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DISCRETE, FINITE REPRESENTATION OF A LINEAR, STATIONARY SYSTEM

By

P. J. Butterly

THE JOHNS HOPKINS UNIVERSITY
CARLYLE BARTON LABORATORY
Charles and 34th Streets
Baltimore 18, Maryland

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Rome Air Development Center
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United States Air Force
Griffiss Air Force Base
New York

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FOREWORD

This report was prepared by Carlyle Barton Laboratory, The Johns Hopkins University, Baltimore, Maryland on Air Force Contract No. AF 30(602)-2597 under Project No. 4505 of Task No. 450501. The work was administered by the Electronic Warfare Laboratory, Rome Air Development Center. Mr. Haywood E. Webb was the project engineer.

PUBLICATION REVIEW

This report has been reviewed and is approved.

Approved: EDWARD N. MUNZER
Chief, Electronic Warfare Laboratory
Directorate of Intelligence & Electronic Warfare

Approved: ROBERT J. QUINN, JR., Col, USAF
Director of Intelligence & Electronic Warfare

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ABSTRACT

This report describes a linear system representation suitable for use with signals of finite dimensionality. This representation takes the form of a matrix with elements which depend both on the choice of signal basis and on the transmission properties of the system. The report is divided into two parts. The first part is devoted to the derivation of the system matrix for an arbitrary system transfer function for several widely-used signal bases. The second part considers the case in which only the system matrix is known. An equivalent representation, expressed in terms of a continuous parameter, is introduced to facilitate approximation of the system transfer function, and transformation formulas are established for a number of cases where they have a simple form.

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1. DISCRETE, FINITE REPRESENTATION OF A LINEAR STATIONARY SYSTEM

1.1. INTRODUCTION

The efficient representation of a given signal class involves the selection of a minimal set of component functions in terms of which every member of the class may be described with acceptable accuracy. In many instances, as, for example, in the representation of signals limited in time and frequency, a finite set of components is sufficient for the complete representation of the class. In order to devise a characterization of a linear, stationary transmission system compatible with this method of signal representation, consider the response of this system to an arbitrary signal chosen from such a signal class. In general, it will not be possible to obtain a complete representation of the response in terms of the original set of component functions; however, the least-mean-square approximation of the response which may be so represented leads to a satisfactory characterization of the system provided the resulting error is small enough[1]. Where it is not, a re-examination of the set of signal components chosen becomes necessary and a larger set may have to be used.

Clearly this situation corresponds to the notion of a linear transformation of the original signal space on to itself and to the representation of the transmission properties of the system by the matrix of the transformation[2]. The elements of this matrix will depend both on the choice of basis - that is, on the signal components, and on the system itself. The derivation and manipulation of this matrix is the subject of Part 1 of this report. The treatment parallels that given in a recent Internal Memorandum[3].

Let $|F\rangle$ be an arbitrary signal chosen from the given signal class. Then $|F\rangle$ may be completely represented in terms of some basis set

$$|\psi_k\rangle, \quad k = 1, 2, \ldots . N$$

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that is, $$|F\rangle = \sum_{k=1}^{N} f_k^{\psi} |\psi_k\rangle \quad .$$

If $|H|$ is the linear operator referred to above, then, following Huggins[1], we can write

$$|H|\psi_k\rangle \sim \sum_{j=1}^{N} h_{jk} |\psi_j\rangle \quad . \qquad (1.1)$$

The left side of this relationship is the true response of the system. The right side is a vector lying in the original signal space. Clearly the N^2 values of h_{jk}, j, k, = 1, 2, ... N, for which the equivalence is in the least-squares sense, constitute a discrete representation of $|H|$ which provides a least-squares approximation of the system response. In order to ascertain those values of h_{jk} let

$$|\epsilon_k\rangle = |H|\psi_k\rangle - \sum_{j=1}^{N} h_{jk} |\psi_j\rangle$$

then, for minimum mean-square error:-

$$\langle \tilde{\epsilon}_\ell | \epsilon_k \rangle = \left[\langle \tilde{\Psi}_\ell | \tilde{H}| - \sum_{j=1}^{N} \langle \tilde{\Psi}_j | h_{j\ell} \right] \left[|H|\psi_k\rangle - \sum_{j=1}^{N} h_{jk} |\psi_j\rangle \right]$$

$$\frac{\partial}{\partial h_{pq}^*} \langle \tilde{\epsilon}_\ell | \epsilon_k \rangle = - \langle \tilde{\Psi}_p | \left[|H|\psi_k\rangle - \sum_{j=1}^{N} h_{jk} |\psi_j\rangle \right]$$

$$= 0 \text{ when } \qquad \langle \tilde{\Psi}_p | H | \psi_k \rangle = \sum_{j=1}^{N} h_{jk} \langle \tilde{\Psi}_p | \psi_j \rangle \quad .$$

Let

$$m_{pk} = \langle \tilde{\Psi}_p | H | \psi_k \rangle$$

$$g_{pj} = \langle \tilde{\Psi}_p | \psi_j \rangle$$

For minimum error,

$$m_{pk} = \sum_{j=1}^{N} h_{jk}\, g_{pj}$$

or, in matrix notation:-

$$\underline{M} = \underline{G}\,\underline{H} \qquad (1.2)$$

and

$$\underline{H} = \underline{G}^{-1}\,\underline{M} \qquad (1.3)$$

$\underline{H}$ is the required system matrix with elements h_{jk} $j = 1, 2, \ldots N$ and $\underline{G}$ is the familiar Gram Matrix.

As an alternative procedure, an orthonormal basis could have been derived by means of a linear transformation of the given basis, in which case the Gram Matrix would reduce to the Unit Matrix. This suggests a relation between the Gram Matrix and the matrix which transforms a given basis to an orthonormal one. This relationship is explored in Appendix 1.

To determine the Error Energy:-

$$\langle \tilde{\varepsilon}_\ell | \varepsilon_k \rangle = \left[| \langle \tilde{\Psi}_\ell | \tilde{H} | - \sum_{i=1}^{N} h^*_{i\ell} \langle \tilde{\Psi}_i | \right] \left[| H | \psi_k \rangle - \sum_{j=1}^{N} h_{jk} | \psi_j \rangle \right]$$

$$= \langle \tilde{\Psi}_\ell | \tilde{H} | H | \psi_k \rangle + \sum_{i=1}^{N} \sum_{j=1}^{N} h^*_{i\ell}\, h_{jk}\, g_{ij} \quad -$$

$$-\sum_{j=1}^{N} h_{jk}\, m^{*}_{j\ell} - \sum_{i=1}^{N} h^{*}_{i\ell}\, m_{ik}$$

$$= \langle \tilde{r}_{\ell} | r_k \rangle + \sum_{i=1}^{N} h^{*}_{i\ell}\, m_{ik} - \sum_{j=1}^{N}\sum_{i=1}^{N} g^{-1}_{ji}\, m_{ik}\, m^{*}_{j\ell}$$

$$-\sum_{i=1}^{N} h^{*}_{i\ell}\, m_{ik}$$

where $|r_k\rangle = |H|\psi_k\rangle$

$$= \langle \tilde{r}_k | r_k \rangle - \sum_{j=1}^{N}\sum_{i=1}^{N} m^{*}_{j\ell}\, g^{-1}_{ji}\, m_{ik} \quad .$$

Let

$$\underline{E} = \underline{U} - \underline{\tilde{M}}\,\underline{G}^{-1}\,\underline{M} \text{ where } u_{\ell k} = \langle \tilde{r}_{\ell} | r_k \rangle \quad .$$

Then the error energy = $\sum_{\ell=1}^{N}\sum_{k=1}^{N} \langle \tilde{\epsilon}_{\ell} | \epsilon_k \rangle$ is just the sum of the elements of $\underline{E}$.

For an orthonormal basis, $\underline{G}^{-1} = \underline{1}$, $\underline{M} = \underline{H}$ in which case:-

Error Energy = $\sum_{k=1}^{N} \langle \tilde{r}_k | r_k \rangle - \sum_{k=1}^{N}\sum_{j=1}^{N} h^{2}_{jk}$, which is the true output energy less the sum of the squares of the matrix elements.

1.2 LINEAR TRANSFORMATIONS AND THE DIAGONAL REPRESENTATION

It is often possible to simplify the system matrix by means of a linear transformation of the signal basis.

Consider any two complete bases $|\psi_i\rangle$ and $|\phi_i\rangle$, $i = 1, 2, \ldots N$ and let

$$|\phi_k\rangle = \sum_{j=1}^{N} c_{jk} |\psi_j\rangle$$

$$|F\rangle = \sum_{k=1}^{N} f_k^{\phi} |\phi_k\rangle$$

$$= \sum_{k=1}^{N} f_k^{\phi} \sum_{j=1}^{N} c_{jk} |\psi_j\rangle$$

$$= \sum_{j=1}^{N} \sum_{k=1}^{N} c_{jk} f_k^{\phi} |\psi_j\rangle \; ;$$

but

$$|F\rangle = \sum_{j=1}^{N} f_j^{\psi} |\psi_j\rangle \; ,$$

$$\therefore \qquad f_j^{\psi} = \sum_{k=1}^{N} c_{jk} f_k^{\phi}$$

or

$$\underline{F}\rangle_{\psi} = \underline{C}\,\underline{F}\rangle_{\phi}$$

and

$$\underline{F}\rangle_{\phi} = \underline{C}^{-1}\,\underline{F}\rangle_{\psi} \quad .$$

If

$$\underline{R}\rangle_{\psi} = \underline{H}_{\psi}\,\underline{F}\rangle_{\psi} \quad ,$$

then

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$$\underline{C}\,\underline{R}\rangle_{\phi} = \underline{H}_{\psi}\,\underline{C}\,\underline{F}\rangle_{\phi}$$
$$\underline{R}\rangle_{\phi} = \underline{C}^{-1}\,\underline{H}_{\psi}\,\underline{C}\,\underline{F}\rangle_{\phi} \quad .$$

But

$$\underline{R}\rangle_{\phi} = \underline{H}_{\phi}\,\underline{F}\rangle_{\phi}$$
$$\underline{H}_{\phi} = \underline{C}^{-1}\,\underline{H}_{\psi}\,\underline{C} \quad . \qquad (2.1)$$

If $\underline{H}_{\psi}$ can be diagonalized, say by the similarity transformation

$$\underline{H}_{s} = \underline{U}^{-1}\,\underline{H}_{\psi}\,\underline{U} \quad , \qquad (2.2)$$

then, from Equations (2.1) and (2.2), it is clear that $\underline{C} = \underline{U}$ gives the linear transformation of the signal basis which will always lead to a diagonal form of the system matrix.

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1.3 EXPONENTIAL BASES

The choice of basis is determined primarily by the nature of the input signal which must be adequately represented. So far, error estimation has been on the assumption that the description of the input vector is complete. For an extremely wide class of signals, however, growing (or decaying) exponentials provide a suitable characterization. Apart from the discontinuity resulting from the signal epoch, these components are eigenfunctions of the system. As will be seen, this leads to a diagonal system matrix with easily computed elements provided the exponents are distinct. The case of a characterization based on a single exponent is discussed separately. A more general discussion of the use of exponential components will be found in Huggins[1] and Lai[4].

A. Forward Components

Let $\psi_k(t) = e^{s_k t}$ $\quad t > 0,\ \sigma_k < 0 \quad s_k = c_k + j\omega_k$

$\psi_k(s) = \dfrac{1}{s - s_k}$ $\quad 0 \le \sigma \quad s = \sigma + j\omega$

$$m_{\ell k} = \int_\Gamma \frac{1}{-s - s_\ell^*} H(s) \frac{1}{s - s_k} \frac{ds}{2\pi j} .$$

From Appendix 2, Γ may be taken as the $j\omega$ axis

$$m_{\ell k} = -\frac{H(-s_\ell^*)}{s_\ell^* + s_k} \text{ for } H(s) \text{ having no poles in right half plane.}$$

Also, $g_{\ell k} = -\dfrac{1}{s_\ell^* + s_k}$

In matrix notation:-

$$\underline{M} = \underline{H}_s \, \underline{G}$$

where

$$\underline{H}_s = \begin{bmatrix} H(-s_1^*) & 0 & 0 & ---- \\ 0 & H(-s_2^*) & 0 & ---- \\ 0 & 0 & H(-s_3^*) & ---- \\ - & - & - & ---- \end{bmatrix}$$

From Equation (1.2)

$$\underline{H}_\psi = \underline{G}^{-1}\, \underline{H}_s\, \underline{G} \tag{3.1}$$

and

$$\underline{H}_s = \underline{G}\, \underline{H}_\psi\, \underline{G}^{-1} \quad . \tag{3.2}$$

From (2.2), the change of basis which will result in a diagonal system matrix is given by:-

$$\underline{C} = \underline{G}^{-1}$$

B. Backward Components

Let $\psi_{\tilde{k}}(t) = e^{s_k t}$ $\qquad t < 0 \quad \sigma_k > 0$

$$\psi_k(s) = \frac{1}{-s+s_k}\ , \qquad \sigma \leq 0$$

$$m_{\tilde{\ell}\tilde{k}} = \langle \tilde{\psi}_{\tilde{\ell}} | H | \psi_{\tilde{k}} \rangle$$

$$= \oint_\Gamma \frac{1}{s+s_\ell^*}\ H(s)\ \frac{1}{-s+s_k}\ \frac{ds}{2\pi j}$$

$$= \frac{H(s_k)}{s_k+s_\ell^*}$$

$$g_{\tilde{\ell}\tilde{k}} = \int_\Gamma \frac{1}{s+s_\ell^*}\ \frac{1}{-s+s_k}\ \frac{ds}{2\pi j}$$

$$= \frac{1}{s_k+s_\ell^*}$$

$$\therefore \quad \underline{M} = \underline{G}\,\underline{H}_s$$

where

$$\underline{H}_s = \begin{bmatrix} H(s_1) & 0 & 0 & \cdots \\ 0 & H(s_2) & 0 & \cdots \\ 0 & 0 & H(s_3) & \cdots \\ 0 & 0 & 0 & \cdots \\ \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots \end{bmatrix}$$

and

$$\underline{H}_\psi = \underline{G}^{-1}\,\underline{M} = \underline{H}_s \qquad , \qquad (3.3)$$

showing, that for backward components this choice of basis leads directly to a diagonal system matrix.

C. Matrix Elements for a Basis Consisting of Laguerre Functions

The reduction of the system matrix to a diagonal form results from the choice of distinct exponents in the basis. There is one important case in which this does not occur, namely, when the input signal is characterized in terms of a single exponent. In this instance, the orthonormal basis derived by the Kautz method[1, 5] has a particularly simple form. Consider the case of $s_k = a, k=1, 2, \ldots N$ (forward components)

$$\psi_k(s) = (-a-a^*)^{1/2} \frac{(s+a^*)^{k-1}}{(s-a)^k} \qquad k = 1, 2, \ldots N$$

$$h_{\ell k} = m_{\ell k} = \langle \tilde{\psi}_\ell | H | \psi_k \rangle$$

$$= (-a-a^*) \int_\Gamma \frac{(-s+a)^{\ell-1}}{(-s-a^*)^\ell} H(s) \frac{(s+a^*)^{k-1}}{(s-a)^k} \frac{ds}{2\pi j}$$

$$= -(-a-a^*) \int_\Gamma \frac{H(s)(s-a)^{\delta-1}}{(s+a^*)^{\delta+1}} \frac{ds}{2\pi j}$$

where $\delta = \ell - k$

$$=(-a-a^*) \frac{1}{\delta!} \frac{\partial^\delta}{\partial s^\delta} \left[(s-a)^{\delta-1} H(s)\right]\Big|_{s=-a^*} \qquad (3.4)$$

$$\delta \geq 0$$

$$= 0 \qquad \delta < 0$$

$$h_o = H(-a^*)$$

$$h_1 = (-a-a^*) H'(-a^*)$$

$$h_2 = \frac{(-a-a^*)}{2} \left[2H'(-a^*) + (-a-a^*) H''(-a^*)\right]$$

$$h_3 = \frac{(-a-a^*)}{3!} \left[6H'(-a^*) + 6(a-a^*) H''(-a^*) + (-a-a^*)^2 H'''(-a^*)\right]$$

etc.

The $\underline{H}$ matrix is therefore triangular and the values of its elements depend only on their distance from the diagonal

If $a = -1/2$

$$h_\delta = \frac{1}{\delta!} \frac{\partial^\delta}{\partial s^\delta} \left[(s+1/2)^{\delta-1} H(s)\right]\Big|_{s=-1/2} \qquad (3.5)$$

$$\delta \geq 0$$

$$= 0 \qquad \delta < 0$$

$$h_o = H(1/2)$$

$$h_1 = H'(1/2)$$

$$h_2 = (1/2)\left[2H'(1/2)+H''(1/2)\right]$$

$$h_3 = (1/6)\left[6H'(1/2)+6H''(1/2)+H'''(1/2)\right] \quad \text{etc.}$$

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1.4. MATRIX ELEMENTS FOR A BASIS CONSISTING OF ELEMENTARY "HOLD" FUNCTIONS

Exponential functions are suitable for characterizing signals of semi-infinite duration especially when there is prior knowledge of the distribution of complex frequencies. They do not lend themselves to an exact description of a time-limited signal such as might result from the application of quantization or pulsed-code techniques. The introduction of compound errors may be avoided by using the so-called "B" basis, defined as follows:-

$$\psi_k(t) = 1 \qquad (k-1) \le t < k$$
$$\qquad = 0 \qquad t < (k-1),\ t \ge k$$

$k = 1, 2, \ldots . N.$

This is an orthonormal basis. Considering forward components only:-

$$\psi_k(s) = \frac{1}{s}(1-e^{-s})\, e^{-(k-1)s} \qquad -\infty < \sigma$$

$$h_{jk} = m_{jk} = \langle \tilde{\psi}_j | H | \psi_k \rangle$$

$$= \int_\Gamma \frac{1-e^{s}}{-s}\, e^{(j-1)s}\, H(s)\, \frac{1-e^{-s}}{s}\, e^{-(k-1)s}\, \frac{ds}{2\pi j}$$

$$= \int_\Gamma \frac{H(s)}{s^2} \left\{ e^{s} + e^{-s} - 2 \right\} e^{\delta s}\, \frac{ds}{2\pi j}\ , \qquad (4.1)$$

where $\delta = j-k$, and Γ may be taken as slightly to the right of the $j\omega$ axis. Again, the values of the elements depend only on distance from the diagonal.

Case 1 $\qquad \delta < 0$

$h_\delta = 0$ since the integrand of (4.1) is analytic to the right of Γ.

Case 2 $\delta = 0$, diagonal terms

$$h_\delta = \int_\Gamma \frac{H(s)}{s^2} e^s \frac{ds}{2\pi j}$$

$$h_\delta = -h^{(-1)}(0) + h^{(-2)}(1) - h^{(-2)}(0), \text{ by Appendix 3.} \tag{4.2}$$

Case 3 $\delta > 0$

$$h_\delta = \int_\Gamma \frac{H(s)}{s^2} \left\{ e^{(\delta+1)s} - 2e^{\delta s} + e^{(\delta-1)s} \right\} \frac{ds}{2\pi j}$$

$$= \int_\Gamma \frac{H(s)}{s^2} e^{(\delta+1)s} \frac{ds}{2\pi j} - 2\int_\Gamma \frac{H(s)}{s^2} e^{\delta s} \frac{ds}{2\pi j} + \int_\Gamma \frac{H(s)}{s^2} e^{(\delta-1)s} \frac{ds}{2\pi j}.$$

Using the result in Appendix 3, we have that

$$h_\delta = \left\{ -(\delta-1)h^{(-1)}(0) + h^{(-2)}(\delta-1) - h^{(-2)}(0) \right\}$$
$$-2\left\{ -\delta h^{(-1)}(0) + h^{(-2)}(\delta) + h^{(-2)}(0) \right\}$$
$$+\left\{ -(\delta+1)h^{(-1)}(0) + h^{(-2)}(\delta+1) - h^{(-2)}(0) \right\}$$

$$h_\delta = h^{(-2)}(\delta+1) - 2h^{(-2)}(\delta) + h^{(-2)}(\delta-1) \quad . \tag{4.3}$$

This completes the determination of matrix elements for signals existing only for positive time. Otherwise, this matrix is one of the four sub-matrices necessary to describe the system. For the other three,

$$h_{\tilde{j}\tilde{k}} = \langle \tilde{\psi}_{\tilde{j}} | H | \psi_{\tilde{k}} \rangle = \langle \psi_j | H | \tilde{\psi}_k \rangle$$

$$\therefore \; h_{\tilde{j}\tilde{k}} = \langle \tilde{\psi}_k | H | \psi_j \rangle = h_{kj} \tag{4.4}$$

$$h_{\tilde{j}k} = \int_\Gamma \frac{1-e^{-s}}{s} e^{-(j-1)s} H(s) \frac{1-e^{-s}}{s} e^{-(k-1)} \frac{ds}{2\pi j}$$

$$\therefore \; h_{\tilde{j}k} = 0 \text{ for all } \tilde{j} \text{ and } k \tag{4.5}$$

$$h_{j\tilde{k}} = \int_\Gamma \frac{1-e^s}{-s} e^{(j-1)s} H(s) \frac{1-e^s}{-s} e^{(k-1)s} \frac{ds}{2\pi j}$$

$$= \int_\Gamma \frac{H(s)}{s^2} \left\{ e^{(\delta-2)s} - 2e^{(\delta-1)s} + e^{\delta s} \right\} \frac{ds}{2\pi j}$$

$$\delta = j - \tilde{k} = j + k$$

$$\delta \geq 2$$

$$h_{j\tilde{k}} = \left\{ -(\delta-2)h^{(-1)}(0) + h^{(-2)}(\delta-2) - h^{(-2)}(0) \right\}$$

$$-2 \left\{ -(\delta-1)h^{(-1)}(0) + h^{(-2)}(\delta-1) - h^{(-2)}(0) \right\}$$

$$\left\{ -\delta h^{(-1)}(0) + h^{(-2)}(\delta) - h^{(-2)}(0) \right\}$$

$$h_{j\tilde{k}} = h^{(-2)}(\delta) - h^{(-2)}(\delta-1) + h^{(-2)}(\delta-2) \qquad (4.6)$$

$$\delta \geq 2$$

A. Effect of Mid-Interval Sampling

It is often more convenient to use the basis defined by:-

$$\psi_k(t) = 1 \qquad (k - 1/2) \leq t < (k + 1/2)$$

$$= 0 \qquad t < (k - 1/2),\ t \geq (k + 1/2)$$

$k = 0, 1, 2, \ldots . N$

$$h_{jk} = \int_\Gamma \frac{1-e^s}{-s} e^{(j-1/2)s} H(s) \frac{1-e^{-s}}{s} e^{-(k-1/2)s} \frac{ds}{2\pi j}$$

$$h_\delta = \int_\Gamma \frac{H(s)}{s^2} e^{\delta s} \left\{ e^s - 2 + e^{-s} \right\} \frac{ds}{2\pi j} \qquad (4.7)$$

$$\delta = j - k$$

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Thus, for a given δ, these matrix elements are the same as those determined in Section 1.4. However, both j and k can now be zero. As before,

$$h_{\tilde{j}\tilde{k}} = h_{kj} \text{ (note } \delta = \tilde{j} - \tilde{k} = k - j)$$

$$h_{\tilde{j}k} = 0 \text{ all } j,\ k, \text{ except } j = k = 0. \quad \delta = \tilde{j} - k = -(j + k) < 0.$$

$$h_{j\tilde{k}} = \int_\Gamma \frac{1}{-s}(1-e^s)\, e^{(j-1/2)s} H(s) \frac{1}{s}(1-e^{-s}) e^{(k+1/2)s} \frac{ds}{2\pi j}$$

$$= \int_\Gamma \frac{H(s)}{s^2} \left\{e^s - 2 + e^{-s}\right\} e^{\delta s} \frac{ds}{2\pi j} \qquad (4.8)$$

where

$$\delta = j - \tilde{k}$$

$$= j + k$$

$$h_{j\tilde{k}} = h^{(-2)}(\delta+1) - 2h^{(-2)}(\delta) + h^{(-2)}(\delta-1) \qquad \text{for } \delta \geq 2$$

from Equation (4.3).

Summarizing, we have

$$h_\delta = 0, \ \delta < 0$$

$$h_\delta = -h^{(-1)}(0) + h^{(-2)}(1) - h^{(-2)}(0), \qquad \delta = 0$$

$$h_\delta = h^{(-2)}(\delta + 1) - 2h^{(-2)}(\delta) + h^{(-2)}(\delta-1), \ \delta > 0,$$

where δ is defined as above for each submatrix.

Only in elements such as $h_{j\tilde{k}}$ does the system matrix differ from that of Section 1.4, although it differs in size by at least one row and column. In integrating the impulse response for insertion in the above formulas arbitrary constants are introduced but do not appear in the final evaluation of the matrix elements.

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1.5 MATRIX ELEMENTS FOR A BASIS CONSISTING OF ELEMENTARY TRIANGULAR FUNCTIONS

A complete description of any function which can be reduced to straight line segments is possible using the elementary triangular component defined by:-

$$\psi_k(s) = \frac{1}{s^2}(e^s - 2 + e^{-s})\, e^{-ks}, \quad k = 0, 1, 2, \ldots N.$$

A. The M Matrix

$$m_{jk} = \int_\Gamma \frac{1}{s^2}(e^{-s}-2+e^s)e^{js}H(s)\frac{1}{s^2}(e^s-2+e^{-s})e^{-ks}\frac{ds}{2\pi j}$$

$$= \int_\Gamma \frac{H(s)}{s^4}\left[e^{s(\delta+2)} - 4e^{s(\delta+1)}+6e^{\delta s}-4e^{s(\delta-1)}+e^{s(\delta-2)}\right]\frac{ds}{2\pi j}. \tag{5.1}$$

Case 1 $\delta < -1$

$m_{jk} = 0$ since the integrand is analytic in the right half plane

Case 2 $\delta = -1$

$$m_{jk} = \int_\Gamma \frac{H(s)}{s^4}\, e^s\, \frac{ds}{2\pi j}$$

$$= -\frac{1}{6}h^{(-1)}(0) - \frac{1}{2}h^{(-2)}(0) - h^{(-3)}(0) - h^{(-4)}(0) + h^{(-4)}(1)$$

by Appendix 3.

Case 3 $\delta = 0$

From Equation (5.1)

$$m_\delta = \int_\Gamma \frac{H(s)}{s^4}\left\{e^{2s} - 4e^s\right\}\frac{ds}{2\pi j}$$

$$= -\frac{8}{6}h^{(-1)}(0) - \frac{4}{2}h^{(-2)}(0) - 2h^{(-3)}(0) - h^{(-4)}(0) + h^{(-4)}(2)$$

$$-4\left[-\frac{1}{6}h^{(-1)}(0) - \frac{1}{2}h^{(-2)}(0) - h^{(-3)}(0) - h^{(-4)}(0) + h^{(-4)}(1)\right]$$

$$= -\frac{2}{3}h^{(-1)}(0) + 2h^{(-3)}(0) + 3h^{(-4)}(0) + h^{(-4)}(2) - 4h^{(-4)}(1) \quad .$$

Case 4 $\qquad \delta = 1$

$$m_\delta = \int_\Gamma \frac{H(s)}{s^4}\left\{e^{3s} - 4e^{2s} + 6e^{s}\right\}\frac{ds}{2\pi j}$$

$$= -\frac{27}{6}h^{(-1)}(0) - \frac{9}{2}h^{(-2)}(0) - 3h^{(-3)}(0) - h^{(-4)}(0) + h^{(-4)}(3)$$

$$-4\left[-\frac{8}{6}h^{(-1)}(0) - \frac{4}{2}h^{(-2)}(0) - 2h^{(-3)}(0) - h^{(-4)}(0) + h^{(-4)}(2)\right]$$

$$+6\left[-\frac{1}{6}h^{(-1)}(0) - \frac{1}{2}h^{(-2)}(0) - h^{(-3)}(0) - h^{(-4)}(0) + h^{(-4)}(1)\right]$$

$$= -\frac{1}{6}h^{(-1)}(0) + \frac{1}{2}h^{(-2)}(0) - h^{(-3)}(0) - 3h^{(-4)}(0) + h^{(-4)}(3)$$

$$-4h^{(-4)}(2) + 6h^{(-4)}(1) \quad .$$

Case 5 $\qquad \delta > 1$

From Equation (5.1)

$$m_\delta = -\frac{(\delta+2)^3}{6}h^{(-1)}(0) - \frac{(\delta+2)^2}{2}h^{(-2)}(0) - (\delta+2)h^{(-3)}(0) - h^{(-4)}(0) + h^{(-4)}(\delta+2)$$

$$-4\left[-\frac{(\delta+1)^3}{6}h^{(-1)}(0) - \frac{(\delta+1)^2}{2}h^{(-2)}(0) - (\delta+1)h^{(-3)}(0) - h^{(-4)}(0) + h^{(-4)}(\delta+1)\right]$$

$$+6\left[-\frac{\delta^3}{6}h^{(-1)}(0) - \frac{\delta^2}{2}h^{(-2)}(0) - \delta h^{(-3)}(0) - h^{(-4)}(0) + h^{(-4)}(\delta)\right]$$

$$-4\left[-\frac{(\delta-1)^3}{6}h^{(-1)}(0) - \frac{(\delta-1)^2}{2}h^{(-2)}(0) - (\delta-1)h^{(-3)}(0) - h^{(-4)}(0) + h^{(-4)}(\delta-1)\right]$$

$$-\frac{(\delta-2)^3}{6}h^{(-1)}(0) - \frac{(\delta-2)^2}{2}h^{(-2)}(0) - (\delta-2)h^{(-3)}(0) - h^{(-4)}(0) + h^{(-4)}(\delta-2)$$

$$m_\delta = h^{(-4)}(\delta+2) - 4h^{(-4)}(\delta+1) + 6h^{(-4)}(\delta) - 4h^{(-4)}(\delta-1) + h^{(-4)}(\delta-2) \quad .$$

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For the other three submatrices related to the double-sided representation,

$$m_{\tilde{j}\tilde{k}} = \int_\Gamma \frac{1}{s^2}\,(e^s-2+e^{-s})\,e^{-js}H(s)\,\frac{1}{s^2}(e^{-s}-2+e^s)e^{ks}\,\frac{ds}{2\pi j}$$

$$= m_{kj}$$

$$m_{\tilde{j}k} = \int_\Gamma \frac{H(s)}{s^4}\left\{e^{2s}-4e^s+6-4e^{-s}+e^{-2s}\right\}\,e^{-(j+k)s}\,\frac{ds}{2\pi j}$$

$$= 0 \text{ all } j > 1,\ k > 1 \qquad .$$

The elements corresponding to $j = 1$, $k = 0$, and $j = 0$, $k = 1$ may be considered as belonging to one of the other submatrices

$$m_{j\tilde{k}} = \int_\Gamma \frac{H(s)}{s^4}\left\{e^{2s}-4e^s+6-4e^{-s}+e^{-2s}\right\}\,e^{(j+k)s}\,\frac{ds}{2\pi j}\ .$$

By defining $\delta = j - \tilde{k} = j + k$ for this submatrix and noting $\delta > 1$, m_δ can be computed from a knowledge of m_{jk}, that is,

$$m_\delta = h^{(-4)}(\delta+2)-4h^{(-4)}(\delta+1)+6h^{(-4)}(\delta)-4h^{(-4)}(\delta-1)+h^{(-4)}(\delta-2).$$

This completes the determination of the $\underline{M}$ matrix. Due to the correlation existing between $|\psi_k\rangle$ and $|\psi_{k+1}\rangle$, triangular matrices do not result from $m_{\tilde{j}\tilde{k}}$ and m_{jk} as in Section 1.4. Apart from this, the general form of the complete matrix is very similar to that derived in Part 1.4B.

B. The $\underline{H}$ Matrix

To obtain the $\underline{H}$ matrix it is necessary to pre-multiply the $\underline{M}$ matrix by $\underline{G}^{-1}$ where $\underline{G}$ is the relevant Gram Matrix. Replacement of $H(s)$ by unity in the previous section leads to a tri-diagonal matrix with elements:

$$g_\delta = 2/3 \qquad \delta = 0$$
$$g_\delta = 1/6 \qquad \delta = \pm 1$$
$$g_\delta = 0 \qquad \delta < -1,\ \delta > 1 \qquad .$$

Inversion of this matrix is carried out in Appendix 4, from which

$$g_{ij}^{-1} = \frac{6}{n+1} \sum_{k=1}^{n} \frac{\sin \frac{ik\pi}{n+1} \sin \frac{kj\pi}{n+1}}{2 + \cos \frac{k\pi}{n+1}} \qquad n = 2N + 1 \quad ,$$

where g_{ij}^{-1} is the element in the ij^{th} position of the Inverted Gram Matrix.

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2. CONTINUOUS REPRESENTATIONS OF SYSTEM MATRICES

2.1 INTRODUCTION

The first part of this report has been concerned with the reduction of a known system transfer function to matrix form. This part considers the converse problem in which only the system matrix is known. Since this matrix contains information which relates only to the finite-dimensional space spanned by the signal basis, it will not generally be possible to recover the original system transfer function of which the given matrix is a discrete representation. In many cases, however, it is possible to find a transfer function which is equivalent to that of the actual system in the sense that it has the same matrix representation with respect to the relevant signal basis.

Consider the case in which this equivalent system transfer function is described by the infinite sum of suitably defined component functions. Formulas derived in Part 1 of this report can then be used to obtain a set of equations relating each known matrix element to the unknown coefficients of the summation. In general, each of the infinite set of coefficients will contribute to every element of the matrix, and the relationship may not be a linear one. As will be seen, however, there is a considerable reduction in complexity when the assumed transfer function is represented in terms of an infinite set of the same component functions as describe the signal. From Equation (1.2) of Part 1, we have

$$\underline{H} = \underline{G}^{-1}\,\underline{M}$$

where $\underline{H}$ is now the known system matrix, $\underline{G}$ the Gram Matrix for the signal basis used, and $\underline{M}$ consists of elements $m_{\ell k}$.

$$\underline{M} = \underline{G}\,\underline{H}$$

so $\underline{M}$ is readily determined.

But

$$m_{\ell k} = \langle \tilde{\psi}_\ell | H | \psi_k \rangle = \int_\Gamma \psi_\ell^*(-s) H(s)\, \psi_k(s) \frac{ds}{2\pi j} ,$$

where $|\psi_k\rangle$, $k = 1, 2, \ldots . N$ is the signal basis

If

$$\hat{H}(s) = \sum_{i=1}^{\infty} A_i\, \psi_i(s) ,$$

then

$$m_{\ell k} = \sum_{i=1}^{\infty} A_i \int_\Gamma \psi_\ell(-s)^* \psi_i(s)\, \psi_k(s) \frac{ds}{2\pi j} . \qquad (1.1)$$

This is a statement of the N^2 equations referred to above. The evaluation of all coefficients A_i which contribute to $m_{\ell k}$ for all ℓ and k less than or equal to N determines $\hat{H}(s)$, giving an equivalent transfer function of which $\underline{H}$ is the matrix representation.

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2.2 EXPONENTIAL EXPANSIONS

Consider a signal class described in terms of forward exponential functions

$$\psi_k(s) = \frac{1}{s-s_k}, \qquad s_k < 0 \quad \sigma \geq 0,\ k = 1, 2, \ldots . N$$

when the s_k are distinct.

Taking the diagonal representation $\underline{H}_s$

$$\underline{H}_s = \underline{G}\,\underline{H}_\psi\,\underline{G}^{-1}$$

where $\underline{H}_\psi$ is the system matrix corresponding to the basis defined above, we have N equations of the form

$$h_{jj} = H(-s_j^*) \qquad j = 1, 2, \ldots . N \qquad . \qquad (2.1)$$

If

$$H(s) = \sum_{i=1}^{\infty} \frac{A_i}{s-s_i}$$

then

$$h_{jj} = \sum_{i=1}^{\infty} \frac{A_i}{-s_j^* - s_i}, \quad j = 1, 2, \ldots . N \qquad . \qquad (2.2)$$

$$= \sum_{i=1}^{\infty} A_i \langle \tilde{\psi}_j | \psi_i \rangle$$

$$= \sum_{i=1}^{\infty} A_i\, g_{ji} \qquad .$$

If $\underline{H}\rangle$ is a column matrix with h_{jj} in the jth row and $\underline{A}\rangle$ is a column matrix with elements A_i, $i = 1, 2, \ldots . N$, then

$$\underline{A}\rangle = \underline{G}^{-1}\,\underline{H}\rangle \qquad . \qquad (2.3)$$

The form of this equation suggests that for the assumed expansion, the coefficients A_i are those which provide a <u>minimum least-squares estimate</u> of H(s). Indeed they have been chosen in such a way that

$$\hat{H}(-s_j^*) = H(-s_j^*) \quad ,$$

which is the criterion for least-squares coefficients for an expansion in terms of a preassigned set of exponents, s_j, $j = 1, 2, \ldots . N$.

An alternative approach is to assume a rational fraction expansion for $\hat{H}(s)$, that is

$$\hat{H}(s) = \frac{\sum_{i=0}^{\infty} b_i' s^i}{\sum_{i=0}^{\infty} a_i' s^i} \quad ,$$

and to make use of the known properties of transfer functions of physical systems. Then, from Equation (2.1), we have:-

$$h_{jj} = \frac{\sum_{i=0}^{N/2-1} b_i'(-s_j^*)^i}{\sum_{i=0}^{N/2} a_i'(-s_j^*)^i} \quad .$$

$$\therefore \sum_{i=0}^{N/2-1} \left\{ b_i h_{jj}^{-1} - a_i \right\} (-s_j^*)^i = (-s_j^*)^{N/2} \qquad (2.4)$$

$$j = 1, 2, \ldots . N$$

where

$$a_i = \frac{a_i'}{a_{N/2}'} \quad \text{and} \quad b_i = \frac{b_i'}{a_{N/2}'}$$

and N is even.

This set of N equations may be solved for a_i, and b_i, $i = 0, 1, 2, \ldots . N/2 - 1$.

An example is given in Appendix 5.

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2.3 LAGUERRE FUNCTION EXPANSIONS

Let

$$\psi_i(s) = L_i(s) = +(-a-a^*)^{1/2} \frac{(s+a^*)^{i-1}}{(s-a)^i}$$

$$i = 1, 2, \ldots . N$$

then, from Equation (1.1):-

$$h_{\ell k} = h_\delta = (-a-a^*)^{3/2} \sum_{i=1}^{\infty} A_i \int_\Gamma \frac{(-s+a)^{\ell-1}(s+a^*)^{i-1}(s+a^*)^{k-1}}{(-s-a^*)^\ell (s-a)^i (s-a)^k} \frac{ds}{2\pi j}$$

$$h_\delta = -(-a-a^*)^{3/2} \sum_{i=1}^{\infty} A_i \int_\Gamma \frac{(s-a)^{\delta-i-1}}{(s+a^*)^{\delta-i+2}} \frac{ds}{2\pi j}$$

$$= -(-a-a^*)^{3/2} \int_\Gamma \left[\frac{A_\delta}{(s-a^*)^2(s-a)} + \frac{A_{\delta+1}}{(s-a)^2(s+a^*)} \right] \frac{ds}{2\pi j}$$

$$= -(-a-a^*)^{3/2} \left[\frac{A_\delta}{(a+a^*)^2} - \frac{A_{\delta+1}}{(-a^*-a)^2} \right]$$

$$= -\cdot \frac{1}{(-a-a^*)^{1/2}} \left\{ A_\delta - A_{\delta+1} \right\} \tag{3.1}$$

$$A_\delta = (-a-a^*)^{1/2} \sum_{i=0}^{\delta-1} h_i \tag{3.2}$$

If $a = -1/2$

$$A_\delta = \sum_{i=0}^{\delta-1} h_i \tag{3.3}$$

To establish that A_δ, $\delta = 1, 2, \ldots . N$ are least-squares coefficients it is sufficient to show that least-square coefficients satisfy Equation (3.1). Consider the orthonormal expansion

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$$H(s) = \sum_{i=1}^{\infty} A_i' \, L_i(s)$$

$$A_\delta' = \langle \tilde{L}_\delta | H \rangle$$

$$A_\delta' - A_{\delta+1}' = \langle \tilde{L}_\delta - \tilde{L}_{\delta+1} | H \rangle$$

$$= (-a-a^*)^{1/2} \int_\Gamma \left[\frac{(-s+a)^{\delta-1}}{(-s-a^*)^\delta} - \frac{(-s+a)^\delta}{(-s-a^*)^{\delta+1}} \right] H(s) \frac{ds}{2\pi j}$$

$$= -(-a-a^*)^{1/2} \int_\Gamma \left[\frac{(s-a)^{\delta-1}}{(s+a^*)^\delta} - \frac{(s-a)^\delta}{(s+a^*)^{\delta+1}} \right] H(s) \frac{ds}{2\pi j}$$

$$= -(-a-a^*)^{1/2} \int_\Gamma \frac{(s-a)^{\delta-1}(s+a^*-s+a)}{(s+a^*)^{\delta+1}} H(s) \frac{ds}{2\pi j}$$

$$= +(-a-a^*)^{3/2} \int_\Gamma \frac{(s-a)^{\delta-1}}{(s+a^*)^{\delta+1}} H(s) \frac{ds}{2\pi j}$$

$$= -(-a-a^*)^{1/2} \int L_j(-s)^* H(s) L_k(s) \frac{ds}{2\pi j}$$

$$= -(-a-a^*)^{1/2} h_\delta$$

Thus a choice of orthogonal Laguerre Functions as basis also leads to least-squares coefficients.

An example is given in Appendix 6.

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2.4 EXPANSIONS IN TERMS OF ELEMENTARY HOLD FUNCTIONS

Let $\hat{H}(s) = \sum_{i=1}^{\infty} A_i \frac{1}{s}(1-e^{-s})\, e^{-(i-1)s}$

then, from Equation (1.1)

$$h_{\ell k} = h_\delta = \sum_{i=1}^{\infty} A_i \int_\Gamma \frac{1}{-s}(1-e^{s})e^{(\ell-1)s}\frac{1}{s}(1-e^{-s})e^{-(i-1)s}\,\frac{1}{s}(1-e^{-s})^{-(k-1)s}\frac{ds}{2\pi j}$$

$$= \sum_{i=1}^{\infty} A_i \int_\Gamma \frac{1}{s^3}\left\{e^{3s}-3e^{s}+3-e^{-s}\right\} e^{(\delta-i)s}\ \frac{ds}{2\pi j}$$

$(\delta-1) \leq -2$, $h_\delta = 0$ since the integrand above has then no poles in the right half plane

$(\delta-1) \geq 1$, $h_\delta = 0$ since the integrand above has then no poles in the left half plane nor at the origin

$$\delta-1 = 1,\ h_\delta = A_{\delta+1}\int_\Gamma \frac{e^{s}}{s^3}\frac{ds}{2\pi j} = \frac{A_{\delta+1}}{2!}$$

$$\delta-1 = 0,\quad h_\delta = A_\delta \int_\Gamma \frac{1}{s^3}\,(e^{2s}-3e^{s})\,\frac{ds}{2\pi j} = \frac{A_\delta}{2!}$$

$$h_\delta = \frac{1}{2}\left\{A_{\delta+1} + A_\delta\right\}$$

$$A_\delta = 2\sum_{i=0}^{\delta-1}(-1)^{\delta-i+1}\ h_i \qquad . \qquad (4.1)$$

This choice of basis does not lead to least squares coefficients.

An example is given in Appendix 7.

2.5. EXPANSIONS IN TERMS OF ELEMENTARY TRIANGULAR FUNCTIONS

From Equation (5.1) of Part 1:

$$m_{jk}=m_\delta=\int_\Gamma \frac{H(s)}{s^4}\left\{e^{(\delta+2)s}-4e^{(\delta+1)s}+6e^{\delta s}-4e^{(\delta-1)s}+e^{(\delta-2)s}\right\}\frac{ds}{2\pi j}$$

$$\psi_i(s)=\frac{1}{s^2}(e^s-2+e^{-s})\,e^{-is}$$

From Equation (1.1):-

$$m_\delta=\sum_{i=1}^{\infty}A_i\int_\Gamma\frac{1}{s^6}\left[e^{(\delta+3)s}-6e^{(s+2)s}+15e^{(\delta+1)s}-20e^{\delta s}+15e^{(\delta-1)s}-6e^{(\delta-2)}+e^{(\delta-3)}\right]e^{-is}\frac{ds}{2\pi j}$$

$i \geq \delta + 3,\ m_\delta = 0$ — since integrand has then no poles in the right half plane.

$i \leq \delta - 3,\ m_\delta = 0$ — since integrand has then no poles in the left half plane nor at the origin.

$i = \delta + 2,\quad m_\delta = A_{\delta+2}\int_\Gamma \frac{e^s}{s^6}\frac{ds}{2\pi j} = \frac{1}{5!}A_{\delta+2}$

$i = \delta + 1,\quad m_\delta = A_{\delta+1}\int_\Gamma \frac{1}{s^6}(e^{2s}-6e^s)\frac{ds}{2\pi j} = \frac{26}{5!}A_{\delta+1}$

$i = \delta,\quad m_\delta = A_\delta\int_\Gamma \frac{1}{s^6}(e^{2s}-6e^{2s}+15e^s)\frac{ds}{2\pi j} = \frac{66}{5!}A_\delta$

$i = \delta - 1,\quad m_\delta = A_{\delta-1}\int_\Gamma \frac{1}{s^6}(e^{4s}-6e^{3s}+15e^{2s}-20e^s)\frac{ds}{2\pi j} = \frac{26}{5!}A_{\delta-1}$

$i = \delta - 2,\quad m_\delta = A_{\delta-2}\int_\Gamma \frac{1}{s^6}(e^{5s}-6e^{4s}+15e^{3s}-20e^{2s}+15e^s)\frac{ds}{2\pi j} = \frac{1}{5!}A_{\delta-2}$

$$m_\delta = \frac{1}{5!}\left[A_{\delta+2}+26A_{\delta+1}+66A_\delta+26A_{\delta-1}+A_{\delta-2}\right] \qquad (5.1)$$

$$A_{\delta+2} = 5!\ m_\delta - 26A_{\delta+1} - 66A_\delta - 26A_{\delta-1} - A_{\delta-2}$$

Since $m_\delta = 0$, $\delta \leq -2$, successive coefficients are readily obtained:-

$$A_1 = 5!\ m_{-1}$$

$$A_2 = 5!\ m_o - 26A_1$$

$$A_3 = 5!\ m_1 - 26A_2 - 66A_1$$

$$A_4 = 5!\ m_2 - 26A_3 - 66A_2 - 26A_1$$

An example is given in Appendix 8.

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CONCLUSION

Part 1 of this report deals with the matrix representation of a known linear system. This representation gives the response of the system which, in terms of the components of the input signal, has minimum error in the least-squares sense, and input-output calculations are reduced to simple operations with matrices. Basic relationships are established for component functions which are not necessarily orthogonal and a brief discussion of the system matrix for exponential basis with distinct or repeated exponents is included. The remainder of Part 1 deals with the computation of matrix elements for signal classes for which two commonly used time representations are appropriate. In one case, the component functions are not orthogonal and the required Inverse Gram Matrix of arbitrary dimensionality is established in an appendix.

One further advantage of the matrix representation is that it retains only those system characteristics which effectively modify the signal. This means that given a system matrix it is not generally possible to infer the overall properties of the actual system. It is possible, however, to determine a transfer function which has the same matrix representation. If this Equivalent Transfer Function is obtained in a suitable form, instrumentation of the discrete system may be achieved by standard synthesis techniques. Further, when an exponential basis is used, an approximation to the actual system transfer function is obtained with expansion coefficients chosen in the least squares sense.

Part 2 of this report deals with the derivation of Equivalent Transfer Functions which are represented by an infinite set of the same components as describe the signal. This restriction

leads to simple expressions relating matrix elements and expansion coefficients for all signal bases considered in Part 1. Examples are given in appendices. It should be noticed that the computation of these coefficients is accompanied by a rapidly-increasing loss of accuracy. Considered in relation to the accuracy with which the matrix elements are known, this factor sets an upper limit on the dimensionality of the discrete system for which this representation is useful.

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APPENDIX I. ORTHONORMALIZATION AND THE GRAM MATRIX

$$g_{\ell k} = \langle \tilde{\psi}_\ell | \psi_k \rangle \quad \text{by definition} \quad .$$

For an orthonormal basis

$$\langle \tilde{\phi}_\ell | \phi_k \rangle = \delta_{\ell k} \quad .$$

Let

$$|\phi_k\rangle = \sum_{i=1}^{N} c_{ik} |\psi_i\rangle$$

$$\langle \tilde{\phi}_\ell | = \sum_{j=1}^{N} \langle \tilde{\psi}_j | c_{j\ell}^{*}$$

$$\langle \tilde{\phi}_\ell | \phi_k \rangle = \delta_{\ell k} = \sum_{i=1}^{N} \sum_{j=1}^{N} c_{j\ell}^{*} c_{ik} g_{ji}$$

$$\underline{1} = \underline{\tilde{C}}\, \underline{G}\, \underline{C}$$

$$\underline{G} = \underline{\tilde{C}}^{-1} \underline{C}^{-1} = \left[\underline{C}\, \underline{\tilde{C}} \right]^{-1} \quad .$$

This gives the relationship between the Gram Matrix and the matrix of the transformation which provides an orthonormal basis.

APPENDIX II. CHOICE OF INTEGRATION PATH IN COMPUTING INNER PRODUCTS

$$\langle \tilde{f} | g \rangle = \int_{-\infty}^{\infty} f(t)^* \, g(t) \, dt = \int_{-\infty}^{\infty} f(t) \, g(t) \, e^{-st} \, dt \Big|_{s=0}, \text{ for } f(t) \text{ real.}$$

Let

$$F(s) = \int_{-\infty}^{\infty} f(t) \, e^{-st} \, dt \; . \quad \gamma_1 < \sigma < \gamma_2, \; s = \sigma + j\omega$$

$$G(s) = \int_{-\infty}^{\infty} g(t) \, e^{-st} \, dt \; . \quad \eta_1 < \sigma < \eta_2$$

$$\langle \tilde{f} | g \rangle = \int_{-\infty}^{\infty} f(t) \left[\int_{c-j\infty}^{c+j\infty} G(p) e^{pt} \frac{dp}{2\pi j} \right] e^{-st} dt \Big|_{s=0}$$

$$= \int_{c-j\infty}^{c+j\infty} G(p) \left[\int_{-\infty}^{\infty} f(t) \, e^{-(s-p)t} \, dt \right] \frac{dp}{2\pi j} \Big|_{s=0}$$

$$= \int_{c-j\infty}^{c+j\infty} G(p) \, F(s-p) \cdot \frac{dp}{2\pi j} \Big|_{s=o}$$

$$\gamma_1 < \mathrm{Re}\,(s-p) < \gamma_2 \quad .$$

For path:-

$$\eta_1 < c < \eta_2$$

also,

$$\gamma_1 < \mathrm{Re}(-p) < \gamma_2, \text{ since } s = 0$$

that is,

$$-\gamma_2 < c < -\gamma_1 \quad .$$

Combining these requirements gives

$$\mathrm{Max}\,(\eta_1, -\gamma_2) < c < \mathrm{Min}\,(\eta_2, -\gamma_1) \quad .$$

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APPENDIX III. TWO USEFUL INTEGRALS

A.

$$\int_{\Gamma} \frac{H(s)}{s^2} e^{\delta s} \frac{ds}{2\pi j} = \int_0^{\delta} (\delta-\lambda)\, h(\lambda)\, d\lambda \quad ,$$

where h(t) is the Inverse Laplace Transform of H(s)

$$\int_{\Gamma} \frac{H(s)}{s^2} e^{\delta s} \frac{ds}{2\pi j} = (\delta-\lambda)h^{(-1)}(\lambda)\Big|_0^{\delta} + \int_0^{\delta} h^{(-1)}(\lambda)d\lambda$$

$$= -\delta h^{(-1)}(0) + h^{(-2)}(\delta) - h^{(-2)}(0) \quad ;$$

where

$$h^{(-1)}(\lambda) = \int h(\lambda)\, d\lambda$$

B.

$$\int_{\Gamma} \frac{H(s)}{s^4} e^{\delta s} \frac{ds}{2\pi j} = \int_0^{\delta} h(\lambda) \frac{(\delta-\lambda)^3}{3!} d\lambda$$

$$= h^{(-1)}(\lambda) \frac{(\delta-\lambda)^3}{3!}\Big|_0^{\delta} + \int_0^{\delta} h^{(-1)}(\lambda) \frac{(\delta-\lambda)^2}{2!} d\lambda$$

$$= -\frac{\delta^3}{3!} h^{(-1)}(0) + h^{(-2)}(\lambda) \frac{(\delta-\lambda)^2}{2!}\Big|_0^{\delta} + \int_0^{\delta} h^{(-2)}(\lambda)(\delta-\lambda)d\lambda$$

$$= -\frac{\delta^3}{3!} h^{(-1)}(0) - \frac{\delta^2}{2!} h^{(-2)}(0) - \delta h^{(-3)}(0) - h^{(-4)}(0) + h^{(-4)}(\delta)$$

$$= -\sum_{i=0}^{\delta} \frac{\delta^i}{i!} h^{(4-i)}(0) + h^{(-4)}(\delta) \quad .$$

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APPENDIX IV. INVERSION OF THE GRAM MATRIX FOR A TRIANGULAR BASIS

From Section 1.4B

$$\underline{G} = \begin{bmatrix} 2/3 & 1/6 & 0 & 0 & 0 & 0 & \cdots \\ 1/6 & 2/3 & 1/6 & 0 & 0 & 0 & \cdots \\ 0 & 1/6 & 2/3 & 1/6 & 0 & 0 & \cdots \\ 0 & 0 & 1/6 & 2/3 & 1/6 & 0 & \cdots \\ 0 & 0 & 0 & 1/6 & 2/3 & 1/6 & \cdots \\ 0 & 0 & 0 & 0 & 1/6 & 2/3 & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \end{bmatrix}$$

Let $\underline{F} = 6\underline{G} - 6\lambda\,\underline{1}$, then

$$\underline{F} = \begin{bmatrix} c & 1 & 0 & 0 & 0 & \cdots \\ 1 & c & 1 & 0 & 0 & \cdots \\ 0 & 1 & c & 1 & 0 & \cdots \\ 0 & 0 & 1 & c & 1 & \cdots \\ 0 & 0 & 0 & 1 & c & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \end{bmatrix}$$

where $c = 4 - 6\lambda$.

To obtain eigenvalues it is required to solve the equation

$$\det \underline{F} = F_n = 0 \quad ,$$

$n = 2N + 1$ where N is the dimensionality of the basis

$$F_n = c\,F_{n-1} - F_{n-2}$$

$$F_{n+1} = cF_n - F_{n-1} \quad .$$

This may be written:

$$\begin{bmatrix} F_{n+1} \\ F_n \end{bmatrix} = \begin{bmatrix} c & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} F_n \\ F_{n-1} \end{bmatrix}$$

$$= \begin{bmatrix} c & -1 \\ 1 & 0 \end{bmatrix}^n \begin{bmatrix} F_1 \\ F_o \end{bmatrix} .$$

In order to raise this matrix to the nth power the Cauchy Integral Formula will be used. Discussion of the general conditions under which this formula may be applied to matrices is beyond the scope of this memorandum, but verification of the result obtained, is, of course, possible.

$$f(\underline{A}) = \int_R \frac{f(z)}{z\underline{1} - \underline{A}} \frac{dz}{2\pi j}$$

where R encloses each zero of $|z\underline{1} - \underline{A}|$ and no singularities of f(z).

$$\begin{bmatrix} c & -1 \\ 1 & 0 \end{bmatrix}^n = \int_R \begin{bmatrix} z-c & 1 \\ -1 & z \end{bmatrix}^{-1} z^n \frac{dz}{2\pi j}$$

$$= \int_R \frac{z^n}{z(z-c)+1} \begin{bmatrix} z & -1 \\ 1 & z-c \end{bmatrix} \frac{dz}{2\pi j} .$$

Let α, α^{-1} be roots of $z(z-c) + 1 = 0$

$$c = \alpha + \alpha^{-1}$$

$$\begin{bmatrix} c & -1 \\ 1 & 0 \end{bmatrix}^n = \frac{\alpha^n}{\alpha - \alpha^{-1}} \begin{bmatrix} \alpha & -1 \\ 1 & \alpha - c \end{bmatrix} + \frac{\alpha^{-n}}{\alpha^{-1} - \alpha} \begin{bmatrix} \alpha^{-1} & -1 \\ 1 & \alpha^{-1} - c \end{bmatrix}$$

$$= \frac{1}{a-a^{-1}} \begin{bmatrix} a^{n+1} - a^{-n-1} & -a^{n} + a^{-n} \\ a^{n} - a^{-n} & -a^{n-1} + a^{-n+1} \end{bmatrix}$$

$$\begin{bmatrix} F_{n+1} \\ F_{n} \end{bmatrix} = \frac{1}{a-a^{-1}} \begin{bmatrix} a^{n+1} - a^{-n-1} & -a^{n} + a^{-n} \\ a^{n} - a^{-n} & -a^{n-1} + a^{-n+1} \end{bmatrix} \begin{bmatrix} a + a^{-1} \\ 1 \end{bmatrix}$$

since

$$F_1 = c = a + a^{-1}$$

$$F_0 = 1$$

For $F_n = 0$

$$(a^{n} - a^{-n})(a + a^{-1}) - a^{n-1} + a^{-n-1} = 0$$

that is

$$a^{n+1} - a^{-n-1} = 0$$

$$a^{2n+2} = 1 = e^{j2\ell\pi} \quad \ell = 1, 2, \ldots . n$$

$$a = e^{j2\ell\pi/2n+2} = e^{j\ell\pi/n+1}$$

$$c = 4 - 6\lambda_\ell = e^{j\ell\pi/n+1} + e^{-j\ell\pi/n+1}$$

$$= 2 \cos \frac{\ell\pi}{n+1}$$

$$\lambda_\ell = \frac{1}{3} \left[2 - \cos \frac{\ell\pi}{n+1} \right]$$

or, if $k = n + 1 - \ell$

$$\lambda_k = \frac{1}{3} \left[2 + \cos \frac{k\pi}{n+1} \right], \quad k = 1, 2, \ldots . N.$$

Coupled Oscillator theory suggests eigenvectors of the form

$$u_m = \sin \frac{mk\pi}{n+1}$$

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For normalized eigenvectors

$$\sum_{m=1}^{n} b \sin \frac{mk\pi}{n+1} = 1 \quad .$$

It can be shown[6] that this requirement is satisfied if

$$b = \sqrt{\frac{2}{n+1}} \quad ,$$

that is,

$$u_{mk} = \sqrt{\frac{2}{n+1}} \sin \frac{mk\pi}{n+1}$$

$$\underline{G}^{-1} = \underline{U}^{-1} \, \underline{\Lambda}^{-1} \, \underline{U}$$

where

$$\underline{\Lambda} = \begin{bmatrix} \lambda_1 & 0 & 0 & \cdots \\ 0 & \lambda_2 & 0 & \cdots \\ 0 & 0 & \lambda_3 & \cdots \\ \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots \end{bmatrix}$$

$$g_{ij}^{-1} = \sum_{k=1}^{N} \sum_{\ell=1}^{n} u_{ik} \lambda_{k\ell}^{-1} u_{\ell j}$$

$$= \sum_{k=1}^{n} u_{ik} \lambda_{kk}^{-1} u_{kj}$$

$$= \frac{6}{n+1} \sum_{k=1}^{n} \frac{\sin \frac{ik\pi}{n+1} \sin \frac{kj\pi}{n+1}}{2 + \cos \frac{k\pi}{n+1}} \quad .$$

This gives the required Inverse Gram Matrix.

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APPENDIX V. EXPONENTIAL EXPANSIONS. EXAMPLE

$\underline{H}_{\phi}$ is given with respect to an orthogonalized harmonic exponential basis by:-

$$\underline{H}_{\phi} = \begin{bmatrix} 1.166667 & 0. & 0 & 0 \\ -0.942809 & 0.833333 & 0 & 0 \\ 0.144338 & -0.898146 & 0.650000 & 0 \\ -0.066667 & 0.282843 & -0.808290 & 0.533333 \end{bmatrix}$$

(i) From Equation (2.3):-

$$\underline{A}\rangle = \begin{bmatrix} 1/2 & 1/3 & 1/4 & 1/5 \\ 1/3 & 1/4 & 1/5 & 1/6 \\ 1/4 & 1/5 & 1/6 & 1/7 \\ 1/5 & 1/6 & 1/7 & 1/8 \end{bmatrix}^{-1} \begin{bmatrix} 1.166667 \\ 0.833333 \\ 0.650000 \\ 0.533333 \end{bmatrix}$$

$$= \begin{bmatrix} 200 & -1200 & 2100 & -1120 \\ -1200 & 8100 & -15120 & 8400 \\ 2100 & -15120 & 29400 & -16800 \\ -1120 & 8400 & -16800 & 9800 \end{bmatrix} \begin{bmatrix} 1.166667 \\ 0.833333 \\ 0.650000 \\ 0.533333 \end{bmatrix}$$

$$= \begin{bmatrix} 1.0009 \\ 0.9945 \\ 0.0008 \\ -0.0064 \end{bmatrix}$$

$$\hat{H}(s) \sim \frac{1.0009}{s+1} + \frac{1.9945}{s+2} + \frac{0.0008}{s+3} - \frac{0.0064}{s+4}$$

(ii) From Equation (2.4):-

$$0.857145\, b_o - a_o + 0.857145\, b_1 - a_1 = 1$$

$$1.200000\, b_o - a_o + 2.400000\, b_1 - 2a_1 = 4$$

$$1.538461\, b_o - a_o + 4.615383\, b_1 - 3a_1 = 9$$

$$1.875000\, b_o - a_1 + 7.500000\, b_1 - 4a_1 = 16 \quad .$$

Solving, we obtain

$$\hat{H}(s) \sim \frac{3.0000s+3.9959}{s^2+2.9985+1.9980} = \frac{0.9969}{s+0.9995} + \frac{2.0032}{s+1.9990} \quad .$$

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APPENDIX VI. LAGUERRE FUNCTION EXPANSIONS EXAMPLE

$\underline{H}_\phi$ is given with respect to a basis consisting of orthogonal Laguerre Functions with $s_k = -1/2$, $k = 1, 2, 3, 4$, by

$$\underline{H}_\phi \quad \begin{bmatrix} 1.46667 & 0 & 0 & 0 \\ -0.76444 & 1.46667 & 0 & 0 \\ -0.34015 & -0.76444 & 1.46667 & 0 \\ -0.16458 & -0.34015 & -0.76444 & 1.46667 \end{bmatrix} .$$

From Equation (3.3)

$$A_1 = 1.46667$$

$$A_2 = 1.46667 - 0.76444 = 0.70223$$

$$A_3 = 0.70223 - 0.34015 = 0.36208$$

$$A_4 = 0.36208 - 0.16458 = 0.1975$$

$$\hat{H}(s) \sim 1.46667\, L_1(s) + 0.70223\, L_2(s) + 0.36028\, L_3(s) + 0.1975\, L_4(s)$$

where

$$L_i(s) = \frac{(s-1/2)^{i-1}}{(s+1/2)^i} \qquad i = 1, 2, 3, 4 \quad .$$

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APPENDIX VII. EXPANSION IN TERMS OF ELEMENTARY HOLD FUNCTIONS EXAMPLE

$\underline{H}_\phi$ is given with respect to the "B" basis by:-

$$\underline{H}_\phi = \begin{bmatrix} 0.93555 & 0 & 0 & 0 \\ 0.77340 & 0.93555 & 0 & 0 \\ 0.19759 & 0.77340 & 0.93555 & 0 \\ 0.06092 & 0.19759 & 0.77340 & 0.93555 \end{bmatrix}$$

then, from Equation (4.1)

$$A_1 = 2h_o = 1.87109$$

$$A_2 = 2h_2 - A_1 = 0.32429$$

$$A_3 = 2h_2 - A_2 = 0.71947$$

$$A_4 = 2h_3 - A_3 = -0.59762$$

$$\hat{H}(s) \sim 1.87109\, B_1(s) - 0.32429\, B_2(s) + 0.71947\, B_3(s) - 0.59762\, B_4(s).$$

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APPENDIX VIII. EXPANSIONS IN TERMS OF ELEMENTARY TRIANGULAR FUNCTIONS. EXAMPLE

A system matrix $\underset{\phi}{\underline{H}}$ is given with respect to a basis consisting of elementary triangular functions by

$$\underset{\phi}{\underline{H}} = \begin{bmatrix} 0.053333 & -0.076667 \\ 1.136667 & 0.356667 \end{bmatrix} .$$

Obtain $\hat{H}(s)$ in the form $\sum_{i=1}^{3} A_i \psi_i(s)$ and show that the matrix corresponding to $\hat{H}(s)$ with respect to the given basis is $\underset{\phi}{\underline{H}}$

$$\underset{\phi}{\underline{M}} = \underset{\phi}{\underline{G}}\underset{\phi}{\underline{H}} = \begin{bmatrix} 2/3 & 1/6 \\ 1/6 & 2/3 \end{bmatrix} \begin{bmatrix} 0.053333 & -0.076667 \\ 1.136667 & 0.356667 \end{bmatrix}$$

$$= \begin{bmatrix} 2.225000 & 0.008333 \\ 0.766667 & 0.225000 \end{bmatrix} .$$

From Equation (5.1)

$$A_1 = 120\,(0.008333) = 0.999960$$

$$A_2 = 120\,(0.225000) - 26(0.999960) = 1.001040$$

$$A_3 = 120\,(0.766667) - 26(1.001040) - 66(0.999960)$$

$$= 0.024360$$

$$H(s) \sim 0.9996\ \psi_1(s) + 1.001040\ \psi_2(s) - 0.024360\ \psi_3(s)$$

From Section 1.5 of Part 1:-

$$h^{(-4)}(0) = 0$$
$$h^{(-4)}(1) = 1/5!\ (0.996)$$
$$h^{(-4)}(2) = 1/5!\ (30.99984)$$
$$h^{(-4)}(3) = 1/5!\ (210.0018)$$

$$m_{-1} = h^{(-4)}(1) = 0.008333$$
$$m_o = h^{(-4)}(2) - 4h^{(-4)}(1) = 0.225000$$
$$m_1 = h^{(-4)}(3) - 4h^{(-4)}(2) + 6h^{(-4)}(1) = 0.765018$$

$$\underline{\hat{H}}_{\phi} = \begin{bmatrix} 2/3 & 1/6 \\ 1/6 & 2/3 \end{bmatrix}^{-1} \begin{bmatrix} 0.225000 & 0.008333 \\ 0.765018 & 0.225000 \end{bmatrix}$$

$$= \begin{bmatrix} 1.6 & -0.4 \\ -0.4 & 1.6 \end{bmatrix} \begin{bmatrix} 0.225000 & 0.008333 \\ 0.765018 & 0.225000 \end{bmatrix}$$

$$\begin{bmatrix} 0.05393 & 0.76667 \\ 1.13403 & 0.35667 \end{bmatrix}$$

which may be compared with $\underline{H}_{\phi}$.