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A STUDY ON THE EFFECT OF A PROGRESSING SURFACE PRESSURE ON A VISCOELASTIC HALF-SPACE

BY

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FOREWORD

The MITRE Corporation is concerned with the survivability of the Air Force Command and Control Systems. It conducts studies in this general area in order to determine the levels at which various systems components fail and investigates various alleviating measures which may be employed to raise the levels of survivability.

One phase of this work is concerned with the behavior of deep underground hard command posts excavated in soil and rock when subjected to nuclear attack.

Among the many problems involved in this area, one of the most important and, perhaps, least understood is the phenomena associated with the transmission of ground shock to the underground command post. It is known that shock loads such as those produced by nuclear weapons will be transmitted through the ground by means of stress waves; however, because of the complexity involved, only skeletal information is presently available that can be applied to the actual design of the underground command post. It is important that basic research in this field be accelerated so that appropriate criteria can be established for the design of installations of interest to the Command-Control Development Division.

At MITRE programs have been initiated to study stress wave propagation in various media because we are interested in many given geographic locations, and since earth materials which vary widely from site to site exhibit different properties and characteristics, many analytical models are necessary to predict the response behavior of the geologic materials at a variety of sites. Some of these models are: linear-elastic, non-linear elastic, elastic-plastic, elastic-locking, visco-elastic, visco-plastic, etc.

Information contained herein is concerned with stress wave propagation phenomena in visco-elastic media. The numerical results given are for the purpose of demonstrating visco-elastic effects and should not be construed as representing the actual behavior of a given rock or soil medium. This is due to a lack of information on the physical visco-elastic constants associated with various types of rock and soil.

This report is a part of a series of studies currently being carried out by MITRE and Paul Weidlinger, Consulting Engineer, New York City.

John J. O'Sullivan

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INTRODUCTION

This report is one of a series of technical discussions and papers concerned with the theory of wave propagation in solids with special applications to ground shock phenomena. It presents theoretical results on the free field effects due to progressing pressure loadings on the surface of a semi-infinite linearly viscoelastic (standard solid) half-space.

The ultimate purpose of this group of papers is to arrive at conclusions for the free field effects due to progressing surface pressure loadings on actual media such as rock or soil with complex properties which are difficult to analyze. The problem of rocklike media has been approached by considering a succession of materials having gradually more complex properties such as 1) an acoustic inviscid fluid, 2) a linear elastic solid, 3) a non-linearly elastic solid [(1)-(3) - See Reference [1]] and a linearly viscoelastic solid in the present report. It is felt that in this manner, certain conclusions which can be drawn for cases of simple properties can be extrapolated for more complex properties by qualitative reasoning.

The paper represents a step in the investigation of free field phenomena in viscoelastic materials but considerable future work remains to be performed in this field. It presents the plane strain solution for the stress distribution produced by the uniform motion of the pressure wave on the surface of the medium. The analysis is based on the assumption that a steady state exists with respect to a coordinate system attached to the moving load and that the velocity of the moving load is greater than that of irrotational and equivoluminal waves in the

medium (superseismic case). Further investigations in which the free field stresses in the trans-seismic and subseismic regions are determined and in which the velocity of the surface pressure varies as a function of time will be required for a more complete picture of the phenomena in question.

Attention must also be paid to the relation of the mathematical model to the expected physical behavior of the material. For those rocks which may be expected to act viscoelastically, much work remains to be done on the experimental determination of the appropriate viscoelastic constants which are required as input parameters in the analysis; such information is not available at the present time.

Numerical results are presented for a hypothetical viscoelastic material and the free field stresses are evaluated. They are compared with the corresponding stresses in the material in its relaxed and unrelaxed elastic state. The results are illustrative only and should not be taken as applying to specific real materials unless experimental investigations show the coincidence of the viscoelastic parameters for the model and the real material.

SECTION I

STRESSES IN A VISCOELASTIC HALF SPACE DUE TO A PROGRESSING SURFACE PRESSURE

BY

JEROME L. SACKMAN

List of Symbols*

C_L (C_L^*): Velocity of propagation of high (low) frequency irrotational waves. $C_L = \left(\frac{\lambda+2\mu}{\rho}\right)^{\frac{1}{2}}$, $C_L^* = \left(\frac{\lambda^*+2\mu^*}{\rho}\right)^{\frac{1}{2}}$.

C_T (C_T^*): Velocity of propagation of high (low) frequency equivoluminal waves. $C_T = \left(\frac{\mu}{\rho}\right)^{\frac{1}{2}}$, $C_T^* = \left(\frac{\mu^*}{\rho}\right)^{\frac{1}{2}}$.

$\bar{f}(p)$: Laplace Transform of $f(t)$.

$H(x)$: Unit step function. $H(x) = 0$, $x < 0$; $H(x) = 1$, $x > 0$.

K: Bulk modulus.

m: Ratio of relaxed to unrelaxed shear modulus. $m = \mu^*/\mu$.

M_L, M_T, M_L^*, M_T^*: Mach numbers: $M_L = \frac{V}{C_L}$, $M_T = \frac{V}{C_T}$, $M_L^* = \frac{V}{C_L^*}$, $M_T^* = \frac{V}{C_T^*}$.

m_L, m_T, m_L^*, m_T^*: Functions of Mach numbers: $m_L = \left(M_L^2 - 1\right)^{\frac{1}{2}}$, etc.

$\bar{m}_L$, $\bar{m}_T$: Transforms of viscoelastic operators: $\bar{m}_L = m_L \left[\frac{p+n\left(\frac{m_L^*}{m_L}\right)^2}{p+n}\right]^{\frac{1}{2}}$

$\bar{m}_T = m_T \left[\frac{p+m\left(\frac{m_T^*}{m_T}\right)^2}{p+m}\right]^{\frac{1}{2}}$.

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* Other symbols are defined as they appear in the text.

n: Ratio of relaxed to unrelaxed moduli. $n = \frac{\lambda^* + 2\mu^*}{\lambda + 2\mu} = 1 - \frac{4}{3}\chi(1-m)$.

p: Laplace transform parameter.

p_o: Intensity of surface pressure.

Q: Viscoelastic operator. $Q = \frac{m + \frac{\partial}{\partial t}}{1 + \frac{\partial}{\partial t}}$.

t: Time.

T: Relaxation time.

u_i: Cartesian components of the displacement vector.

V: Velocity of steadily moving surface pressure.

$\bar{x}, \bar{y}, \bar{z}$: Fixed, rectangular, Cartesian, space coordinates.

x, y, z: Space coordinates attached to the moving load.

δ_{ij}: Kronecker delta: $\delta_{ij} = 1,\ i = j;\ \ \delta_{ij} = 0,\ \ i \neq j$.

δ, δ^*: Functions of the Mach numbers: $\delta = (m_T^2 - 1)^2 + 4m_L m_T$, $\delta^* = (m_T^{*2} - 1)^2 + 4m_L^* m_T^*$.

Δ: Viscoelastic operator: $\Delta = (\bar{m}_T^{\,2} - 1)^2 + 4\bar{m}_L\bar{m}_T$.

ξ, η: Nondimensional space coordinates: $\xi = \frac{x}{VT}$, $\eta = \frac{z}{VT}$.

$\mathcal{K}$: Ratio of propagation velocity of high frequency equivoluminal waves to high frequency irrotational waves. $\mathcal{K} = \frac{\mu}{\lambda + 2\mu}$.

λ (λ^*): Unrelaxed (relaxed) Lamé constant.

$\tilde{\lambda}$: Viscoelastic Lamé operator. $\tilde{\lambda} = K - \frac{2}{3}\tilde{\mu}$.

μ (μ^*): Unrelaxed (relaxed) shear modulus.

$\tilde{\mu}$: Viscoelastic shear operator. $\tilde{\mu} = \mu\left[\frac{m + T\frac{\partial}{\partial t}}{1 + T\frac{\partial}{\partial t}}\right]$.

ν: Poisson's ratio for the unrelaxed material.

ρ: Density of the material.

σ_{ij} : Cartesian components of the stress tensor.

$\sigma_{ij\varphi}$ ($\sigma_{ij\psi}$): Portion of stress components due to the irrotational (equivoluminal) potential.

Φ : Scalar (or irrotational) potential.

Ψ_1 : Vector (or equivoluminal) potential.

Ψ : The y component of Ψ_1 .

$\dot{f}$: $\frac{\partial}{\partial t} f$.

$f_{,x}$: $\frac{\partial}{\partial x} f$.

Introduction.

This paper considers the plane strain problem of finding the stress distribution produced by the uniform motion of a step pressure on the surface of a linear viscoelastic (standard-solid) half-space (Fig. 1). The speed of the load is assumed to be greater than the velocities of plane irrotational and equivoluminal waves in the medium (superseismic case). Eliminating the effect of initial conditions, it is assumed that the load is moving in from $\bar{x} = \infty$, such that a steady state exists with respect to a coordinate system attached to the moving load. The equivalent problem for the elastic case has been treated in Ref. [1].

The solution for arbitrary load distribution can be obtained by superposition from the solution for the step pressure. Although not considered here, the case of tangential loads on the surface of the half-space could be treated in the same manner.

Formulation.

Let $(\bar{x}, \bar{y}, \bar{z})$ denote fixed, rectangular, Cartesian, space coordinates, while (x, y, z) denote space coordinates attached to the front of the moving load $p_o H(x)$ (Fig. 1), where p_o is the pressure intensity and H is the unit step function. The uniformly distributed pressure moves over the surface of the half-space in the negative x direction with a speed V. At the time $t = 0$, the two coordinate systems are taken to be in coincidence so that

$$x = \bar{x} + Vt \,, \quad y = \bar{y} \,, \quad z = \bar{z} \tag{1}$$

The stress-displacement relations in a homogeneous, isotropic medium which is elastic in bulk and viscoelastic in shear is

$$\sigma_{ij} = \tilde{\lambda}\, u_{k,k}\, \delta_{ij} + \tilde{\mu}(u_{i,j} + u_{j,i}) \qquad (2)$$
$$i,j,k = \bar{x},\bar{y},\bar{z}$$

with

$$\tilde{\lambda} = K - \frac{2}{3}\tilde{\mu} \qquad (3a)$$

where K is the bulk modulus and $\tilde{\mu}$ is the shear operator* Selecting the standard-solid model:

$$\tilde{\mu} = \mu \frac{m + T\frac{\partial}{\partial t}}{1 + T\frac{\partial}{\partial t}} \qquad (3b)$$

where μ is the "unrelaxed" and $m\mu = \mu^*$ $(0 \leq m < 1)$ is the relaxed shear modulus, while T is a relaxation time (see Ref. 2).

The equations of motion of a continuum in the absence of body forces are, for small displacements,

$$\sigma_{ij,j} = \rho \ddot{u}_i \qquad (4)$$

where ρ is the density. Eqs. (2) and (4) can be combined to give the differential equations on the displacements:

$$(\tilde{\lambda} + \tilde{\mu})u_{j,ji} + \tilde{\mu}\, u_{i,jj} = \rho \ddot{u}_i \qquad (5)$$

By means of the Helmholtz resolution, the displacement vector can be separated into an irrotational and equivoluminal part

$$u_i = \Phi_{,i} + \epsilon_{ijk}\Psi_{k,j} \quad ; \quad \Psi_{j,j} = 0 \qquad (6)$$

where ϵ_{ijk} is the alternating tensor, and where the irrotational part comes from the scalar potential Φ and the equivoluminal part from the vector potential Ψ_i.

* The usual convention of summation over repeated subscripts is adopted.

Equation (5) will be satisfied if Φ and Ψ_i satisfy the following equations:

$$(\tilde{\lambda} + 2\tilde{\mu})\,\Phi_{,jj} = \rho\,\ddot{\Phi} \tag{7a}$$

$$\tilde{\mu}\,\Psi_{i,jj} = \rho\,\ddot{\Psi}_i \tag{7b}$$

where

$$\left.\begin{aligned}
\tilde{\lambda} + 2\tilde{\mu} &= (\lambda + 2\mu)\left(\frac{n + T\frac{\partial}{\partial t}}{1 + T\frac{\partial}{\partial t}}\right) \\
\lambda &= K - \frac{2\mu}{3} \\
n &= 1 - \frac{4}{3}\varkappa(1-m) \\
\varkappa &= \frac{\mu}{\lambda + 2\mu} = \frac{1-2\nu}{2(1-\nu)}
\end{aligned}\right\} \tag{7c}$$

ν being Poisson's ratio for the unrelaxed material. Note that for $-1 \leq \nu \leq \frac{1}{2}$, $0 \leq n < 1$.

For plane strain in the $(\bar{x}, \bar{z})$ plane, a suitable form for the potentials is

$$\Phi = \Phi(\bar{x}, \bar{z}, t)\,, \quad \Psi_{\bar{x}} = \Psi_{\bar{z}} = 0\,, \quad \Psi_{\bar{y}} = \Psi(\bar{x}, \bar{z}, t) \tag{8}$$

Since the solution is a steady state solution in the (x, z) coordinate system, transforming to this system leads to equation independent of t. Thus, by the use of Eqs. (1) and (8), Eqs. (7a) and (7b) give:

$$\left[\frac{C_L^{*2} + C_L^2\frac{\partial}{\partial \xi}}{1 + \frac{\partial}{\partial \xi}}\right]\left[\frac{\partial^2\Phi}{\partial \xi^2} + \frac{\partial^2\Phi}{\partial \eta^2}\right] = V^2\,\frac{\partial^2}{\partial \xi^2}\Phi \tag{9a}$$

$$\left[\frac{C_T^{*2} + C_T^2\frac{\partial}{\partial \xi}}{1 + \frac{\partial}{\partial \xi}}\right]\left[\frac{\partial^2\Psi}{\partial \xi^2} + \frac{\partial^2\Psi}{\partial \eta^2}\right] = V^2\,\frac{\partial^2}{\partial \xi^2}\Psi \tag{9b}$$

where

$$\left.\begin{aligned} C_L^2 &= \frac{\lambda + 2\mu}{\rho} \\ C_L^{*2} &= \frac{\lambda^* + 2\mu^*}{\rho} = \frac{K + \frac{4}{3}\mu^*}{\rho} = nC_L^2 \\ C_T^2 &= \frac{\mu}{\rho} \quad ; \quad C_T^{*2} = \frac{\mu^*}{\rho} = mC_T^2 \\ \xi &= \frac{x}{VT} \quad ; \quad \eta = \frac{z}{VT} \end{aligned}\right\} \tag{9c}$$

The quantity C_L (C_L^*) represents the velocity of propagation of high (low) frequency, plane, irrotational waves in the standard-solid medium, while C_T (C_T^*) represents the propagation velocity for high (low) frequency, plane, equivoluminal waves.

The boundary conditions to be satisfied by Φ and Ψ are determined from the traction conditions at the surface of the half space, $\eta = 0$:

$$\sigma_{xz}(\xi,0) = 0 \tag{10a}$$

$$\sigma_{zz}(\xi,0) = - p_o H(\xi) \tag{10b}$$

Utilizing Eqs. (1), (2), (6) and (9), Eqs. (10) can be written in terms of Φ and Ψ. At $\eta = 0$:

$$(VT)^2 \frac{\sigma_{xz}}{\mu} = 2Q \frac{\partial^2}{\partial\xi\partial\eta}\Phi - [M_T^2 - 2Q] \frac{\partial^2}{\partial\xi^2}\Psi = 0 \tag{11a}$$

$$(VT)^2 \frac{\sigma_{zz}}{\mu} = [M_T^2 - 2Q] \frac{\partial^2}{\partial\xi^2}\Phi + 2Q \frac{\partial^2 \Psi}{\partial\xi\partial\eta} = - (VT)^2 \frac{p_o}{\mu} H(\xi) \tag{11b}$$

where

$$M_T = \frac{V}{C_T} \quad ; \quad Q = \frac{m + \frac{\partial}{\partial\xi}}{1 + \frac{\partial}{\partial\xi}} \tag{11c}$$

The formulation of the problem is not yet complete, for there is still a further condition to be specified which arises from physical considerations. Because the load moves with a velocity which is greater than the propagation velocities in the standard-solid medium, no disturbances can ever get ahead of the load. Thus for $\xi < 0$, the medium is undisturbed (see Fig. (1)). Actually, since the propagation velocities of irrotational waves have a definite maximum value (C_L), it is seen physically that there will be a straight line of demarcation between the disturbed and undisturbed portion of the medium - the Mach line (or wave front) for irrotational disturbances, indicated on Fig. (1) by the line OI. Irrotational disturbances can exist only behind this line. Similarly, equivoluminal disturbances can exist only behind the equivoluminal wave front or Mach line, OE in Fig. (1). Thus for $\xi < 0$ quiescent conditions exist, and the solution will have non-vanishing values only for $\xi > 0$, permitting the use of a Laplace transform in ξ.

The formulation of the problem is now complete. Equations (9) must be solved subject to the conditions at $\eta = 0$, given by Eqs. (11), and to the condition of quiescence for $\eta < 0$. Having obtained Φ and Ψ, the displacements and stresses may be computed from Eqs. (6) and (2), respectively, utilizing Eqs. (1) and (8).

Formal Solution.

The solution to the problem can be written formally as a complex integral by the use of the Laplace transform, defined by

$$\bar{f}(p,\eta) = \int_0^\infty f(\xi,\eta)e^{-p\xi}\, d\xi \tag{12}$$

with the inversion

$$f(\xi,\eta) = \frac{1}{2\pi i}\int_{\gamma-i\infty}^{\gamma+i\infty} \bar{f}(p,\eta)e^{p\xi}\,dp \tag{13}$$

Applying this transform, and utilizing the condition of quiescence for $\xi < 0$, Eqs. (9) yield

$$\frac{\partial^2\bar{\Phi}}{\partial\eta^2} - p^2\bar{m}_L^{\,2}\,\bar{\Phi} = 0 \tag{14a}$$

$$\frac{\partial^2\bar{\Psi}}{\partial\eta^2} - p^2\bar{m}_T^{\,2}\,\bar{\Psi} = 0 \tag{14b}$$

where

$$\left.\begin{aligned}
&\bar{m}_L = m_L\left[\frac{p+n\left(\frac{m_L^*}{m_L}\right)^2}{p+n}\right]^{1/2}, \quad \mathrm{Re}\,\bar{m}_L > 0\\
&\bar{m}_T = m_T\left[\frac{p+m\left(\frac{m_T^*}{m_T}\right)^2}{p+m}\right]^{1/2}, \quad \mathrm{Re}\,\bar{m}_T > 0\\
&m_L = (M_L^2 - 1)^{1/2}\ ;\quad m_T = (M_T^2 - 1)^{1/2}\\
&m_L^* = (M_L^{*2} - 1)^{1/2}\ ;\quad m_T^* = (M_T^{*2} - 1)^{1/2}\\
&M_L = \frac{V}{C_L}\ ;\quad M_L^* = \frac{V}{C_L^*}\ ;\quad M_T^* = \frac{V}{C_T^*}
\end{aligned}\right\} \tag{14c}$$

Equations (14a, b) have the solutions

$$\bar{\Phi} = Ae^{-p\bar{m}_L\eta} + A'e^{p\bar{m}_L\eta} \tag{15a}$$

$$\bar{\Psi} = Be^{-p\bar{m}_T\eta} + B'e^{p\bar{m}_T\eta} \tag{15b}$$

where A, A', B, B' are arbitrary constants. The condition of quiescence for $\xi < 0$ will be satisfied if

$$A' = B' = 0 \tag{15c}$$

The constants A and B are determined from the transforms of the boundary conditions, Eqs. (11):

$$2 \frac{\partial}{\partial \eta} \bar{\Phi} - p(\bar{m}_T^{\,2} - 1) \bar{\Psi} = 0 \tag{16a}$$

$$p(\bar{m}_T^{\,2} - 1) \bar{\Phi} + 2 \frac{\partial}{\partial \eta} \bar{\Psi} = - \frac{1}{p^2}\left(\frac{p+1}{p+m}\right) (VT)^2 \left(\frac{p_o}{\mu}\right) \tag{16b}$$

These equations give for A and B:

$$A = - \left[\frac{p_o (VT)^2}{\mu \Delta}\right] \left[\frac{p+1}{p^3(p+m)}\right] \left[\bar{m}_T^{\,2} - 1\right] \tag{17a}$$

$$B = 2 \left[\frac{p_o (VT)^2}{\mu \Delta}\right] \left[\frac{p+1}{p^3(p+m)}\right] \bar{m}_L \tag{17b}$$

where

$$\Delta = (\bar{m}_T^{\,2} - 1)^2 + 4\, \bar{m}_L \bar{m}_T \tag{17c}$$

Thus $\bar{\Phi}$ and $\bar{\Psi}$ are determined, and the transforms of the stress components can now be found.

<u>Stress Components at the Wave Fronts.</u>

From Eqs. (2) and (6) it is seen that the stress components may be written in two portions

$$\sigma_{ij} = \sigma_{ij\varphi} + \sigma_{ij\psi} \tag{18}$$

where $\sigma_{ij\varphi}$ results from the irrotational potential Φ and $\sigma_{ij\psi}$ from the equivoluminal potential Ψ.

Utilizing the results of the preceding sections, the transforms of these quantities are:

$$\frac{\bar{\sigma}_{xx\varphi}}{p_o} = -\left[\frac{1}{p\Delta}\right]\left[\bar{m}_T^{\,2} + 1 - 2\bar{m}_L^{\,2}\right]\left[\bar{m}_T^{\,2} - 1\right]e^{-p\bar{m}_L\eta} \tag{19a}$$

$$\frac{\bar{\sigma}_{xx\psi}}{p_o} = \left[\frac{1}{p\Delta}\right]\left[4\bar{m}_L\bar{m}_T\right]e^{-p\bar{m}_T\eta} \tag{19b}$$

$$\frac{\bar{\sigma}_{xz\varphi}}{p_o} = \left[\frac{1}{p\Delta}\right]\left[2\bar{m}_L(\bar{m}_T^{\,2} - 1)\right]e^{-p\bar{m}_L\eta} \tag{19c}$$

$$\frac{\bar{\sigma}_{xz\psi}}{p_o} = -\left[\frac{1}{p\Delta}\right]\left[2\bar{m}_L(\bar{m}_T^{\,2} - 1)\right]e^{-p\bar{m}_T\eta} \tag{19d}$$

$$\frac{\bar{\sigma}_{zz\varphi}}{p_o} = -\left[\frac{1}{p\Delta}\right]\left[\bar{m}_T^{\,2} - 1\right]^2 e^{-p\bar{m}_L\eta} \tag{19e}$$

$$\bar{\sigma}_{zz\psi} = -\bar{\sigma}_{xx\psi} \tag{19f}$$

$$\frac{\bar{\sigma}_{yy\varphi}}{p_o} = -\left[\frac{1}{p\Delta}\right]\left[\bar{m}_T^{\,2} - 2\bar{m}_L^{\,2} - 1\right]\left[2(\bar{m}_T^{\,2} - 1)\right]e^{-p\bar{m}_L\eta} \tag{19g}$$

$$\bar{\sigma}_{yy\psi} = 0 \tag{19h}$$

Note the formal correspondence between these results and the results for the elastic body. If the Laplace transform (in the variable $\xi = \frac{x}{VT}$) of the

elastic solution [Ref. (1) - Eqs. (51-53)] is taken, and then m_T and m_L in the elastic solution are replaced by $\bar{m}_T$ and $\bar{m}_L$, Eqs. (19) are obtained.

Because of the complexity of the inverted forms of $\sigma_{ij\varphi}$ and $\sigma_{ij\psi}$, it is useful to obtain closed form expressions for these quantities and their first derivatives at the wave fronts. These expressions are obtained by considering the asymptotic expansions of the quantities $e^{pm_L\eta}\,\bar{\sigma}_{ij\varphi}$ and $e^{pm_T\eta}\,\bar{\sigma}_{ij\psi}$. As shown in the Appendix, the first two coefficients of the expansion in powers of $\frac{1}{p}$ of $e^{pm_L\eta}\,\bar{\sigma}_{ij\varphi}$ represent the values of $\sigma_{ij\varphi}$ and $\frac{\partial}{\partial\xi}\sigma_{ij\varphi}$ just behind the irrotational wave front. A similar situation exists for the equivoluminal portion of the stress components at the equivoluminal wave front.

Immediately behind the irrotational wave front (i.e. at $\xi = m_L\eta^+$), these values are:

$$\frac{\sigma_{xx\varphi}}{p_o} = -\left(\frac{1}{\delta}\right)\left(M_T^2 - 2m_L^2\right)\left(m_T^2 - 1\right)e^{-\alpha\eta} \tag{20a}$$

$$\frac{\partial}{\partial\xi}\sigma_{xx\varphi} = \sigma_{xx\varphi}\left[\frac{M_T^2(1-m) - 2M_L^2(1-n)}{M_T^2 - 2m_L^2} + \frac{M_T^2(1-m)}{M_T^2 - 2} - \frac{2a}{\delta} + b\eta\right] \tag{20b}$$

$$\frac{\sigma_{xz\varphi}}{p_o} = \left(\frac{2m_L}{\delta}\right)\left(m_T^2 - 1\right)e^{-\alpha\eta} \tag{20c}$$

$$\frac{\partial}{\partial\xi}\sigma_{xz\varphi} = \sigma_{xz\varphi}\left[\frac{M_T^2(1-m)}{m_T^2 - 1} + \frac{M_L^2(1-n)}{2m_L^2} - \frac{2a}{\delta} + b\eta\right] \tag{20d}$$

$$\frac{\sigma_{zz\varphi}}{p_o} = -\frac{(m_T^2 - 1)^2}{\delta} e^{-\alpha\eta} \tag{20e}$$

$$\frac{\partial}{\partial\xi}\sigma_{zz\varphi} = \sigma_{zz\varphi}\left[\frac{2M_T^2(1-m)}{M_T^2 - 1} - \frac{2a}{\delta} + b\eta\right] \tag{20f}$$

$$\frac{\sigma_{yy\varphi}}{p_o} = -\left(\frac{2}{\delta}\right)\left(m_T^2 - 1\right)\left(M_T^2 - 2M_L^2\right) e^{-\alpha\eta} \tag{20g}$$

$$\frac{\partial}{\partial\xi}\sigma_{yy\varphi} = \sigma_{yy\varphi}\left[\frac{M_T^2(1-m) - 2M_L^2(1-n)}{M_T^2 - 2M_L^2} + \frac{M_T^2(1-m)}{m_T^2 - 1} - \frac{2a}{\delta} + b\eta\right] \tag{20h}$$

where

$$\left.\begin{aligned}
\delta &= (m_T^2 - 1)^2 + 4\, m_L m_T \\
\alpha &= \frac{M_L^2(1-n)}{2m_L} \\
a &= M_T^2(m_T^2 - 1)\,(1 - m) + \frac{m_T}{m_L} M_L^2\,(1 - n) + \frac{m_L}{m_T} M_T^2\,(1 - m) \\
b &= \left[\frac{1-n}{\delta m_L}\right]\left[M_L^2(1+3n) - 4n\right]\left[\frac{M_L}{m_L}\right]^2
\end{aligned}\right\} \tag{20i}$$

Immediately behind the equivoluminal wave front (i.e. at $\xi = m_T\eta^+$) the values and first derivatives of $\sigma_{ij\psi}$ are:

$$\frac{\sigma_{xx\psi}}{p_o} = \frac{4m_L m_T}{\delta} e^{-\beta\eta} \tag{21a}$$

$$\frac{\partial}{\partial \xi} \sigma_{xx\psi} = \sigma_{xx\psi} \left[\frac{M_L^2(1-n)}{2m_L^2} + \frac{\beta}{m_T} - \frac{2a}{\delta} + c\eta \right] \tag{21b}$$

$$\frac{\sigma_{xz\psi}}{p_o} = - \left(\frac{2m_L}{\delta} \right) \left(m_T^2 - 1 \right) e^{-\beta\eta} \tag{21c}$$

$$\frac{\partial}{\partial \xi} \sigma_{xz\psi} = \sigma_{xz\psi} \left[\frac{M_L^2(1-n)}{2m_L^2} + \frac{M_T^2(1-m)}{m_T^2 - 1} - \frac{2a}{\delta} + c\eta \right] \tag{21d}$$

$$\sigma_{zz\psi} = - \sigma_{xx\psi} \tag{21e}$$

$$\frac{\partial}{\partial \xi} \sigma_{zz\psi} = - \frac{\partial}{\partial \xi} \sigma_{xx\psi} \tag{21f}$$

$$\sigma_{yy\psi} = \frac{\partial}{\partial \xi} \sigma_{yy\psi} = 0 \tag{21g}$$

where

$$\left. \begin{aligned} \beta &= \frac{M_T^2(1-m)}{2m_T} \\ c &= \left[\frac{1-m}{8m_T} \right] \left[M_T^2(1+3m) - 4m \right] \left[\frac{M_T}{m_T} \right]^2 \end{aligned} \right\} \tag{21h}$$

Note that the jumps at the two wave fronts may be written as

$$\sigma_{ij\varphi} = \sigma_{ij\varphi}^{El} \, e^{-\alpha\eta}$$

$$\sigma_{ij\psi} = \sigma_{ij\psi}^{El} \, e^{-\beta\eta}$$

The quantities $\sigma_{ij\varphi}^{El}$ and $\sigma_{ij\psi}^{El}$ are the equivalent jumps in an elastic medium having elastic moduli equal to the unrelaxed moduli of the standard solid.

Stress Components Far Behind the Wave Fronts.

To obtain the value of the stress components far behind the wave front, the final value theorem for the Laplace transform is utilized:

$$\lim_{p \to 0} p\, \bar{\sigma}_{ij} = \lim_{\xi \to \infty} \sigma_{ij}$$

The values obtained are (for $\xi \to \infty$):

$$\frac{\sigma_{xx\varphi}}{p_o} = -\left(\frac{1}{\delta^*}\right)\left(m_T^{*2} - 1\right)\left(M_T^{*2} - 2m_L^{*2}\right) \tag{22a}$$

$$\frac{\sigma_{xx\psi}}{p_o} = \frac{4m_L^* m_T^*}{\delta^*} \tag{22b}$$

$$\frac{\sigma_{xz\varphi}}{p_o} = \frac{2m_L^*}{\delta^*}\left(m_T^{*2} - 1\right) \tag{22c}$$

$$\sigma_{xz\psi} = -\sigma_{xz\varphi} \tag{22d}$$

$$\frac{\sigma_{zz\varphi}}{p_o} = -\frac{\left(m_T^{*2} - 1\right)^2}{\delta^*} \tag{22e}$$

$$\sigma_{zz\psi} = -\sigma_{xx\psi} \tag{22f}$$

$$\frac{\sigma_{yy\varphi}}{p_0} = -\left(\frac{2}{\delta^*}\right)\left(m_T^{*2} - 1\right)\left(M_T^{*2} - 2M_L^{*2}\right) \tag{22g}$$

$$\sigma_{yy\psi} = 0 \tag{22h}$$

where

$$\delta^* = \left(m_T^{*2} - 1\right)^2 + 4m_L^* m_T^* \tag{22i}$$

These values correspond to the values of the stress components which would occur in an elastic medium having elastic moduli equal to those of the totally relaxed standard-solid. This fact might have been foreseen from purely physical reasoning.

Approximate Expressions for the Stress Components.

By utilizing the expressions for the jumps and first derivatives of the stresses behind the wave fronts, and for the values of the stresses as $\xi \to \infty$, approximate expressions for the stress components for all values of ξ may be constructed. For instance, having the value of $\sigma_{ij\varphi}$ and $\frac{\partial}{\partial \xi}\sigma_{ij\varphi}$ at the irrotational wave front, and of $\sigma_{ij\varphi}$ as $\xi \to \infty$, a suitable curve may be interpolated to connect the value of $\sigma_{ij\varphi}$ (with proper slope) at the wave front and the asymptotic value of $\sigma_{ij\varphi}$ at $\xi \to \infty$. An exponential type of response is typical of linear viscoelastic phenomena, and such a curve if therefore suggested. This suggestion is supported in Ref. [3] where it is shown that such an approximation is succesful in an analogous problem (wave propagation in a rod of standard-solid behavior).

Appendix

To show that the values and derivatives of $\sigma_{ij\varphi}$ immediately behind the irrotational wave front can be determined from the coefficients of the asymptotic expansion of $e^{pm_L\eta}\,\bar{\sigma}_{ij\varphi}(p,\eta)$, Heaviside's series expansion method is employed (Ref. [4]).

Suppose $e^{pm_L\eta}\bar{\sigma}_{ij\varphi}(p,\eta)$ can be expanded in the series :

$$e^{pm_L\eta}\bar{\sigma}_{ij\varphi}(p,\eta) = \sum_{n=1}^{\infty} A_n(\eta)\, p^{-n}.$$

Inverting both sides

$$\sigma_{ij\varphi}(\xi + m_L\eta,\eta) = H(\xi)\sum_{n=1}^{\infty}\frac{A_n(\eta)\,\xi^{n-1}}{(n-1)!}$$

where the right hand side has been inverted term by term (justified in Ref. [4]), and the left hand side has been inverted by use of the shift theorem with the knowledge that $\sigma_{ij\varphi}(\xi,\eta) \equiv 0$ for $\xi < m_L\eta$. Thus

$$\sigma_{ij\varphi}\,(\xi,\eta) = H(\xi - m_L\eta)\sum_{n=1}^{\infty}\frac{A_n(\eta)\,(\xi-m_L\eta)^{n-1}}{(n-1)!}$$

But the infinite sum represents the Taylor series expansion in ξ, about $\xi = m_L\eta^+$, of $\sigma_{ij\varphi}$ (continued analytically at $\xi = m_L\eta^+$).

Hence

$$A_{n+1}(\eta) = \frac{\partial^n}{\partial\xi^n}\,\sigma_{ij\varphi}\,(\xi,\eta)\Big]_{\xi = m_L\eta^+}.$$

Similarly if

$$e^{pm_T\eta}\,\bar{\sigma}_{ij\psi} = \sum_{n=1}^{\infty} B_n(\eta)\, p^{-n}$$

then

$$B_{n+1}(\eta) = \frac{\partial^n}{\partial \xi^n}\ \sigma_{ij\psi}(\xi,\eta)\Big]_{\xi = m_T\eta^+}$$

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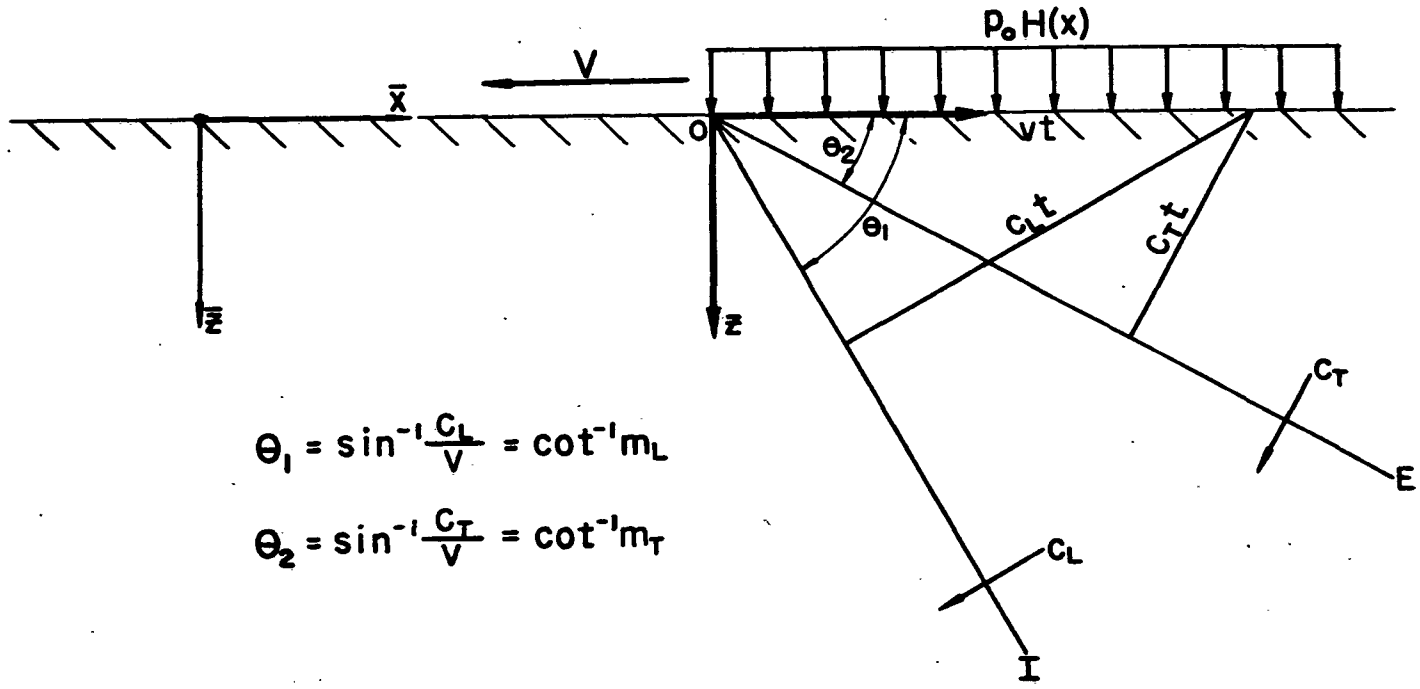
3. Bleich, H.H. and Sackman J.L., "An Approximation in Problems of Viscoelastic Wave Propagation", Technical Report No. 10, Contract Nonr 266(34), Columbia University, September 1960.

4. Carslaw, H.S. and Jaeger, J.C., "Operational Methods in Applied Mathematics", 2nd Ed., Cambridge University Press, 1953, § 123.



$$\theta_1 = \sin^{-1}\frac{c_L}{V} = \cot^{-1} m_L$$

$$\theta_2 = \sin^{-1}\frac{c_T}{V} = \cot^{-1} m_T$$

FIG. 1

SECTION II

APPROXIMATIONS IN PROBLEMS OF VISCOELASTIC WAVE PROPAGATION

BY

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AND

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Approximations in Problems of Viscoelastic Wave Propagation.

Problems involving stress wave propagation in viscoelastic materials can generally be treated by integral transform methods, but the complete numerical evaluation of the inversion integral becomes in many cases very complex and lengthy. It is usually possible, however, to use asymptotic methods to obtain the value of the stress $\sigma(t_o)$ and its time derivative $\dot{\sigma}(t_o)$ at the time of arrival, t_o, of the stress wave at a particular point in the body; in many cases, these initial values may be obtained in closed form. In addition, the long term solution for the stress, $\sigma(\infty)$, can also be determined in a simple manner, either from the physical situation or from an asymptotic evaluation of the inversion integral.

Once the values of $\sigma(t_o)$, $\dot{\sigma}(t_o)$ and $\sigma(\infty)$ have been found, it is possible to interpolate a curve for the stress σ. Bleich and Sackman have presented an exponential interpolation in Reference [1]. Their interpolation is based on the reasoning that since an exponential type of response is typical for linear viscoelastic media, an exponential interpolation is appropriate. Figures (1a, b) show typical stresses σ (at some point in the body) to which the interpolation may be applied as a function of an actual or non-dimensional time t. The complete history of the stress is given by Eq. (1) of Reference [1]:

$$\left.\begin{aligned} \sigma &= 0 & t &< t_o \\ \sigma &= \sigma_\infty - (\sigma_\infty - \sigma_o)\exp\left[\frac{-\dot{\sigma}_o(t-t_o)}{\sigma_\infty - \sigma_o}\right] & t &\geq t_o \end{aligned}\right\} \qquad (1)$$

where t_o, σ_o, $\dot{\sigma}_o$ and σ_∞ are evaluated first for the problem under consideration.

In the application of the above interpolation to practical problem, it is also possible to encounter cases in which $(\sigma_\infty - \sigma_o)$ is opposite in sign to $\dot{\sigma}_o$. For such cases, the exponential interpolation of Eq. (1) is obviously invalid since the exponential would be raised to a positive power in time. Figures (2a, b) show typical stress-time relations of this type. An interpolation in the form of the product of a linear polynomial in t multiplied by an exponential in t can be used in such a case. The complete history of the stress is then given by:

$$\left.\begin{aligned} \sigma &= 0 \qquad\qquad t < t_o \\ \sigma &= \sigma_\infty - \left[\left(\sigma_\infty - \sigma_o\right)\left(1 + \frac{t-t_o}{T}\right) - \dot{\sigma}_o(t-t_o)\right] e^{-\left(\frac{t-t_o}{T}\right)} \qquad t \geq t_o \end{aligned}\right\} \quad (2)$$

In the case of Eq. (1), the interpolation was uniquely determined by the three values σ_o, $\dot{\sigma}_o$ and σ_∞. In the present interpolation, Eq. (2), these three quantities and an additional constant T are required. The value of T cannot be determined unless an additional asymptotic value is obtained from the inversion integral of the problem; this in general would be very difficult to do. It suggests itself, however, to chose the decay time of the viscoelastic medium as a suitable value of the constant T.

The interpolations of Eqs. (1) and (2) may be utilized to obtain stresses which are produced by time varying pressure loadings P(t) acting on the viscoelastic body, once the stresses which are produced by a unit

step pressure loading are evaluated. Let the quantities σ_o, $\dot{\sigma}_o$ and σ_∞ be the stresses at a point in the viscoelastic body which are produced by a unit step pressure loading. The corresponding stress-time history $\sigma(t)$ which is produced by the loading $P(t)$ is obtained through the use of Duhamel's integral and Eq. (1) or Eq. (2).

For an applied pressure of the form

$$P(t) = P_o\left(1 - \frac{t}{D_p}\right)\left[\bar{A}\, e^{-\frac{\bar{\alpha}t}{D_p}} + \bar{B}\, e^{-\frac{\bar{\beta}t}{D_p}}\right] \tag{3}$$

the stress $\sigma(t)$ produced at a point in the body becomes, using Eq. (1):

$$\sigma(\tau) = 0 \qquad\qquad \tau < 0$$

$$\begin{aligned}\sigma(\tau) = {} & P_o\bar{A}\left\{\frac{\dot{\sigma}_o}{\left(\frac{\bar{\alpha}}{D_p} - k\right)^2}\left\{\left(\frac{\bar{\alpha}-kD_p-1}{D_p}\right)e^{-k\tau} + \left[\frac{1}{D_p} + \left(\frac{\bar{\alpha}}{D_p} - k\right)\left(\frac{\tau}{D_p} - 1\right)\right]e^{-\frac{\bar{\alpha}\tau}{D_p}}\right\}\right\} + \\ & + P_o\bar{B}\left\{\frac{\dot{\sigma}_o}{\left(\frac{\bar{\beta}}{D_p} - k\right)^2}\left\{\left(\frac{\bar{\beta}-kD_p-1}{D_p}\right)e^{-k\tau} + \left[\frac{1}{D_p} + \left(\frac{\bar{\beta}}{D_p} - k\right)\left(\frac{\tau}{D_p} - 1\right)\right]e^{-\frac{\bar{\beta}\tau}{D_p}}\right\}\right\} + \\ & + P_o\sigma_o\left[1 - \frac{\tau}{D_p}\right]\left[\bar{A}\, e^{-\frac{\bar{\alpha}\tau}{D_p}} + \bar{B}\, e^{-\frac{\bar{\beta}\tau}{D_p}}\right] \qquad \tau \geq 0\end{aligned} \tag{4}$$

where $\tau = t - t_o$ is measured from the time of arrival, t_o, of the wave at the point and

$$k = \frac{\dot{\sigma}_o}{\sigma_\infty - \sigma_o} \tag{5}$$

Repeating the procedure with Eq. (2), the stress at a point in the body due to the time decaying pressure loading of Eq. (3) becomes:

$$\sigma(\tau) = 0 \qquad \tau < 0 \tag{6a}$$

$$\sigma(\tau) = \frac{P_o \bar{A}}{\left[\frac{T\bar{\alpha}}{D_p} - 1\right]^2} \left\{ \left\{ \Delta + b(\tau - T) \right\} \left\{ \left[\frac{T}{D_p} (\bar{\alpha} - 1) - 1 \right] e^{-\frac{\tau}{T}} + \left[\frac{T}{D_p} \left(1 + \frac{\tau}{T} \left[\frac{T\bar{\alpha}}{D_p} - 1 \right] \right) - \frac{T\bar{\alpha}}{D_p} + 1 \right] e^{-\frac{\bar{\alpha}\tau}{D_p}} \right\} - \right.$$

$$- \frac{Tb}{\frac{T\bar{\alpha}}{D_p} - 1} \left\{ \left[\frac{T}{D_p} (\bar{\alpha} - 2) - 1 \right] e^{-\frac{\tau}{T}} + \left[\frac{T}{D_p} \left(\left[\frac{\bar{\alpha}T}{D_p} - 1 \right]^2 \left[\frac{\tau}{T} \right]^2 + 2 \left[\frac{\bar{\alpha}T}{D_p} - 1 \right] \left[\frac{\tau}{T} \right] + 2 \right) - \right.$$

$$\left. \left. - \left(\frac{T\bar{\alpha}}{D_p} - 1 \right) \left(1 + \left[\frac{T\bar{\alpha}}{D_p} - 1 \right] \frac{\tau}{T} \right) \right] e^{-\frac{\bar{\alpha}\tau}{D_p}} \right\} \right\} + \tag{6b}$$

$$+ \frac{P_o \bar{B}}{\left[\frac{T\bar{\beta}}{D_p} - 1\right]^2} \left\{ \left\{ \Delta + b(\tau - T) \right\} \left\{ \left[\frac{T}{D_p} (\bar{\beta} - 1) - 1 \right] e^{-\frac{\tau}{T}} + \left[\frac{T}{D_p} \left(1 + \frac{\tau}{T} \left[\frac{T\bar{\beta}}{D_p} + 1 \right] \right) - \frac{T\bar{\beta}}{D_p} + 1 \right] e^{-\frac{\bar{\beta}\tau}{D_p}} \right\} - \right.$$

$$- \frac{Tb}{\frac{T\bar{\beta}}{D_p} - 1} \left\{ \left[\frac{T}{D_p} (\bar{\beta} - 2) - 1 \right] e^{-\frac{\tau}{T}} + \left[\frac{T}{D_p} \left(\left[\frac{\bar{\beta}T}{D_p} - 1 \right]^2 \left[\frac{\tau}{T} \right]^2 + 2 \left[\frac{\bar{\beta}T}{D_p} - 1 \right] \left[\frac{\tau}{T} \right] + 2 \right) - \right.$$

$$\left. \left. - \left(\frac{T\bar{\beta}}{D_p} - 1 \right) \left(1 + \left[\frac{T\bar{\beta}}{D_p} - 1 \right] \frac{\tau}{T} \right) \right] e^{-\frac{\bar{\beta}\tau}{D_p}} \right\} \right\} +$$

$$+ P_o \sigma_o \left(1 - \frac{\tau}{D_p} \right) \left[\bar{A} e^{-\frac{\bar{\alpha}\tau}{D_p}} + \bar{B} e^{-\frac{\bar{\beta}\tau}{D_p}} \right] \qquad \tau \geq 0$$

where

$$\left.\begin{aligned} \Delta &= \sigma_\infty - \sigma_o \qquad &(a) \\ \text{and} \qquad & \\ b &= \frac{\Delta}{T} - \dot{\sigma}_o \qquad &(b) \end{aligned}\right\} \qquad (7)$$

The interpolations considered should prove convenient in different problems of wave propagation in viscoelastic media. It can be applied in two - or three dimensional problems where two distinct signals, similar to P and S waves in elastic media are received at a point. In such cases, each component of the signal may be approximated by an expression similar to those given by Eq. (1), (2), (4) or (6). The applications of these interpolations to the stresses produced by a progressing surface pressure on the surface of a viscoelastic half-space [Section I of this report] are given in Section III.

REFERENCE.

1. "An Approximation in Problems of Viscoelastic Wave Propagation" by H.H. Bleich and J.L. Sackman, Technical Report No. 10, Contract Nonr-266(34), Columbia University, September 1960.

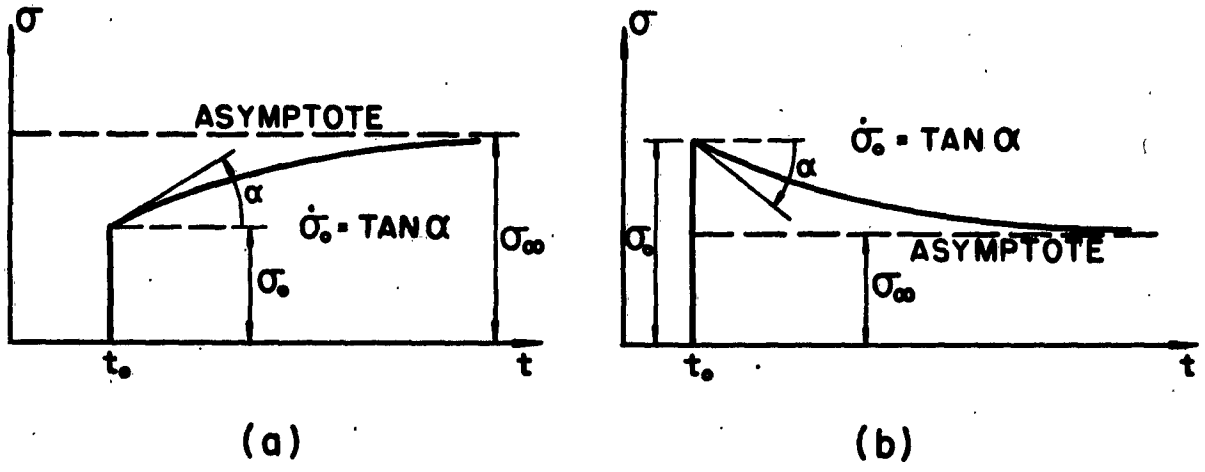
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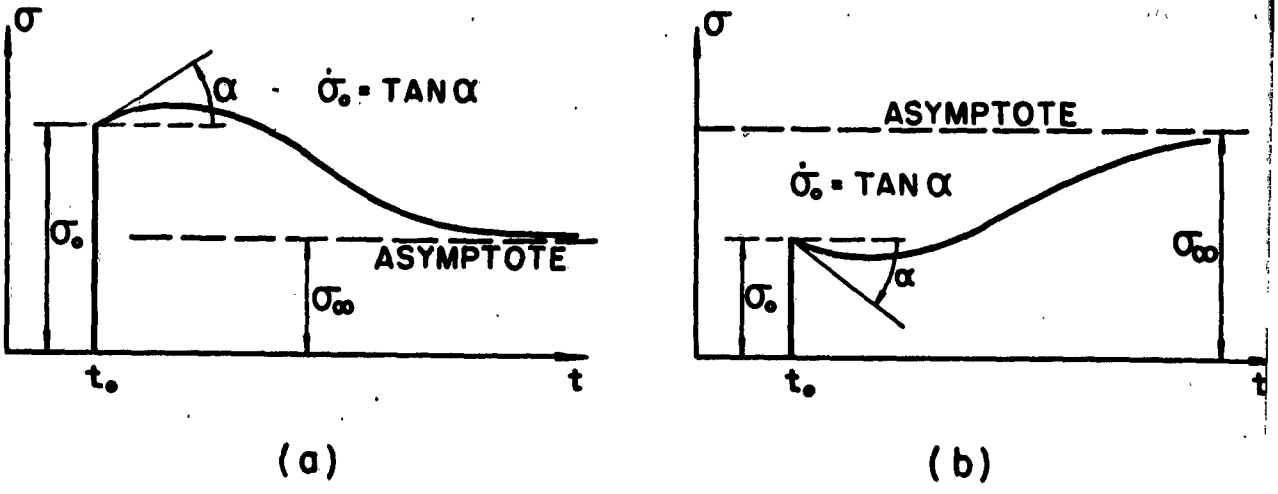
FIG. 1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(b)

FIG. 2

SECTION III

EVALUATION OF THE STRESSES IN A VISCOELASTIC HALF-SPACE WHICH ARE PRODUCED BY A PROGRESSING TIME-DECAYING SURFACE PRESSURE

BY

MELVIN L. BARON

AND

RAYMOND PARMES

EVALUATION OF THE STRESSES IN A VISCOELASTIC HALF-SPACE WHICH ARE PRODUCED BY A PROGRESSING TIME-DECAYING SURFACE PRESSURE.

1. Introduction.

The stresses which are produced in a linear viscoelastic (standard solid) half-space by the uniform motion of a step pressure on its surface have been studied in Section I for the case in which the velocity of the traveling surface pressure is greater than the velocities of plane irrotational and equivoluminal waves in the medium (superseismic case). Specific results were presented which allow the determination of the quantities σ_o, $\dot{\sigma}_o$ and σ_∞ corresponding to both the irrotational and the equivoluminal waves produced in the medium by the unit step pressure on the surface. The quantities σ_o, $\dot{\sigma}_o$ and σ_∞ can be used in conjunction with the interpolations which are presented in Section II to determine the stress history at a point in the medium produced by a time-decaying surface pressure [Eqs. (4) or (6), Section II].

The present Section presents numerical results for the stress components σ_{xx}, σ_{zz} and σ_{xz} at various points in the viscoelastic medium [Fig. (1)] which are produced by the time-decaying surface pressure shown in Fig. (2). [See Eq. (3), Section II]. For comparison purposes, the corresponding stresses which would be produced in an elastic medium having a shear modulus equal to the unrelaxed shear modulus of the viscoelastic medium are also shown in each case.

The stresses produced by the irrotational and the equivoluminal waves in the viscoelastic media are first shown separately. In addition, the total stresses at a point are evaluated by superimposing the irrotational and the equivoluminal stresses with an appropriate time delay.

Due to the absence of reliable information on the appropriate viscoelastic constants for real materials, a set of viscoelastic constants has been chosen for an illustrative example only. The results should not be taken as applying to a specific real material unless the coincidence of the viscoelastic parameters for the real material and those used in the illustrative example can be shown. The numerical values of the material and the load constants which were used in the computations are tabulated below.

(a) Material Constants - Viscoelastic Medium.

μ = Unrelaxed shear modulus = 1.200 (10^6) lb/in^2.

μ^* = Relaxed shear modulus = $\mu/2$ = 0.600 (10^6) lb/in^2.

ρ = Mass density of medium = 5.186 lb.sec^2/ft^4.
(w = 167 lb/ft^3).

ν = Poisson's ratio of unrelaxed body = 0.25.

T = Relaxation time for viscoelastic medium = 15 ms.

(b) Material Constants - Linearly Elastic Media.

ρ = Mass density of medium = 5.186 lb.sec^2/ft^4

ν = Poisson's ratio = 0.25.

C_p = Velocity of P waves = 10,000 ft/sec [unrelaxed μ].

C_s = Velocity of S waves = 6,000 ft/sec [unrelaxed μ].

μ = Shear modulus = μ (unrelaxed) = 1.2 (10^6) lb/in^2.

or

μ = Shear modulus = μ (relaxed) = 0.6 (10^6) lb/in^2.

The elastic medium for which μ (unrelaxed) is used will be referred to as Elastic-Unrelaxed; that for which μ (relaxed) is used will be referred to as Elastic-Relaxed.

(c) Constants for Surface Pressure Distribution [Fig. (2)].

P_o = Peak pressure = 2000 lb/in^2.

V = Uniform velocity of traveling wave = 12,000 ft/sec.

Relaxation time of surface pressure = 30 ms.

2. Numerical Results.

Consider the geometry shown in Fig. (1) in which the point A is located at a depth z = 500 ft and I and E represent the irrotational and the equivoluminal wave fronts respectively which are produced by the surface pressure shown in Fig. (2).

Figures (3)-(5) show the stresses σ_{xx}, σ_{zz} and σ_{zx} which are produced at the point A (z = 500 ft) by the irrotational wave in the viscoelastic medium. The value of the abscissa, $\tau = 0$, represents the arrival time of the wave at the point. The results were obtained using an interpolation of the type shown in Eq. (4), Section III and the input parameters σ_o, σ_∞ and $\dot{\sigma}_o$ were computed from Eq. (20) and Eq. (22) of Section I. In each case, the corresponding stress in an elastic body whose shear modulus is equal to the shear modulus of the unrelaxed viscoelastic body is also shown. It is seen that the high peak stress which is predicted by linear elastic theory is considerably attenuated by the viscoelastic medium. However, at later times, the stresses in the viscoelastic body may be higher than those in the linearly elastic body.

The corresponding stresses are shown in Figs. (5)-(6) for the case of the equivoluminal wave. The value of the abscissa $\tau = 0$ represents the arrival time of the equivoluminal wave at the point A. Interpolations of the type given by Eq. (4) were used in evaluating $\sigma_{xx} = -\sigma_{zz}$ and σ_{zx}. In all cases, it is seen that the peak stresses which are predicted by linear elastic theory are significantly attenuated by the viscoelastic medium. In addition, the viscoelastic stresses are lower in magnitude than the corresponding elastic stresses over the entire time history.

The complete stress-time histories which are produced at the point A can be evaluated by superimposing the stresses shown in the previous figures with an appropriate delay time. Figures (7)-(9) show the total stresses σ_{xx}, σ_{zz} and σ_{zx} for both the viscoelastic and the elastic [unrelaxed] media where the time $\tau = 0$ represents the arrival time of the irrotational wave I at the point in question. The peak stresses which occur at the significant times, i.e. the arrival times of the irrotational and equivoluminal waves at the point, are considerably attenuated by the viscoelastic medium. Intermediate values of the stresses may be larger than those for the elastic [unrelaxed] medium.

It should be noted that the significant attenuations of peak stresses by the viscoelastic medium are purely a function of the depth z and do not occur for relatively shallow depths. To illustrate this, Figs. (10)-(11) show the magnitude (not the direction) of the principal stresses at points located at depths $z = 50$ ft and $z = 500$ ft, for the irrotational and the equivoluminal waves respectively. Interpolations of the type given by Eq. (4) were used in evaluating σ for the irrotational wave while those of Eq. (6) were required for the case of the equivoluminal wave. For the stresses produced by the irrotational wave, it is seen that the attenuation of the peak stresses in the neighborhood of $\tau = 0$ is very small for shallow depths but increases significantly with depth.

For the ratios V/C_p encountered in this example, the values of the principal irrotational elastic stresses obtained using the unrelaxed shear modulus are practically equal to the corresponding stresses obtained by using the relaxed shear modulus (*), and consequently only a single elastic stress curve is presented.

(*) See Appendix A, "Theoretical Studies in Ground Shock Phenomena", by M.L. Baron, H.H. Bleich, P. Weidlinger; The MITRE Corporation, Report SR-19, October 1960.

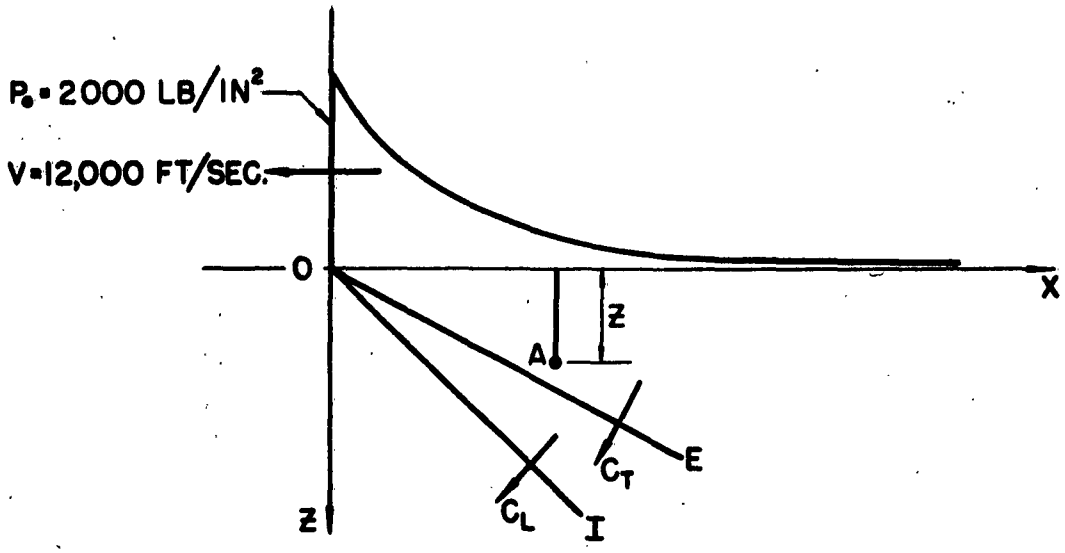

FIG. 1

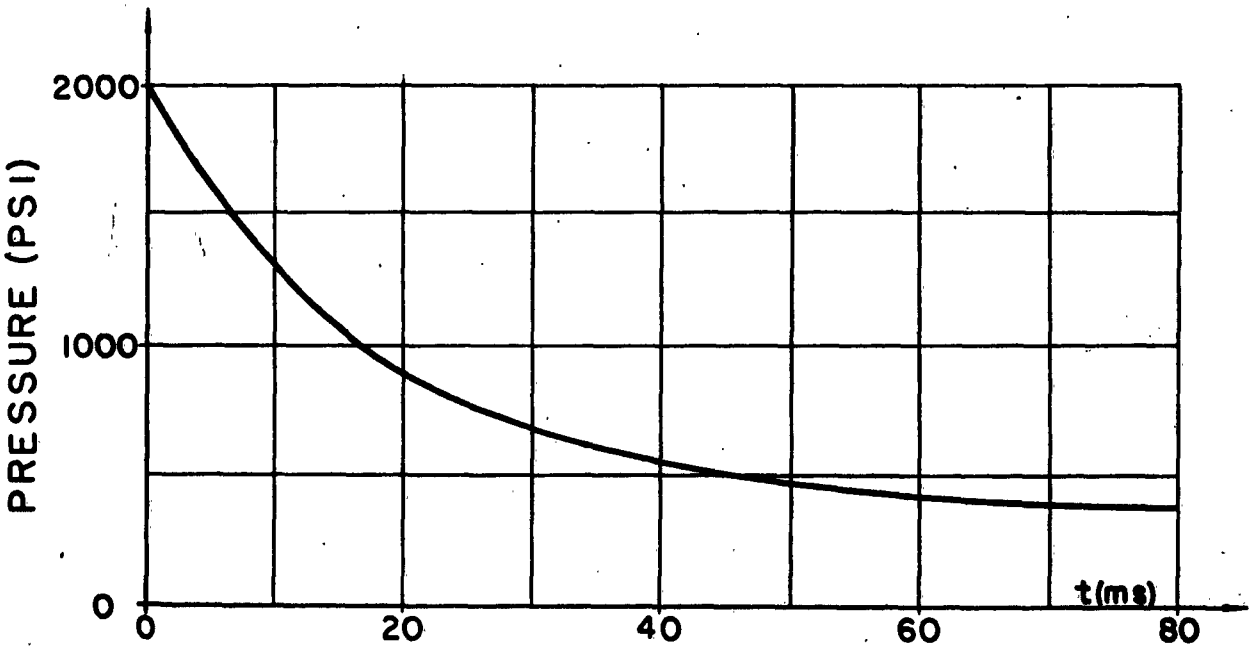

FIG. 2 SURFACE PRESSURE-TIME HISTORY

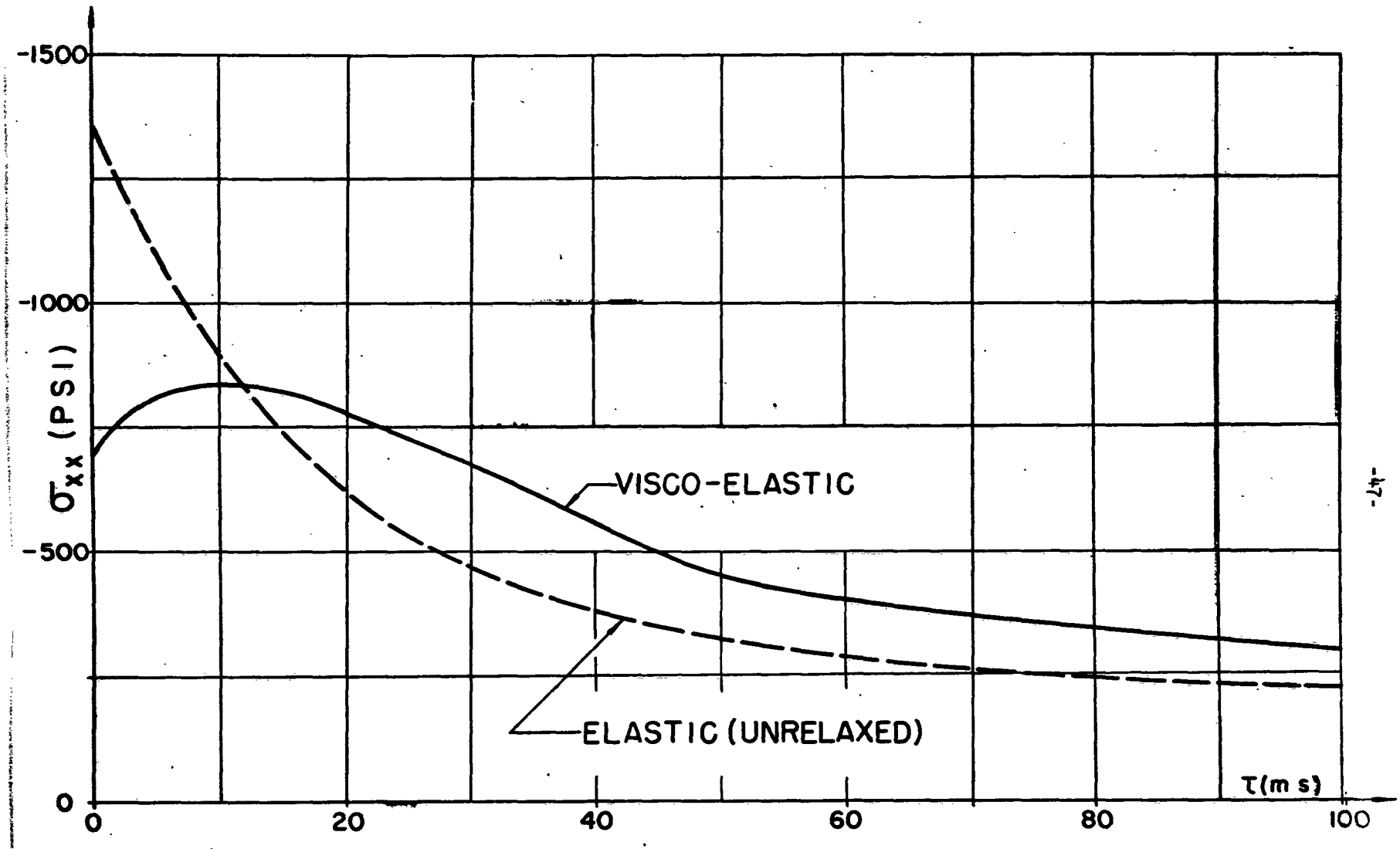

FIG. 3 STRESS σ_{xx} (IRROTATIONAL WAVE): Z=500 FT.

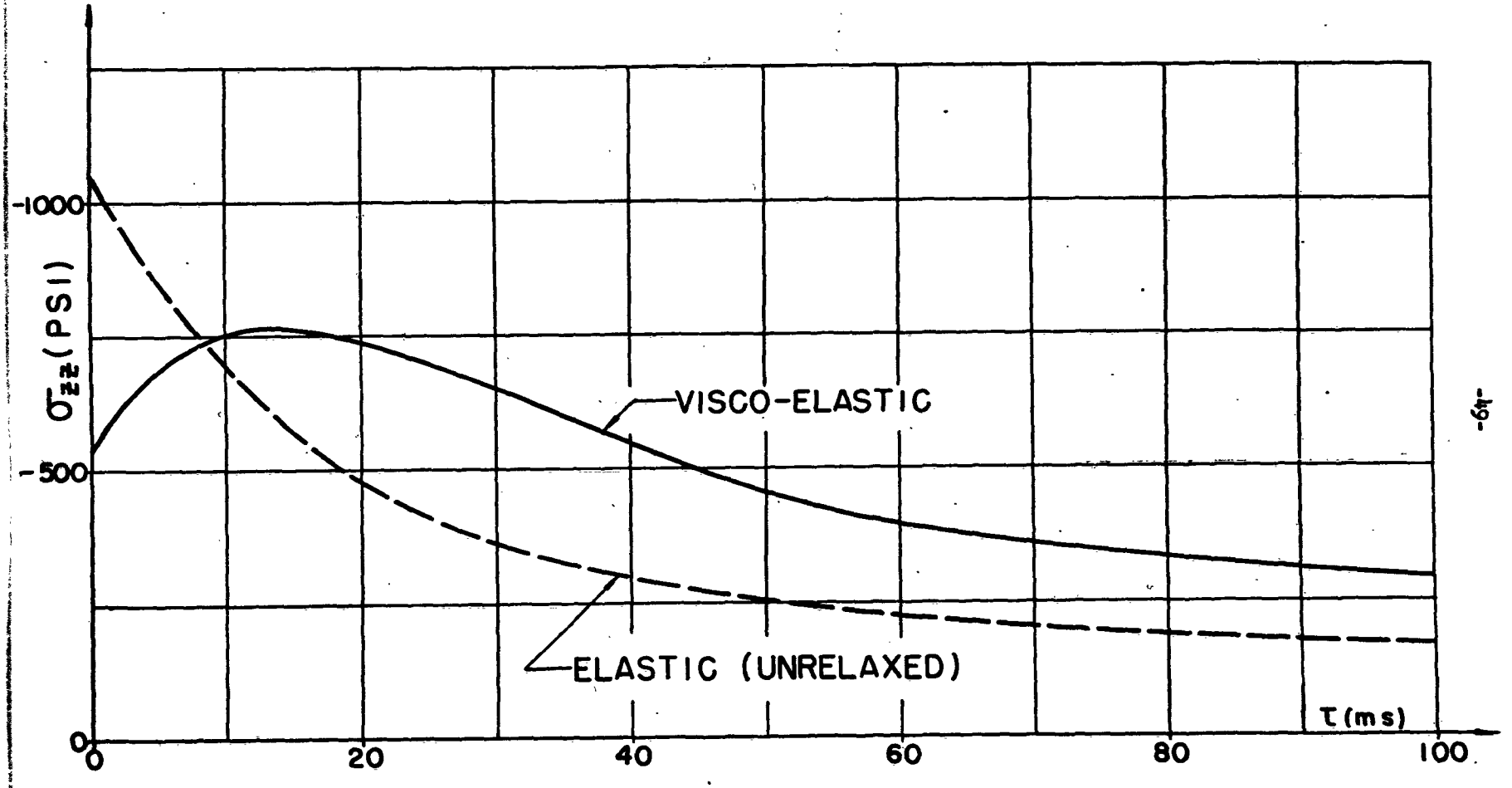

FIG. 4 STRESS σ_{zz} (IRROTATIONAL WAVE): Z=500 FT.

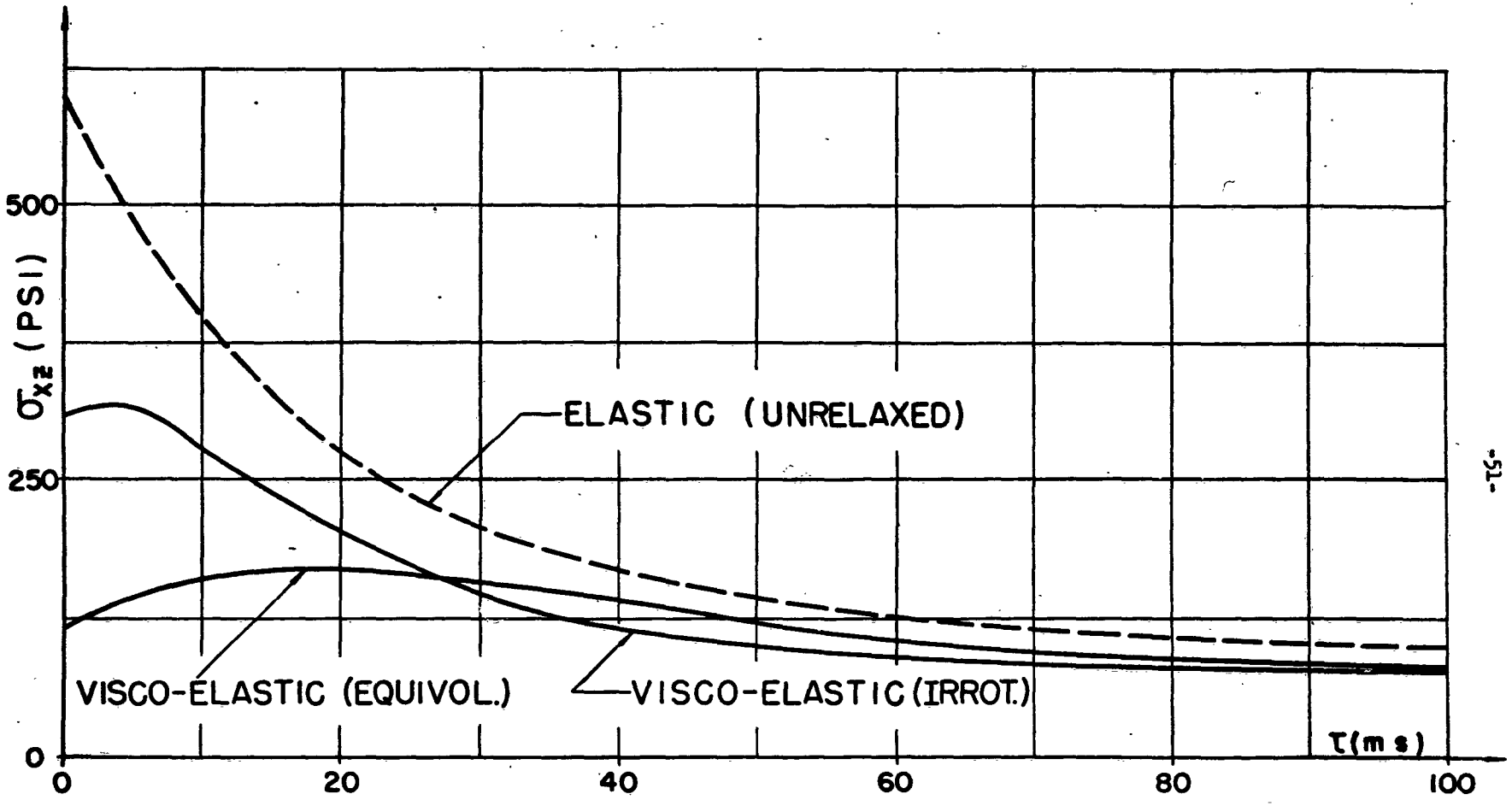

FIG. 5 STRESS $\begin{cases} \sigma_{xz} \text{ (IRROTATIONAL WAVE)} \\ -\sigma_{xz} \text{ (EQUIVOLUMINAL WAVE)} \end{cases}$ z=500FT.

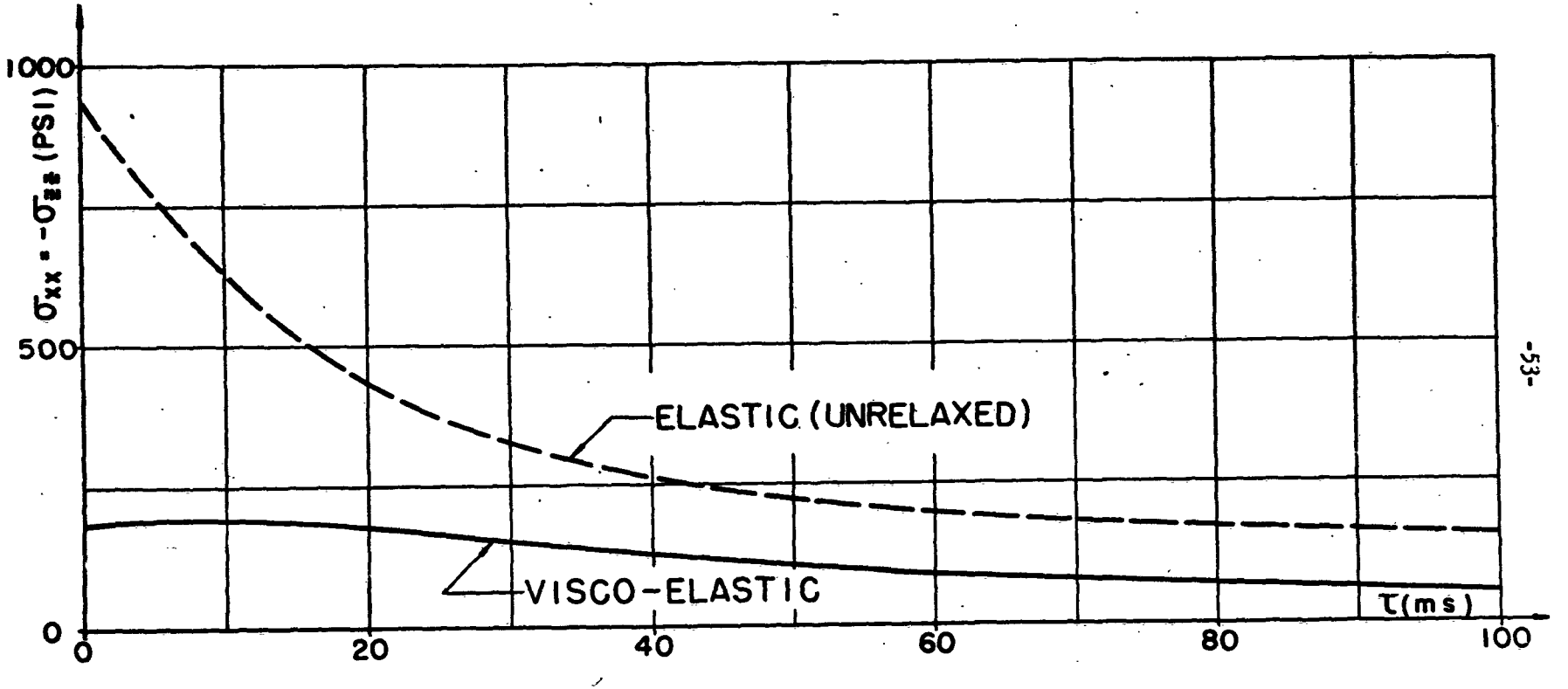

FIG. 6 STRESSES $\begin{cases} \sigma_{xx} \\ -\sigma_{zz} \end{cases}$ (EQUIVOLUMINAL WAVE) : Z=500FT.

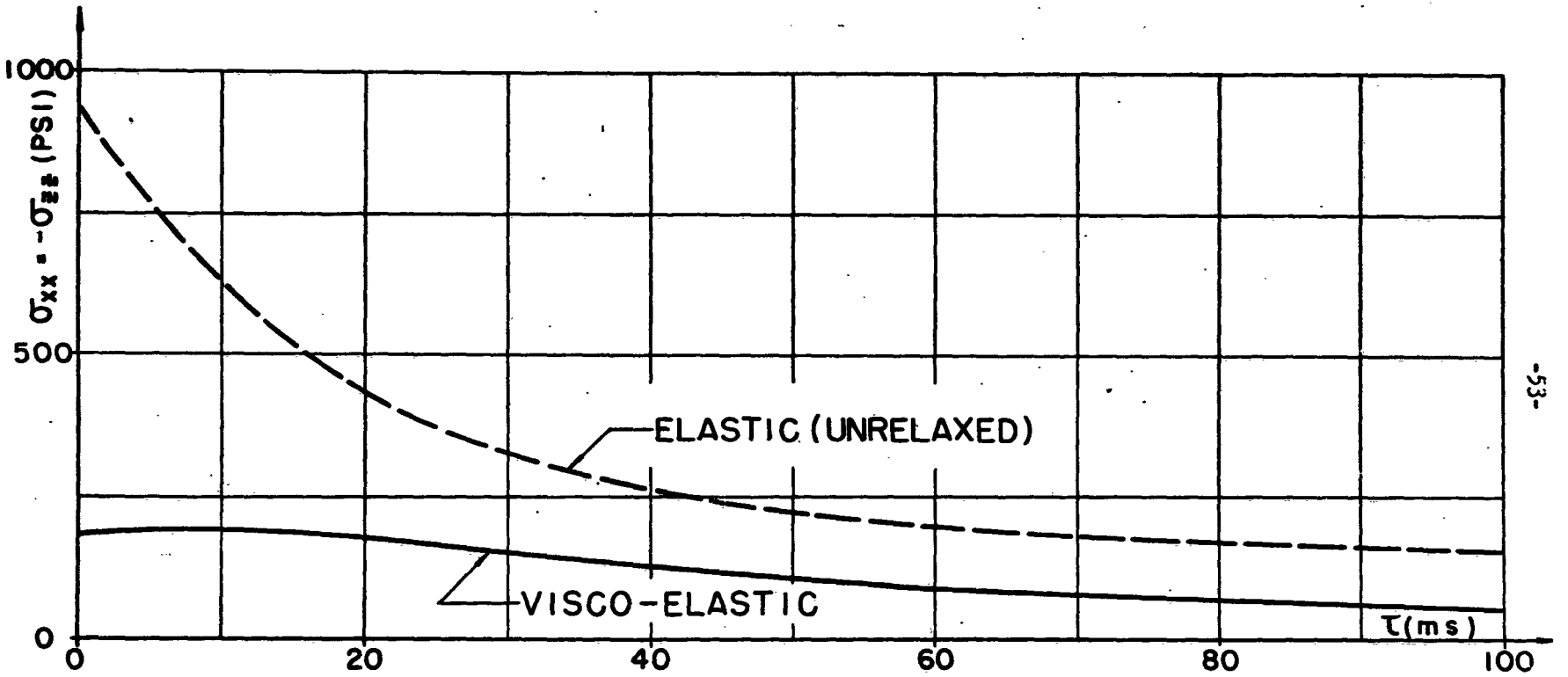

FIG. 6 STRESSES $\begin{cases} \sigma_{xx} \\ -\sigma_{zz} \end{cases}$ (EQUIVOLUMINAL WAVE) : Z=500FT.

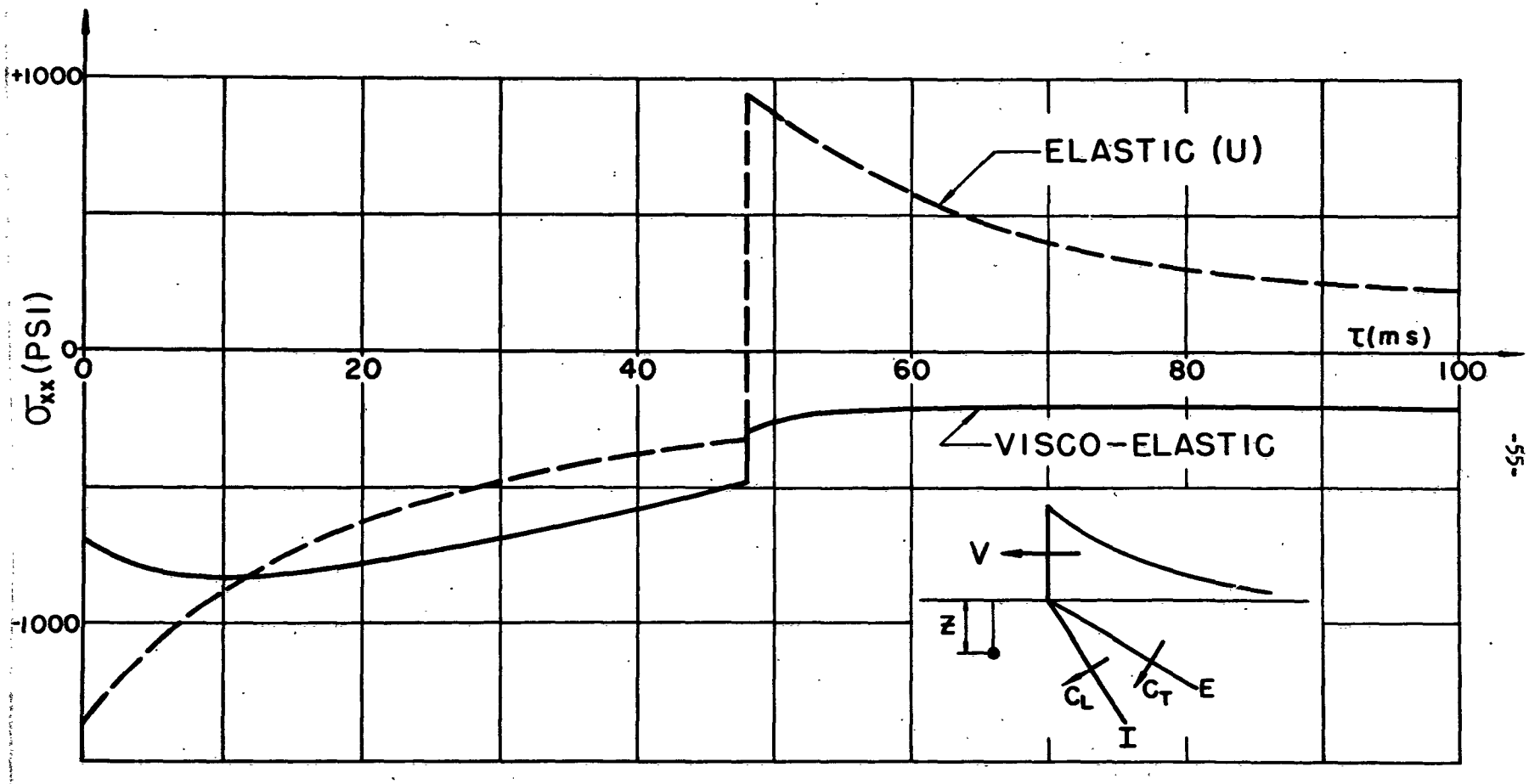

FIG. 7 TOTAL STRESS σ_{xx} : Z = 500FT.

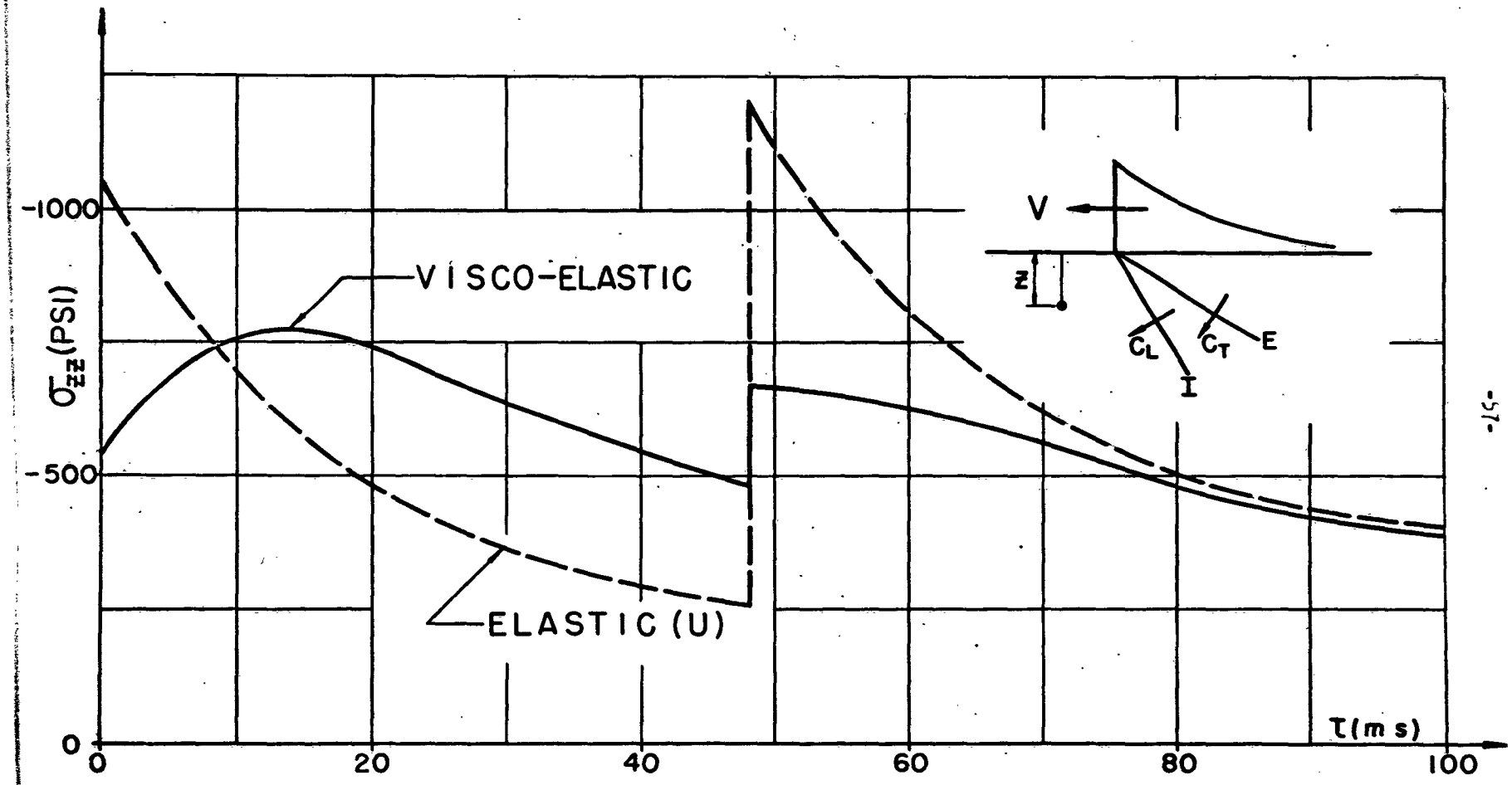

FIG.8 TOTAL STRESS σ_{zz} : Z=500FT.

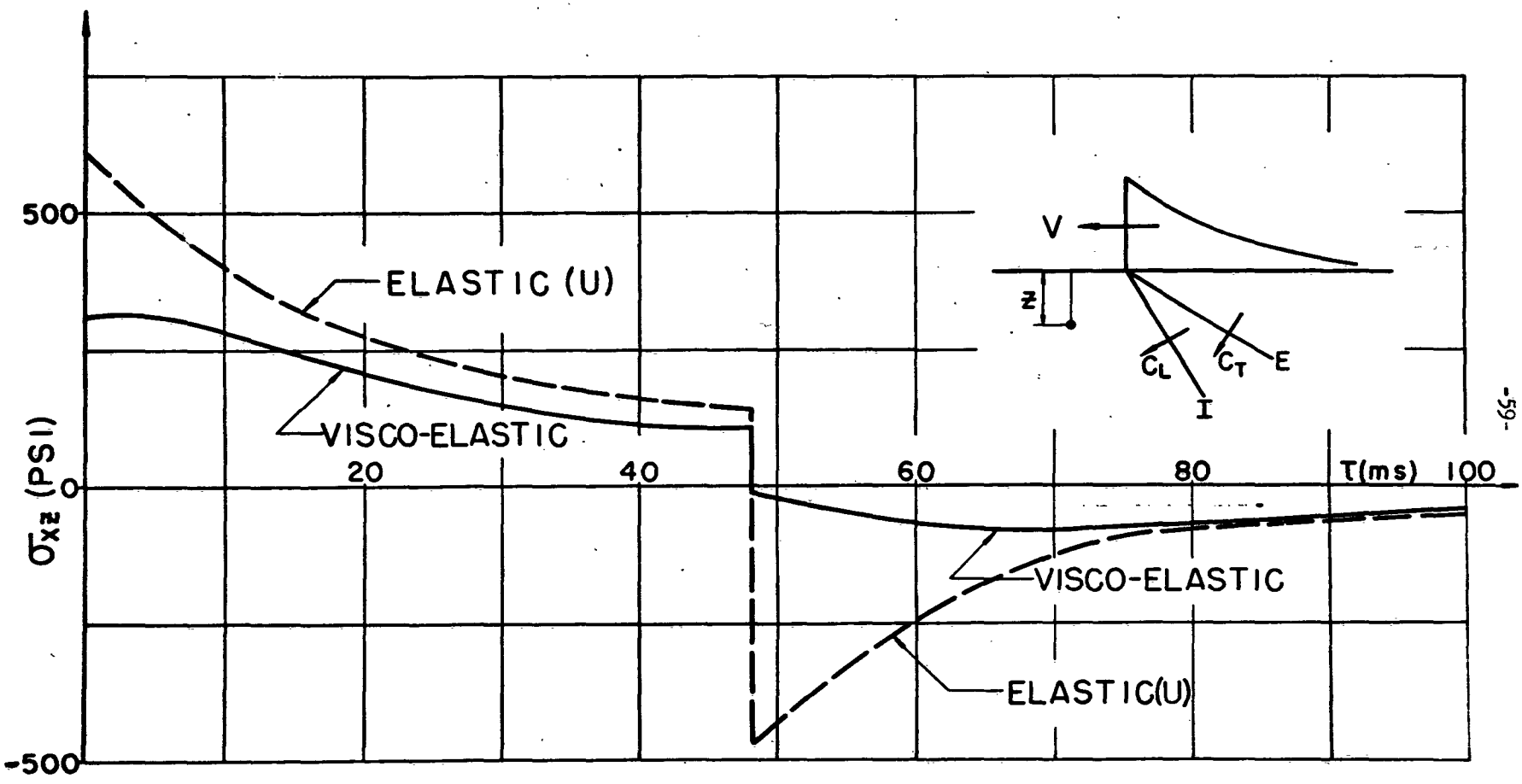

FIG. 9 TOTAL STRESS σ_{xz} : Z = 500 FT.

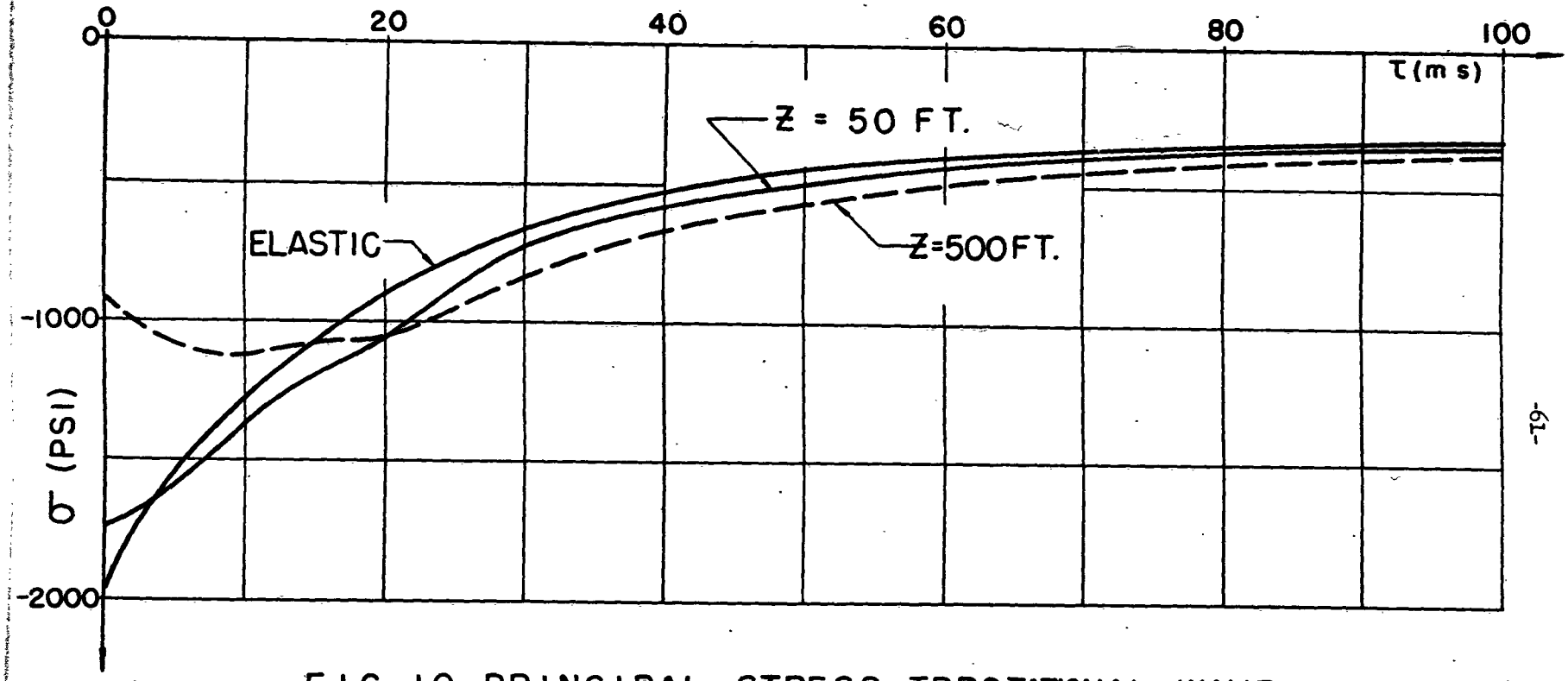

FIG. 10 PRINCIPAL STRESS-IRROTATIONAL WAVE

Z=50FT, 500FT.

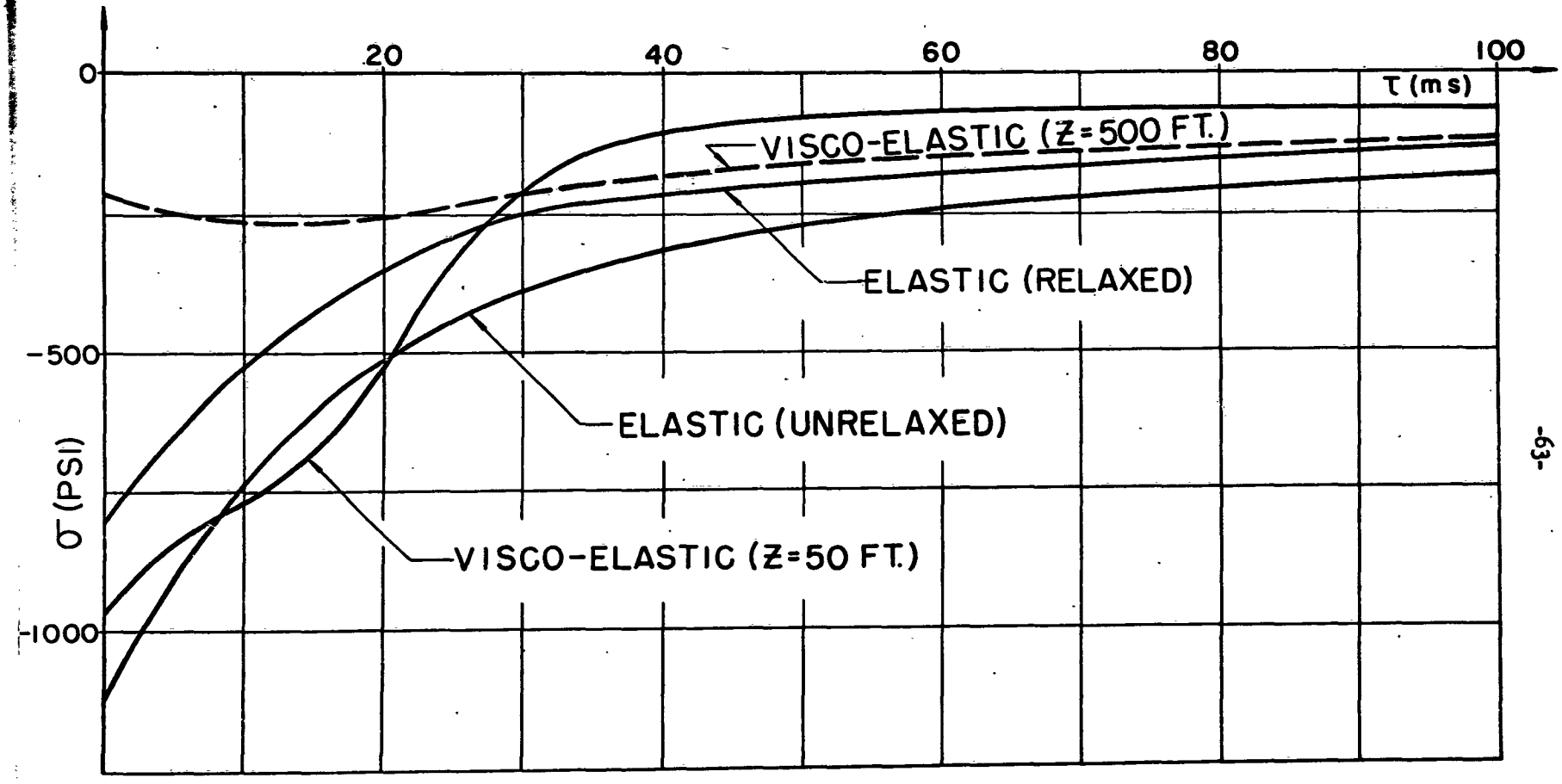

FIG. 11 PRINCIPAL STRESS-EQUIVOLUMINAL WAVE

Z = 50 FT, 500 FT.