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BACK SCATTERING OF A PLANE ELECTROMAGNETIC WAVE FROM AN INFINITE CYLINDER OF PLASMA WITH SMALL AXIAL GRADIENTS AND ARBITRARY RADIAL GRADIENTS OF ELECTRON DENSITY - NORMAL INCIDENCE

By F. Lape
M. Abela
S. Mariano

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BACK SCATTERING OF A PLANE ELECTROMAGNETIC WAVE FROM AN INFINITE CYLINDER OF PLASMA WITH SMALL AXIAL GRADIENTS AND ARBITRARY RADIAL GRADIENTS OF ELECTRON DENSITY - NORMAL INCIDENCE

By F. Lane
M. Abele
S. Mariano

1 DESCRIPTION AND FORMULATION OF PROBLEM

The problem of scattering of plane electromagnetic waves by cylindrical plasma structures without axial variation of plasma properties and with or without radial variation has been amenable to more or less classical diffraction analysis and has been treated by several investigators over the past few years 1, 2, 3, 4.

Axial uniformity of the plasma structure simplifies these analyses tremendously by rendering them essentially two-dimensional. Even under the case of obliquely incident plane waves, the axial dependence of the solution is set by the axial dependence of the incident wave, and the axial coordinate effectively disappears from the problem.

The introduction of axial property variation of the plasma adds essentially to the complexity of the problem and requires a three-dimensional treatment.

Moreover, axial dependence of plasma properties alters essentially the nature of scattered field. An obliquely incident plane wave exhibits no back scattering from an axially uniform plasma structure, whereas axial nonuniformity allows the possibility of back scatter. A normally incident plane wave exhibits purely radial scattering from an axially uniform plasma structure, whereas axial nonuniformity introduces axial as well as radial scattering (hence the essentially 3-dimensional nature of the problem).

In an effort to simulate more closely the characteristics of ionized missile wakes in their effects on incident electromagnetic radiation, one is eventually forced to introduce axial nonuniformity since the axially uniform configuration is considerably over-idealized. The present report describes an approximate diffraction calculation for cylindrical plasma structures having arbitrary (axisymmetric) radial property variation and weak axial property variation (small axial gradients of electron density).

A plane wave is assumed normally incident upon a cylindrical configuration such that the incident wave has its electric vector parallel to the cylinder axis (fig. 1a). The electron density distribution in the plasma is assumed to be axisymmetric arbitrarily distributed radially, but slowly varying axially (fig. 1b).

Under harmonic time-dependence $e^{-i\omega t}$ the field equations appropriate to the problems are usually given in the form:

\[ \nabla \times \mathbf{B} = i\omega \mu_0 \mathbf{H} \]  

\[ \nabla \times \mathbf{H} = -i\omega \epsilon_0 \mathbf{E} + \frac{Z E}{m(\nu^2-i\omega)} \]  

where $\mathbf{E}$ and $\mathbf{H}$ are respectively the electric and magnetic field vectors, $\mu_0$, $\epsilon_0$ are respectively the magnetic and dielectric permeabilities in a vacuum, $n$ is electron number density, $e$ is electron mass, and $\nu$ is the collision frequency for electrons with neutral particles.

Introducing plasma frequency $\omega_p$, defined by

\[ \omega_p^2 = \frac{2}{\epsilon_0 m} \]
Eq. (2) takes the form
\[ \nabla \times \mathbf{H} = -\omega \epsilon_0 \mathbf{E} \left( 1 - \frac{\omega^2}{\omega^2 + \mu^2} \left( 1 - j \frac{\omega}{\omega^2 + \mu^2} \right) \right) \] (4)

The field equations (1)-(4) are supplemented by the statement that the problem is forced by an incident plane wave of the form
\[ \mathbf{E}_{\text{inc}} = \begin{cases} 0 & \text{for } \omega \leq \omega_0 \\ \epsilon_0 \mathbf{E}_0 e^{-j \omega t - j k_0 z} & \text{for } \omega > \omega_0 \end{cases} \] (5)

where
\[ k_0 = \frac{\omega}{c} \sqrt{\epsilon_0 \mu_0} \] (6)

and by the condition that a scattered field results in the vacuum assumed to exist outside the plasma structure, with continuity required between tangential field components (\( H \) and \( E \)) across the plasma-vacuum interface
\[ \left[ \mathbf{E}^T \right]_{\text{tang}} = \left[ \mathbf{H}^T \right]_{\text{tang}} = 0 \] (7)

where \([ ]\) signifies jump in a quantity across the interface. Moreover, the scattered field must satisfy the radiation or outgoing-wave condition at large distances from the cylinder.

II. SOLUTION METHOD
A. Review of Axially Uniform Plasma Case

In the case of an axially uniform plasma, the scattered field, as well as the field inside the plasma, has a purely axial electric vector and, since normal incidence is assumed, no \( z \)-dependence enters the problem. Thus \( \mathbf{E} \) is divergenceless and
\[ \nabla \times \nabla \times \mathbf{E} = -\Delta \mathbf{E} = k_0^2 \mathbf{E} \left( 1 - \frac{\omega^2}{\omega^2 + \mu^2} \right) \] (8)

Furthermore, since \( \mathbf{E} \) is purely axial and, this equation (8) applies component by component in Cartesian coordinates, the equation of the problem, under axial uniformity of plasma structure, reduces to
\[ \Delta \mathbf{E} + k_0^2 \left( 1 - P \right) \mathbf{E} = 0 \] (9)

where \( \mathbf{E} \) represents \( \mathbf{E}_0 \) and where
\[ P = \frac{\omega_p^2}{\omega^2 + \mu^2} \left( 1 - \frac{\omega}{\omega_p} \right) \] (10)

In the vacuum, of course, \( P \) reduces to zero.

The magnetic field is expressible, via equation (1), by
\[ \mathbf{H} = \frac{1}{j \omega \mu_0} \nabla \times \mathbf{E} \] (1)

both inside and outside the plasma.

*A subscript \( u \) on an equation number indicates that the equation holds only in the case of an axially uniform plasma and not in the presence of axial nonuniformity.
Switching, for convenience, to cylindrical polar coordinates \((r, \theta, z)\), the incident wave (5) takes the form (dropping the harmonic time-dependence henceforth)

\[
E_{\text{inc.}} = E_0 e^{-i\omega t} \sum_{m=-\infty}^{\infty} c_m H_m(\beta r) e^{-im\phi} J_m(k r) e^{im\phi}
\]

This must be (and is), of course, a solution of (9) with \(P(r) = 0\). The scattered electric field must then have the form

\[
E_{\text{scat.}} = \sum_{m=-\infty}^{\infty} c_m H_m(\beta r) e^{-im\phi}
\]

(12)

since this is the general solution of (9) with \(P = 0\) which satisfies the radiation condition at large \(r\). Here the \(c_m\) are constant coefficients, and the axially uniform plasma problem then reduces to the problem of determining these coefficient \(c_m\). The magnetic field, corresponding to purely axial electric field is given by

\[
\vec{H} = \frac{1}{i \omega} \begin{pmatrix} H_r \\ H_\theta \\ H_z \end{pmatrix} = \frac{1}{i \omega} \begin{pmatrix} 1 \frac{\partial E}{\partial r} \\ -\frac{\partial E}{\partial \theta} \\ \frac{\partial}{\partial z} \end{pmatrix}
\]

(13)

(This holds both inside and outside the plasma.)

The electric field inside the plasma is purely axial and is given by an expression of the form

\[
E = \sum_{m=-\infty}^{\infty} f_m(r) e^{-im\phi}
\]

(14)

where the functions \(f_m\) depend upon the nature of the radial dependence of the plasma electron density. In the case where the plasma is collisionless and uniform radially, the \(f_m\) reduce to Bessel Functions \((J_m(r)\) for the underdense case and \(K_m(r)\) for the overdense case) with argument \(r k \sqrt{1 - \alpha^2}\). For electron density varying radially as \(1/r^2\), the \(f_m\) are given by Bessel Functions of non-integer order \((1, 2)\) while for more general radial distributions of electron density, a combination of such functions may be applicable \((1, 2)\) or numerical solution of the radial equation may be required \((3, 4)\).

At any rate, the general internal electric field is of the form (14). Moreover, boundedness of the field at the centerline dictates that each \(f_m\) involved exactly one undetermined coefficient. Thus there remain, for each \(\theta\)-harmonic (each \(m\)) two undetermined coefficients \((c_m, \text{ and the undetermined coefficient in } f_m)\). The continuity of \(E\) (\(E\) axial) and \(H_\theta\) across the plasma-vacuum interface then give precisely the two conditions necessary to fix the undetermined coefficient in \(f_m\) and the coefficient \(c_m\) in the scattered field. Determination of the \(c_m\) completes the description of the scattered field from which, in turn, the scattered energy can be computed.

B. Extension to Axially Nonuniform Plasma

Now the mode of reasoning introduced for the case of an axially varying plasma is as follows:

The form (12) of the axially uniform problem suggests that the scattered field may be considered as arising from a superposition of line multipoles, that is, multipoles uniformly distributed with respect to \(z\) over the centerline of the plasma cylinder. (The representation is, of course, valid only outside the plasma). Associated with \(m = 0\) is the line source, with \(m = \pm 1\) is the line
dipole, etc. The strength of each of these multipoles is given by the appropriate
coefficient $c_m$ which depends on the details of the radial distribution of electron
density. The effects of the radial distribution of electron density (and collision
frequency) thus become lumped into a single coefficient $c_m$ for each order $m$ of
multipole.

This suggests that, as an approximation for slowly axially varying plasma
configurations, a sort of "strip-theory" could be invoked in which the scattered
field would continue to be expressed as (12), but each $c_m$ would be considered a
function of $z$, this function to be that appropriate to the two-dimensional problem
with plasma properties given by those of the actual configuration at the local value
of $z$. This has several obvious deficiencies. First, the resulting field is no
longer a solution of the vacuum field equations.

Second, the electric field would persist in being purely axial and the
magnetic field, purely transverse; and this is not the true state of affairs. Third,
the axial field dependence would be purely that of the plasma or some function
thereof and would not represent the axial propagation which actually occurs. In
fact, the only conditions really met by the strip-theory solution would be the jump
conditions across the vacuum-plasma interface and the radial radiation condition.

Suppose, however, that instead of a simple strip-theory as mentioned
above, one invokes a modified form of strip-theory in which the line multipole
density is made $z$-dependent and the field is computed from the resulting $z$-
dependent multipole distributions. That is, the $c_m$'s are interpreted as multiple
intensities and those multipole intensities are made $z$-dependent in accord-
ance with a local or strip-theory value of $c_m$. Then the total three-dimensional
scattered field is calculated as having arisen from this $z$-dependent line dis-
tribution of multipole intensities. The resulting field then (1) satisfies the
vacuum field equations (2) exhibits axial propagation (3) has all three electric
vector components (but still lacks an axial magnetic vector component) (4)
satisfies the radiation condition. The chief failings are that the magnetic field
persists in remaining purely transverse, (as will be seen subsequently) and the
interface conditions are violated, the degree of violation depending upon the
degree of nonuniformity of the axial plasma dependence. These failings may
appear to render the method questionable; however several arguments lend
justification to the procedure. First, the solution does reduce to the correct one
as the axial dependence of plasma properties approaches zero. Second, the
method can be shown to correspond to the first step in an iterative procedure
based upon an integral-equation formulation of the problem. Third, the exact
solution (in the axially nonuniform plasma case) should be representable by some
distribution (in a pair of distributions, as will be shown) of line multipoles along
the centerline, and we are computing an approximation to the correct distributions.

In the vacuum, outside the cylindrical plasma structure, the field equations
(1), (2) reduce to the classical Maxwell's equations for harmonic time dependence.

\[ \nabla \times \mathbf{D} = \omega \varepsilon_0 \mathbf{H} \]  
(1)

\[ \nabla \times \mathbf{H} = -\omega \mu_0 \mathbf{E} \]  
(15)

Utilizing the ideas of Stratton (p. 393 and it.) it can be shown that a
general solution to these field equations is expressible through the use of two
independent scalar quantities $\varphi$ and $\chi$. (This can even be done inside the
plasma when the plasma has purely $z$-al nonuniformity but is independent of
$x,y$ or $r, \theta$.) The general solution in terms of scalars $\varphi$ and $\chi$ may be expressed
as follows:
Next, we seek line-multipole solutions to the system (16) (17) (18). As is well-known, the simple source solution to the reduced wave equation in three-space is given, for a \( e^{-i\omega t} \) time-dependence, by

\[
\frac{i\omega \rho}{\rho}
\]

Where \( \rho \) is distance between source and field point.

A centered line distribution of sources is then obviously still a solution of the reduced wave equation and is given by

\[
\int_{-\infty}^{\infty} f(\zeta) e^{i\omega \rho} d\zeta
\]

where

\[
\rho = \sqrt{x^2 + y^2 + (z - \zeta)^2} = \sqrt{r^2 + (z - \zeta)^2}
\]

and where \( f(\zeta) \) gives the source density and must satisfy such conditions that the integral (20) exists. It is well-known that (20) reduces to a multiple of the symmetric Hankel Function \( H_\omega^{(1)}(k_\omega \rho) \) when \( f(\zeta) \) is constant, independent of \( \zeta \).

Letting \( (r, \theta) = r \sin \phi \),

\[
\int_{-\infty}^{\infty} e^{i\omega \rho} d\rho \int_{-\infty}^{\infty} e^{ik_\omega \rho \cos \theta} d\tau = i\pi H_\omega^{(1)}(k_\omega \rho)
\]

Thus (20) is an extension of the axially symmetric (simple line source) two-dimensional field \( H_\omega^{(1)}(k_\omega \rho) \) to the \( z \)-dependent or three-dimensional case with the density function \( f(\zeta) \) remaining to be appropriately determined.

The higher-order multipoles may now be easily derived from the simple source distribution (20). Obviously if \( u(x, y, z) \) is a solution of the reduced wave equation, then so is
\[ \frac{\partial}{\partial x^m} \frac{\partial}{\partial y^m} u(x, y) = 0 \]

Suppose we introduce the notation

\[ I_n(x, y) = \frac{1}{2\pi} \int_{-\pi}^{\pi} f_\alpha(\zeta) e^{i\zeta x} e^{-i\zeta y} d\zeta \]

(24)

where \( \alpha = 0, 1, 2 \ldots \).

The \( I_n \) are all evidently solutions of the reduced wave equation (under suitable integrability restrictions on the \( f_\alpha(\zeta) \)). Moreover,

\[ \frac{\partial}{\partial x^m} \frac{\partial}{\partial y^m} \theta \]

Thus

\[ I_n = \frac{1}{2\pi} \int_{-\pi}^{\pi} f_\alpha(\zeta) e^{i\zeta x} e^{-i\zeta y} d\zeta \]

(26)

Note that if all the \( f_\alpha(\zeta) \) are 1 for all \( n \), then

\[ I_0 = H_{10}^1(k_0 x) \]

(27)

and, using mathematical induction,

\[ I_n = e^{i\zeta x} H_{10}^1(k_0 x) \]

(28)

utilizing the recurrence relations \( \gamma \) for the Hankel Functions. Thus, in the special case where the density functions are constant, the \( I_n \) reduce to the two-dimensional solutions to the reduced wave equation

\[ e^{i\zeta x} H_{10}^1(k_0 x) \]

(30)

Hence, for \( \zeta \)-dependent density functions \( f_\alpha \), the integrals \( I_n \) must represent three-dimensional solutions to the reduced wave equation corresponding to a distribution of an \( n \)th order multipole along the centerline, the density being given by the \( f_\alpha(\zeta) \).

Similarly, if we introduce

\[ I_{-n} = \frac{1}{2\pi} \int_{-\pi}^{\pi} f_\alpha(\zeta) e^{-i\zeta x} e^{i\zeta y} d\zeta \]

(31)

it follows that when \( f_\alpha = 1 \) for all \( n \), then

\[ I_{-n} = e^{-iax} H_{10}^1(k_0 x) \]

(32)

so that the expressions \( I_{-n} \) of (31) constitute the extension to three-dimensionality of the negative-index terms in the two-dimensional expansion (12a).
\[ I_n = \left( \frac{1}{k_0} \right)^n e^{-i \alpha \theta} \left[ \left( \frac{\partial}{\partial r} \right)^n \frac{1}{r} \right] \left( \frac{\partial}{\partial r} \right)^{n-1} \cdots \left( \frac{\partial}{\partial r} \right) \frac{1}{r} \int_0^\infty f_n \left( \frac{r}{\rho} \right) \frac{1}{\rho} d\rho d\zeta \]

\[ = \left( \frac{-1}{k_0} \right)^n e^{-i \alpha \theta} \left[ \left( \frac{\partial}{\partial r} \right)^{n-1} \frac{1}{r} \right] \cdots \left[ \left( \frac{\partial}{\partial r} \right)^{n-1} \frac{1}{r} \right] \left( \frac{\partial}{\partial r} \right) \frac{1}{r} \int_0^\infty f_n \left( \frac{r}{\rho} \right) \frac{1}{\rho} d\rho d\zeta \]  

(33)

and similarly

\[ I_n = \left( \frac{r}{k_0} \right)^n e^{-i \alpha \theta} \left[ \left( \frac{\partial}{\partial r} \right)^n \frac{1}{r} \right] \int_0^\infty f_n \left( \frac{r}{\rho} \right) \frac{1}{\rho} d\rho d\zeta \]

(34)

Now the expressions (33) (34) can be given in a still more convenient form in terms of spherical Hankel functions \( h_n^{(1)} (k_0 \rho) \) (see, for example, Stratton (3) p. 404 and ff.) These functions may be defined as follows:

\[ h_n^{(1)} (k_0 \rho) = \frac{1}{\sqrt{2 \pi}} e^{i k_0 \rho} W_{n+\frac{1}{2}}^{(1)} (k_0 \rho) \]

(35)

Consequently it is easily seen that

\[ h_n^{(1)} (k_0 \rho) = e^{i k_0 \rho} \frac{1}{i k_0} \]

(36)

Therefore, from (33) we see that

\[ I_0 = \frac{k_0}{\pi} \int_0^\infty f_0 (\xi) h_0^{(1)} (k_0 \rho) d\xi \]

(37)

Now we resort again to mathematical induction:

\[ I_n = e^{i \alpha \theta} \frac{k_0}{\pi} \int_0^\infty f_0 \left( \frac{r}{\rho} \right) h_n^{(1)} (k_0 \rho) f_n (\xi) d\xi \]

(38)

Then, by (33)

\[ I_{n+1} = -\frac{1}{k_0} \int_0^\infty f_{n+1} (\xi) \left( \frac{\partial}{\partial r} \right) \frac{1}{r} \int_0^\infty f_n \left( \frac{r}{\rho} \right) h_n^{(1)} (k_0 \rho) f_{n+1} (\xi) d\xi \]

(39)

\[ = -\frac{1}{\pi} e^{i (n+1) \theta} \int_0^\infty f_{n+1} (\xi) \left\{ \frac{n+1}{\rho} h_n^{(1)} (k_0 \rho) f_{n+1} (\xi) d\xi \right\} \]

(40)

\[ = e^{i (n+1) \theta} \frac{k_0}{\pi} \int_0^\infty f_{n+1} (\xi) \left( \frac{r}{\rho} \right) h_n^{(1)} (k_0 \rho) f_{n+1} (\xi) d\xi \]

(41)

Hence,

\[ I_{n+1} = -\frac{1}{\pi} e^{i (n+1) \theta} \int_0^\infty f_{n+1} (\xi) \left( \frac{n+1}{\rho} \right) h_n^{(1)} (k_0 \rho) f_{n+1} (\xi) d\xi \]

(42)
and it is proven that (38) holds:

$$I_n = e^{i\theta} \frac{k_o}{\sigma} \int_{-\infty}^{\infty} \left( \frac{r}{\rho} \right)^n h_n^{(1)}(k_o \rho) f_n(\xi) d\xi$$

(38)

$$n = 0, 1, 2, \ldots$$

By inspection of (34) we see that

$$I_{-n} = e^{-i\theta} \frac{k_o}{\sigma} \int_{-\infty}^{\infty} \left( \frac{r}{\rho} \right)^n h_n^{(1)}(k_o \rho) f_n(\xi) d\xi$$

(43)

$$n = 1, 2, 3, \ldots$$

Now the $I_n$, $I_{-n}$, given by (38) and (43) constitute three-dimensional solutions to the reduced wave equation constructed from linear distributions of multipoles of order $n$, the $f_n(\xi)$ giving the line density of the distribution. Recalling the system (16), (17), (18), giving a general solution to the vacuum field equations, we see that two solutions, $\varphi$ and $X$, of the reduced wave equation are required. These solutions are then introduced as follows:

$$\varphi = \sum_{n=-\infty}^{\infty} e^{i\theta} \varphi_n(\tau, z)$$

$$X = \sum_{n=-\infty}^{\infty} e^{i\theta} X_n(\tau, z)$$

(44)

where

$$\varphi_n = \frac{k_o}{\sigma} \int_{-\infty}^{\infty} \left( \frac{r}{\rho} \right)^n h_n^{(1)}(k_o \rho) f_n(\xi) d\xi$$

$$n = 0, 1, 2, \ldots$$

(45a)

$$\varphi_{-n} = (-1)^n \frac{k_o}{\sigma} \int_{-\infty}^{\infty} \left( \frac{r}{\rho} \right)^n h_n^{(1)}(k_o \rho) f_{-n}(\xi) d\xi$$

$$n = 1, 2, 3, \ldots$$

(45b)

$$X_n = \frac{k_o}{\sigma} \int_{-\infty}^{\infty} \left( \frac{r}{\rho} \right)^n h_n^{(1)}(k_o \rho) \xi_n(\xi) d\xi$$

$$n = 0, 1, 2, 3, \ldots$$

(45c)

$$X_{-n} = (-1)^n \frac{k_o}{\sigma} \int_{-\infty}^{\infty} \left( \frac{r}{\rho} \right)^n h_n^{(1)}(k_o \rho) \xi_{-n}(\xi) d\xi$$

$$n = 1, 2, 3, \ldots$$

(45d)

or

$$\begin{cases} 
\varphi_n = \frac{k_o}{\sigma} \int_{-\infty}^{\infty} \left( \frac{r}{\rho} \right)^n h_n^{(1)}(k_o \rho) f_n(\xi) d\xi & \text{all } n \\
X_n = \frac{k_o}{\sigma} (-1)^n \frac{m+n}{2} \int_{-\infty}^{\infty} \left( \frac{r}{\rho} \right)^n h_n^{(1)}(k_o \rho) \xi_n(\xi) d\xi & \text{all } n 
\end{cases}$$

(45e)

Then the corresponding vacuum scattered fields are given by

$$\vec{E}_{scat.} = \sum_{n=-\infty}^{\infty} e^{i\theta} \varphi_n(\tau, z)$$

$$\begin{pmatrix} \frac{\partial \varphi_n}{\partial r} \\
\frac{\partial \varphi_n}{\partial z} \\
0 \\
\frac{\partial^2 \varphi_n}{\partial z^2} + k_o^2 \varphi_n 
\end{pmatrix}$$

(46)

$$\vec{H}_{scat.} = \sum_{n=-\infty}^{\infty} e^{i\theta} X_n(\tau, z)$$

$$\begin{pmatrix} \frac{\partial X_n}{\partial r} \\
\frac{\partial X_n}{\partial z} \\
0 \\
\frac{\partial^2 X_n}{\partial z^2} + k_o^2 X_n 
\end{pmatrix}$$

(47)
Now expressions (44) through (47) have the capability of providing the exact vacuum scattered field if the multipole densities \( f_n \), \( g_n \) could be computed exactly. Instead of attempting this exact computation of the densities, we resort, at this point, to an approximation based on a strip theory, the approximation becoming exact as the axial nonuniformity of the plasma reduces to zero. Inspection of (12), (13), and (28) or (30) shows that, in the limit of an axially uniform plasma, the \( E \)-field is purely axial, the \( H \)-field is transverse \( (H_y = 0) \) and therefore

\[
\begin{align*}
    f_n &= 0, \quad \text{all } n \\
    g_n &= \text{constant} = -\frac{c_n}{k_0}, \quad \text{all } n
\end{align*}
\]

(48)

where the \( c_n \) are computed in accordance with the discussion following eq. (14) and contain all the influences of the radial distribution of plasma properties. The type of strip theory proposed herein then goes as follows: for the axially nonuniform plasma, provided that the nonuniformity is gradual (all gradients of plasma properties are small) the functions \( f_n \) and \( g_n \) are to be given by

\[
\begin{align*}
    f_n &= 0, \quad \text{all } n \\
    g_n(\zeta) &= -\frac{c_n(\zeta)}{k_0}
\end{align*}
\]

(49)

where \( c_n(\zeta) \) is the value of the 2-dimensional \( c_n \) that corresponds to the plasma properties at \( \zeta \). The scattered field is then given by (46) and (47) with the \( c_n \) all vanishing, with the \( X_n \) given by the last two of expressions (45), and with the \( g_n \) given by (49). The resulting \( E \)-field will then have all three components, but the \( H \)-field will remain purely transverse, as in the axially uniform case. Moreover the resulting \( E \) and \( H \) scattered fields will be solutions of the vacuum field equations and will satisfy the radiation condition since they are synthesized from line distributions of multipoles. Having the scattered field, obtained by the preceding approximate process, and knowing, as we do, the incident field, the total vacuum field is described. Then the internal (plasma) field is completely determined by the tangential \( E_{\theta} \) and \( E_z \) components of vacuum total electric field and the requirement of continuity of these components across the plasma-vacuum interface. If we were to compute the internal field and check the continuity of the tangential components of the magnetic field across the interface we would find that, if the scattered field is exactly correct, then tangential magnetic field continuity will be realized; and if the scattered field is not exact, then tangential magnetic field continuity will be violated. It is in this violation that the error in our approximation appears, the error, of course, going to zero as axial plasma nonuniformity is reduced to zero.

C. Integral Equation Method

There is an alternative method of arriving at the approximation developed in the preceding section. This involves the formulation of the scattering problem as a vector integral equation. Since this alternative formulation sheds some further light on the implications of the approximation introduced in the multipole method, is is worthwhile to reproduce the integral equation formulation at this point.

Stratton and Cho have developed a vector analog of Green's theorem which expresses the electric and magnetic fields in terms of surface integrals involving surface values of the fields themselves and volume integrals involving spatial distributions of current and charge. For our purposes, we assume the plasma
electron density to be "smeared out" at the plasma-vacuum interface. That is,
we assume that the electron density changes continuously (together with its
gradient) over a thin layer from zero in the vacuum to its edge value in the
plasma. This eliminates the plasma-vacuum interface surface from the problem
and allows us to concern ourselves only with volume integrals plus the limiting
surface integrals on a surface which is extended to infinity. In this case,
Stratton and Chu derive the result:

\[ \mathbf{E}(p) = \frac{1}{4\pi} \left( \frac{\epsilon_0 k_0}{p} \nabla \times J \left( \frac{e^{ik_0 p}}{p} \right) + \frac{1}{\epsilon_0} \nabla \times \mathbf{g} \right) \times \mathbf{g} \cdot \nabla \mathbf{g} \right) \right] dV_g + \int_S \mathbf{g} \times \nabla \mathbf{g} \right) dS \]

(50)

\[ \mathbf{E}(p) = \frac{1}{4\pi} \left( \frac{\epsilon_0 k_0}{p} \nabla \times J \left( \frac{e^{ik_0 p}}{p} \right) + \frac{1}{\epsilon_0} \nabla \times \mathbf{g} \right) \times \mathbf{g} \cdot \nabla \mathbf{g} \right) \right] dV_g + \int_S \mathbf{g} \times \nabla \mathbf{g} \right) dS \]

(51)

(Where \( g \) is integration-variable point, \( p \) is field-variable point, \( p \) is distance
between \( p \) and \( g \), and \( S \) is a surface integral over bounding surface \( S \) of volume \( V_g \))
as a solution to the system:

\[ \nabla \times \mathbf{E} = \mu_0 \mathbf{J} \]

(52)

\[ \nabla \times \mathbf{H} = -\mu_0 \nabla \times \mathbf{E} \cdot \mathbf{J} \]

\[ \epsilon_0 \nabla \cdot \mathbf{E} = \mathcal{Q} = \text{charge density} \]

\[ \mathcal{J} = \nabla \times \mathbf{E} \]

Comparison of (52) with (1) and (4) of the present report shows that we may
interpret the plasma in terms of equivalent current and charge densities,

\[ \mathcal{J} = \mu_0 \frac{2}{\omega^2} \mathbf{J}_v \left( 1 - \frac{1}{\omega^2} \right) \]

(53)

\[ \mathcal{J} = \mu_0 \mathbf{P} \left( r, s \right) \mathbf{E} \]

with \( P = \frac{\omega^2}{\mu_0^2} \).

Thus, keeping in mind the smeared out plasma interface we may write for the
plasma problem, the analog of the Stratton and Chu relations:

\[ \mathbf{E}(p) = \frac{1}{4\pi} \left( \nabla \times \mathbf{J} \left( \frac{e^{ik_0 p}}{p} \right) + \frac{1}{\epsilon_0} \nabla \times \mathbf{g} \right) \times \mathbf{g} \cdot \nabla \mathbf{g} \right) \right] dV_g + \int_S \mathbf{g} \times \nabla \mathbf{g} \right) dS \]

(55)

\[ \mathbf{H}(p) = \frac{1}{4\pi} \left( \nabla \times \mathbf{E} \left( \frac{e^{ik_0 p}}{p} \right) \right) \times \mathbf{g} \cdot \nabla \mathbf{g} \right) \right] dV_g + \int_S \mathbf{g} \times \nabla \mathbf{g} \right) dS \]

(56)

Now if \( \mathbf{E} \) and \( \mathbf{H} \) represent the total field, then the surface integrals \( \int_S \)
should be such as to vanish as the surface \( S \) becomes indefinitely large except
for the contribution of the incident wave which violates the radiation condition.
This incident wave should contribute, through the surface integral, exactly
the quantity $\mathbf{E}_{\text{inc}}$ to (55) and $\mathbf{H}_{\text{inc}}$ to (56). Thus we have, as the surface $S$
recedes to infinity,

$$
\mathbf{E}_{\text{total}} = \mathbf{E}_{\text{inc}} + \frac{1}{4\pi} \int_{\text{plasma volume}} \left\{ \mathbf{E}(g) - \frac{\mathbf{E}_{\text{inc}}(p) \cdot \mathbf{E}(g)}{\mathbf{g}} \mathbf{g} \cdot \mathbf{E}(g) - \frac{\mathbf{E}_{\text{inc}}(p) \cdot \mathbf{E}(g)}{\mathbf{g}} \mathbf{g} \cdot \mathbf{E}(g) \right\} dV_g
$$

(57)

$$
\mathbf{H}_{\text{total}} = \mathbf{H}_{\text{inc}} + \frac{1}{4\pi} \int_{\text{plasma volume}} \left\{ i\omega \mathbf{E}(g) \mathbf{g} \times \mathbf{E}(g) + \mathbf{E}(g) \cdot \mathbf{g} \frac{\mathbf{E}_{\text{inc}}(p) \cdot \mathbf{E}(g)}{\mathbf{g}} \mathbf{g} \cdot \mathbf{E}(g) \right\} dV_g
$$

(58)

Actually (58) follows directly from (1) and (57).

Eq. (57) appears to be a vector integro-differential equation for the electric field but can be reduced to an ordinary vector integral equation by making use of (54). From Eq. (54) we see that

$$
\mathbf{E} \cdot \mathbf{E} = \mathbf{E} \cdot \mathbf{g} + \mathbf{g} \cdot \mathbf{E}
$$

(59)

so that

$$
\mathbf{E} \cdot \mathbf{E} = \frac{\mathbf{E} \cdot \mathbf{g} + \mathbf{g} \cdot \mathbf{E}}{\mathbf{g} \cdot \mathbf{g}}
$$

(60)

Therefore we obtain a vector integral equation for the electric field:

$$
\mathbf{E}(p) = \mathbf{E}_{\text{inc}}(p) + \frac{1}{4\pi} \int_{\text{plasma volume}} \left\{ \mathbf{E}(g) - \frac{\mathbf{E}_{\text{inc}}(p) \cdot \mathbf{E}(g)}{\mathbf{g}} \mathbf{g} \cdot \mathbf{E}(g) - \frac{\mathbf{E}_{\text{inc}}(p) \cdot \mathbf{E}(g)}{\mathbf{g}} \mathbf{g} \cdot \mathbf{E}(g) \right\} dV_g
$$

(61)

Now the field point $p$ in (61) may lie inside or outside the plasma. The incident field $\mathbf{E}_{\text{inc}}$ is given everywhere by

$$
\mathbf{E}_{\text{inc}} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} e^{-ik_0 r}
$$

(62)

It will be shown that (61) can be related, for the case where point $P$ lies outside the plasma, to the multipole distribution formulation of the preceding section; but first a few remarks are in order concerning the system (61) itself. If (61) is to be solved as an integral equation system then, of course, points $p$ inside the plasma must be considered. Furthermore, in the limit where the plasma is axially uniform, $P$ depends only upon radius $r$ and not upon $z$. Then it is evident from inspection of (61) and (62) that a purely axial electric field will satisfy (61) and that the second term disappears from the integrand. It is equally evident, however, that as small axial gradients in plasma properties occur ($\frac{\mathbf{E}}{\mathbf{g}}$ small, non-zero) then all three components must appear in $\mathbf{E}$, though the axial component $\mathbf{E}_z$ may be small. Correspondingly, all three components will appear in $\mathbf{H}$, though the axial component may be small.
The relation between eq. (61), for points \( p \) outside the plasma, and the line multipole distribution approach may be seen through the following development: the free-space Green's function or simple-pole solution for the reduced wave equation \( \frac{e^{ik_0 \rho}}{\rho} \) can be expanded in spherical wave functions.

\[
\frac{e^{ik_0 \rho}}{\rho} = e^{ik_0} \sum_{n=0}^{\infty} (2n+1) \sum_{m=0}^{n} \epsilon_m \frac{(n-m)!}{(n+m)!} \cos \left[ m \left( \frac{\rho}{R_G} - \theta_G \right) \right] \nu_n(\cos \psi_p) \nu_n(\cos \phi_g) \begin{cases} j_n(k_0 R_g) h_n^{(1)}(k_0 \rho) \vspace{10pt} \\
 h_n^{(1)}(k_0 R_g) j_n(k_0 \rho) \vspace{10pt} \\
 k_0 R_g < R_F \\
 k_0 R_g > R_F \end{cases} \tag{63}
\]

where \( \epsilon = 1, m = 0 \)

\( \epsilon = 2, m > 0 \)

\( j_n \) are the ordinary spherical Bessel functions

\( h_n^{(1)} \) are the spherical Hankel functions

\( \nu_n \) are the associated Legendre Polynomials

\( R \) is radius in spherical coordinates

\( (,) , (,),(,) \) represent field point and source point, respectively

\( \rho \) is distance between points \( p \) and \( g \)

\( \theta \) is longitude angle

\( \phi \) is colatitude angle

Now suppose we choose, as origin for the spherical coordinate system appropriate to expression (63), the point \( (x = 0, y = 0, z = z_g) \) i.e., the point on the plasma center line in the \( (x,y) \) plane of the source point \( g \). Then

\[
\begin{align*}
R_G &= r = \text{cylindrical radius} \\
\theta_G &= \psi / 2 \\
R_F &= \sqrt{(x_p - x_g)^2 + (y_p - y_g)^2} \\
\psi_p &= \arctan \frac{r_p}{\theta_p} = \arccos \frac{r_p}{R_F} \\
\theta_p &= \arcsin \frac{\theta_p}{R_F}
\end{align*}
\tag{64}
\]

and, for points \( g \) inside the plasma and \( p \) outside the plasma, \( R_p > R_F \). With this choice, the expression (63) reduces to

\[
\frac{e^{ik_0 \rho}}{\rho} = e^{ik_0} \sum_{n=0}^{\infty} (2n+1) \sum_{m=0}^{n} \epsilon_m \frac{(n-m)!}{(n+m)!} \cos \left[ m \left( \frac{\rho}{R_G} - \theta_G \right) \right] \nu_n(\cos \psi_p) \nu_n(\cos \phi_g) \begin{cases} j_n(k_0 R_g) h_n^{(1)}(k_0 \rho) \vspace{10pt} \\
 h_n^{(1)}(k_0 R_g) j_n(k_0 \rho) \vspace{10pt} \\
 k_0 R_g < R_F \\
 k_0 R_g > R_F \end{cases} 
\tag{65}
\]

Now eq. (61) in its present form must be interpreted in terms of Cartesian field components \( E_x, E_y, E_z \) rather than cylindrical components. In order to convert to a form suitable for cylindrical components we make linear combinations of the \( x \) and \( y \) components of (61) to form \( E_x \) and \( E_y \) at the field point \( p \) and, at the same time, express the internal \( E_x(g), E_y(g) \) as linear combinations of \( E_x(g) \). \( E_y(g) \). Also we note that the gradient \( \nabla_g \) in the second term of the integrand may be replaced by \( -\nabla_p \) (since it operates on \( \rho \)) and, if desired, taken outside the integral. The resulting cylindrical form is
\[
\begin{align*}
\begin{pmatrix}
E_r(p) \\
E_\theta(p) \\
E_z(p)
\end{pmatrix} &= \begin{pmatrix} 0 \\
0 \\
E_{zinc}(p)
\end{pmatrix} - \frac{1}{4\pi} \int \frac{k_0 P(g) E_r(g) \sin(\theta - \theta_\circ)}{g^2} dV_g + \frac{v}{g P(g) E_z(g)} \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} \frac{\sin(n \varphi)}{n} P_n(\cos \varphi) J_0(k_0 g) h_n(k_0 R_p) \\
&\quad \left\{ \begin{array}{l}
\frac{E_r^m(g) \cos \theta - E_\theta^m(g) \sin \theta_\circ}{1 - P(g)} \\
\frac{E_\theta^m(g) \cos(\theta - \theta_\circ) + E_r^m(g) \sin \theta_\circ}{1 - P(g)} \\
\frac{E_z^m(g)}{1 - P(g)}
\end{array} \right.
\end{align*}
\]

Expanding the electric field components in Fourier (with respect to \( \theta \)) components, noting that \( p \) depends on \( \theta_p \) only through the combination \( \theta_\circ - \theta_p \), defining

\[
\theta_0 = \theta - \theta_\circ
\]

and introducing expression (65) for \( e^{i k_0 g} \), there results, for each \((m)^{th}\)

Fourier component of the electric field vector:

\[
\begin{align*}
\begin{pmatrix}
E_r^m(p) \\
E_\theta^m(p) \\
E_z^m(p)
\end{pmatrix} &= \begin{pmatrix} 0 \\
0 \\
E_{zinc}(p)
\end{pmatrix} - \frac{1}{4\pi} \int_{\text{plasma boundary}} \int_{0}^{2\pi} \int_{0}^{\infty} \frac{e^{i k_0 g} e^{i \theta_0}}{g} dV_g d\varphi d\theta_0 \\
&\quad \left\{ \begin{array}{l}
\frac{E_r^m(g) \cos \theta - E_\theta^m(g) \sin \theta_\circ}{1 - P(g)} \\
\frac{E_\theta^m(g) \cos(\theta - \theta_\circ) + E_r^m(g) \sin \theta_\circ}{1 - P(g)} \\
\frac{E_z^m(g)}{1 - P(g)}
\end{array} \right.
\end{align*}
\]

It should be noted that an integration by parts, with respect to \( \theta_0 \), is equivalent to a replacement of the gradient \( \frac{1}{r_\rho} \hat{n}_\rho \) by \( + \frac{im}{r_\rho} \) in the second component of the second term of the integrand.

We will content ourselves with exhibiting the correspondence between (68) and the multiple formulation for the \( z \)-component \( E_z \) of the field. The argument for the \( E_r \) and \( E_\theta \) components is slightly more cumbersome, but proceeds similarly.

Performing the \( \theta_0 \) integration and interchanging the order of \( \nu \) and \( n \) summations, we find

\[
E_z^m(p) = E_{zinc}^m(p) - \frac{ik_0}{2} \int_{r_g}^{r_p} dV_g + \frac{v_\rho P(g) E_z(g)}{1 - P(g)} \sum_{n=0}^{\infty} \frac{(2n+1)(n-1)!}{n!} P_n(\cos \varphi) J_{n+1}(k_0 g) h_n(k_0 R_p)
\]

(continued)
Now, by using the recurrence relations satisfied by the spherical Bessel functions and by the associated Legendre polynomials, together with the geometric relations given in eq. (64), we find that:

\[
\frac{3}{2n}\Phi_n^m(\cos \varphi_p) b_n^{(1)}(k_0 R_p) = k_0 \left\{ \Phi_n^{m+1} h_n^{(1)}(n-m+1) - \Phi_n^{m-1} h_n^{(1)}(n+1) \right\}
\]

Moreover, the associated Legendre polynomials \(P_n^m\) cut off or vanish when \(m > n\). Thus, any expression of the form:

\[
P \left[ \frac{m}{n} \right] (\cos \varphi_p) b_n^{(1)}(k_0 R_p), n \neq m
\]

is obtainable with a combination of operations of the form

\[
\frac{3}{2n}\Phi_n^m(\cos \varphi_p) b_n^{(1)}(k_0 R_p)
\]

upon \(P \left[ \frac{m}{n} \right] (\cos \varphi_p) b_n^{(1)}(k_0 R_p)\). That is, the first partial leads to

\[
P \left[ \frac{m}{m+1} \right] h \left[ \frac{m+1}{m} \right]^{(1)}
\]

The second partial leads to

\[
P \left[ \frac{m}{m+2} \right] h \left[ \frac{m+2}{m} \right]^{(1)} \text{ and } P \left[ \frac{m}{m} \right] h \left[ \frac{m}{m} \right]^{(1)}
\]

The third partial leads to

\[
P \left[ \frac{m}{m+3} \right] h \left[ \frac{m+3}{m} \right]^{(1)} \text{ and } P \left[ \frac{m+1}{m+1} \right] h \left[ \frac{m+1}{m+1} \right]^{(1)}
\]

etc.

Thus all terms in (69) of the form

\[
P \left[ \frac{m}{m} \right] (\cos \varphi_p) b_n^{(1)}(k_0 R_p)
\]

are expressible as linear combinations of operations of the form

\[
\frac{3}{2n}\Phi_n^m(\cos \varphi_p) b_n^{(1)}(k_0 R_p)
\]

upon a single term of the form \(P \left[ \frac{m}{m} \right] h \left[ \frac{m}{m} \right]^{(1)}\).

Utilizing this result, and performing the integration with respect to \(r_g\), the expression (69) may be put in the symbolic form:

\[
E_n^m(p) = E_{z_\text{inc.}}^m(p) + \int s_g \sum_{m=0}^{\infty} A_m(z_g) \frac{3}{2n} P \left[ \frac{m}{m} \right] (\cos \varphi_p) b_n^{(1)}(k_0 R_p)
\]

(71)

Then a series of integrations by parts, assuming suitable behavior at \(a \rightarrow \infty\), permits this to be written in the still more elementary form

\[
E_n^m(p) = E_{z_\text{inc.}}^m(p) + \int s_g F(z_g) \frac{3}{2n} \left( \frac{2n!}{R_p^{(2n)}} \right) b_n^{(1)}(k_0 R_p)
\]

(72)
where \( F(z_g) = \sum_{n=0}^{\infty} \frac{(z_g)^n}{\gamma^n} A_n(z_g) \)  
(73)

and where use has been made of the fact that

\[
\rho = \frac{m}{p} (\cos \varphi p) = \frac{(2n)!}{2^n n!} (\sin \varphi p)^m
\]

and \( \sin \varphi p = \frac{r_p}{R_p} \)  
(74)

Now, of course, the function \( F(z_g) \) is not known and contains the influence of the field \( E \) itself inside the plasma, but nevertheless the expression (72) shows the connection between the integral equation and multipole methods. It is easily seen that (72) is strictly equivalent to the multipole form for \( E \) expressed through relations (45) and (46). (Note that in (45) and (46) the required operation \( \frac{\partial^2}{\partial \rho^2} \) acts only \( \rho \), which is equal to \( R_p \), and hence may be replaced by \( \frac{\partial^2}{\partial \rho^2} \) which, in turn, permits integration by parts and a re-interpretation of the density function \( g_0 \). Thus the integral equation and the multipole formulation have been shown (at least for the \( \rho \)-component of the electric field) to be equivalent for the exterior of the plasma, i.e., for the scattered field. The arguments for the \( r \)- and \( \theta \)-components are similar but more tedious.

Now the strip-theory method of choosing the functions \( f_0 \) and \( g_0 \) in the multipole formulation may be interpreted as the first step in an iterative solution of the integral equation formulation. It is to be recalled that \( f_0 \) and \( g_0 \) interpreted as line multipole distribution densities in the multipole formulation, were to be computed in accordance with a strip theory whereby they were to be related to the values of the 2-dimensional coefficients \( c_n \) that correspond to the local (same value of \( z \)) plasma properties. This has the following interpretation in the integral equation formulation of eq. (61). If we insert, as a first trial or approximation in the integrand of (61) the values of \( E \) which correspond to the two dimensional solution appropriate to the plasma properties at the local value of \( z \), then the next approximation, given by \( E(p) \) in (61) corresponds to our multipole-strip-theory approximation. In other words the multipole-strip-theory approximation is equivalent to the result of one iteration on the integral equation formulation, starting from a first trial based on the simplest strip theory (2-dimensional results based on the local-\( z \) plasma properties).

The question that arises immediately is: will an iterative process applied to the integral equation (61) be convergent? Evidently, for a dilute enough plasma (\( P \) small everywhere) the iteration will converge. In fact, the first iteration corresponding to an initial trial of \( E = E_{\text{incident}} \) is precisely the "electron-scattering" solution, in which each electron is presumed to act as though influenced only by the incident field and not by the fields scattered by the other electrons. When the plasma is dense, on the other hand, an iterative procedure will probably diverge. However it should be noted that (1) we are starting from an initial trial which is not the incident wave but, rather, is the simple strip-theory field; (2) we are restricting the plasma to have weak axial dependence; and (3) the first iteration (and the zero-th, as well) is exact as the axial dependence goes to zero. In effect, we are relying upon these considerations to lend justification to the strip-theory approximation proposed herein.
III. SCATTERED ENERGY AND BACK SCATTERING

The scattered energy flow is calculated by means of the Poynting vector $\vec{S}_{\text{scat}}$, which gives the intensity of energy flow of the scattered field. For harmonic time dependence, the mean intensity of energy flow in the scattered field is given by

$$ \langle S_{\text{scat}} \rangle = \frac{1}{2} R_s \left( E_{\text{scat}}^* \cdot H_{\text{scat}} \cdot \right) $$

(75)

where ($\ast$) denotes complex conjugate. Hence the radial component of scattered energy flow is given by

$$ S_{\text{scat}}^r = \frac{1}{2} \arg \left( E_{\text{scat}}^* \cdot H_{\text{scat}} \cdot \right) $$

(76)

In our strip theory approximation, since the $\varphi_n$ are all zero, $H_z$ is zero and

$$ S_{\text{scat}}^z = \frac{1}{2} \arg \left( E_{\text{scat}}^* \cdot H_{\text{scat}} \cdot \right) $$

(77)

However, since the $X_m$ are solutions of the reduced wave equation with $e^{im\varphi}$ type dependence, therefore the $X_m$ satisfy

$$ X_{mzz} + k_n^2 X_m + X_{mz} + \frac{1}{r} \frac{\partial}{\partial r} X_m = 0 $$

(78)

Thus, (79) may be replaced by

$$ S_{\text{scat}}^z = \frac{1}{2} \arg \left[ \sum_{n=-\infty}^{\infty} \frac{e^{-im\varphi}}{r} \left( \frac{d^2}{dr^2} X_m - \frac{2}{r} \frac{\partial}{\partial r} X_m \right) \right] $$

(79)

since $r$ is real, the last term in the first summation may be dropped, contributing as it does only a pure imaginary quantity to the product, and the final expression for $S_{\text{scat}}^z$ becomes

$$ S_{\text{scat}}^z = \frac{1}{2} \arg \left[ \sum_{n=-\infty}^{\infty} \frac{e^{-im\varphi}}{r} \left( \frac{d^2}{dr^2} X_m - \frac{2}{r} \frac{\partial}{\partial r} X_m \right) \right] $$

(80)

Now $X_n$ and $X_m$ are given by (45). However, we are actually interested only in the far field ($k_0 r \gg 1$) scattered energy so that only large values of $k_0 r$ are pertinent. Thus we replace the spherical Hankel functions in (45) by their asymptotic representations:

$$ h_n^{(1)} (k_0 r) \approx \frac{\pi}{2k_0 r} \frac{e^{-ik_0 r}}{r} e^{-\imath \pi/2} g_n (\vartheta) d\vartheta $$

(81)

Thus

$$ X_n \approx \frac{1}{r} \int_{\vartheta=0}^{\pi} \left( \frac{\pi}{2} \right)^n e^{\imath k_0 r \cdot e^{-\imath \pi/2}} g_n (\vartheta) d\vartheta $$

(82)

In the far field ($r \rightarrow \infty$)

$$ X_n \approx \frac{1}{r} \int_{\vartheta=0}^{\pi} \left( \frac{\pi}{2} \right)^n e^{\imath k_0 r \cdot e^{-\imath \pi/2}} g_n (\vartheta) d\vartheta $$

(83)
These can be put into a single form; valid for all $m$:

$$
\frac{s^2 \chi_n}{x_n} - \frac{s^2 \chi}{x_n} = \frac{k_n^2}{x_n} e^{-\text{inv}/2} \int_{-\infty}^{\infty} \frac{r}{|r| + 2} e^{i k_0 \rho} \delta_n (l) \, dl \tag{84}
$$

All $n$

In a similar manner, we find that

$$
\frac{\delta X_m}{\delta r} = \frac{k_0}{v} e^{\text{inv}/2} \int_{-\infty}^{\infty} \frac{r}{|r| + 2} e^{-i k_0 \rho} \delta_m (l) \, dl \tag{85}
$$

All $m$

Thus the radial component of scattered energy flow in the far field is given by:

$$
S_{\text{scat.}} = \frac{1}{2} \Re \left\{ \frac{1}{\lambda k_0} \sum_{m=\pm \infty} e^{i m \phi} e^{-i m / 2} \frac{k_n^2}{x_n} \int_{-\infty}^{\infty} \frac{r}{|r| + 2} e^{i k_0 \rho} \delta_n (l) \, dl \right\} \tag{86}
$$

Combining (87) and (88) we find

$$
\sigma_B [p, \, \rho_p] = \frac{2 \pi \sigma_{\text{scat.}} \rho_p}{\pi k_0} \Re \left\{ \sum_{m=\pm \infty} e^{-i m / 2} \int_{-\infty}^{\infty} \frac{r}{|r| + 2} e^{i k_0 \rho} \delta_m (l) \, dl \sum_{m=\pm \infty} e^{-i m / 2} \int_{-\infty}^{\infty} \frac{r}{|r| + 2} e^{i k_0 \rho} \delta_m (l) \, dl \right\} \tag{89}
$$
We now insert the result of the strip theory approximations

\[ s_n (\xi) = \frac{c_n (\xi)}{k_0} \]  

(49)

and, noting that the \( c_n \) will be proportional to the \( n \)th Fourier coefficient of the incident wave, we define coefficients \( \gamma_n \) by

\[ c_n (\xi) = \gamma_n (\xi) E_0 e^{-i\xi/2} \]  

(90)

In terms of the \( \gamma_n \), the back scattering cross section \( G_B \) becomes:

\[
G_B(x_1, x_2) = \frac{2\pi}{w} R^2 \sum_{n=\infty}^{\infty} \left( \frac{1}{2} \right)^n \left( -1 \right)^n \int_{-\infty}^{\infty} \left[ \frac{\xi}{p} \right] \frac{1}{m+1} e^{i\xi \rho} \gamma_n (\xi) d\xi
\]

(91)

IV. NUMERICAL COMPUTATION

We consider a family of plasma structures whose properties, in terms of \( P (r, z) \), are given by

\[ P (r, z) = \frac{P_0 (r)}{2} \left[ 1 + \tanh \left( \frac{a k_0 z}{2} \right) \right] \]  

(92)

These configurations have the property of vanishing strength as \( z \to -\infty \) and approaching an axially uniform but radially varying structure

\[ P \to P_0 (r) \]

as \( z \to +\infty \). The coefficient \( a \) is a parameter which governs the axial gradients or the rate of approach of the plasma properties to their asymptotic values. It is assumed that for an axially uniform plasma with radial variation of \( P \) given by \( CP_0 (r) \) (where \( C \) is a constant) the scattered field (the \( c_m \) coefficients) is calculable. Then the \( \gamma_m \) which enter into eq. (91) for the back-scattered energy correspond to this set of \( c_m \). The "constant" \( C \) is now, however, related to axial location by the factor \( \frac{1 + \tanh a k_0 z}{2} \). This gives the strip-theory \( z \)-dependence of the \( \gamma_m \).

Now, as \( z \to -\infty \) the expression (92) for \( P \) approaches zero exponentially

\[ P (r, z) \to P_0 (r) e^{2a k_0 z} \to 0 \]  

(93)

as \( z \to -\infty \). This makes the infinite range integrals in the expression (91) for \( G_B \) rapidly convergent in the negative-\( \xi \) direction. In the positive-\( \xi \) direction, however, the \( \gamma_m \) do not vanish but, rather, settle down to a constant...
The value of \( \gamma_n(\infty) \) is known from the \( c_n \) corresponding to an axially uniform plasma with \( P=P_0(r) \). The integral in (98) can be evaluated easily for large radius \( r \). Letting

\[
\tau = r \sinh \theta
\]

and

\[
\int_{\tau=\infty}^{\infty} \int_{T=0}^{\infty} e^{+\pm ik_0 r \cosh \theta} \frac{\tau^\nu+1}{\sqrt{\tau^2+\lambda^2}} \gamma_n(T+\xi_p) d\tau d\theta
\]

For large \( \tau \), the method of stationary phase gives

\[
\int_{\tau=\infty}^{\infty} \int_{T=0}^{\infty} e^{+\pm ik_0 r \cosh \theta} \gamma_n(T+\xi_p) d\tau d\theta
\]

Thus the required integrals reduce to two exponentially decaying sub-integrals plus expressions of the form \( I_\pm \) in (101).

As a matter of interest, the availability of the \( I_\pm \) terms permits computation of the asymptotic values (for large \( s \)) of the scattering cross section. The asymptotic value of the integral \( \int_{\tau=\infty}^{\infty} e^{+\pm ik_0 \sqrt{\tau^2+\lambda^2}} d\tau \) is simply \( 2 \pi \) for large positive \( s \) and zero \( (0) \) for large negative \( s \). Moreover, the form of \( I_\pm \), as shown in Eq. (101), indicates that the positive-\( s \) asymptote will be independent of the radius \( r \) of the receiver.

A digital computer program has been developed to perform the various phases of the foregoing computation, leading ultimately to \( \sigma_B \), on the IBM 7090 computer. This program utilizes one subroutine to generate the \( \gamma_n \) functions, which occur in
the integrands of the infinite-range integrals. This subroutine depends upon the type of radial distribution $P_0(r)$ assumed for the plasma electron density. Two options have been programmed for this subroutine, one of which treats a radially uniform plasma, and the other a central uniform core surrounded by a radially nonuniform annulus with electron density varying as $1/r^2$ and matching up with the core value at the core radius (see Fig. 1c). This corresponds to the configuration (with respect to radius) treated in Ref. (9) except that collisions were not included in the present program. As noted earlier, the $\gamma_n$ functions are derived from the radial variation but depend on axial location $\zeta$ insofar as $\zeta$ influences the level or strength of the distribution. The form of the $\gamma_n$, as derived in Ref. (9), involves combinations of Bessel functions $J_\nu$, $Y_\nu$, $I_\nu$ of both integer and non-integer orders. The necessary Bessel function generation was accomplished by utilizing an existing IBM subroutine. It should be noted that the present program must permit the core plasma to be underdense or overdense, depending upon $z$, whereas Ref. (9) treated only overdense cores.

Integrations were performed using numerical quadratures with a continuous test for mesh size. It was found that the integrands exhibited sharp oscillations for small $\zeta$ values and then smoothed out as $\zeta$ increased in magnitude. In order to avoid unnecessarily lengthy machine runs, a test was introduced into the integration procedure to allow the step size to increase when the integrand smooths out and to cut the step size wherever necessary for accuracy in the region where the integrand is sharply oscillatory. The two series were summed, continuing to increasing values of the summation index, until a stopping criterion was satisfied. This stopping criterion required three successive partial sums to agree within a preset tolerance. A similar stopping criterion was used to determine the cutoff point for the integrations.

The complete calculations, proceeding from a prescribed configuration and a specified receiver location to a computed value of $\sigma_R$, is quite lengthy. This is due to the fact that the integration steps must be quite small in the $\zeta$-region where the integrand oscillation is violent (steps of the order of $\Delta(k_0\zeta)$ = .005) whereas the integrations must extend to values of $k_0\zeta$ of the order of 1000. In the regions where the integrand is smooth, step sizes $\Delta(k_0\zeta)$ of the order of 1.5 were permitted. Moreover, for the configuration treated, approximately thirty terms were required in each of the two summations indicated in Eq. (91). As a consequence of this lengthy running time, only a limited number of calculations could be performed.

One configuration was chosen for the sample runs, as described by the following parameter values (see Eq. (92)):

\[ P_0(r) = \begin{cases} 2 & 0 < k_0r < k_0r_c \\ \frac{r}{r_c}^2 & k_0r_c < k_0r < k_0r_p \\ 0 & k_0r > k_0r_p \end{cases} \]

\[ \sigma_a = 0.015 \]

\[ k_0r_c = 10 \]

\[ k_0r_p = 20 \]
The $P_0$ value of 2 inside the plasma core, in view of eq. (92) implies that the core changes from underdense to overdense at $z = 0$. The $c^2$-value of 0.0115 was chosen so that the plasma reaches 99% of its asymptotic value at a $z$-value corresponding to $k_0 z = 200$ (and is reduced to 1% at a $z$-value corresponding to $k_0 z = -200$). The core radius and outer plasma radius values are given by $k_0 r_c = 10$ and $k_0 r_p = 20$ respectively. It should be noted that all lengths are dimensionless and are actually scaled with respect to signal wavelength, e.g.,

$$k_0 r_p = \frac{2\pi}{\lambda} r_p$$

For this plasma configuration a series of calculations of $\Omega_B$ (again computed in dimensionless form, $\Omega_B$) was performed. The results are illustrated by Figs. 3a and 3b and by Table I. At receiver radial distances (from the plasma configuration centerline) corresponding to $k_0 r = 1500$ and $k_0 r = 1000$, a survey was made of the back-scattering cross section $\Omega_B$ as the receiver's axial position $z$ is varied.

As expected, the values of $\Omega_B$ approach their asymptotes for magnitudes of $z$, both upstream and downstream, which are comparable to the value, $k_0 z = \pm 200$, at which the plasma itself has effectively attained its asymptotic form. For the larger receiver distance ($k_0 r = 1500$) the scattering has practically achieved its respective asymptotic values for $k_0 z = +400$ and for $k_0 z = -300$. For the smaller receiver distance ($kr=1000$) the asymptotes have effectively been attained at $k_0 z = +300$ and $k_0 z = -200$. The positive $z$ asymptote (computed from the $1^\pm$ as noted earlier) has a magnitude $k_0^\sigma = 29.47$ in both cases, implying that this is the back-scattering cross section for the asymptotic configuration given by

$$P(r) = \begin{cases} \frac{2}{r} & 0 < k_0 r < 10 \\ \frac{1}{r^2} & 10 < k_0 r < 20 \\ 0 & k_0 r > 20 \end{cases}$$

The wavelike irregularity with respect to axial location, in the back scattering at the closer receiver is quite noticeable, whereas the more distant receiver exhibits a smoother behavior as it traverses an axial path relative to the plasma configuration. This smoothing trend should continue in the direction of increasing receiver radius and, while a more severe wave may build up as receiver distance is decreased, the asymptotic approximation used in the calculation of the spherical Hankel functions does not permit a closer approach without an attendant loss of accuracy. The waviness in the axial dependence of the return is undoubtedly caused by a spatially oscillatory diffraction pattern superimposed on the monotonic behavior $\frac{1}{r} (1 + \tan h k_0 z)$ characterising the plasma itself. This diffraction pattern could be likened to a lobe pattern for the scattered field, the more distant receiver sensing this pattern in a more diffuse form with respect to linear distance, and the closer receiver sensing it in a more concentrated form spatially.

Figs. 2a and 2b illustrate (by means of plots of successive partial sums) the mode of convergence of the series for the axial electric field $E_z$ (for both real and imaginary components) for both receiver distances, $k_0 r = 1000$ (Fig. 2a) and $k_0 r = 1500$ (Fig. 2b). The mode of convergence is seen to be quite erratic.
in both cases until the summation index exceeds a value of about 20, following
which, the partial sums settle down rapidly. In this connection one remark is
of interest: some attempts have been made by other organizations recently to
calculate approximately the scattering from overdense turbulent plasmas using
a sort of Born scattering but assuming the electrons excited in accordance with
the deterministic field corresponding to the mean plasma configuration. In
these calculations only the zero-order (\( m = 0 \) in the summation) term was used.

Fig. 2 illustrates the magnitude of the error that can be introduced by stopping
at any index value short of the converged values.

The results presented apply for the specific sample configuration treated.
The computer program which has been developed for this calculation, however,
has the capability to handle other configurations and, with appropriate sub-
stitution of the \( \gamma_m \) subroutine, other radial distributions of electron density. As
noted earlier, a \( \gamma_m \) subroutine is available for treating radially uniform distribu-
tions. Other, more general radial distributions might require numerical
solution of the radial differential equation for each (\( \theta \)) Fourier component, but
this involves no conceptual difficulties. The radial solution would be performed
by finite-difference technique and would replace the Bessel function generation
appropriate to the \( \gamma_m \)-calculation for the present electron density radial
distribution.

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**LIST OF SYMBOLS**

- $c$  
  light speed $= 1/\sqrt{\varepsilon_0 \mu_0}$

- $e$  
  electron charge

- $E$  
  electric field

- $E_0$  
  incident electric field intensity

- $H$  
  magnetic field

- $k_0$  
  $= \omega/c$

- $m$  
  electron mass

- $n$  
  electron number density

- $P(r, z) = \frac{\omega_p^2}{\omega^2 + \omega_p^2} \left(1 - \frac{\omega_p^2}{\omega^2}\right)$

- $r, \theta, z$  
  cylindrical coordinates

- $R, \theta, \phi$  
  spherical polar coordinates

- $x, y, z$  
  cartesian coordinates

- $r_c$  
  plasma core radius

- $r_p$  
  external plasma radius

- $S$  
  Poynting vector

- $\varepsilon_0$  
  dielectric permeability of vacuum

- $\mu_0$  
  magnetic permeability of vacuum

- $\omega$  
  signal frequency

- $\omega_p$  
  plasma frequency $= \sqrt{\frac{e^2 n}{m \varepsilon_0 \mu_0}}$

- $\nu$  
  collision frequency (electrons with neutral particles)

- $\sigma_B$  
  back scattering cross section

- $\lambda$  
  signal wave length $= 2\pi/k_0$

- $\varphi, \chi$  
  defined by Eqs. (16), (17), (18)

- $\rho$  
  distance between field and source points

- $\{ \}$  
  cartesian components of a vector

- $\{ \}$  
  cylindrical components of a vector
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**Table I**

Numerical Values of Back-Scattering Cross Sections

**Fig. 1** Axial Variation of Centerline Value of Electron Density
FIG. 3b BACK SCATTERING CROSS SECTION VERSUS AXIAL POSITION OF RECEIVER FOR DIMENSIONLESS RADIAL DISTANCE $k_{0r} = 1500$

END