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Analysis of Transients and Stability in an Idealized Two-Level Laser System

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There have been some recent discussions of the stability and transient behavior (spiking, etc.) in various lasers. The analysis of three- or four-level laser systems involves several coupled differential equations which do not yield analytical solutions.

It is the purpose of this note to point out a relatively simple analytical solution to an idealized two-level laser system. Such a two-level system bears a resemblance to the diode laser system.

The equations describing the idealized two-level system shown in Fig. 1 may be written as follows

\[
\begin{align*}
\frac{dn_1}{dt} &= -A_n - B_{n0} + R + B_{p0}, \\
\frac{dp}{dt} &= -B_{p0} + B_{n0} - \alpha p + \lambda n,
\end{align*}
\]

where \(n_1\) and \(n_0\) are the populations of the lower and upper states (electrons per unit volume).

\(p(t)\) is the radiation field intensity (photons per unit volume per second). [For a unit volume laser with the radiation in one plane, \(p(t)\) is proportional to \(s(t)^2\), where \(s(t)\) is the reflectance of the walls.]

\(n_1 + n_0 = \xi (\text{constant})\), where \(\xi = \alpha / \alpha p\).

\(J\) is the coefficient of spontaneous emission.

\(A\) is the coefficient of stimulated emission. (The frequency dependence of \(B\), like \(s(t)\), is neglected.)

The addition of (1) to (2) leads to

\[
\frac{dp}{dt} - \frac{dn_1}{dt} = R - \alpha p.
\]

If a steady-state condition of the laser system does exist, then both \(dn_1/dt\) and \(dp/dt\) must be zero leading to

\[
R = \alpha p.
\]

This indicates that if a steady state exists (to be shown), then the pump rate \(R\) must equal the loss rate \(\alpha p\).

For (1) and (2) and rearrange:

\[
\begin{align*}
\frac{dn_1}{dt} &= R + B_{pc} - A_{n1} - 2B_{p0}, \\
\frac{dp}{dt} &= -\alpha p - B_{pc} + A_{n1} + 2B_{p0}.
\end{align*}
\]

\(n_1 \) shall be simply \(n \) from here on.

The last term in Eqs. (5) and (6) being a cross term prevents a general solution in closed form, so a subterfuge is in order.

Define \(\rho_0 = R/\alpha p\), as the pump rate.

\[
\rho_0 = \frac{\alpha R + B_{pc}}{A + 2Bp}.
\]

These values are chosen so that in a steady-state condition \(n_1 = n_0, p = \rho_0\). Now

\[
\begin{align*}
\rho &= \rho_0 + (\rho - \rho_0) \\
\alpha n &= \alpha n_0 + \alpha \rho_0 + \rho \alpha n + (\rho - \rho_0)(n - n_o).
\end{align*}
\]

When \(\rho \to \rho_0\) and \(n \to n_0\) (i.e., near the steady-state position), the last term in (9) is small and may be neglected. This approximation will hold at points far from the steady state but will permit the determination of the stability of the system near the point of steady-state operation. If the laser is stable about this point, the subsequent solutions will describe its behavior.

Inserting the approximation in Eqs. (5) and (6), we obtain:

\[
\begin{align*}
\frac{dn_1}{dt} &= R + 2B_{p0} + (Bc - 2B_{n0}) - (A + 2Bp)n, \\
\frac{dp}{dt} &= -2Bp - (\alpha + Bc - 2B_{n0})p + (A + 2Bp)n.
\end{align*}
\]

Rewrite Eq. (10) and differentiate

\[
\rho - \frac{1}{Bc - 2B_{n0}} \frac{dn}{dt} + (A + 2Bp)n - (R + 2B_{p0}n).
\]

and

\[
\frac{d\rho}{dt} - \frac{1}{(Bc - 2B_{n0})(A + 2B_{p0})} \frac{dn}{dt} - (A + 2Bp) \frac{dn}{dt} = 0.
\]

Use (10a) and (10b) to remove \(\rho\) dependence in (11),

\[
\begin{align*}
\frac{dn}{dt} + (A + 2Bp + \alpha + Bc - 2B_{n0}) \frac{dn}{dt} + (A + 2Bp)n \\
\frac{d^2n}{dt^2} + D \frac{dn}{dt} + En + F = 0.
\end{align*}
\]

Solving this in the usual manner:

\[
\left(\frac{d}{dt} + D + \sqrt{D^2 - 4E}\right) \left(\frac{d}{dt} + D - \sqrt{D^2 - 4E}\right)n + F = 0.
\]
The general solution is then given by

\[ n = c_1 \exp \left( \frac{-D + \sqrt{D^2 - 4E}}{2} \right) + c_2 \exp \left( \frac{-D - \sqrt{D^2 - 4E}}{2} \right) \]

Equation (14) may now be inspected to determine the nature of the solution in the vicinity of the steady-state position. The steady-state position for (14) is just \( n = F/E \) and the rate at which any deviations from the steady state will be damped out is just \( \exp \left( \frac{-D}{2} \right) \). If \( 4E > D^2 \), then the second part of the exponential \( \left[ \sqrt{(D^2 - 4E)/2} \right] \) is of the form \( \exp (icot) \) and the approach to steady-state operation will be a damped oscillation with the frequency given by \( \omega = \sqrt{(4E - D^2)/2} \). If \( D^2 > 4E \), then (14) is critically damped and shows no oscillation whatsoever. Since \( D \) and \( E \) are both positive constants, the exponentials in (14) cannot become positive, therefore the laser cannot become unstable.

It may be concluded, then, that this two-level laser would achieve a steady-state operation with constant pump rate. After a momentary disturbance of the steady state by some means, such as injection of a light pulse or addition of a pulse to the pump, the laser would return to its steady-state operation with a damped oscillation transient. The same transient is expected when the laser pump is first turned on.