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MEMORANDUM REPORT NO. 1466
APRIL 1963

ON THE QUASI-LINEAR SUBSTITUTION METHOD FOR MISSILE MOTION CAUSED BY STRONGLY NONLINEAR STATIC MOMENT

Charles H. Murphy

RDT & E Project No. 1M010501A005

BALLISTIC RESEARCH LABORATORIES

ABERDEEN PROVING GROUND, MARYLAND

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A B E R D E E N P R O V I N G G R O U N D, M A R Y L A N D

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MEMORANDUM REPORT NO. 1466

CHMurphy/jdk
Aberdeen Proving Ground, Md.
April 1963

ON THE QUASI-LINEAR SUBSTITUTION METHOD FOR MISSILE MOTION CAUSED BY STRONGLY NONLINEAR STATIC MOMENT

ABSTRACT

An improved quasi-linear substitution method is developed to treat properly the influence of a cubic static moment on the modal damping of a missile acted on by quite general nonlinear damping and Magnus moments. The predictions of this method are compared for various special cases with those of the more accurate but much more complicated perturbation method. The new quasi-linear theory predicts boundary curves for planar motion, almost circular motion and almost planar motion which are quite close to those of the perturbation theory. An original result of the theory is that all planar singular points for a non-spinning missile whose moment coefficients are only functions of the total angle of attack are nodes. That is, almost planar motion with amplitude close to that of a stable planar limit motion will tend to that motion.

TABLE OF CONTENTS

	Page
ABSTRACT	3
TABLE OF SYMBOLS	7
1. INTRODUCTION	11
2. THE IMPROVED SUBSTITUTION QUASI-LINEAR SOLUTION	11
3. PLANAR MOTION	19
4. ALMOST CIRCULAR MOTION	21
4.1 CUBIC MAGNUS MOMENT	24
4.2 CUBIC DAMPING MOMENTS AND ZERO SPIN	26
5. ALMOST PLANAR MOTION	29
6. SUMMARY	35
REFERENCES	36
DISTRIBUTION	45

TABLE OF SYMBOLS

a_{jk}	coefficients defined by Equation (17)
C_D	drag coefficient
C_{L_α}	lift coefficient
C_{M_α}	static moment coefficient
$C_{M_{\dot\alpha}}$, C_{M_q}	damping moment coefficients
$C_{M_{p\alpha}}$	Magnus moment coefficient
$E(k)$	complete elliptic integral of the second kind
H	$\frac{\rho S \ell}{2m} \left[\gamma C_{L_\alpha} - C_D - k_t^{-2} \left(C_{M_q} + \gamma C_{M_{\dot\alpha}} \right) \right]$
H_o	$\left[H \right]_{\xi = \xi' = 0}$
H_2	cubic damping moment coefficient
I_x	axial moment of inertia
$I_y = I_z$	transverse moments of inertia
$K(k)$	complete elliptic integral of the first kind
K_j	amplitude of the jth mode
k	modulus of the elliptic integrals
k_a	axial radius of gyration, $\sqrt{I_x/m\ell^2}$
k_t	transverse radius of gyration, $\sqrt{I_y/m\ell^2} = \sqrt{I_z/m\ell^2}$
ℓ	reference length
M	$\frac{\gamma \rho S \ell}{2m} \left[k_t^{-2} C_{M_\alpha} - C'_{L_\alpha} \right]$
$\tilde{M}$	part of M which is function of δ^2
M^*	part of M which is function of $(\delta^2)'$

M_o	$\left[M\right]_{\xi = \xi' = 0}$
$\hat{M}_o$	$M_o - P^2/4$
M_2	cubic static moment coefficient
M_{11}	non-linear moment coefficient
m	mass
m	$\dfrac{M_2 \delta_{max}^3}{\hat{M}_o \delta_{max}}$
m_2	$\dfrac{M_2}{\hat{M}_o}$
P	gyroscopic spin, $P = \dfrac{I_x}{I_y} \dfrac{p\ell}{V}$
$\hat{P}$	$P/\left\|\hat{M}_o\right\|^{1/2}$
p	axial component of angular velocity
r_j	defined by Equation (87)
S	reference area
s	dimensionless distance along flight path
T	$\dfrac{\rho S \ell}{2m}\left[C_{L_\alpha} + k_a^{-2} \; C_{M_{p\alpha}}\right]$
T_o	$\left[T\right]_{\xi = \xi' = 0}$
T_2	cubic Magnus moment coefficient
u, v, w	components of velocity
V	magnitude of velocity, $V = \sqrt{u^2 + v^2 + w^2}$
γ	cosine of total angle of attack
δ	$\left\|\xi\right\|$

θ	angle the flight path makes with respect to the vertical
λ_j	damping coefficient of the jth mode
λ_j^*	defined by Equation (12)
ξ	$\frac{v + iw}{V}$
ρ	air density
σ	coefficient of exponential density function
$\tilde{\sigma}$	$\sigma \ell \cos\theta$
ϕ_j	phase angle of the jth mode
$\hat{\phi}$	$\phi_1 - \phi_2$

Superscript

'	derivative with respect to arclength, s
-	complex conjugate
~	quantity related to non-rotating coordinate system

Subscripts

c	quantities evaluated for circular motion
p	quantities evaluated for planar motion

1. INTRODUCTION

In Reference 1, three different quasi-linear methods were described and their different predictions of the nutational frequency for a missile with a cubic static moment were compared with the exact result obtained by the use of elliptic integrals. The best of the three was called the substitution method and was employed to obtain the combined effect of a cubic static moment, varying air density and both linear and nonlinear damping moments. Although this approach was not as accurate as the perturbation method[2,3] which uses the exact elliptic function solution for no damping as the initial approximation, it did give trends with a significant reduction in the necessary algebraic work.

One difficulty with the substitution method was use of a rather strange condition on the damping of the modal amplitudes:

$$K_1' e^{i\phi_1} + K_2' e^{i\phi_2} = 0 \tag{1}$$

A re-examination of this question has revealed a different substitution method which yields the same expression for the frequencies but does not make use of Equation (1). This improved substitution method does lead to different expressions for modal damping which are the same as those for the earlier substitution method for planar motion but are closer to those obtained from the perturbation method for almost circular motion. It is the purpose of this report to describe this new quasi-linear method.

2. THE IMPROVED SUBSTITUTION QUASI-LINEAR SOLUTION

The equation for the pitching and yawing motion of symmetric missiles can be written in the form[2]

$$\tilde{\xi}'' + (H - \frac{\gamma'}{\gamma} - iP)\tilde{\xi}' - (M + iPT)\tilde{\xi} = 0 \tag{2}$$

where the various symbols are defined in the Table of Symbols.

$$\therefore \quad \tilde{\xi}'' + (H_o - iP)\tilde{\xi}' - (M_o + iPT_o)\tilde{\xi}$$

$$= - \left[H - H_o - \frac{\gamma'}{\gamma}\right] \tilde{\xi}' + \left[(M - M_o) + iP(T - T_o)\right] \tilde{\xi} \tag{3}$$

where the zero subscript denotes the value of the aerodynamic quantities for zero amplitude motion.

The left side of Equation (3) is the linearized version of Equation (2) and has a solution in the form

$$\tilde{\xi} = K_1 e^{i\phi_1} + K_2 e^{i\phi_2} \tag{4}$$

The primary interest of a missile designer is usually the behavior of the modal amplitudes K_j. This can conveniently be described by their logarithmic derivatives

$$\frac{K_j'}{K_j} = \lambda_j \tag{5}$$

For linear moments, the λ_j's are constants but when the moments are nonlinear the various quasi-linear analyses obtain λ_j's which are functions of K_1 and K_2.

We now differentiate Equations (4) twice, substitute in Equation (3) and solve for the frequency and damping of the first mode.

$$(\phi_1')^2 - P\phi_1' + M_o - \lambda_1(\lambda_1 + H_o) - \lambda_1'$$

$$- i \left[(2\phi_1' - P)\lambda_1 + H_o\phi_1' - PT_o + \phi_1''\right]$$

$$= \left[H - H_o - \frac{\gamma'}{\gamma}\right] \left[(\lambda_1 + i\phi_1') + (\lambda_2 + i\phi_2') \frac{K_2}{K_1} e^{-i\hat{\phi}}\right]$$

$$- \left[(M - M_o) + iP(T - T_o)\right] \left[1 + \frac{K_2}{K_1} e^{-i\hat{\phi}}\right] \tag{6}$$

$$- \left\{ \left[(\phi_2')^2 - P\phi_2' + M_o - \lambda_2(\lambda_2 + H_o) - \lambda_2' \right] \right.$$

$$\left. - i \left[(2\phi_2' - P)\lambda_2 + H_o\phi_2' - PT_o + \phi_2'' \right] \right\} \left(K_2/K_1 \right) e^{-i\hat{\phi}} \tag{6}$$

where $\hat{\phi} = \phi_1 - \phi_2$

For linearized motion, all the terms on the right of Equation (6) vanish except for the term in braces. Since the left side of the linearized Eq. (6) is constant or slowly varying due to density or Mach number induced variations of M_o, H_o, and T_o and spin or velocity variations in P, it can only equal the periodic term in $e^{-i\hat{\phi}}$ on the right side if both sides are zero. This condition yields the usual linear relations. The nonlinear terms on the right, however, will in general be periodic but their average or d. c. components will not necessarily be zero. This average will directly affect the terms on the left side and it is the assumption of the quasi-linear method that the average is the <u>only</u> influence of the nonlinearities on the frequencies and damping exponents. We, therefore, average Equation (6) over a period of nutation* and neglect the small damping term in comparison with M_o in the real part of this equation.

$$(\phi_1')^2 - P\phi_1' + M_o - i \left[(2\phi_1' - P)\lambda_1 + H_o\phi_1' - PT_o + \phi_1'' \right]$$

$$= \frac{1}{2\pi} \int_o^{2\pi} \left(H - H_o - \frac{\gamma'}{\gamma} \right) \left[\phi_1' + \phi_2' \left(\frac{K_2}{K_1} \right) e^{-i\hat{\phi}} \right] d\hat{\phi}$$

$$- \frac{1}{2\pi} \int_o^{2\pi} \left[(M - M_o) + iP(T - T_o) \right] \left[1 + \frac{K_2}{K_1} e^{-i\hat{\phi}} \right] d\hat{\phi} \tag{7}$$

γ which is the cosine of the total angle of attack can be related to $|\xi|$, the magnitude of the sine of the total angle of attack by the relation

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* Ballisticians frequently use the terms nutation and precession to distinguish two modal oscillations. Nutation in this report has the classical meaning assigned by top theory, i.e., the variation of the amplitude of the total angle, $|\xi|$.

$$\gamma^2 = 1 - \delta^2 \tag{8}$$

where $\delta^2 = |\xi|^2 = \xi \bar{\xi}$

δ^2 can, then, be computed from Equation (4)

$$\delta^2 = K_1^2 + K_2^2 + K_1 K_2 (e^{i\hat{\phi}} + e^{-i\hat{\phi}})$$

$$= K_1^2 + K_2^2 + 2K_1 K_2 \cos \hat{\phi} \tag{9}$$

Since

$$\frac{\gamma'}{\gamma} = \frac{(\gamma^2)'}{2\gamma^2} = \frac{-(\delta^2)'}{2(1 - \delta^2)} \quad , \tag{10}$$

we see that γ'/γ is an odd function of $\hat{\phi}$ and, therefore, only affects the real part of Equation (7). If we make the assumption that H and T are functions of δ^2 and, thereby, are even functions of $\hat{\phi}$, the following equations may be obtained from Equation (7).

$$(\phi_1')^2 - P\phi_1' + \frac{1}{2\pi} \int_0^{2\pi} \left\{ \left(\frac{\gamma'}{\gamma}\right) \phi_2' \left(\frac{K_2}{K_1}\right) \sin \hat{\phi} + M \left[1 + \frac{K_2}{K_1} \cos \hat{\phi}\right] \right\} d\hat{\phi} = 0 \tag{11}$$

$$\lambda_1 = \lambda_1^* - \frac{\phi_1''}{2\phi_1' - P} \tag{12}$$

where $\lambda_1^* = \dfrac{-1}{2\pi(2\phi_1' - P)} \displaystyle\int_0^{2\pi} \left\{ H \left[\phi_1' + \phi_2' \frac{K_2}{K_1} \cos \hat{\phi}\right] \right.$

$$\left. - PT \left[1 + \frac{K_2}{K_1} \cos \hat{\phi}\right] + M \frac{K_2}{K_1} \sin \hat{\phi} \right\} d\hat{\phi}$$

Similar expressions apply for the other mode.

This result differs from that of the substitution method of Reference 1 in the presence of $2\phi_1' - P$ in the denominators of λ_1 and λ_1^*. In that report, $\phi_1' - \phi_2'$ appeared. This is the same as $2\phi_1' - P$ for planar motion but differs for other motions.

For small geometrical angles ($\gamma' \doteq 0$) and a cubic static moment ($M = M_o + M_2\delta^2$), Equation (11) and the corresponding equation for the other mode yield frequency relations which are identical with those of Reference 1.

$$\phi_1' = \frac{P}{2} + \sqrt{-\hat{M}_o \left[1 + m_2(K_1^2 + 2K_2^2)\right]} \tag{13}$$

$$\phi_2' = \frac{P}{2} - \sqrt{-\hat{M}_o \left[1 + m_2(2K_1^2 + K_2^2)\right]} \tag{14}$$

where $\hat{M}_o = M_o - \frac{P^2}{4}$

$$m_2 = \frac{M_2}{\hat{M}_o}$$

Equation (13) can be differentiated and substituted in Equation (12) with the result:

$$\lambda_1 \left[1 + \frac{m_2K_1^2}{2\left[1 + m_2(K_1^2 + 2K_2^2)\right]}\right] + \lambda_2 \left[\frac{2m_2K_2^2}{2\left[1 + m_2(K_1^2 + 2K_2^2)\right]}\right]$$

$$= \lambda_1^* - \left[\frac{M_o' - P'\phi_1'}{4\hat{M}_o} + \frac{M_2'm_2(K_1^2 + 2K_2^2)}{4M_2}\right]\left[1 + m_2(K_1^2 + 2K_2^2)\right]^{-1}$$

$$= \lambda_1^* - \left(\frac{M_o' - P'\phi_1'}{4\hat{M}_o}\right) + \left[\frac{M_o' - P'\phi_1'}{4\hat{M}_o} - \frac{M_2'}{4M_2}\right]\frac{m_2(K_1^2 + 2K_2^2)}{1 + m_2(K_1^2 + 2K_2^2)} \tag{15}$$

A similar equation can be derived for the other mode.

$$\lambda_1 \left[\frac{2m_2K_1^2}{2\left[1 + m_2(2K_1^2 + K_2^2)\right]} \right] + \lambda_2 \left[1 + \frac{m_2K_2^2}{2\left[1 + m_2(2K_1^2 + K_2^2)\right]} \right]$$

$$= \lambda_2^* - \left(\frac{M_o' - P'\phi_2'}{4\hat{M}_o} \right) + \left[\frac{M_o' - P'\phi_2'}{4\hat{M}_o} - \frac{M_2'}{4M_2} \right] \frac{m_2(2K_1^2 + K_2^2)}{1 + m_2(2K_1^2 + K_2^2)} \tag{16}$$

Equations (15-16) may now be solved simultaneously for λ_1 and λ_2.

$$\lambda_j = a_{j1} \left[\lambda_1^* - \frac{M_o'}{4\hat{M}_o} \right] + a_{j2} \left[\lambda_2^* - \frac{M_o'}{4\hat{M}_o} \right] + a_{j3} \left[\frac{M_o'}{\hat{M}_o} - \frac{M_2'}{M_2} \right] + a_{j4} \left(\frac{P'\phi_j'}{\hat{M}_o} \right) \tag{17}$$

where the a_{jk}'s are defined in the Table.

TABLE

General	Quasi-Planar Motion $K_1 = K_2 = K;\ m = 4m_2K^2$	Almost Circular Motion $K_2 \ll K_1;\quad m = m_2K_1^2$
$a_{11} = \frac{2}{d}\left[1 + m_2(K_1^2 + 2K_2^2)\right]\left[2 + m_2(4K_1^2 + 3K_2^2)\right]$	$\frac{2(4 + 3m)(8 + 7m)}{(8 + 9m)(8 + 5m)}$	$\frac{2(1 + m)}{2 + 3m}$
$a_{12} = -\left(\frac{4}{d}\right)(m_2K_2^2)\left[1 + m_2(2K_1^2 + K_2^2)\right]$	$\frac{-4m(4 + 3m)}{(8 + 9m)(8 + 5m)}$	0
$a_{13} = \left(\frac{1}{2d}\right)\left[2m_2(K_1^2 + 2K_2^2) + m_2^2(4K_1^4 + 7K_1^2K_2^2 + 4K_2^4)\right]$	$\frac{3m}{2(8 + 9m)}$	$\frac{m}{2(2 + 3m)}$
$a_{14} = \left(\frac{1}{2d}\right)\left[2 + m_2(4K_1^2 + 3K_2^2) - 2\left(\frac{\phi_2^{\bullet}}{\phi_1^{\bullet}}\right)(m_2K_2^2)\right]$	$\frac{2\left[8 + 7m - 2m\left(\frac{\phi_2^{\bullet}}{\phi_1^{\bullet}}\right)\right]}{(8 + 9m)(8 + 5m)}$	$\frac{1}{2(2 + 3m)}$
$a_{21} = -\left(\frac{4}{d}\right)(m_2K_1^2)\left[1 + m_2(K_1^2 + 2K_2^2)\right]$	$\frac{-4m(4 + 3m)}{(8 + 9m)(8 + 5m)}$	$\frac{-2m(1 + m)}{(2 + 3m)(1 + 2m)}$

TABLE (Cont.)

General	Quasi-Planar Motion $K_1 = K_2 = K;\quad m = 4m_2K^2$	Almost Circular Motion $K_2 \ll K_1;\quad m = m_2K_1^2$
$a_{22} = \left(\frac{2}{d}\right)\left[1 + m_2(2K_1^2 + K_2^2)\right]\left[2 + m_2(3K_1^2 + 4K_2^2)\right]$	$\frac{2(4 + 3m)(8 + 7m)}{(8 + 9m)(8 + 5m)}$	1
$a_{23} = \left(\frac{1}{2d}\right)\left[2m_2(2K_1^2 + K_2^2) + m_2^2(4K_1^4 + 7K_1^2K_2^2 + 4K_2^4)\right]$	$\frac{3m}{2(8 + 9m)}$	$\frac{m(1 + m)}{(2 + 3m)(1 + 2m)}$
$a_{24} = \left(\frac{1}{2d}\right)\left[2 + m_2(3K_1^2 + 4K_2^2) - 2\left(\frac{\phi_1'}{\phi_2'}\right) m_2K_1^2\right]$	$\frac{2\left[8 + 7m - 2m\left(\frac{\phi_1'}{\phi_2'}\right)\right]}{(8 + 9m)(8 + 5m)}$	$\frac{2 + 3m - 2m\left(\frac{\phi_1'}{\phi_2'}\right)}{4(2 + 3m)(1 + 2m)}$

$$d = 4 + 14m_2(K_1^2 + K_2^2) + m_2^2(12K_1^4 + 21K_1^2K_2^2 + 12K_2^4)$$

3. PLANAR MOTION

In the Table, values of the a_{jk}'s are computed for quasi-planar motion, i.e., nutation between zero and δ_{max}. For this motion, $K_1 = K_2 = K$, $\delta_{max} = K_1 + K_2 = 2K$ and the ratio of the maximum value of the cubic term to the linear term is

$$m = \frac{M_2 \delta^3_{max}}{\hat{M}_o \delta_{max}} = 4m_2 K^2 \tag{18}$$

There are three types of cubic static moments which can cause a periodic motion. These are shown in Figure 1 and may be described as follows[2]:

(a) Stable at small angles; more stable at larger angles ($\hat{M}_o < 0$, $M_2 < 0$); quasi-planar motion of all amplitudes possible ($m > 0$).

(b) Stable at small angles; less stable at large angles ($\hat{M}_o < 0$, $M_2 > 0$); amplitude of quasi-planar motion must be small enough to assure positive M ($-1 < m < 0$).

(c) Unstable at small angles; stable at large angles ($\hat{M}_o > 0$, $M_2 < 0$); amplitude of quasi-planar motion must be large enough to ensure positive <u>average</u> M ($m < -2$).

Pure planar motion occurs when spin is zero. With zero spin, a number of simplifications are possible.

$$M_o = \hat{M}_o \tag{19}$$

$$\phi_1' = -\phi_2' = \sqrt{-M_o\left(1 + \frac{3m}{4}\right)} \tag{20}$$

$$\lambda_1^* = \lambda_2^* = \lambda^* \tag{21}$$

$$\lambda^* = -\frac{1}{4\pi}\int_0^{2\pi}\left\{H\left[1 - \cos\hat{\phi}\right] + \frac{M}{\phi_1'}\sin\hat{\phi}\right\} d\hat{\phi} \tag{22}$$

$$\delta^2 = 2K^2(1 + \cos \hat{\phi}) \tag{23}$$

$$(\delta^2)' = - 4\phi_1' K^2 \sin \hat{\phi} \tag{24}$$

$$\lambda_1 = \lambda_2$$

$$= \frac{2(4 + 3m)\lambda^* - 8\left(\frac{M_o'}{4M_o}\right) - 6m\left(\frac{M_2'}{4M_2}\right)}{8 + 9m} \tag{25}$$

If the cause of the variation in the coefficients is changing air density due to entering or leaving an exponential atmosphere[1],

$$\frac{M_o'}{M_o} = \frac{M_2'}{M_2} = \frac{\rho'}{\rho} = \tilde{\sigma} \tag{26}$$

$$\therefore \lambda_1 = \lambda_2 = \frac{2(4 + 3m)\left(\lambda^* - \frac{\tilde{\sigma}}{4}\right)}{8 + 9m} \tag{27}$$

Equation (27) was essentially derived by Coakley, Laitone and Mass[4] and predicts that the amplitude of planar motion for a type (b) moment has an upper bound imposed by the requirement $- 8/9 < m$. For linear damping, however, the more exact perturbation method of Reference 2 yields the relation:[1,3]

$$\lambda_1 = \lambda_2 = \frac{- b_o}{2(1 + m)} \left[\frac{H_o}{2} + \frac{\tilde{\sigma}}{4}\right] \tag{28}$$

where

$b_o = 4(1 + m)a_2 - 2ma_4$ types (a) and (c)

$= 2(2 + m - 2a_2 - ma_4)$ type (b)

$$a_2 = k_p^{-2} \left[1 - E_p/K_p \right]$$

$$a_4 = (1/3)\, k_p^{-2} \left[2(1 + k_p^2) a_2 - 1 \right]$$

$K_p = K(k_p)$ complete elliptic integral of the first kind

$E_p = E(k_p)$ complete elliptic integral of the second kind

$$k_p^2 = \frac{m}{2(1 + m)} \qquad \text{types (a) and (c)}$$

$$= \frac{-m}{2 + m} \qquad \text{type (b)}$$

The coefficients of $\left[\frac{H_o}{2} + \frac{\tilde{\sigma}}{4} \right]$ in Equations (27) and (28) are compared for the three types of moments in Figures (2-4). With the exception of the vicinity of $m = -8/9$, the quasi-linear substitution value is a reasonable approximation of the more exact but quite complicated perturbation result.

4. ALMOST CIRCULAR MOTION

For almost circular motion, one modal amplitude is much larger than the other and any static moment can be approximated by a cubic in the vicinity of the amplitude of the circular motion. In the Table, the coefficients, a_{jk}, are computed for $K_2 << K_1$. (The coefficients for $K_1 << K_2$ may be obtained by interchanging K_1 and K_2.) As in the case of planar motion, a number of simple relations can be written for $K_2 << K_1$ and constant spin.* $(P' = 0)$

$$\phi_1' = \frac{P}{2} + \sqrt{-\hat{M}_o(1 + m)} \tag{29}$$

$$\phi_2' = \frac{P}{2} - \sqrt{-\hat{M}_o(1 + 2m)} \tag{30}$$

---

* If $K_1 << K_2$, the frequency equations are

$$\phi_1' = \frac{P}{2} + \sqrt{-\hat{M}_o(1 + 2m)}; \quad \phi_2' = \frac{P}{2} - \sqrt{-\hat{M}_o(1 + m)} \ .$$

$$\lambda_1^* = \frac{-1}{4\pi\sqrt{-\hat{M}_o(1+m)}} \int_0^{2\pi} \left[H\phi_1' - PT \right] d\hat{\phi} \tag{31}$$

$$\lambda_2^* = \frac{1}{4\pi\sqrt{-\hat{M}_o(1+2m)}} \int_0^{2\pi} \left\{ H \left[\phi_2' + \phi_1' \frac{K_1}{K_2} \cos\hat{\phi} \right] - PT \left[1 + \frac{K_1}{K_2} \cos\hat{\phi} \right] - M_2 \frac{K_1}{K_2} \sin\hat{\phi} \right\} d\hat{\phi} \tag{32}$$

$$\lambda_1 = \frac{4(1+m)\left[\lambda_1^* - \frac{M_o'}{4\hat{M}_o}\right] + m\left[\frac{M_o'}{\hat{M}_o} - \frac{M_2'}{M_2}\right]}{2(2+3m)} \tag{33}$$

$$\lambda_2 = \frac{-2m(1+m)}{(2+3m)(1+2m)} \left[\lambda_1^* - \frac{M_o'}{4\hat{M}_o}\right] + \lambda_2^* - \frac{M_o'}{4\hat{M}_o} + \frac{m(1+m)}{(2+3m)(1+2m)} \left[\frac{M_o'}{\hat{M}_o} - \frac{M_2'}{M_2}\right] \tag{34}$$

Note that Equation (32) contains the very large quantity K_1/K_2. For most nonlinearities, the averaging process of the integral formally yields a magnitude for λ_2^* of the order of 10^{-2} and certainly much less than one. This need for the average of a large periodic term to vanish would lead us to expect Equation (32) to be less accurate than Equation (31). As we shall see, this is a correct conjecture.

The exact elliptic integral solution[2] places the following limitations on m:

type (a) no limitation ($m > 0$).

type (b) only possible circular motions are those for which $-2/3 < m < 0$.

type (c) only possible circular motions are those for which M is negative ($m < -1$).

The presence of $2 + 3m$ in the denominator of Equation (33) is the first time a quasi-linear approach has been able to indicate the completely unexpected limitation on circular motions for a type (b) moment which was previously obtained through the use of elliptic integrals. This result is the first evidence of the value of the improved substitution method in comparison with that of Reference 1. The fallacious indication of trouble for $m = -1/2$ in Equation (34) reinforces our concern for the value of the expressions for damping of the small modal amplitude.

In order to derive an estimate for the accuracy of Equations (33-34), we will now consider two special cases for which the coefficients are constants ($M_0' = M_2' = 0$).

$$\lambda_1 = \frac{2 + 2m}{2 + 3m} \lambda_1^* \tag{35}$$

$$\lambda_2 = \frac{-2m(1 + m)}{(2 + 3m)(1 + 2m)} \lambda_1^* + \lambda_2^* \tag{36}$$

These cases were treated by the perturbation method in Reference 2 and the validity of our results will be determined by comparison with results of that method.

4.1 CUBIC MAGNUS MOMENT

In the first example, we will consider a spinning missile with a cubic Magnus moment ($T = T_o + T_2\delta^2$)

$$\lambda_1^* = \frac{-H_o\left[\frac{P}{2} + \sqrt{-\hat{M}_o(1+m)}\right] + P\left[T_o + T_2K_1^2\right]}{2\sqrt{-\hat{M}_o(1+m)}}$$

$$= -\left[\frac{1}{4}\left(2H_o - \left|1+m\right|^{-1/2}\hat{P}(2T_o - H_o + 2T_2K_1^2)\right]\right. \tag{37}$$

$$\lambda_2^* = \frac{H_o\left[\frac{P}{2} - \sqrt{-\hat{M}_o(1+2m)}\right] - P\left[T_o + 2T_2K_1^2\right]}{2\sqrt{-\hat{M}_o(1+2m)}}$$

$$= -\left(\frac{1}{4}\right)\left[2H_o + \left|1+2m\right|^{-1/2}\hat{P}(2T_o - H_o + 4T_2K_1^2)\right] \tag{38}$$

where $\hat{P} = \dfrac{P}{\left|\hat{M}_o\right|^{1/2}}$

When the damping coefficient of the large mode as given by Equations (35) and (37) is compared with that obtained from the perturbation method[2], we find them to be identical! A comparison of the damping coefficients for the small mode reveals that they do differ. A measure of the magnitude of this difference can be obtained by considering the conditions for a circular motion singularity which is a stable node. The location of the singularity is given by

$$\lambda_1 = -\left(\frac{1}{2}\right)\left(\frac{1+m}{2+3m}\right)\left[2H_o - \left|1+m\right|^{-1/2}\hat{P}(2T_o - H_o + 2T_2K_1^2)\right] = 0 \tag{39}$$

Since small circular motion must grow and large circular motion must decay,

$$2H_o - \left|1 + m\right|^{-1/2} \hat{P}(2T_o - H_o) < 0 \tag{40}$$

$$\hat{P}T_2 < 0 \tag{41}$$

These inequalities naturally are equivalent with those of Reference 2. The final requirement is that in the vicinity of the singular point, almost circular motion will approach circular motion.

$$\lambda_2 = \lambda_2^* < 0 \tag{42}$$

By the use of Equations (38-39), T_2 may be eliminated from Inequality (42).

$$2\left[1 + 2\left|\frac{1 + m}{1 + 2m}\right|^{1/2}\right] H_o - \left|1 + 2m\right|^{-1/2} \hat{P}(2T_o - H_o) > 0 \tag{43}$$

But Inequalities (40) and (43) for m outside the interval (-1, -1/2) may be combined to yield

$$2\left|1 + m\right|^{1/2} H_o < \hat{P}(2T_o - H_o) < 2\left[\left|1 + 2m\right|^{1/2} + 2\left|1 + m\right|^{1/2}\right] H_o \tag{44}$$

Outside the forbidden interval of $-1 < m < -1/2$ Inequality (44) requires that H_o be positive and

$$\left|1 + m\right|^{1/2} < \frac{\hat{P}(2T_o - H_o)}{2H_o} < \left|1 + 2m\right|^{1/2} + 2\left|1 + m\right|^{1/2} \tag{45}$$

The corresponding inequality from Reference 2 is*

$$\left| 1 + m \right|^{1/2} < \frac{\hat{P}(2T_o - H_o)}{2H_o} < \frac{|6 + 7m|}{2} \left| 1 + m \right|^{-1/2} \tag{46}$$

These bounds are compared in Figure 5. As can be seen from the figure, the upper bound is a good approximation.

4.2 CUBIC DAMPING MOMENTS AND ZERO SPIN

For a nonspinning missile, it has been shown[5] that there are two cubic damping moments which can affect the modal amplitudes:**

$$H = H_o + (H_2 + M_{11})\delta^2 \tag{47}$$

$$M = M_o + M_2\delta^2 + M_{11}(\delta^2)' \tag{48}$$

Since we are concerned with almost circular motion ($K_1 >> K_2$), a much more general moment can be considered. This moment will be approximated by Equations (47-48) for almost circular motion. This moment can be written in terms of the aerodynamic moment coefficients as***

$$C_m + iC_n = - i \left[(c_o + c_2 \delta^2 + c^*) \xi + d \xi' \right] \tag{49}$$

---

* The $\tilde{P}$ of that report is $\left| 1 + m \right|^{1/2} \hat{P}$ for $m > - 2/3$ and $- \hat{P} \left| 1 + m \right|^{1/2}$ for $m < - 1$.

** The presence of M_{11} in (47) is due to its definition as the coefficient of $\xi^2 \xi'$ in Reference 5.

*** The good approximation $\xi' = i (q + ir) \ell V^{-1}$ has been made in Equation (49) so that C_{M_q} and $C_{M_{\dot{\alpha}}}$ appear as a sum.

where $c^* = c^*\left((\delta^2)'\right)$ is a function of $(\delta^2)'$

$d = d\,(\delta^2)$ is a function of δ^2

$c^*(0) = 0$ and

$d\,(0) = C_{M_{qo}} + C_{M_{\dot{\alpha}o}}$

The coefficients of the differential equation of pitching and yawing motion assume the form

$$H = \frac{\rho S \ell}{2m}\left[C_{L_\alpha} - C_D - k_t^{-2}\,d\right] = H(\delta^2) \tag{50}$$

$$M = \frac{\rho S \ell}{2m}\,k_t^{-2}\left[c_o + c_2\delta^2 + c^*\right] = M_o + M_2\delta^2 + M^*\left((\delta^2)'\right) \tag{51}$$

If H and M^* are differentiable functions, they can be expanded about the circular motion with amplitude δ_c and amplitude derivative $(\delta^2)'_c = 0$.

$$H = H_c + \left[\frac{dH}{d\delta^2}\right]_c (\delta^2 - \delta_c^2) \tag{52}$$

$$M = M_o + M_2\delta^2 + \left[\frac{dM^*}{d(\delta^2)'}\right]_o (\delta^2)' \tag{53}$$

where $H_c = H(\delta_c^2)$

$$\left[\frac{dH}{d\delta^2}\right]_c = \left[\frac{dH}{d\delta^2}\right]_{\delta^2 = \delta_c^2}$$

$$\left[\frac{dM^*}{d(\delta^2)'}\right]_o = \left[\frac{dM^*}{d(\delta^2)'}\right]_{(\delta^2)' = 0}$$

$$\delta^2 = K_1^2 + K_2^2 + 2K_1K_2 \cos \hat{\phi}$$

$$(\delta^2)' = -2K_1K_2 (\phi_1' - \phi_2') \sin \hat{\phi}$$

Equations (52-53) are essentially of the same form as Equations (47-48) but allow us to consider much more complicated moments.

For these moments and almost circular motion,

$$\lambda_1^* = -(\tfrac{1}{2}) \left\{ H_c + \left[\frac{dH}{d\delta^2}\right]_c (K_1^2 - \delta_c^2) \right\} \tag{54}$$

$$\lambda_2^* = \lambda_1^* + (\tfrac{1}{2}) \left\{ \left[\frac{dH}{d\delta^2}\right]_c \sqrt{\frac{1 + m_c}{1 + 2m_c}} + \left[\frac{dM^*}{d(\delta^2)'}\right]_o \left[1 + \sqrt{\frac{1 + m_c}{1 + 2m_c}}\right] \right\} K_1^2 \tag{55}$$

The actual damping exponent for the larger mode can be obtained from Equations (35) and (54) and is exactly that given by the perturbation method.

$$\lambda_1 = -\left(\frac{1 + m_c}{2 + 3m_c}\right) \left[H_c + \left[\frac{dH}{d\delta^2}\right]_c \left(K_1^2 - \delta_c^2\right) \right] \tag{56}$$

The conditions on λ_1 for a stable node at $K_1 = \delta_c$ are

$$H_c = 0 \tag{57}$$

$$\left[\frac{dH}{d\delta^2}\right]_c > 0 \tag{58}$$

According to Equation (36) the actual damping of the smaller mode near the singularity ($\lambda_1^* \doteq 0$) is λ_2^*. For a stable node this must be negative.

$$\therefore \left[\frac{dH}{d\delta^2}\right]_c \sqrt{\frac{1 + m_c}{1 + 2m_c}} + \left[\frac{dM^*}{d(\delta^2)'}\right]_o \left[1 + \sqrt{\frac{1 + m_c}{1 + 2m_c}}\right] < 0 \tag{59}$$

or

$$\left[\frac{dM^*}{d(\delta^2)'}\right]_o \Big/ \left[\frac{dH}{d\delta^2}\right]_c < \frac{-1}{1 + \sqrt{\frac{1 + 2m_c}{1 + m_c}}} \tag{60}$$

The corresponding inequality derived from the perturbation method is

$$\left[\frac{dM^*}{d(\delta^2)'}\right]_o \Big/ \left[\frac{dH}{d\delta^2}\right]_c < -\left[\frac{1 + m_c}{2 + 3m_c}\right] \tag{61}$$

These upper bounds are compared in Figure 6. It is interesting to note that the M^* function is necessary for a circular limit cycle, i.e. $M^* = 0$ does not satisfy Inequality (61).

5. ALMOST PLANAR MOTION

Another important special type of motion is almost planar motion ($K_1 \doteq K_2$). For this case somewhat more lengthy algebra is required but fairly general results are attainable. For planar motion $K_1 = K_2 = K_p$ and

$$\delta_p^2 = 2K_p^2(1 + \cos \hat{\phi}) \tag{62}$$

$$(\delta^2)_p' = -2K_p^2\hat{\phi}_p' \sin \hat{\phi} \tag{63}$$

$$\hat{\phi}_p' = \sqrt{-M_o(4 + 3m_p)} \tag{64}$$

where $m_p = \dfrac{4M_2K_p^2}{M_o}$

If H and M^* are differentiable functions, they can be expanded about planar motion with amplitude $2K_p$.

$$H = H_p + \left[\frac{dH}{d\delta^2}\right]_p \left[\delta^2 - \delta_p^2\right] \tag{65}$$

$$M^* = M_p^* + \left[\frac{dM^*}{d(\delta^2)'}\right]_p \left[(\delta^2)' - (\delta^2)_p'\right] \tag{66}$$

where the p subscript denotes quantities evaluated for planar motion with $\delta^2 = \delta_p^2$ and $(\delta^2)' = (\delta^2)_p'$.

For a nonspinning missile,

$$\lambda_1^* = \frac{-1}{4\pi}\int_0^{2\pi}\left\{H\left[1 + \frac{\phi_2' K_2}{\phi_1' K_1}\cos\hat{\phi}\right] + M^*\left(\frac{K_2}{\phi_1' K_1}\right)\sin\hat{\phi}\right\}d\hat{\phi} \tag{67}$$

In order to determine the character of a planar singularity amplitude plane, differential equations should be derived for the neighborhood of the singularity. The variables ϵ_1 and ϵ_2 are introduced for this purpose and squares of these variables will be omitted in comparison with the variables themselves.

$$K_j = K_p(1 + \epsilon_j) \tag{68}$$

$$\delta^2 - \delta_p^2 = 2K_p^2(1 + \cos\hat{\phi})\ (\epsilon_1 + \epsilon_2) \tag{69}$$

$$\begin{aligned}\hat{\phi}' &= \phi_1' - \phi_2' \\ &= (1/2)\left[\sqrt{-M_o\left[4 + m_p(3 + 2\epsilon_1 + 4\epsilon_2)\right]} + \sqrt{-M_o\left[4 + m_p(3 + 4\epsilon_1 + 2\epsilon_2)\right]}\right] \\ &= \hat{\phi}_p'\left[1 + \frac{3m_p}{8 + 6m_p}(\epsilon_1 + \epsilon_2)\right]\end{aligned} \tag{70}$$

$$\begin{aligned}(\delta^2)' - (\delta^2)_p' &= \left[-2K_1K_2\hat{\phi}'\sin\hat{\phi}\right] - \left[-2K_p^2\hat{\phi}_p'\sin\hat{\phi}\right] \\ &= -K_p^2\hat{\phi}_p'\sin\hat{\phi}\left(\frac{8 + 9m_p}{4 + 3m_p}\right)(\epsilon_1 + \epsilon_2)\end{aligned} \tag{71}$$

$$\frac{\phi_2' K_2}{\phi_1' K_1} = -1 + \left[\frac{4 + 2m_p}{4 + 3m_p}\right] (\epsilon_1 - \epsilon_2) \tag{72}$$

$$\therefore \quad \lambda_1^* = \lambda_p^* - \frac{1}{4\pi} \int_0^{2\pi} \left\{ H_p \left(\frac{4 + 2m_p}{4 + 3m_p}\right) (\epsilon_1 - \epsilon_2) \cos\hat{\phi} \right.$$

$$+ 2 \left[\frac{dH}{d\delta^2}\right]_p K_p^2 (\epsilon_1 + \epsilon_2) \sin^2\hat{\phi} - \frac{2M_p^* \left[4(1 + m_p)\epsilon_1 - (4 + m_p)\epsilon_2\right] \sin\hat{\phi}}{\hat{\phi}_p'(4 + 3m_p)}$$

$$\left. - 2 \left[\frac{dM^*}{d(\delta^2)'}\right]_p K_p^2 \left(\frac{8 + 9m_p}{4 + 3m_p}\right) (\epsilon_1 + \epsilon_2) \sin^2\hat{\phi} \right\} d\hat{\phi} \tag{73}$$

where $\lambda_p^* = -\frac{1}{4\pi} \int_0^{2\pi} \left[(H_p)(1 - \cos\hat{\phi}) + (M_p^*)\left(\frac{2 \sin\hat{\phi}}{\hat{\phi}_p'}\right)\right] d\hat{\phi}$

The terms involving $\left[\frac{dH}{d\delta^2}\right]_p$ and M_p^* may be integrated by parts and the result reduced by routine algebra so that Equation (73) assumes the much simpler form

$$\lambda_1^* = \lambda_p^* - \frac{[H]_1}{4(4 + 3m_p)} \left\{ \left[(8 + 5m_p)\epsilon_1 + m_p\epsilon_2\right] + \left[m_p\epsilon_1 + (16 + 11m_p)\epsilon_2\right] r_o \right.$$

$$\left. - \left[(16 + 17m_p)\epsilon_1 + 7m_p\epsilon_2\right] r_2 \right\} \tag{74}$$

where $r_j = \dfrac{-\left[\frac{dM^*}{d(\delta^2)'}\right]_j K_p^2}{[H]_1}$, $j = 0,2$.

$$\left[\frac{dM^*}{d(\delta^2)'}\right]_j = \frac{1}{\pi}\int_0^{2\pi}\left[\frac{dM^*}{d(\delta^2)'}\right]_p \cos j\hat{\phi}d\hat{\phi} \quad , \qquad j = 0,2 \quad .$$

$$[H]_1 = \frac{1}{\pi}\int_0^{2\pi} H_p \cos \hat{\phi}d\hat{\phi}$$

Similarly

$$\lambda_2^* = \lambda_p^* - \frac{[H]_1}{4(4 + 3m_p)}\left\{\left[m_p\epsilon_1 + (8 + 5m_p)\epsilon_2\right] + \left[(16 + 11m_p)\epsilon_1 + m_p\epsilon_2\right] r_o - \left[7m_p\epsilon_1 + (16 + 17m_p)\epsilon_2\right] r_2\right\} \tag{75}$$

The numerical subscript on the outside of the bracketed expressions in the definition of λ_j identifies that expression as a particular Fourier cosine coefficient. It is quite surprising that the influence of H on λ_1 is completely determined by its first order Fourier cosine coefficient. The influence of M^*, however, is specified by the zeroth and second order Fourier cosine coefficients of its first derivative. These coefficients are computed for fixed modal amplitudes, K_j, and therefore, are functions of these amplitudes.

For a planar singularity $\lambda_p^* = 0$ and λ_j can be computed from the following special form of Equation (17)

$$\lambda_j = a_{j1}\lambda_1^* + a_{j2}\lambda_2^* \tag{76}$$

where $a_{11} = a_{22} = \dfrac{2(4 + 3m_p)(8 + 7m_p)}{(8 + 9m_p)(8 + 5m_p)}$

$$a_{12} = a_{21} = \frac{-4m_p(4 + 3m_p)}{(8 + 9m_p)(8 + 5m_p)}$$

$$\therefore \quad \lambda_1 = - \frac{[H]_1}{2(8 + 5m_p)(8 + 9m_p)} \left[b\epsilon_1 + a\epsilon_2\right] \tag{77}$$

$$\lambda_2 = - \frac{[H]_1}{2(8 + 5m_p)(8 + 9m_p)} \left[a\epsilon_1 + b\epsilon_2\right] \tag{78}$$

where $a = - m_p(8 + 3m_p) + (128 + 200m_p + 75m_p^2)\ r_o$

$$- (24m_p + 15m_p^2)\ r_2$$

$$b = 64 + 96m_p + 33m_p^2 - 3m_p(8 + 5m_p)\ r_o$$

$$- (128 + 248m_p + 105m_p^2)\ r_2$$

The differential equation for solution curves in the vicinity of a planar singularity in the amplitude plane is

$$\frac{d\epsilon_2}{d\epsilon_1} = \frac{\lambda_2}{\lambda_1} = \frac{a\epsilon_1 + b\epsilon_2}{b\epsilon_1 + a\epsilon_2} \tag{79}$$

According to Reference 6, the singularity must be either a saddle point or a node. It is a node if $a^2 - b^2$ is negative and a saddle if $a^2 - b^2$ is positive.

$$a^2 - b^2 = (a + b)(a - b)$$

$$= - 4(8 + 5m_p)(8 + 9m_p) \left[2(2 + m_p) - (r_o + r_2)(8 + 5m_p)\right]$$

$$\times \left[(4 + 3m_p)(1 + 2r_o) - 4(2 + 3m_p)\ r_2\right] \tag{80}$$

Note that if r_o and r_2 vanish and m_p is outside the interval $(-2, -8/9)$, $a^2 - b^2$ is negative. Thus, if the aerodynamic moment coefficients are functions of δ^2 alone and not functions of $(\delta^2)'$, all planar singularities

are nodes and almost planar motions near a planar singular motion will tend to the planar singular motion if neighboring planar motions tend to the planar singular motion.*

Another interesting special case is that when the moment coefficients are precisely those defined by Equations (47-48)

$$\left[H\right]_1 = 2(H_2 + M_{11})K_p^2 \tag{81}$$

$$\left[\frac{dM^*}{d(\delta^2)'}\right]_o = 2M_{11} \tag{82}$$

$$\left[\frac{dM^*}{d(\delta^2)'}\right]_2 = 0 \tag{83}$$

$$\therefore \quad r_o = -\frac{M_{11}}{H_2 + M_{11}} \tag{84}$$

$$r_2 = 0 \tag{85}$$

Therefore, a planar singularity is a node if

$$(8 + 9m_p)(8 + 5m_p)(4 + 3m_p)(1 + 2r_o)\left[4 + 2m_p - r_o(8 + 5m_p)\right] > 0 \tag{86}$$

When m_p is outside the interval (- 2, - 8/9) this Inequality is equivalent to

$$-1 < \frac{-M_{11}}{H_2} < \frac{4 + 2m_p}{12 + 7m_p} \tag{87}$$

---

* For circular singularities, Equation (60) shows the circular singularities are always saddles if the moment coefficients are functions of δ^2 alone, i.e.,

$$M_{11} = \left[\frac{dM^*}{d(\delta^2)'}\right]_c = 0$$

In Reference 3, the perturbation method was applied to almost planar motion and after considerable tedious algebra, inequalities like (87) were obtained. The lower bounds are identical but the perturbation method's upper bound is expressed in terms of complete elliptic integrals and differs from that of Inequality (87) when $m_p \neq 0$. These two boundary curves are compared in Figure 7. The much more easily derived bound of Inequality (87) is surprisingly good.

SUMMARY

1. A quasi-linear substitution method has been derived on plausible assumptions and compared with the more exact results of the more complicated perturbation method.

2. The planar motion predictions are good when m is not near - 8/9.

3. The damping of the dominant mode of almost circular motion is exactly predicted while the damping of the other mode yields approximately correct stability boundaries.

4. The character of planar singularities can be reasonably well determined by this method.

5. In view of the above, the algebraically much simpler quasi-linear method can be used to obtain approximate stability boundaries in the presence of a strongly nonlinear static moment.

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CHARLES H. MURPHY

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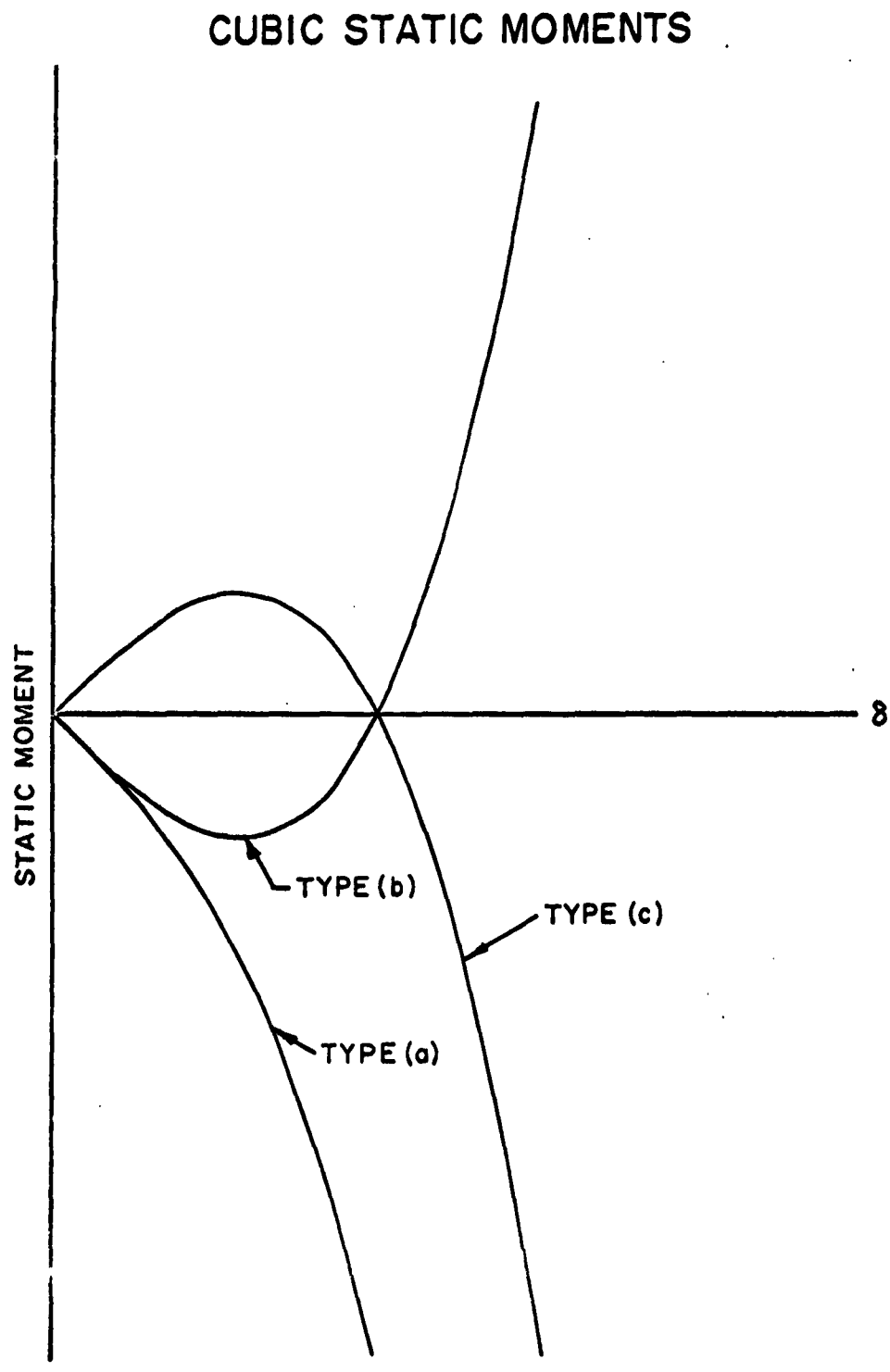

Fig. 1

COMPARISON OF PLANAR DAMPING MOMENT COEFFICIENTS TYPE (a)

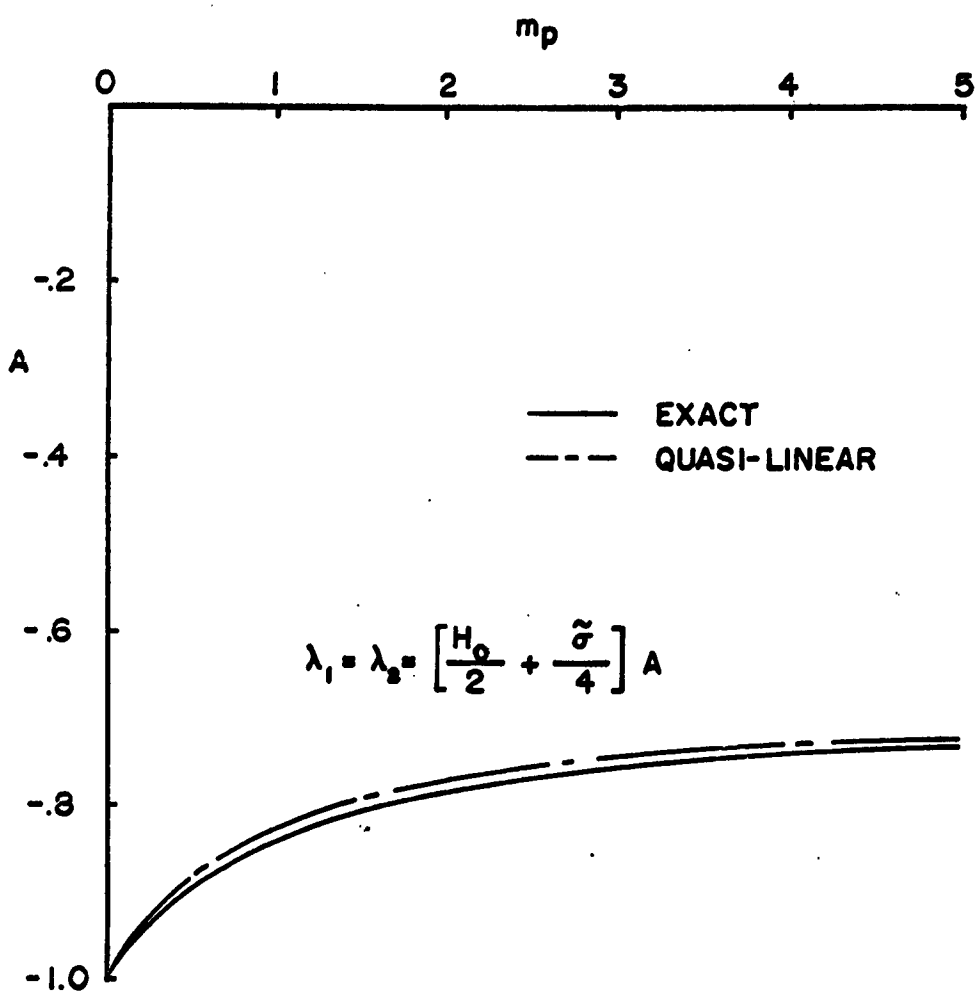

FIG. 2

COMPARISON OF PLANAR DAMPING MOMENT COEFFICIENTS TYPE (b)

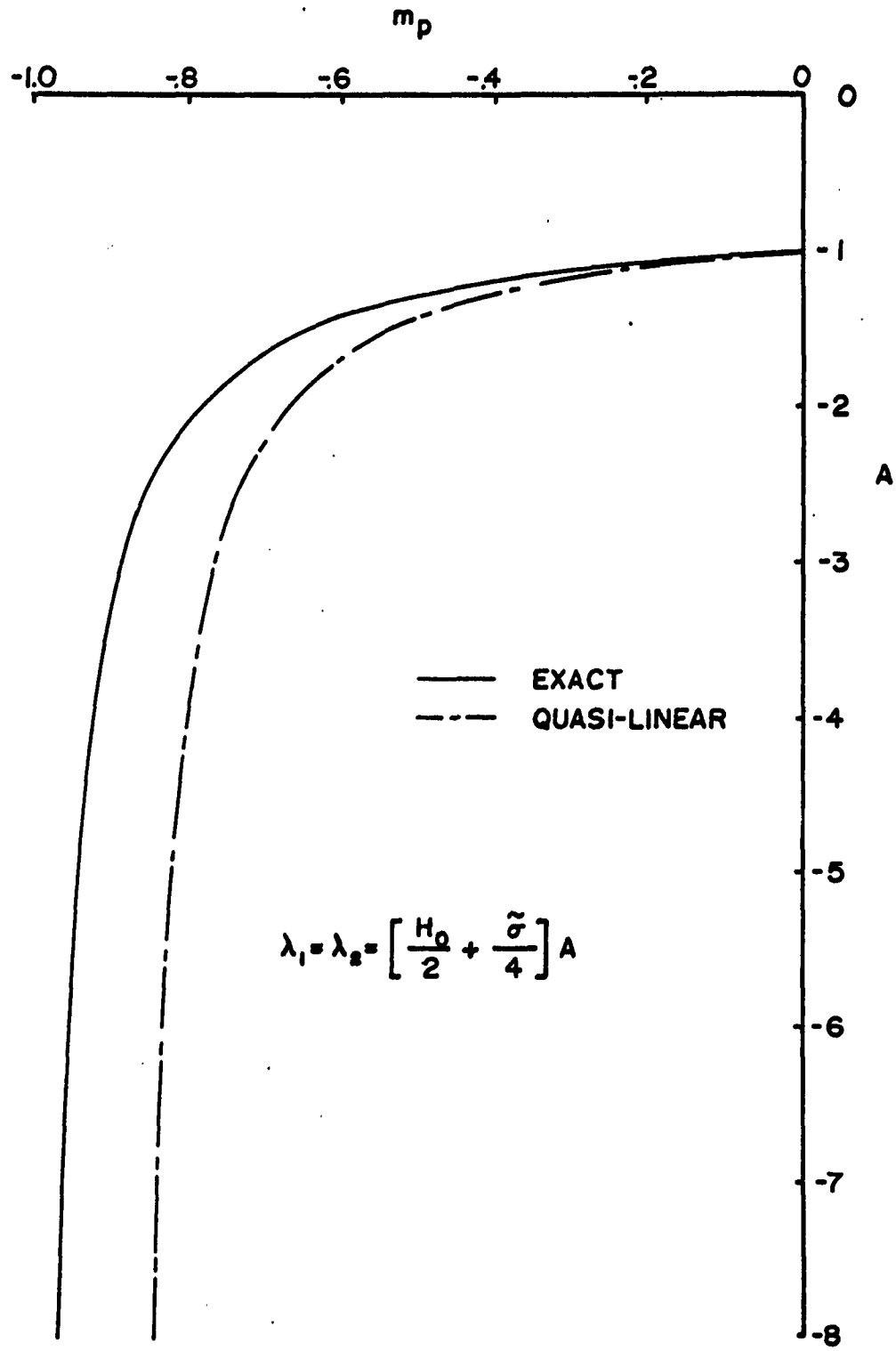

FIG. 3

COMPARISON OF PLANAR DAMPING MOMENT COEFFICIENTS TYPE (c)

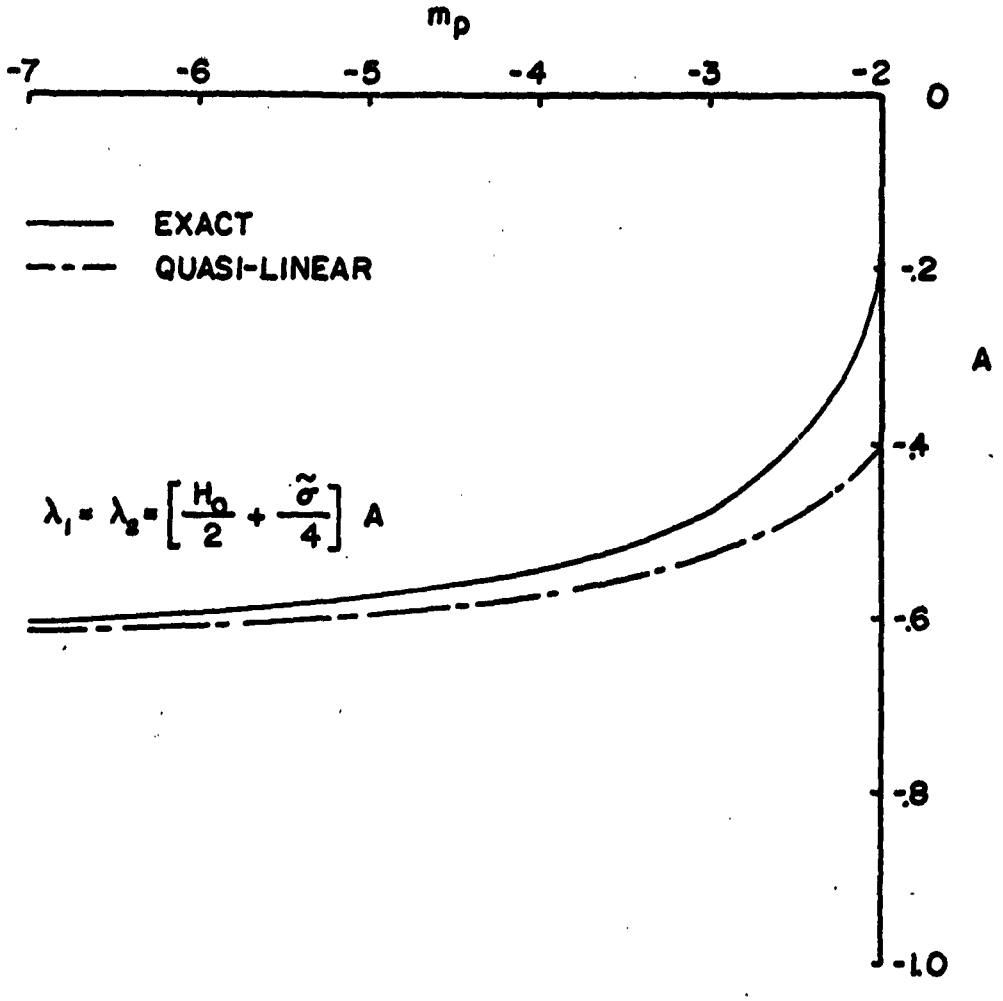

FIG. 4

COMPARISON OF UPPER BOUNDS FOR CIRCULAR NODES

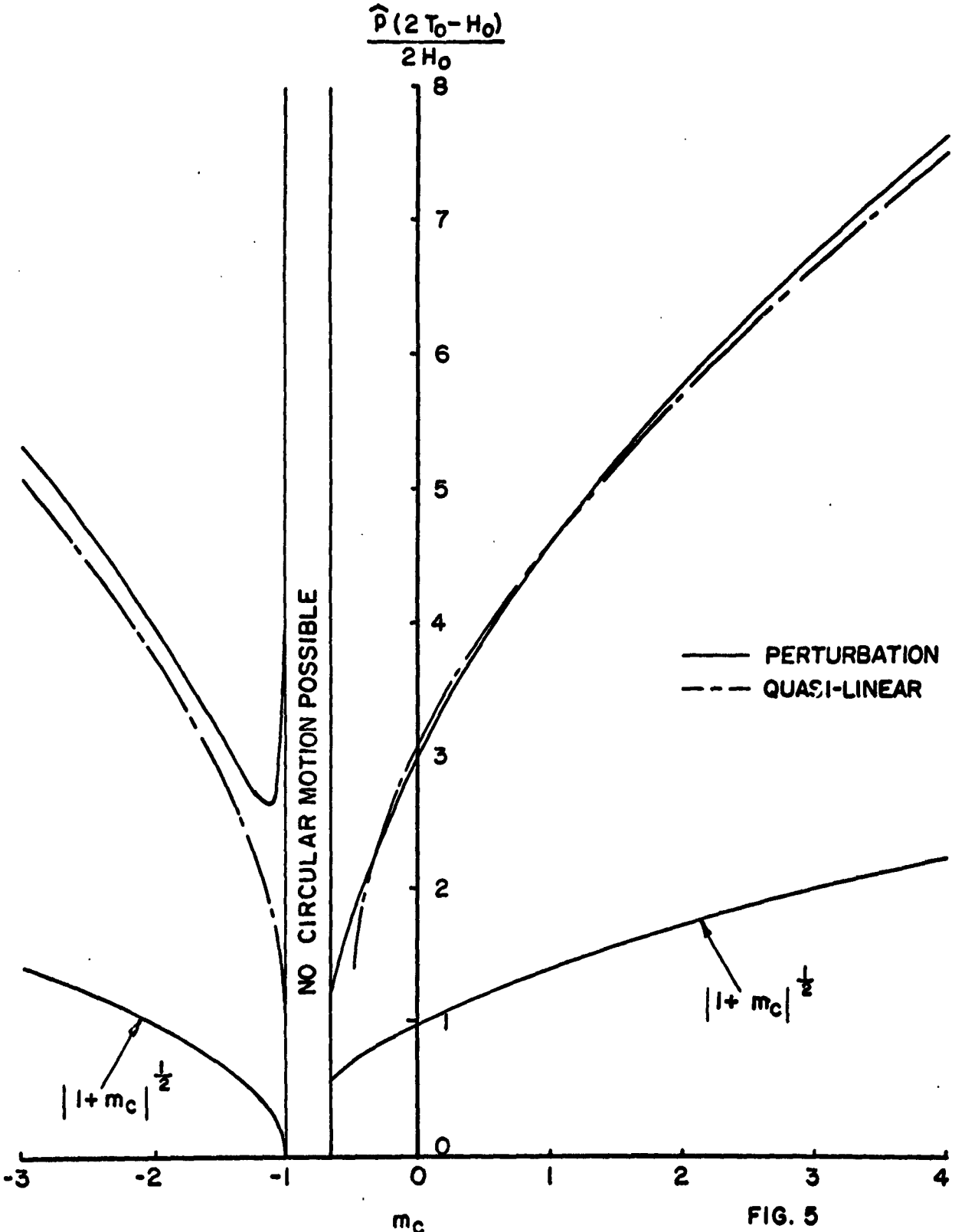

FIG. 5

CIRCULAR SINGULARITIES
FOR
NONLINEAR DAMPING MOMENTS

$$\left[\frac{dM}{d\delta^2}\right]_c > 0$$

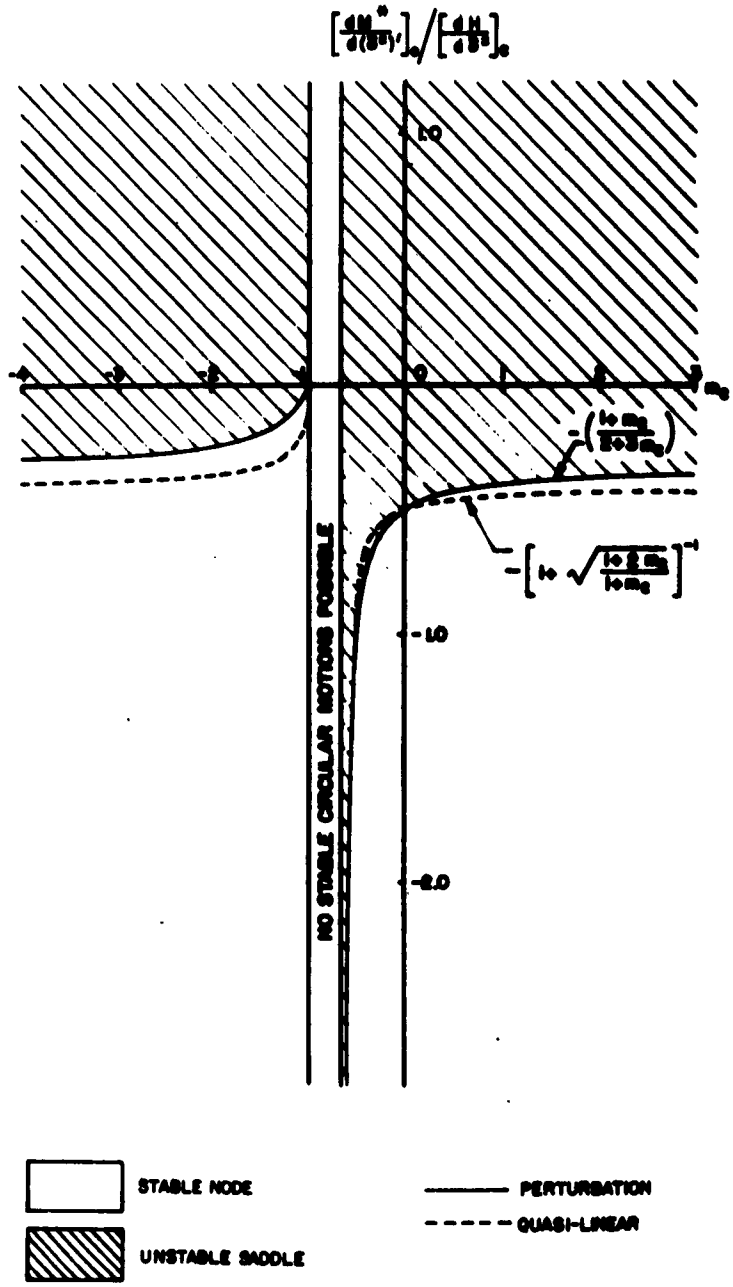

FIG. 8

COMPARISON OF UPPER BOUNDS
FOR PLANAR NODES

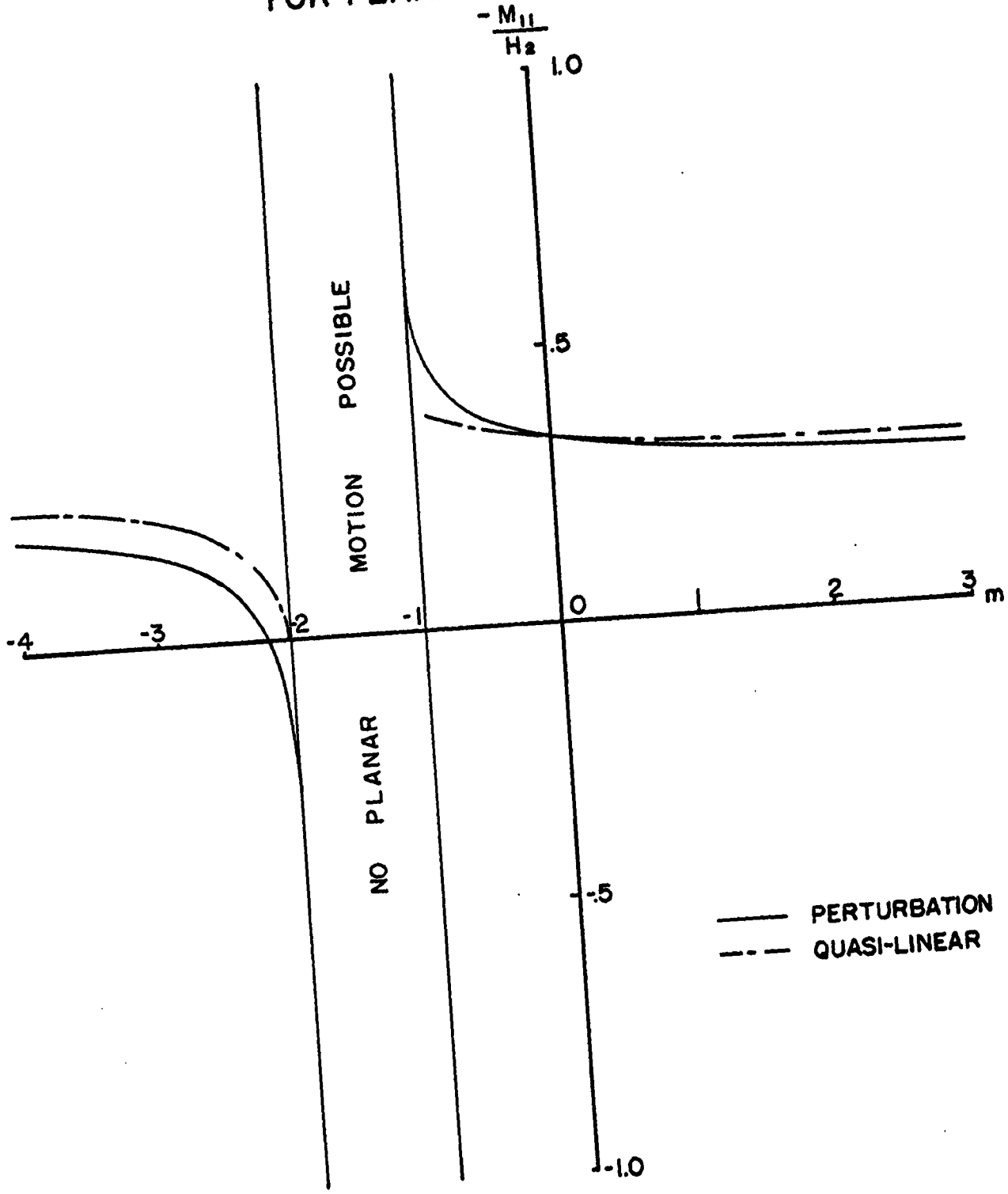

FIG. 7

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Missiles - Exterior ballistics
Missile motion - Mathematical analysis

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