for a given activity, the most qualified estimator to choose is the mode, since it is the best estimate of the completion time for the activity. The optimist, i.e. the most optimistic estimate, is the greatest amount of time the activity may be reasonably expected to consume. Another, Min, is the amount of time the activity is most likely to require. Finally, he pessimist, or the most pessimistic estimate, t, which is the greatest amount of time the activity can be expected to consume, being completely unknown. Hence, t, t, and Min form the upper and lower bounds, respectively, of the variation of tv, and Min is the mode of the density function. The probability density function is assumed to have the value zero at P and to attain a single peak at Min.

Several well known mathematical functions exhibit the properties we have just assumed for the distribution of tv. The Beta function

\[ f(t) \propto k(t-a)^{x-1} (b-t)^{y-1} \]  

with parameters \( a, b, x, \) and \( y, \) was chosen as the model for the Ritz system activity time because it could be specified to fit the constraints of the problem. The three estimates \( t, M, \) and \( P, \) specified three of the four bounds of the Beta function. The remaining parameter was specified by constructively requiring the required function, \( f(t), \) to equal
One-sixth of the range is a frequently used estimator for the
standard deviation of unimodal frequency distributions, hence it is
considered to be a reasonable assumption. For example, the standard normal
distribution, truncated at ±1.64, has its standard deviation equal to
1/6 the range. 7 As a result of the above requirements, one of the
parameters, say, X*, becomes a function of the relative position of M in
the range between 0 and 1. Determination of this parameter requires
the solution of a cubic equation (see Appendix I). The exact determination
of the mean, or expected value of X*, symbolized by ε, is a tedious
process, since ε is a function of 2, 0, 0, 0, 0, 0, 0. The following reasonably
close linear approximation for the mean was adopted as a standard:

\[ ε \approx \frac{6X + 61 + P}{6} \]  

The probability density function thus assumed has a shape similar
to that shown in Figure 4. This distribution will be referred to subse-
sequently as the PERT Beta distribution.

![Figure 4](image)

A Typical PERT Beta Density Function

A more detailed discussion of the PERT Beta distribution is included in
Appendix I.
For purposes of predicting the critical path, the three estimates are obtained by means of the weighted average formula, (2), for the mean. The expected value, or mean, \( t_e \), is used as a deterministic value of \( t^* \) in the prediction of the critical path. \( t_e \) lies between \( t \) and the midpoint of \( (2,3) \), and is separated from \( t \) by 1/3 the distance between \( t \) and the midpoint. In this manner, account is taken of the usual tendency for activities to be optimistic about placing \( t \). The fundamental reason, however, for using \( t_e \) instead of \( t \) as an estimator for \( t^* \) lies in probability theory, rather than in an effort to correct for this supposed bias.

In order to determine \( t^* \), two new random variables, \( T^* \) and \( T_e^* \), are defined. \( T^* \) is associated with a given event, and is the elapsed time between the beginning event and the occurrence of the given event. \( T^* \) is the end of the random variables, \( t^* \), for the activities on the project, up to the event. At this point in the development of the basis TV system an assumption is made that the longest path to an event \( T^* \) is the path whose mean length is greatest. The mean length of \( T^* \) is the sum of the means, \( t_e \), of the activities on the path. Then, by an exception, the mean, \( t_e \), of \( T^* \), is the largest of the mean lengths of all paths to the event. Operating on the basis of the same assumption, the further assumption is made that \( T^* \) is approximately normal and used. This assumption is justified since we assume we know the longest path to the event, and since this path is the sum of a random variable. In fact, the Central Limit Theorem of probability theory asserts its

The normality assumption, and the Central Limit Theorem of probability theory asserts its
The distribution of \( T \) is now completely specified by \( T_0 \) and \( F(T) \) in view of the necessity of \( T^* \).

We may associate a random variable, \( T^* \), with an activity as well as an event. We define \( T^* \) for an activity as the elapsed time until the activity is completed. Note that \( T^* \) for an activity is not necessarily equal to \( T^* \) for the event following the activity. \( T^* \) for the event is equal to the minimum of the \( T^* \)’s for the activities incident to it, that is, terminating at the event.

The random variable, \( T_{10}^* \), for an event is defined as the latest time the event can occur and not delay the scheduled completion of the project. \( T_{10}^* \) is calculated by subtracting from \( T_0 \) the sum of the \( T^* \)’s for all the activities on the longest path from the event to the end event. Again the assumption is made that this longest path will be the path of the event with length \( \sum_{\text{eq}} \) is longest. Thus the value of \( T_{10}^* \), written \( T_{10}^* = (T_0 - \sum_{\text{eq}}) \), where the sum is the maximum of all such sums for paths between the given event and the end event. The variance of \( T_{10}^* \), \( \sigma^2(T_{10}^*) = \sigma^2(T^*) \), along the path with greatest sum length.

Another variable, \( T_{11}^* \), may also be associated with an activity. \( T_{11}^* \) for an activity is defined as the latest elapsed time since the beginning event at which the activity must be completed in order not to delay the scheduled completion of the project. Notice that \( T_{11}^* \) for any activity is the same as the \( T_{10}^* \) for the event following the activity.

It is apparent that the sum of the two paths used to determine \( T_{10}^* \) and \( T_{11}^* \) for a given event/activity is the longest path from the beginning to the end which contains that event/activity. From our definition of slack, we obtain the following relationship for slack, \( S_T \):
\[ K^* = D_0 - \left( D_0 - T_{L*} \right) - T_{I*} - T^* \]

then the mean value, \( K_g \), of slack is the difference, \( T_L - T_g \). Notice that this expression for \( K_g \) depends upon the assumptions made in determining \( T_L \) and \( T_g \).

The probability of meeting a scheduled completion date. We have developed the random variable, \( T^* \), whose distribution we have specified, by means of simplifying assumptions, to be normally distributed, with mean \( 0 \) and variance, \( \sigma^2(T^*) \). The probability, \( P_S \), that an event will occur before some scheduled completion date is simply \( P[T^* < T] \), where \( T \) is the elapsed time from the beginning event to the scheduled completion date. In particular, the probability, \( P_D \), that the project will be completed before the directed date is \( P[D < D^*] \), where the duration, \( D \), is a random variable equal to \( T^* \) for the end event. These probabilities can be obtained from a table of values for the normal random variable.

The completed critical path is that path whose mean value of slack, \( K_g \), is zero. All events or activities on this path will have the value \( K_g \), which will be less than the \( K_g \) for all other events/activities.

This completes the description of the basic PERT system which is fairly in general use. In practice, the network representing the project is drawn, events and activities are assigned identification numbers, and are assigned to specific organizational for responsibility.

Supervisors responsible for completion of each activity then submit the estimates, \( D, M, \) and \( P \). Several computer programs have been developed which take the activity network, preceding and succeeding event
numbers, and the three time estimates as inputs, then perform the network calculations. [5,7,15,20] The output lists the activities in any predetermined ordering, giving \( t_u \), \( t_g \), the probability, \( P_g \), of meeting \( t_G \), \( \sigma^2(t_u) \), \( \sigma^2(t_g) \). The reports may list the predicted critical path, and several other paths in increasing order of \( t_g \). Management may take appropriate action by scheduling of resources if the probability of meeting the overall project LOB is unacceptable small. In order to "buy time" additional resources are allocated, and additional managerial attention directed to those activities on the predicted critical path, and to those which seem slack is relatively small. Resources may be shifted from paths with greater \( t_g \) to paths with lesser \( t_g \). In more drastic reorganization of the network is required, placing of activities in parallel rather than in series paths, or eliminating activities whose products are not entirely essential to the project.

The results of original PROTOPA program may be the basis for establishing an overall project schedule. This schedule specifies scheduled completion times, \( t_G \), for activities, and scheduled times of occurrence, \( t_u \), for certain events.

While the project is in progress, revised inputs are submitted periodically, showing percent of completion, and revised estimates. In the case of a partially completed activity, these estimates should be considerably more reliable than those submitted before work was commenced. These inputs are fed to the computer, and another network calculation is run, as before, except that completed activities are noted and the actual completion time is used in the network rather than an estimated time. The results of these runs may necessitate schedule changes, or/for any of the actions previously mentioned.
A. Critique of the Basic PERT System

The system which we have just described tends to identify and call
attention to a single path, the Predicted Critical Path. This path,
though actually uncertain, is deterministic establishe by virtue of
fixed assumptions. In real networks, the activities on this "critical
path" will have probabilities of actually being critical which less than
unity. Furthermore, other activities, not on the path, may have prob-
abilities of being critical larger than some activities on the "critical
path." Hence, the system, by focusing attention on the single path,
emphasizes activities which are actually more important.

The assumptions which were made in determining $E_T$ and $T_S$ lead to
certain results in these figures, and in $E_T$. By means of the same
assumptions, the probability that an event is completed before a CEC is
also optimistic.

$E_T$ is the random variable representing the elapsed time to occur-
currence of an event. Suppose there be a different path to the event,
the random variables, $p_{11}$, $p_{21}$, $\ldots$, $p_{nn}$, be the lengths of these n
paths. Note $E_T$ is the minimum of these $n$ variables, but the minima of a
set of random variables is not normally distributed, even if the indi-
vidual variables themselves are normal. In particular, $E_T$ does not have
a distribution of the path with the largest mean, nor does $E_T$ have
the variance of this $p_i$. Furthermore, the $p_i$ are neither independent
nor identically distributed, hence the calculation of the distribution
function of this sum is too complicated to be solvable by analytical
methods.

In general, the probability that an event occurs before some
specific scheduled date is the probability that none of the possible paths overrun the scheduled date. By the PERT assumptions, we considered only one path, namely that with the greatest mean length. Obviously, if there are several parallel paths with nearly the same mean length, then by considering only the probability that one of the paths does not overrun, we are being quite optimistic for our result.

For example, suppose there are three parallel and independent paths, each approximately equally likely to be the critical path, and each having a probability of being less than \( P_D \) of about 0.50. Then \( P_D \), the probability that the project is completed prior to the EDD, is the probability that none of the path lengths exceed \( P_D \). Hence, \( P_D = (0.5)^3 = 0.125 \).

The PERT procedure would consider only the path with the largest mean length, and could calculate \( P_D = 0.50 \). For networks with several parallel paths of comparable length, the error would be quite large, in this example. In networks with only one predominately long path, the error would be negligible.

In order to obtain a more general prediction of criticality than the single path prediction, we need to calculate the probability that each activity will be on the critical path. We shall call this probability the Criticality Index, written \( P_C \). Computation of \( P_C \) involves the concept of determining the probability that a particular \( P_C \) will be the minimum of a set of \( n \) random variables, \( \{p_j\} \). As before, these \( p_j \) are not independent nor identically distributed, hence this computation is not feasible analytically. It is not, however, difficult to determine \( P_C \) by the Monte Carlo method. This method also readily yields \( P_C, T_L, \) and \( P_D \) without resorting to the assumptions which were so
troublesome, but necessary, in the analytic approach. A detailed discussion of this method, and suggestions for specific use of the criticality index will form a substantial portion of this study.

The beta probability distribution was chosen as the distribution for t for an activity. Recall that in specifying this distribution the variance of the distribution was arbitrarily chosen to be \((\frac{\alpha \cdot \beta}{6})^2\). By this assumption, we have specified the degree of uncertainty with which the estimator makes his three estimates. In other words, we are laying the difference between the case where the estimator may predict, with a high degree of confidence, that t will occur within a few time units of \(\bar{t}\), and the case where \(\bar{t}\) has a higher likelihood of occurring near the extremes of the range of t. The probability distribution for the former case should have a shape similar to that shown in Figure 5. While the latter case would be better represented by Figure 6. It is not unreasonable to assume that, under certain conditions supervisors will have a basis for estimates with a much greater degree of accuracy, even though the range of possible values of \(\bar{t}\) might be quite large. Under other conditions, the supervisor might be unable to place \(\bar{t}\) with an appreciable degree of confidence. In the interest of more accuracy in predicting the system behavior, perhaps a choice of density functions should be available to the estimator to help describe his confidence in his estimates. This would not require a knowledge of probability theory on the part of the supervisor. He could simply be asked to state his uncertainty as to the relative position of \(\bar{t}\) on a numerical scale, with three descriptive choices. His choice would lead to a corresponding probability distribution. Appendices I, II, III and IV
A technique of providing this flexibility in choice of density function, i.e., with the desired degree of variance.

Figure 5
A Density Function with Small Variance

Figure 6
A Density Function with Large Variance
5. The Monte Carlo Technique

We have seen that the completion time, $t_e$, for any activity in a proposed project is a random variable. As the project is performed, each activity is completed in some actual time, $t_a$, only after completion of the entire project can actual values of $t$, slack, and elapsed time, $t_r$, for events and activities be fixed, and the critical path determined.

The Monte Carlo method calculates average values of these parameters by repeatedly simulating the performance of the entire network, and statistically averaging the results. The simulation of the project is accomplished as follows:

A random value of $t$ for each activity in the network is drawn from a probability distribution for $t_a$ for the activity. These random values of $t$, called realizations of $t_a$, are drawn in such a way that in repeated drawings from a specific probability distribution, the number of realizations falling within a segment of fixed length on the $t$ axis is proportional to the height of the probability density curve over that segment. Each activity has its own unique probability distribution, determined by the three estimates, hence, for one simulation of the project, a realization of $t_e$ drawn for an activity is a completion time which would have resulted from a normal performance of the activity. One simulation of the project consists of drawing a value of $t$ for each activity of the network. Calculations of the network are then performed, using the values of $t$ instead of the $t_e$ used in the analytical method. $t$ and $t_r$ for each activity are calculated, and then the critical path for that simulation is determined. For each activity a tally is incremented
if the activity was actually on the critical path. Another tally is in-
corrected for each scheduled event if the event was accomplished prior
to its scheduled completion date.

The above simulation is repeated many times and the values of t, T,
and g for each activity are added to cumulative totals after each simu-
lation of the project. After an adequate number of simulations, these
cumulative totals and tallies are divided by the number of simulations,
yielding the average values, t^g, g^g, and T^g for each activity and the
probability that the activity will be on the critical path. For all
scheduled events, the probability that the event was completed prior to
the scheduled date is also computed. The observed variance of these
statistics is also available if desired, as well as an approximate proba-
bility distribution for project duration.

The kind of coupling the probability distributions of the activity
times for calculations of t^g is discussed in detail in appendices I, II,
III, and IV. Using the above techniques, it is not difficult to provide
a choice of probability distributions for t^g in order to account for
varying degrees of uncertainty in activities, as was discussed earlier.

The computer codes designed to implement the basic PERT system may
be adapted, without great difficulty, to perform the Monte Carlo calcula-
tion. The author recently modified the North American Aviation PERT
code (T code in this paper). This program, designated for the CDC 7600,
will now evaluate a network of 1500 activities without requiring external
input data. [16]

The Monte Carlo technique is not subject to the pitfalls of inde-
pendence, assumptions of normality due to the Central Limit Theorem, and
assumption of a particular longest path, etc., which plague the analytic computation. The inaccuracies inherent in the Monte Carlo technique are due to non-randomness in the random number generator, and perhaps, failure to perform a sufficient number of replications to permit the law of large numbers to obtain. Excellent random number generators are available, providing results will within the limit of accuracy imposed by the law of large numbers. Statistical methods exist by which to determine the number of replications required in order to obtain the desired degree of confidence in the results. [17]

The system is limited in accuracy only by the random error in estimating activity times, and in approximating the correct probability distribution for use.

The Monte Carlo technique requires less computer running time than the analytic method. An efficiently coded routine requires approximately 300 microseconds to draw a random number and then the calculated solution, to transform it into a realization of the system. For details, see Appendix I. The approximate increase in time required, if the Monte Carlo technique is used in lieu of the standard analytic method, may be estimated by the product

\[ \text{No. of activities} \times \text{No. of replications} \times (750 + 10^{-6} \text{ sec}) \]

For example, a network consisting of 1000 activities would require 2.5 million of additional computer running time if a Monte Carlo calculation of 1000 replications were used instead of the standard technique. errors inherent in this technique would seem to render this increase time negligible.
8. The Extension to FERT/COST [10]

An extension of the basic FERT system, FERT/COST, was adopted 1 July 1969 as the basic standard for management time and cost control systems by USAF and USA. FERT/COST extends the original FERT system to provide cost estimation and control features in addition to the procedure for prediction and control of time performance. Two additional optional features, a Time-Cost supplement, and a Resource Allocation supplement were also promulgated.

FERT/COST adds to the basic system the following capabilities:

1. Detailed estimation of project cost, broken down by work packages, sub-systems, etc.

2. Provision of a consolidated estimate of required costs for manpower and other resources as an aid to scheduling and procurement.

3. Provision of an in-progress cost reports, showing current cost status and revised cost estimates in comparison with budgeted expenditures and contract estimates.

The optional Time-Cost supplement provides a method of estimating flexible schedules for project completion, with their estimated costs, and in evaluation of the technical risks associated with each schedule. These schedules are designed to provide completion in accordance with these criteria: (1) the most efficient use of time and resources; (2) completion by the EOD; (3) earliest feasible completion.

The Resource Allocation supplement outlines a procedure for scheduling a project to meet a EOD in an optimal manner with respect to costs, utilization of manpower.
The concept of operation of PERT/COST is essentially as follows:

A. **Project Breakdown.** The project is broken down into activities and work packages, and graphically represented by a network. For the purposes of cost reporting, estimating, and accounting, several activities may be combined into a "work package", when the activities involved are small, and detailed cost reporting and estimating of these activities would involve unnecessary expense. In general, a work package does not comprise a closed network. That is, it may not be encapsulated and represented by a single activity due to the presence of events inherent to the work package which are connected by activities to other work packages. In this case, the work package must be broken down into its component activities and events for the network time calculations. The cost estimating and expenditure reporting section of the system may utilize the single work package as the smallest organizational division.

Certain work packages may not appear on the project network. In general, these are the functions of management not directed toward the accomplishment of specific activities and events. For example, the forecasting, purchasing, or management travel costs for the project would be difficult to represent as activities in the network, however, the cost of these work packages may be estimated and reported, and treated in the cost analysis portion of the PERT/COST system.

B. **Time Calculations.** Time estimates are made for each activity, and the network calculations performed. The project duration is compared with the PCD, and if necessary, any of the following actions taken in order to reduce the overall project duration:

1. Rescheduling the network by increasing the parallelism of activities.
(b) Commit additional resources to selected activities, or reallocate resources within the network.

c) Change or delete activities.

c. **Hiring/Leaving.** A breakdown by skills of manpower required for an activity/work package may be included with the time and cost estimates. Computer programs now in existence assemble this information and prepare a report and display showing the project manpower requirements, by skills, plotted against time. [7, 11] This information provides project management with a forecast of manpower needs, and a basis for rescheduling activities in order to best utilize available manpower. Scheduling a large project without considering the overall use of manpower skills tends to vary unrealistic manpower use. The time period may require a lower level for a certain skill for in excess of that available, while the following time period takes little use of that skill. In performing the necessary rescheduling, the time constraints of the network and effectiveness of activities must be considered. Those activities least likely to be critical are the obvious candidates for rescheduling for the purpose of leveling manpower requirements. Computer programs now in existence, supplementing standard PERT routines, perform this rescheduling under the constraints of minimum availability of manpower and with consideration of slack. [7, 11]

D. **Relining the network.** A scheduled duration, $t_0$, is predetermined for each activity. These times may be less than, equal to, or greater than $t_0$, depending on the situation and management policy. Using the $t_0$ for each activity, the network is computed and values of $S_g$ and $S_r$, corresponding to $T_g$ and $T_r$, are determined. $S_g$ represents the earliest
date on which an activity/event can be scheduled for completion. \( S_e \)
represents the latest date on which an activity/event may be scheduled
for completion without causing a schedule slippage for the project com-
pletion. \( S_g \) and \( S_L \) are computed in exactly the same manner as were \( T_g \)
and \( T_L \), when \( T_g \) is used as activity time instead of \( T_e \). As a final step
in scheduling, an elapsed time, \( T_g \), is imposed at each event. \( T_g \)
is chosen to lie between \( S_g \) and \( S_L \), and may be translated into a firm sched-
uling completion date for activities immediately preceding the event, and
a starting date for activities immediately following the event.

II. Completing the Project. After the schedule is finalized, cost
estimates are prepared for each work package, and a budget prepared for
financing funds.

III. Monitoring Progress of the Project. As the project progresses,
isos are submitted from activities/work packages, indicating progress,
progress estimates, funds committed to date, and new estimates for cost.
The information forms inputs to the YERT/COST computer program, which
performs the network calculations, and prepares reports showing new
values for \( T_g, T_L, T_R, P_D, \) and the new critical path, as well as current
cost estimates in comparison with estimates and budget. The new projected
cost curve is also generated and compared with estimates and budget.

With this information, management may reschedule, reallocate, re-
build, or take whatever action as may be appropriate to the situation.

The Time-Cost Option Supplement. The purpose of this supplement is
an outline of procedures by which a project manager may prepare three
alternative project schedule proposals, and evaluate the technical risks
associated with each. By technical risk, we mean the gamble of
performance, time, and cost incurred by departing from the best development technique in order to meet the schedule.

First, a plan is formulated designed to meet the project ECO, as proposed by the controlling agency.

Next, a More Efficient plan is formulated, in which the best development techniques are used in order to reduce technical risks, and most efficient use is made of manpower and resources, resulting in greater duration for the project. Total cost is generally lower for this plan.

Finally, a Shortest Time plan is proposed. The purpose of such a plan is to explore the feasibility of scheduling the project for a shorter schedule than that proposed by the contracting agency in order to benefit from any strategic benefits which might accrue from such early completion. In order to arrive at the shortest time plan, the schedule is required by allocation of additional resources, permitting activities which should normally go in sequence, eliminating activities, and 2) using technical approaches to the problems of design and construction.

All such actions serve to increase the technical risks involved, and most of the increase cost.

The three options are presented to the contracting agency with estimates of cost and time, and with a comparison and evaluation of the technical risks associated with each plan.

The PERT/COST system is the primary tool for obtaining the time and cost estimates needed to prepare the three schedule proposals.

The Resource Allocation Supplement: This supplement provides a procedure by which project managers may schedule the project in the optimal manner with regard to the time-cost tradeoff. The concept of operation
is as follows.

For any activity there may exist several feasible schedules, involving different time-cost relationships. (See Figure 7.) Efficient use of manpower and existing resources and machinery may result in schedule A. Further time extension to B might result in higher costs due to the effect of fixed costs. By use of overtime, hiring additional manpower, or purchase of additional machinery or space, time-cost combination C or D might be achieved.

![Figure 7](image)

**Figure 7**

Time-Cost Plans for an Activity

Schedule A would be the most economical for this activity, hence would be the time estimated for the activity for the first network calculation. In a similar manner, most economical estimates are obtained, with corresponding cost estimates, from the other activities. The network calculation is then performed, and the project duration, D, compared.
with $B_g$. If $D > B_g$, selected activities on the critical path must be shortened by resorting to a higher cost-shorter time work schedule.

For each activity on the critical path, determine the increase in cost, divided by the decrease in time that would occur in moving to each shorter time-cost point. This is the slope of the time-cost curve for the corresponding time reduction. The activity with the lowest value of slope is then chosen as the activity to reschedule.

If this time reduction in the critical path exceeds the difference in slack between the critical path and the smallest slack not on the critical path, it becomes necessary to recompute the network and determine a new critical path. The above process is repeated as often as necessary until $D < B_g$.

Slack paths are then reexamined to see if any activities may be extended to lower cost points without going critical. If fixed costs were not a part of the activity estimates, notice may be taken of the fact that the most economical point to operate is usually to the left of the minimum cost point with respect only to direct costs. After determining the most economical plan with respect to direct costs, a test is made to determine if further time reduction below the ECD would result in lower overall costs. This is done by adding on the fixed costs for the project and attempting further reduction in time. If further time reduction results in lower overall cost, then further time reduction is performed until total costs reach a minimum.

The schedule is then adjusted as necessary to reach an optimum leveling of resources and other resources within the existing constraints considering total cost.
A method for Resource Allocation, using the Criticality Index

In Section 4, we pointed out the error in the concept of a deterministic prediction of the critical path. We also showed the optimistic bias inherent in the assumption that the longest path to an event will be the path whose mean length is greatest. The Monte Carlo technique does not depend on this assumption, hence, with this technique we can calculate $P_C$, $T_g$, and $P_D$ with accuracy limited only by the supervisor's estimates and our approximation of the true probability distributions of $r_i$ for the activities. We shall now develop a system for optimum resource allocation utilizing the criticality index obtained from the Monte Carlo calculation.

In order to use this method, we must require that a work package be a closed network. A closed network is a network with a single beginning and a single end event. Activities external to the closed network cannot be incident to, or emanate from, events within the closed network other than the beginning or end event. (See Figure 8.) In this example, if no activities not included in the work package should be incident to, or emanate from events B, C, or D, then the network would not be closed.

The restriction is implicit in the resource allocation supplement of the standard PERT/COST system, although not stated in the PERT/COST manual. Notice that the closed network may be integrated and represented by a single activity connecting events A and E. Some PERT computer programs now in existence are capable of integrating such sub-networks for simplification of the network for the benefit of higher level of management. [15]
A Closed Network

The next requisite that the supervisor in charge of each work package submit is any time-cost plans he deems feasible for his work package completion. These plans are represented by points on the time-cost curve in the same manner as previously discussed in connection with the Resource Utilization Supplement. No points to the right of the minimum cost point are submitted. The time-cost information is represented as a set of points, (time, cost), in descending order of time and corresponding ascending order of cost. For example:

- 14 weeks, $40,000
- 12 weeks, $60,000
- 10 weeks, $90,000

The time corresponding to the minimum cost point is taken as the most likely estimate, $t$, of activity completion time. The additional time estimates, $\delta$ and $\varphi$, are also submitted for the minimum cost plan.
These three entities and the corresponding cost are the basis of the preliminary calculation of the network. From this calculation, the probability, \( P_D \), of meeting the project ECD, is determined. Recall that this probability is considerably more valid as a result of the Monte Carlo technique.

If \( P_D \) is unacceptably small, the network must be compressed by redesigning the network layout and/or allocating more resources to selected work packages in order to achieve shorter completion time. If, after whatever network redesign is considered feasible, the network must be further compressed, the resource allocation phase of the program is performed. This phase consists of the following logical steps:

1. For each time-cost plan of each activity having positive \( P_C \), a following figure of merit is computed.

\[
MTB = P_C \frac{\Delta C}{\Delta T}
\]

where \( \Delta T \) is the reduction in time achieved by using the T-C plan being considered, over that used in the previous network calculation. \( \Delta C \) is the net, ending increase in cost. MTB may be considered as the expected amount of time which will be "bought" per unit cost by selecting the T-C plan being considered.

2. Form 3-tuples consisting of each MTB, its corresponding \( \Delta T \), and its activity identification. Order this list in descending order of MTB.

3. Start from the top of the list, a sufficient number of reduced T-C plans so that the sum of these \( \Delta T \)'s does not exceed a predetermined percentage, \( p \), of \( D \). The best value for \( p \) may be determined experimentally. The reduction is made from the top of the list, but only one
plan, that with the largest \( \Delta t \), is retained for any one activity.

4. The activity chosen for time reduction, the time corresponding to the new T-C plan is taken as the new value of \( \Delta t \). The estimates for \( p \) and \( q \) are revised downward by multiplying by the ratio, \( \frac{q_{\text{new}}}{q_{\text{old}}} \) (This procedure may be subject to question; another method could be to require \( p \) and \( q \) estimates for each T-C plan.)

5. With the new time and cost estimates for the accelerated activities, the network is again calculated, and \( P_b \) determined. All activities will have a new \( P_b \). The new duration, and \( P_b \) are again evaluated. If still unacceptable, the process may be continued until either an acceptable \( P_b \) is attained, or the budgetary limit is reached. The schedule is finalized on the basis of the time plans used in the last network calculation, after consideration of manpower and other resource leveling.

By the above method, resources are allocated where, probabilistically, they can be expected to contribute cost effectively to shortening project duration. The requirement for several feasible time-cost plans may serve the additional purpose of forcing line supervisory personnel to consider the time-cost relationship more carefully, allowing operation at the most efficient point on the time-cost curve whenever feasible.

An optimum value for \( p \) can be determined with a few experimental runs of the program. \( p \) should be chosen as large as possible in order to reduce the number of network calculations required, but small enough so that successive calculations produce a smooth compression of the network. Operation of the program will tend to decrease the differences between the MSA's for the activities, which is equivalent to operating the activities at the same level of expected marginal utility. This is
a well known principle of optimization.

The inputs to the system, three time estimates and a T-C function
specified by several 2-cupla, are not complicated, and are compatible
with the concept of progress and cost monitoring envisioned in the TMT/
COST system.

A flow diagram of the logic used in this resource allocation method
is shown in Appendix V.
8. A Military Application of the PERT Network Approach

One of the most outstanding features of the PERT approach is the graphical portrayal of the project by means of the network. By this device, relationships between the various tasks may be clearly visualized. The insight thus gained by the managers of the project, into these complex relationships, entitles them to function with greatly increased effectiveness in their managerial capacity.

A military operation is in many ways similar to a development project. The operation may be divided into a number of tasks, with many interrelationships existing between individual tasks. Hence, the network may serve the same useful function in this application as in industrial development.

The phases of military planning known as "The Development of the Plan," and "Supervision of the Planned Action," are most readily benefited by the network approach. In the formalized military planning sequence, the Development of the Plan phase has been preceded by the "Estimate of the Situation," in which the assigned mission has been studied, possible outcomes based on enemy capabilities and alternative courses of action have been analyzed in a game theoretic matrix, and a decision has been reached regarding a general course of action to be pursued.

The problem involves developing a complete, detailed plan for the operation, including organizing the available forces and assigning to each task unit the appropriate tasks which make up the general plan. At this stage, detailed planning is done in which training, acquisition of intelligence, movement, communications, logistics, and battle action are

35
The plan begins with an initial concept, which may be represented by a relatively simple network, whose tasks are stated in broad terms of accomplishment. As the planning becomes more detailed, these initial tasks may be subdivided into networks of lower-order tasks. This process may be done at lower echelons of command, upon receipt by them of their superior’s directive.

The network will aid in establishing a firm schedule for the operation, in much the same manner as is done in industrial projects. It may also be a valuable aid for conducting briefings for subordinates, since it will enable them to better appreciate the relationship of their own task to the whole operation.

During the operation, the network may serve as a display, on which current progress is indicated as reported. The effects of delays and failures may be more readily evaluated and corrected.

The initial concept of a plan to execute an amphibious landing in enemy-held territory might be represented by the network shown in Figure 9. A network may also be used to lay out the basic strategy of a protracted military campaign. In this usage, tasks would be represented by military operations, such as: "Seize Island B," or "Cut supply line A." The procedure is similar to that employed in the operation just discussed, except that it takes place over an extended period of time. In many cases, proposed actions are tentative, or the selection of alternatives may depend on future developments, such as the success or failure of a previous endeavor, or on the subsequent enemy reaction. We recognize this need for decision at a particular point in the network.
by showing a "Decision box" in lieu of an event at that point. The deci-
sion box has more than one task emanating from it; the decision, to be
made at some later date, determining which task(s) to be performed. The
network then proceeds to show the planned action to be taken for each
alternative. Figure 10 is an example of a portion of such a strategy
network. [33,34]
APPENDIX I

Monte Carlo Techniques with the Beta Distribution

The model for activity completion time, \( t^* \), chosen by the originators of PERT, is the Beta probability distribution, characterized by its density function:

\[
X(t) = \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)} t^{\alpha-1}(1-t)^{\beta-1}, \quad 0 < t < 1
\]

which may reduce this to the standard form of the Beta distribution by the following transformation:

\[
x^* = \frac{t - \alpha}{\beta - \alpha}
\]

The probability density function of \( x^* \) is:

\[
x^*_n(x) = \frac{1}{n} x^{n-1}, \quad 0 < x < 1
\]

The variable \( x^* \) is equivalent to \( t^* \), where \( \alpha = 0 \) and \( \beta = 1 \).

The two parameters, \( \alpha \) and \( \beta \), specify the upper and lower limits of the variation of \( t^* \). It is now desired to specify the remaining two parameters, \( \alpha \) and \( \beta \), so that the mode of the density function occurs at \( x^* \), and so that the distribution has the desired variance.

Let \( \alpha = \frac{\beta - \alpha}{\beta} \) denote the mode of \( x^* \). By finding the root of \( x^*_n(x) \), we obtain:

\[
x^* = \frac{\alpha}{\alpha + \beta}
\]

The variance of \( x^* \) is given by:

\[
\sigma^2(x^*) = \frac{(\alpha + 1)(\beta + 1)}{(\alpha + \beta + 2)(\alpha + \beta + 3)}
\]

Now suppose we desire that the standard deviation, \( \sigma(x^*) \), be expressed by \( (\frac{t^* - 1}{d}) \), where \( d \) is an arbitrary constant. Then \( \sigma(x^*) = \frac{1}{\alpha + \beta} \), by
requiring that $S^2(x^*) = \frac{1}{d^2}$, we obtain the following relationship in $x^*$ and $m$.

$$x^2 + (d^2 m^3 - d^2 m^2 + 7m) x^2 + (16 - d^2) m^2 x^2 - (d^2 - 12) m = 0$$

Now $x^*$ can be determined as a function of $m$ and $d$ by the above equation.

With $x$ determined, $f$ is specified by solving equation (8):

$$f = \frac{x(1-m)}{m}$$

The mean of $x^*$ is given by:

$$E(x) = \frac{x^* + 1}{x^* + d}$$

Since $x^*$ must be determined from the cubic equation, (10), for each value of $m$, the computation of $E(x)$ is cumbersome. In the PERT Beta model, $d$ was chosen as six. For this value of $d$, $E(x)$ is approximately linear in $m$, and may be approximated by the following relation:

$$E(x) = \frac{4m + 1}{6}$$

The transformation to $x^*$ yields:

$$E(x) = \frac{6 + 6M + p}{6}$$

**Perturbed Sampling from the Beta Distribution.** There are several good random number generating routines which yield random numbers from the uniform (0,1) distribution. We shall use the technique of the Probability Integral Transformation to transform the random number, $u$, drawn from the uniform (0,1) distribution, to a corresponding sample from whatever other probability distribution we desire. [3]

This transformation depends on the following theorem:
For any random variable, v, having the probability distribution function, \( F_{v_1} \), and the density function, \( f_{v_1} \), define the random variable, \( u = F_{v_1} \), that is:

\[
u = F_{v_1}(v) = \int_{-\infty}^{v} f_{v_1}(t) \, dt
\]  

(15)

then, \( u \) is a uniformly distributed random variable on the interval \((0,1)\).

Hence, we may use the inverse function, \( v = F_1^{-1}(u) \), to transform the sample, \( u \), from the uniform \((0,1)\) distribution to the corresponding sample, \( v \), from the desired distribution.

The distribution function:

\[
F_{v_2}(x) = F_1 \left( \int_{0}^{x} y^{\infty} (1-y)^{d-1} \, dy \right)
\]

(16)

for the data variable, \( x \), must be calculated by numerical integration.

In order to utilize the procedure given above for sampling from this distribution, we may store the tabulated distribution functions in the memory of the computer and use a table look-up routine to enter the table with the random sample, \( u \), and determine the corresponding value for \( v \). A complete distribution function must be stored for each of the increments of \( n \), and for a limited number of values of \( d \).

The input card for an activity specifies the choice of \( d \) which best represents the estimator's uncertainty. Also specified are the three estimates, \( \hat{r}, \hat{h}, \) and \( \hat{p} \). With \( d \) and \( h \) specified, the proper table may be selected, \( u \) is produced by the random number generator. The corresponding value of \( u \) is obtained from the table, then transformed into a realization of \( v \) by the following transformations.
\[ t = (P - d) x + \theta \]  

Tables I through IV are tables of distribution functions for the standard beta distribution, with \( d \) taking on values of 4 through 8, respectively. A complete function is given for each of 11 values of \( n \).  

As an example of the use of the procedure given above, suppose \( d = 4, 6, \) and \( 8 \) were 12, 15, and 20 weeks, respectively, and the selection, \( d \), 6, 12 weeks. Then \( n = \frac{12-12}{8} = .375 \). Suppose we draw the random number, \(.795\), from the uniform random number generator. Entering Table III for \( n = .3 \), with the argument, \(.795\), we obtain \( x = .50 \). From the table for \( n = .4 \), linear interpolation yields \( x = .572 \). Interpolating between these two values of \( x \), we obtain \( x = .554 \) for \( n = .375 \). The corresponding realization of \( t \) is then \( 8x + 12 = 16.4 \).  

An efficiently coded routine designed to perform the above table searches and calculations, requires approximately 750 microseconds.  

Figures 11, 12, and 13 are families of curves of the probability density functions for \( x \) corresponding to \( d = 4, 6, \) and 8, respectively.
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</table>

**Table I**

\[
F_x(x) = \frac{\int_0^x (1-y)^A \, dy}{\int_0^1 (1-y)^A \, dy}
\]

$\alpha$ is the position of the mode

---

**Best Available Copy**
### Distribution Function of the Standard Beta Function with $D = 5$

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### Table II

\[
F_x(x) = \frac{\int_0^x y^A (1-y)^G \, dy}{\int_0^1 y^A (1-y)^G \, dy}
\]

$x$ is the position of the mode.
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<th>$G=3.595$</th>
<th>$G=3.781$</th>
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<td>$=.000$</td>
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<td>0.722</td>
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**Table III**

\[
F_X(x) = \frac{\int_0^x y^{\lambda} (1-y)^G \, dy}{\int_0^1 y^{\lambda} (1-y)^G \, dy}
\]

$x$ is the position of the mode.

47
### Table IV

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</tr>
<tr>
<td>1.00</td>
<td>1.20</td>
</tr>
</tbody>
</table>

The table provides values for the function \( x_0 \) and \( y_0 \) as \( x \) varies from 0.05 to 1.00 in increments of 0.05.
<table>
<thead>
<tr>
<th>( \frac{\Delta}{\kappa} )</th>
<th>.008</th>
<th>.020</th>
<th>.041</th>
<th>.082</th>
<th>.162</th>
<th>.325</th>
<th>.650</th>
<th>.130</th>
<th>.260</th>
<th>.520</th>
<th>.100</th>
<th>.200</th>
<th>.400</th>
<th>.800</th>
<th>1.600</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \frac{\Delta}{\kappa} )</td>
<td>.000</td>
<td>.090</td>
<td>.185</td>
<td>.370</td>
<td>.740</td>
<td>1.480</td>
<td>2.960</td>
<td>5.920</td>
<td>11.840</td>
<td>23.680</td>
<td>47.360</td>
<td>94.720</td>
<td>189.440</td>
<td>378.880</td>
<td>757.760</td>
</tr>
</tbody>
</table>

Table V

\[
F_x(x) = \frac{\int_0^x A (1-y)^6 \, dy}{\int_0^1 y A (1-y)^6 \, dy}
\]

This is the position of the mode.
Figure 12

Data Density Functions for $d = 6$

Nodes at 0, .1, .2, .3, .4, and .5
Figure 13

Beta Density Functions for \( d = 8 \)

Values at 0, .1, .2, .3, .4, and .5
APPENDIX II

A Simpler Method for Obtaining the Beta Distribution

Since the Monte Carlo method does not require that the variance of \( P \) be calculated and applied in order to evaluate \( P_D \), we may consider disregarding with the requirement that the variance of \( P \) be constant over the range of \( m \). This requirement is not necessarily a natural one. The variance of the triangular distribution, for example, defined naturally by \( m \), is given by:

\[
\sigma^2(m) = \frac{1}{18} \left( \frac{5m^3 + 6m^2 + 6m + 1}{m^2} \right)
\]

We note that the variance for this distribution varies from a maximum of \( \frac{\pi^2}{18} \) with \( m \) at the extreme of its range, to \( \frac{1}{18} (2m)^2 \) with \( m \) at the extreme. The variance of the SinCos function, (Appendix IV), another function naturally specified by the three parameters, behaves in the same general manner.

We may simplify the determination of the beta function parameters, including the necessity to solve the cubic equation, (10), by the following procedure:

Let the variance of the symmetric beta function, \( m = 0.5 \), by

\[
\sigma^2(m) = \frac{1}{18} \left( \frac{5m^3 + 6m^2 + 6m + 1}{m^2} \right)
\]

Where \( d \) is an arbitrary positive constant \( \geq \sqrt{12} \), or

\[
\sigma^2(m) \bigg|_{m = 0.5} = \frac{1}{d^2}
\]

For the case where \( m = 0.5 \), equations (8), (9), and (20) yield the following:
\[ \alpha = \beta = \frac{d^2}{8} - \frac{12}{d} \]  
\[ \gamma = \frac{d^2}{8} + \frac{2}{d} \]  

Then, for any value of \( d \) in its range, let \( \alpha \) and \( \gamma \) be determined as follows:

\[ \gamma = \sigma, \quad 0 \leq m \leq 0.5 \]

\[ \alpha = \sigma, \quad 0.5 \leq m \leq 1 \]

The remaining parameter may be determined by equation (8). The variance of \( \omega \), as determined by equation (5), then varies with \( m \) as shown in Figure 14.

**Figure 14**

Change of Variance with Mode Position

The mean, \( E[b] \), may be calculated directly by equation (12). The following close linear approximations for \( E[b] \), for several values of \( d \),
show the relative weights given to d, m, and p in the determination of the expected value, hence give an intuitive feeling for the relationship between d and the amount of uncertainty with which m is located. 

\[ d \quad \frac{\gamma [d]}{2 \gamma [n] + \lambda + 2p} \quad 6 \quad \frac{d + 4n + p}{6} \quad 8 \quad \frac{d + 4n + p}{8} \]

By this relatively simple procedure, the parameters of the Beta function for any desired values of d may be determined for the purpose of the Monte Carlo calculations. The Probability distribution functions needed for the transformation from \( u^2 \) to \( x^2 \), may be extracted from a table of the incomplete Beta function. [1]
The triangular probability density function is naturally defined by the three time estimates, \( \theta \), \( \mu \), and \( \psi \). Since the distribution function of this density may be calculated by a simple formula, the transformation from a sample, \( u \), drawn from a uniform \((0,1)\) distribution, to a sample from the desired triangular density, may be accomplished by a single, simple calculation. Then this procedure can be used, a considerable saving in computer running time is effected.

The technique is illustrated below for the random variable, \( y^* \), having the general triangular distribution with lower limit, \( \theta \), upper limit, \( \psi \), and mode, \( \mu \).

![Figure 15: Triangular Density Function](image)

The density function of \( y^* \) is:

\[
\begin{align*}
    f_{y^*}(y) &= \frac{2(y-\theta)}{(\psi-\theta)(\psi-\mu)} & \text{if } \theta \leq y \leq \mu \\
    &= \frac{2(\psi-y)}{(\psi-\theta)(\psi-\mu)} & \text{if } \mu \leq y \leq \psi \\
    &= 0 & \text{elsewhere}
\end{align*}
\]

(25)
The distribution function, \( F_y(y) = \int_{-\infty}^{y} f_y(x) \, dx \), is:

\[
F_y(y) = \begin{cases} 
\frac{(y-d)^2}{(p-0)(p-0)}, & 0 < y < d \\
\frac{y^2}{(p-0)(p-0)} + 2(p-y) + d^2, & d < y < P \\
0, & y < d \\
1, & y > P.
\end{cases}
\]

Now let \( u = F_y(y) \), and solve for the inverse function, \( y = F_p^{-1}(u) \).

As a result:

\[
y = d + \sqrt{u(p-d)(p-0)}, \quad 0 < u < \frac{d-0}{p-0} \\
y = \frac{\sqrt{1-u(p-0)(p-d)}}{p-0}, \quad \frac{d-0}{p-0} < u < 1.
\]

If we draw a random number, \( u \), from the uniform \((0,1)\) distribution, estimate \( y \) by means of the function, (27), then \( y \) has the desired distribution.

The mean, \( \mu y \), is given exactly by:

\[
\mu y = \frac{d + \frac{p + d}{2}}{2}.
\]

The variance, \( \sigma^2(y) \), is given by:

\[
\sigma^2(y) = 2\left( \frac{p-d}{6} \right)^2 - \frac{(p-0)(p-0)}{18}.
\]

As the right varies from 0, when it is at either extreme, to \( \left( \frac{p}{6} \right)^2 \), when it is at the midpoint.
APPENDIX IV

The SinCos Function.

Another Probability Function Suitable for PERT
Monte Carlo Calculations

Another probability function which is completely specified by its
mean and mode is the following, which we shall call the SinCos
function, for obvious reasons.

The probability density function is:

\[
\begin{align*}
    f_{\text{SinCos}}(x) &= \frac{\pi}{2(p-0)} \sin \left( \frac{\pi (x-\mu)}{2(p-0)} \right), & 0 \leq x \leq p \\
    &= \frac{\pi}{2(p-0)} \cos \left( \frac{\pi (x-\mu)}{2(p-0)} \right), & \mu \leq x \leq p \\
    &= 0, & \text{elsewhere}
\end{align*}
\]

This function has the following shape, where \( \mu \) can take any position between 0 and \( p \).

Figure 16

The SinCos Density Function
The distribution function, \( P_{x_0}(x) \), is:

\[
P_{x_0}(x) = \begin{cases} 
0, & x < 0 \\
\frac{1}{P} \left[ 1 - \cos \left( \frac{\pi (x - 0)}{2(x_0 - 0)} \right) \right], & 0 \leq x \leq x_0 \\
\frac{1}{P} \left[ 1 - \sin \left( \frac{\pi x}{2(x_0 - P)} \right) \right], & x_0 \leq x \leq P \\
1, & x > P
\end{cases}
\] (31)

Setting \( u = P_{x_0}(z) \), and solving for \( z \), we obtain:

\[
z = 0 + \frac{2(x_0 - 0)}{\pi} \cos^{-1} \left[ 1 - \frac{1}{(P - 0)} \right], \quad 0 \leq u \leq \frac{1}{P}
\]

\[
z = \frac{2(x_0 - 1)}{\pi} \sin^{-1} \left[ \frac{u - 0}{u - \frac{1}{P}} \right], \quad \frac{1}{P} \leq u \leq 1
\] (32)

This inverse function transforms a random sample, \( u \), from the uniform \((0,1)\) distribution to the corresponding sample, \( z \), from the \( \sin \cos \) distribution.

The mean, \( \mu \), of the \( \sin \cos \) distribution is:

\[
\mu = \frac{(7 - 3)(7 - 1) + (7 - 1) + (7 - 3)}{17}
\] (33)

Considering as a weighted average, we see that the extremes, 0 and 1, receive more weight than \( x_0 \).

The variance, \( \sigma^2 \), is:

\[
\sigma^2 = \frac{1}{17} \left[ (7 - 3)^2 + (7 - 1)^2 - 2 \left( \frac{7^2 - 16\pi^2}{4\pi^2} \right) \right]
\] (34)

In a form for numerical comparison with the PERT beta variance:

\[
\sigma^2 = 2.06 \left( \frac{7 - 3}{9} \right)^2 - 0.366 \left( \frac{7 - 1}{9} \right) \] (35)
We note that \( \sigma^2 (\alpha) \) is 1.68 \( \pm 0.1 \) times the PERT beta variance, \( \sigma^2_{\beta} \), where \(-1 \leq K \leq 1\), depending on the relative position of \( \alpha \).

Although, with the \( \text{SinCos} \) function, the transformation from \( x \) to \( y \) can be accomplished by a single transformation equation, computer running time may be as great, or greater, than the table look-up routine, due to the number of operations required to calculate the \( \text{Sin}^{-1} \) or the \( \text{Cos}^{-1} \).
APPENDIX V

PICT CHART FOR RESOURCE ALLOCATION USING CRITICALITY INDEX.

For each T-C plan, form the following computer "word".

<table>
<thead>
<tr>
<th>Activity No.</th>
<th>ETB</th>
<th>ΔT</th>
<th>ΔG</th>
</tr>
</thead>
</table>

START

N = NO. OF T-C PLANS FOR THIS ACT
K = 2

GO TO JTH ACTIVITY

J = J+1

NO

CALCULATE ETB FOR KTH PLAN

K = N

YES

GENERATE PLUS INDEX, PLACE ACTIVITY IN DO LIST

NO

CHEDULE ORI.

YES

RETURN FROM CALCULATION

J = J

STOP

PLACE ACTIVITY IN DO LIST

T = T + ΔT

YES

IS THIS ACTIVITY IN DO LIST?

NO

REMOTE PROCESS FOR THIS ACT.

YES

IS THIS ACTIVITY IN DO LIST?

NO

T = T + ΔT

YES

RETURN FROM CALCULATION

J = J + 1

NO
BIBLIOGRAPHY


20. Control Data Corporation, Program Evaluation Review Technique; CDC, Minneapolis.