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THE STEADY-STATE DIFFRACTION OF ELECTROMAGNETIC RADIATION BY AN OBSTACLE IN AN INHOMOGENEOUS ANISOTROPIC CONDUCTING MEDIUM

Calvin H. Wilcox

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ABSTRACT

This paper deals with the diffraction of time-harmonic electromagnetic radiation by perfectly conducting obstacles immersed in an inhomogeneous, anisotropic, conducting medium. A mathematical formulation of the problem is presented which is applicable to obstacles of arbitrary shape and to a very general class of media, and the existence, uniqueness and continuous dependence on the data of the solution is demonstrated.

THE STEADY-STATE DIFFRACTION OF ELECTROMAGNETIC RADIATION BY AN OBSTACLE IN AN INHOMOGENEOUS ANISOTROPIC CONDUCTING MEDIUM

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§1. Introduction. This paper deals with the diffraction of time-harmonic electromagnetic radiation by perfectly conducting obstacles immersed in an inhomogeneous, anisotropic, conducting medium. A mathematical formulation of the problem is presented which is applicable to obstacles of arbitrary shape and to a very general class of media, and the existence, uniqueness and continuous dependence on the data of the solution is demonstrated.

Electromagnetic fields are represented by pairs of vector fields, $E(x, t)$ (the electric field) and $H(x, t)$ (the magnetic field), which are described here by their (real-valued) components $E_j(x, t)$, $H_j(x, t)$ $(j = 1, 2, 3)$ relative to a fixed rectangular coordinate system. The symbol $x = (x_1, x_2, x_3)$ denotes a point in Euclidean space R^3 and t is a time coordinate. Time-harmonic electromagnetic fields have the form

$$(1.1) \qquad E_j(x, t) = \mathrm{Re}\,\{e^{-i\omega t} E_j(x)\}, \quad H_j(x, t) = \mathrm{Re}\,\{e^{-i\omega t} H_j(x)\}$$

where ω is a real frequency and

$$E_j(x) = E_j^1(x) + iE_j^2(x), \quad H_j(x) = H_j^1(x) + iH_j^2(x)$$

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are the complex-valued components of vector fields $E(x)$, $H(x)$ which are independent of t. Maxwell's equations for time-harmonic fields in an inhomogeneous, anisotropic, conducting medium filling a domain $\Omega \subset R^3$ have the form

$$(1.2) \qquad \left.\begin{aligned} (\nabla \times H)_j - (i\omega\epsilon_{jk} + \sigma_{jk}) E_k &= J_j \\ (\nabla \times E)_j + i\omega\mu_{jk} H_k &= K_j \end{aligned}\right\} \quad \text{in } \Omega \quad .$$

Here the summation convention is used (repeated indices are summed from 1 to 3), $(\nabla \times H)_1 = \partial H_3/\partial x_2 - \partial H_2/\partial x_3$, etc. The functions $\epsilon_{jk} = \epsilon_{jk}(x)$, $\mu_{jk} = \mu_{jk}(x)$ and $\sigma_{jk} = \sigma_{jk}(x)$ are real-valued and represent the components of the dielectric, magnetic permeability, and electric conductivity tensors, respectively. J_j and K_j are complex-valued and represent the electric and magnetic current densities.

The quadratic form

$$\frac{1}{2}(\epsilon_{jk}(x)\, E_j(x,t)\, E_k(x,t) + \mu_{jk}(x)\, H_j(x,t)\, H_k(x,t))$$

defines the energy density for (real-valued) solutions of Maxwell's equations in an inhomogeneous, anisotropic medium. Hence, the tensors ϵ_{jk} and μ_{jk} are assumed to be positive definite. The conductivity tensor σ_{jk} must also be positive definite if the medium is to be dissipative (energy is absorbed, rather than created, in it).

A perfect conductor (of electricity) is characterized by the property that the tangential component of the electric field vanishes on its surface. Thus, if a perfectly conducting obstacle $O \subset R^3$ is immersed in a medium occupying $\Omega = R^3 - O$ then

$$N \times E = 0 \quad \text{on} \quad \partial\Omega \ , \tag{1.3}$$

where N is a normal vector on $\partial\Omega$.

The diffraction problem considered here (the steady-state diffraction problem) asks for the time-harmonic fields (1.1) generated by prescribed time-harmonic electric and magnetic current densities

$$J(x,t) = \mathrm{Re}\,\{e^{-i\omega t}\, J(x)\}, \ K(x,t) = \mathrm{Re}\,\{e^{-i\omega t}\, K(x)\}$$

acting in the presence of a prescribed obstacle O . Maxwell's equations (1.2) and the boundary condition (1.3) are necessary conditions on the solution. However, they do not, in general, determine the solution uniquely. Indeed, if the medium is homogeneous and isotropic and $\partial\Omega$ has sharp edges it is known that (1.3) must be supplemented by an "edge condition" to obtain uniqueness [1]. Moreover, if Ω is unbounded a "condition at infinity" is needed to obtain uniqueness [5, 7]. A complete formulation of the steady-state diffraction problem is given below in §2 . The discussion in the remainder of this section is intended to motivate the final formulation.

The time-average of the energy density for a time-harmonic electromagnetic field is

$$\mathrm{Re}\,\{\epsilon_{jk} E_j \bar{E}_k + \mu_{jk} H_j \bar{H}_k\} \quad ,$$

where $\bar{E}_k = E_k^1 - i E_k^2$ is the complex conjugate of E_k, etc. Most edge conditions have been based on the physical principle that the energy in bounded portion of space should finite; i.e.,

$$(1.4) \qquad \mathrm{Re} \int_{\Omega \cap C} (\epsilon_{jk} E_j \bar{E}_k + \mu_{jk} H_j \bar{H}_k)\, dx < \infty \qquad (dx = dx_1\, dx_2\, dx_3)$$

for each bounded set $C \subset R^3$ [3, 4] . This condition eliminates the possibility that point-or line-sources of energy might reside in a sharp edge. (These would clearly lead to non-uniqueness unless their distribution and strengths were specified.) For homogeneous, isotropic media, (1.4) has been used to derive restrictions on the singularities in E and H that can occur at an edge. The latter were then used to prove the uniqueness of the solution [3]. In this paper (1.4) is used directly in the formulation of the problem and in the existence and uniqueness theorems.

The Silver-Müller radiation condition has been used to obtain uniqueness in the steady-state diffraction problem for bounded obstacles in a homogeneous, isotropic medium [5, 6, 8] . It is a condition which guarantees that the solution behaves like an "outgoing wave" at large distances from the obstacle. The form of the condition depends strongly on the form of Maxwell's equations for homogeneous, isotropic media and no such conditions are known for inhomogeneous or anisotropic media. However, the author has shown that for homogeneous,

isotropic dissipative media (i.e., media with a positive electrical conductivity) the Silver-Müller radiation condition implies that the field components tend to zero exponentially at infinity [8, p. 120]. In particular, it follows that

$$\text{(1.5)} \qquad \operatorname*{Re} \int_{|x| \geq R} (\epsilon_{jk} E_j \bar{E}_k + \mu_{jk} H_j \bar{H}_k)\, dx < \infty$$

for R sufficiently large. This is meaningful for inhomogeneous, anisotropic media. Moreover, it is plausible that in dissipative media there is an energy balance between the energy introduced by the source fields J and K and the energy dissipated in the medium, so that the time-average energy is finite.

Conditions (1.4) and (1.5) are used as "edge condition" and "condition at infinity" below. They can be combined conveniently into the single condition that the total (time-average) energy in the medium is finite:

$$\text{(1.6)} \qquad \operatorname{Re} \int_{\Omega} (\epsilon_{jk} E_j \bar{E}_k + \mu_{jk} H_j \bar{H}_k)\, dx < \infty \quad .$$

Notice that

$$\text{(1.7)} \qquad E_j \bar{E}_k = (E_j^1 + i E_j^2)(E_k^1 - i E_k^2) = (E_j^1 E_k^1 + E_j^2 E_k^2) + i(E_j^2 E_k^1 - E_j^1 E_k^2)$$

so that (1.6) can also be written

$$\text{(1.8)} \qquad \int_{\Omega} \{\epsilon_{jk}(E_j^1 E_k^1 + E_j^2 E_k^2) + \mu_{jk}(H_j^1 H_k^1 + H_j^2 H_k^2)\}\, dx < \infty \quad .$$

§2. Formulation of the Diffraction Problem. The tensors ϵ_{jk}, μ_{jk} and σ_{jk} are assumed to have the following properties.

(2.1) $\epsilon_{jk}(x)$, $\mu_{jk}(x)$ and $\sigma_{jk}(x)$ are bounded, Lebesgue-measurable functions of $x \in \Omega$, and

(2.2) $\epsilon_{jk}(x)$, $\mu_{jk}(x)$ and $\sigma_{jk}(x)$ are uniformly positive definite in Ω; i.e., there exist positive constants ϵ_m, μ_m and σ_m such that

$$\epsilon_{jk}(x)\xi_j \xi_k \geq \epsilon_m |\xi|^2, \quad \mu_{jk}(x)\xi_j \xi_k \geq \mu_m |\xi|^2, \quad \sigma_{jk}(x)\xi_j \xi_k \geq \sigma_m(\epsilon_{jk}(x)\xi_j \xi_k)$$

for all $x \in \Omega$ and all real ξ_j $(|\xi|^2 = \xi_1^2 + \xi_2^2 + \xi_3^2)$.

Conditions (2.1) and (2.2) imply that there exist finite constants ϵ_M, μ_M and σ_M such

$$(2.3) \quad \epsilon_{jk}(x)\xi_j \xi_k \leq \epsilon_M |\xi|^2, \mu_{jk}(x)\xi_j \xi_k \leq \mu_M |\xi|^2, \sigma_{jk}(x)\xi_j \xi_k \leq \sigma_M(\epsilon_{jk}(x)\xi_j \xi_k)$$

for all $x \in \Omega$ and all real ξ_j . The parts of conditions (2.1) and (2.2) applying to ϵ_{jk} and μ_{jk} imply that the conditions (2.2) and (2.3) on σ_{jk} are equivalent to the conditions

$$\sigma'_m |\xi|^2 \leq \sigma_{jk}(x)\xi_j \xi_k \leq \sigma'_M |\xi|^2 \quad .$$

However, the first form proves to be more convenient.

The tensors ϵ_{jk}, μ_{jk} and σ_{jk} are not assumed to be symmetric. However, (2.1), (2.2) and (2.3) imply that their antisymmetric parts are bounded relative

to their symmetric parts; i.e., there exist finite constants a such that

$$(2.4)\qquad \begin{cases} |(\epsilon_{jk}(x) - \epsilon_{kj}(x))\,\xi_j\,\eta_k| \le a\,\epsilon_{jk}(x)(\xi_j\,\xi_k + \eta_j\,\eta_k) \\ |(\mu_{jk}(x) - \mu_{kj}(x))\,\xi_j\,\eta_k| \le a\,\mu_{jk}(x)(\xi_j\,\xi_k + \eta_j\,\eta_k) \\ |(\sigma_{jk}(x) - \sigma_{kj}(x))\,\xi_j\,\eta_k| \le a\,\sigma_{jk}(x)(\xi_j\,\xi_k + \eta_j\,\eta_k) \end{cases}$$

for all $x \in \Omega$ and all real ξ_j and η_j. All the results given below require, for their proofs, that (2.4) should hold with sufficiently small values of a, namely

$$(2.5)\qquad a < \frac{1}{2}\frac{\sigma_m}{\sigma_M}\frac{\sigma_m}{\sqrt{\sigma_m^2 + 4\omega^2} + 2|\omega|} .$$

The formulation of the diffraction problem given below makes use of several classes of vector fields on Ω. To define them let

$$L_2(\Omega) = \{A : A(x) \text{ is Lebesgue-measurable on } \Omega,\ \int_\Omega |A(x)|^2\,dx < \infty\}$$

denote the Lebesgue class of square-integrable, complex-valued vector fields on Ω. Here $A(x) = A^1(x) + iA^2(x)$ has complex-valued components $A_j(x) = A_j^1(x) + iA_j^2(x)$ and

$$|A(x)|^2 = |A^1(x) + iA^2(x)|^2 = (A^1(x) + iA^2(x))\cdot(A^1(x) - iA^2(x)) = |A^1(x)|^2 + |A^2(x)|^2 .$$

Notice that, because of (2.1), (2.2) and (2.3), a (Lebesgue-measurable) electromagnetic field has finite energy if and only if $E \in L_2(\Omega)$ and $H \in L_2(\Omega)$. The class of vector fields $A \in L_2(\Omega)$ for which $\nabla \times A \in L_2(\Omega)$ is needed in the formulation of the diffraction problem given below. Its definition is motivated by the identity

$$(2.6) \qquad \int_\Omega A \cdot \nabla \times \Phi \, dx - \int_\Omega \Phi \cdot \nabla \times A \, dx = \int_{\partial\Omega} N \times A \cdot \Phi \, dS \ ,$$

which is valid if A and Φ are continuously differentiable and $\partial\Omega$ is sufficiently smooth.

<u>Definition</u>. Let $A \in L_2(\Omega)$. Then $\nabla \times A$ exists and equals $B \in L_2(\Omega) \Longleftrightarrow$

$$\int_\Omega A \cdot \nabla \times \Phi \, dx = \int_\Omega B \cdot \Phi \, dx \quad \text{for all} \quad \Phi \in C_0^\infty(\Omega) \ .$$

Here $C_0^\infty(\Omega)$ denotes the class of vector fields on Ω which have continuous derivatives of all orders and vanish outside a compact subset of Ω. $\nabla \times A$ is unique, if it exists, because $C_0^\infty(\Omega)$ is dense in $L_2(\Omega)$. The notations

$$L_2(\nabla x; \Omega) = \{A\colon A \text{ and } \nabla \times A \text{ are in } L_2(\Omega)\}$$

and

$$L_2^0(\nabla x; \Omega) = L_2(\nabla x; \Omega) \cap \{A\colon \int_\Omega A \cdot \nabla \times B \, dx = \int_\Omega B \cdot \nabla \times A \, dx \text{ for all } B \in L_2(\nabla x; \Omega)\}$$

are used below. Notice that "$A \in L_2^0(\nabla x; \Omega)$" generalizes the boundary condition "$N \times A = 0$ on $\partial\Omega$". Indeed, if A and $\nabla \times A$ are continuous in the closure of Ω and $\partial\Omega$ is smooth then $A \in L_2^0(\nabla x; \Omega)$ implies

$$\int_{\partial\Omega} N \times A \cdot \Phi \, dS = 0$$

for all Φ which are continuously differentiable in the closure of Ω, and it follows that $N \times A = 0$ on $\partial\Omega$.

A formulation of the diffraction problem which is applicable to arbitrary domains Ω, and media satisfying (2.1) and (2.2), is contained in the

<u>Definition</u>. Fields E and H define a <u>strict solution</u> of the steady-state diffraction problem for the domain Ω and source fields $J \in L_2(\Omega)$ and $K \in L_2(\Omega) \Longleftrightarrow E \in L_2^0(\nabla x; \Omega)$, $H \in L_2(\nabla x; \Omega)$ and Maxwell's equations (1.2) hold almost everywhere in Ω.

Notice that the generalized boundary condition, together with the combined "edge condition" and "condition at infinity" (1.6), are contained in the definitions of $L_2^0(\nabla x; \Omega)$ and $L_2(\nabla x; \Omega)$.

§3. <u>The Energy Inequality.</u> The fields defined by

$$J_j \equiv (\nabla \times H)_j - (i\omega\epsilon_{jk} + \sigma_{jk}) E_k \quad ,$$

(3.1)

$$K_j \equiv (\nabla \times E)_j + i\omega\mu_{jk} H_k \quad ,$$

are in $L_2(\Omega)$ for every $E \in L_2^0(\nabla x; \Omega)$ and $H \in L_2(\nabla x; \Omega)$, by (2.1). Thus every pair $E \in L_2^0(\nabla x; \Omega)$, $H \in L_2(\nabla x; \Omega)$ defines a strict solution of the diffraction problem, and the correspondence $E, H \to J, K$ defines a linear operator on $L_2(\Omega) \times L_2(\Omega)$, with domain $L_2^0(\nabla x; \Omega) \times L_2(\nabla x; \Omega)$. The main theorem in this paper is an "a priori" estimate which implies that this operator is bounded. It will be called

<u>Theorem 1 (The Energy Inequality).</u> Let $\omega(\neq 0)$ be a real number and let ϵ_{jk}, μ_{jk} and σ_{jk} satisfies (2.1), (2.2), (2.4) and (2.5) . Then there exists a constant C, depending on ω and the bounds for ϵ_{jk}, μ_{jk} and σ_{jk} only, such that

$$\text{(3.2)} \qquad \mathrm{Re} \int_\Omega (\epsilon_{jk} E_j \bar{E}_k + \mu_{jk} H_j \bar{H}_k)\, dx \leq C \int_\Omega (|J|^2 + |K|^2)\, dx$$

for all $E \in L_2^0(\nabla x; \Omega)$, $H \in L_2(\nabla x; \Omega)$, with J and K defined by (3.1) . Indeed, (3.2) holds with

$$\text{(3.3)} \qquad C = 4\, \mathrm{Max}(1/\epsilon_m \;\; 1/\mu_m) \frac{\sigma_m^2 + 4\omega^2}{\omega^2(\sigma_m^2 - 8\omega a \sigma_M - 4a^2 \sigma_M^2)} \quad .$$

It is shown below that this number is positive when (2.5) holds. Theorem 1 and (2.2) imply

Corollary 1. Under the same hypotheses,

$$(3.4) \qquad \int_\Omega (|E|^2 + |H|^2)\,dx \le C_1 \int_\Omega (|J|^2 + |K|^2)\,dx$$

where $C_1 = C\,\mathrm{Max}(1/\epsilon_m, 1/\mu_m)$ and C is given by (3.3) .

The proof of Theorem 1 is based on two lemmas concerning the bilinear form

$$(3.5) \qquad I(E,H) = \int_\Omega (E_j \bar{J}_j - \bar{H}_j K_j)\,dx \quad ,$$

which may be stated as follows.

Lemma 1. Under the hypotheses of Theorem 1, there exists a positive constant m, depending on ω and the bounds for ϵ_{jk}, μ_{jk} and σ_{jk} only, such that

$$(3.6) \qquad m\,\mathrm{Re} \int_\Omega (\epsilon_{jk} E_j \bar{E}_k + \mu_{jk} H_j \bar{H}_k)\,dx \le |I(E,H)|$$

for all $E \in L_2^0(\nabla x;\Omega)$, $H \in L_2(\nabla x;\Omega)$. Indeed, (3.6) holds with

$$(3.7) \qquad m^2 = \omega^2 \frac{\sigma_m^2 - 8|\omega|\,a\sigma_M - 4a^2\sigma_M^2}{\sigma_m^2 + 4\omega^2} \quad .$$

Lemma 2. Under the same hypotheses

$$(3.8) \qquad |I(E,H)| \le 2\,\mathrm{Max}(1/\sqrt{\epsilon_m}, 1/\sqrt{\mu_m})\left(\mathrm{Re}\int_\Omega \epsilon_{jk} E_j \bar{E}k + \mu_{jk} H_j \bar{H}_k\,dx\right)^{\frac{1}{2}}\left(\int_\Omega |J|^2 + |K|^2 dx\right)^{\frac{1}{2}}.$$

Lemmas 1 and 2 imply Theorem 1 with $C = 4\,\mathrm{Max}(1/\epsilon_m, 1/\mu_m)/m^2$ which, with (3.7), gives (3.3) .

<u>Proof of Lemma 1</u>. Substituting (3.1) in (3.5) gives

$$I(E,H) = \int_\Omega \{E_j(\nabla\times\bar{H})_j - \bar{H}_j(\nabla\times E)_j + i\omega\epsilon_{jk} E_j \bar{E}_k - \sigma_{jk} E_j \bar{E}_k - i\omega\mu_{jk} \bar{H}_j H_k\}\,dx \ .$$

The sum of the first two integrals vanishes because $E \in L_2^0(\nabla x;\Omega)$, $H \in L_2(\nabla x;\Omega)$. Thus

$$I(E,H) = \int_\Omega \{(-\sigma_{jk} + i\omega\epsilon_{jk})\, E_j\, \bar{E}_k - i\omega\mu_{jk}\, \bar{H}_j\, H_k\}\,dx \ .$$

If

$$E^+_{jk} = E^1_j E^1_k + E^2_j E^2_k, \quad E^-_{jk} = E^2_j E^1_k - E^1_j E^2_k \ ,$$

$$H^+_{jk} = H^1_j H^1_k + H^2_j H^2_k \ , \quad H^-_{jk} = H^2_j H^1_k - H^1_j H^2_k \ ,$$

then (see (1.7))

$$E_j \bar{E}_k = E^+_{jk} + i\, E^-_{jk}, \quad H_j \bar{H}_k = H^+_{jk} + i\, H^-_{jk} \ , \tag{3.9}$$

and therefore

$$I(E,H) = -\int_\Omega (\sigma_{jk} E^+_{jk} + \omega\epsilon_{jk} E^-_{jk} + \omega\mu_{jk} H^-_{jk})\,dx - i\int_\Omega (\sigma_{jk} E^-_{jk} - \omega\epsilon_{jk} E^+_{jk} + \omega\mu_{jk} H^+_{jk})\,dx \ .$$

Thus

$$|I(E,H)|^2 = (\int_\Omega \sigma_{jk} E^+_{jk}\,dx)^2 + 2\omega \int_\Omega \sigma_{jk} E^+_{jk}\,dx(\int_\Omega \epsilon_{jk} E^-_{jk} + \mu_{jk} H^-_{jk}\,dx) + \underline{\omega^2(\int_\Omega \epsilon_{jk} E^-_{jk} + \mu_{jk} H^-_{jk}\,dx)^2}$$

$$\underline{+(\int_\Omega \sigma_{jk} E^-_{jk}\,dx)^2} - 2\omega \int_\Omega \sigma_{jk} E^-_{jk}\,dx(\int_\Omega \epsilon_{jk} E^+_{jk} - \mu_{jk} H^-_{jk}\,dx) + \omega^2(\int_\Omega \epsilon_{jk} E^+_{jk} - \mu_{jk} H^+_{jk}\,dx)^2 .$$

Dropping the underlined terms gives the inequality

$$|I(E,H)|^2 \geq (\int_\Omega \sigma_{jk} E^+_{jk}\,dx)^2 + 2\omega \int_\Omega \epsilon_{jk} E^-_{jk}\,dx \int_\Omega \sigma_{jk} E^+_{jk}\,dx + 2\omega \int_\Omega \mu_{jk} H^-_{jk}\,dx \int_\Omega \sigma_{jk} E^+_{jk}\,dx$$

(3.10)

$$-2\omega \int_\Omega \sigma_{jk} E^-_{jk}\,dx \int_\Omega \epsilon_{jk} E^+_{jk}\,dx + 2\omega \int_\Omega \sigma_{jk} E^-_{jk}\,dx \int_\Omega \mu_{jk} H^+_{jk}\,dx + \omega^2(\int_\Omega \epsilon_{jk} E^+_{jk} - \mu_{jk} H^+_{jk}\,dx)^2 .$$

Now, by (2.4)

$$|\epsilon_{jk}(E^2_j E^1_k - E^1_j E^2_k)| = |(\epsilon_{jk} - \epsilon_{kj}) E^2_j E^1_k| \leq a\, \epsilon_{jk}(E^1_j E^1_k + E^2_j E^2_k) \quad ;$$

i.e.,

$$|\epsilon_{jk} E^-_{jk}| \leq a\, \epsilon_{jk} E^+_{jk}, \quad \text{or} \quad -a\, \epsilon_{jk} E^+_{jk} \leq \epsilon_{jk} E^-_{jk} \leq a\, \epsilon_{jk} E^+_{jk} \quad .$$

When multiplied by ω and integrated over Ω this gives

$$-a|\omega| \int_\Omega \epsilon_{jk} E^+_{jk}\,dx \leq \omega \int_\Omega \epsilon_{jk} E^-_{jk}\,dx \leq a|\omega| \int_\Omega \epsilon_{jk} E^+_{jk}\,dx \quad .$$

Moreover, (2.4) implies exactly similar results for μ_{jk} and σ_{jk}. Combining these with (3.10) gives

$$|I(E,H)|^2 \geq (\int_\Omega \sigma_{jk} E^+_{jk}\,dx)^2 - 4|\omega| a \int_\Omega \sigma_{jk} E^+_{jk}\,dx (\int_\Omega \epsilon_{jk} E^+_{jk} + \mu_{jk} H^+_{jk}\,dx) + \omega^2 (\int_\Omega \epsilon_{jk} E^+_{jk} - \mu_{jk} H^+_{jk}\,dx)^2 .$$

Eliminating σ_{jk} by means of (2.2) and (2.3) then gives

$$(3.11)\quad |I(E,H)|^2 \geq \sigma_m^2 (\int_\Omega \epsilon_{jk} E^+_{jk}\,dx)^2 - 4|\omega| a \sigma_M \int_\Omega \epsilon_{jk} E^+_{jk}\,dx (\int_\Omega \epsilon_{jk} E^+_{jk} + \mu_{jk} H^+_{jk}\,dx)$$
$$+ \omega^2 (\int_\Omega \epsilon_{jk} E^+_{jk} - \mu_{jk} H^+_{jk}\,dx)^2 .$$

Notice that, by (3.9),

$$\mathrm{Re} \int_\Omega (\epsilon_{jk} E_j \bar{E}_k + \mu_{jk} H_j \bar{H}_k)\,dx = \int_\Omega (\epsilon_{jk} E^+_{jk} + \mu_{jk} H^+_{jk})\,dx .$$

Hence (3.6) is equivalent to

$$|I(E,H)|^2 \geq m^2 (\int_\Omega \epsilon_{jk} E^+_{jk} + \mu_{jk} H^+_{jk}\,dx)^2 .$$

Thus if

$$\alpha = \int_\Omega \epsilon_{jk} E^+_{jk}\,dx, \quad \beta = \int_\Omega \mu_{jk} H^+_{jk}\,dx ,$$

then (3.6) follows from (3.11) and the inequality

$$(3.12)\qquad \sigma_m^2 \alpha^2 - 4|\omega| a \sigma_M \alpha(\alpha+\beta) + \omega^2 (\alpha-\beta)^2 \geq m^2 (\alpha+\beta)^2 .$$

The proof of Lemma 1 will be completed by showing that (3.12) holds for <u>all</u> real α and β when m is given by (3.7) . Indeed, (3.12) is equivalent to

$$(\sigma_m^2 - 2|\omega|\delta + \omega^2 - m^2)\alpha^2 - 2(|\omega|\delta + \omega^2 + m^2)\alpha\beta + (\omega^2 - m^2)\beta^2 \geq 0, \quad \delta = 2a\sigma_M \quad .$$

This is true for all α and β if and only if

$$\sigma_m^2 - 2|\omega|\delta + \omega^2 - m^2 \geq 0 \tag{3.13}$$

and the discriminant

$$(\sigma_m^2 - 2|\omega|\delta + \omega^2 - m^2)(\omega^2 - m^2) - (|\omega|\delta + \omega^2 + m^2)^2 \geq 0 \quad .$$

The last inequality is equivalent to

$$(\sigma_m^2 - 4|\omega|\delta - \delta^2)\omega^2 - (\sigma_m^2 + 4\omega^2)m^2 \geq 0 \quad . \tag{3.14}$$

This has <u>positive</u> solutions m^2 provided

$$f(\delta) \equiv \sigma_m^2 - 4|\omega|\delta - \delta^2 = \sigma_m^2 + 4\omega^2 - (\delta + 2|\omega|)^2 > 0 \quad ;$$

i.e., provided δ lies between 0 and the larger root of $f(\delta) = 0$:

$$0 < \delta = 2a\sigma_M < -2|\omega| + \sqrt{\sigma_m^2 + 4\omega^2} = \frac{\sigma_m^2}{\sqrt{\sigma_m^2 + 4\omega^2} + 2|\omega|} \quad :$$

This inequality is equivalent to (2.5) . Assuming it holds, (3.14) has solutions

$$0 < m^2 \leq \omega^2 \frac{\sigma_m^2 - 4|\omega|\delta - \delta^2}{\sigma_m^2 + 4\omega^2}, \qquad \delta = 2a\sigma_M \quad .$$

Conversely, (3.14) implies (3.12) provided (3.13) holds. Now, if m^2 has its largest allowable value (3.7) ,

$$\sigma_m^2 - 2|\omega|\delta + \omega^2 - m^2 = \frac{(\sigma_m^2 + 2\omega^2 - |\omega|\delta)^2}{\sigma_m^2 + 4\omega^2} > 0 \quad ,$$

because

$$\delta < \frac{\sigma_m^2}{\sqrt{\sigma_m^2 + 4\omega^2} + 2|\omega|} < \frac{\sigma_m^2 + 2\omega^2}{|\omega|} \quad .$$

This completes the proof of Lemma 1 .

<u>Proof of Lemma 2.</u> Notice that, by (2.2) ,

$$|E|^2 = |E^1|^2 + |E^2|^2 \leq \frac{1}{\epsilon_m}\epsilon_{jk}(E_j^1 E_k^1 + E_j^2 E_k^2) = \frac{1}{\epsilon_m}\epsilon_{jk} E_{jk}^+ \quad .$$

Thus

$$|I(E,H)| = |\int_\Omega (E_j \bar{J}_j - \bar{H}_j K_j)\,dx| \leq \int_\Omega |E_j \bar{J}_j|\,dx + \int_\Omega |\bar{H}_j K_j|\,dx$$

$$\leq \int_\Omega |E||J|\,dx + \int_\Omega |H||K|\,dx \leq (\int_\Omega |E|^2 dx)^{\frac{1}{2}} (\int_\Omega |J|^2 dx)^{\frac{1}{2}} + (\int_\Omega |H|^2 dx)^{\frac{1}{2}} (\int_\Omega |K|^2 dx)^{\frac{1}{2}}$$

$$\leq (\frac{1}{\epsilon_m}\int_\Omega \epsilon_{jk} E_{jk}^+ dx)^{\frac{1}{2}} (\int_\Omega |J|^2 dx)^{\frac{1}{2}} + (\frac{1}{\mu_m}\int_\Omega \mu_{jk} H_{jk}^+ dx)^{\frac{1}{2}} (\int_\Omega |K|^2 dx)^{\frac{1}{2}}$$

$$\leq \frac{1}{\sqrt{\epsilon_m}}(\int_\Omega \epsilon_{jk} E_{jk}^+ + \mu_{jk} H_{jk}^+ dx)^{\frac{1}{2}} (\int_\Omega |J|^2 + |K|^2 dx)^{\frac{1}{2}} + \frac{1}{\sqrt{\mu_m}}(\int_\Omega \epsilon_{jk} E_{jk}^+ + \mu_{jk} H_{jk}^+ dx)^{\frac{1}{2}} (\int_\Omega |J|^2 + |K|^2 dx)^{\frac{1}{2}}$$

$$\leq 2\,\mathrm{Max}(1/\sqrt{\epsilon_m}\,,\, 1/\sqrt{\mu_m})(\int_\Omega \epsilon_{jk} E_{jk}^+ + \mu_{jk} H_{jk}^+ dx)^{\frac{1}{2}} (\int_\Omega |J|^2 + |K|^2\, dx)^{\frac{1}{2}} \quad ,$$

This is equivalent to (3.8), and proves Lemma 2.

§4. The Fundamental Theorems. Let E, H and E', H' be strict solutions of the diffraction problem corresponding to source fields J, K and J', K', respectively. Then the differences $E - E'$, $H - H'$ define a strict solution with source fields $J - J'$, $K - K'$, because Maxwell's equations (1.2) are linear. Hence, Corollary 1 implies

Corollary 2. Under the hypotheses of Theorem 1,

$$\|E - E'\|^2 + \|H - H'\|^2 \leq C_1(\|J - J'\|^2 + \|K - K'\|^2) , \tag{4.1}$$

where $\| \cdots \|$ denotes the $L_2(\Omega)$ norm.

Corollary 2 asserts the continuous dependence of strict solutions on their "data", the source fields J, K in $L_2(\Omega)$. An immediate consequence is

Corollary 3 (The Uniqueness Theorem). The diffraction problem has at most one strict solution corresponding to data J and K in $L_2(\Omega)$.

Indeed, if E, H and E', H' are strict solutions with the same data J, K then (4.1) with $J' = J$, $K' = K$ implies $E' = E$, $H' = H$. Corollary 2 also plays a key role in the proof of

Theorem 2 (The Existence Theorem). Let Ω be an arbitrary domain in R^3, let $\omega\,(\neq 0)$ be a real number, and let ϵ_{jk}, μ_{jk} and σ_{jk} satisfy (2.1), (2.2), (2.4) and (2.5). Then the corresponding steady-state diffraction problem has a (unique) strict solution for every pair of source fields J and K in $L_2(\Omega)$.

The proof makes use of the following (apparently) weaker notion of solution.

<u>Definition</u>. Fields E, H define a <u>solution with finite energy</u> of the steady-state diffraction problem with source fields J and K in $L_2(\Omega) \Longleftrightarrow$ E and H are in $L_2(\Omega)$ (i.e., the electromagnetic field has finite energy) and

$$(4.2)\quad \begin{cases} \int_\Omega \{H_j(\nabla\times\Phi)_j - i\omega\epsilon_{kj} E_j \Phi_k - \sigma_{kj} E_j \Phi_k - J_j \Phi_j\}\, dx = 0 \ , \\ \int_\Omega \{E_j(\nabla\times\Psi)_j + i\omega\mu_{kj} H_j \Psi_k - K_j \Psi_j\}\, dx = 0 \ , \end{cases}$$

for every $\Phi \in L_2^0(\nabla x;\Omega)$ and $\Psi \in L_2(\nabla x;\Omega)$.

The usefulness of this notion stems from

<u>Lemma 3</u>. Fields E, H define a solution with finite energy $\Longleftrightarrow$ E, H define a strict solution.

<u>Proof</u>. (The implication "$\Longrightarrow$") Note that $C_0^\infty(\Omega) \subset L_2^0(\nabla x;\Omega) \subset L_2(\nabla x;\Omega)$. Hence, identities (4.2) imply

$$\int_\Omega H\cdot\nabla\times\Phi\, dx = \int_\Omega (i\omega\epsilon_{jk} E_k + \sigma_{jk} E_k + J_j)\Phi_j\, dx \ ,$$

and

$$\int_\Omega E\cdot\nabla\times\Psi\, dx = \int_\Omega (-i\omega\mu_{jk} H_k + K_j)\Psi_j\, dx$$

for all Φ and Ψ in $C_0^\infty(\Omega)$, where the fields $i\omega\epsilon_{jk} E_k + \sigma_{jk} E_k + J_j$ and $-i\omega\mu_{jk} H_k + K_j$ are in $L_2(\Omega)$. This implies (i) $\nabla\times H$ and $\nabla\times E$. exist in $L_2(\Omega)$ and (ii) Maxwell's equations (1.2) hold almost everywhere in Ω . In particular

E and H are in $L_2(\nabla x; \Omega)$. Finally, the second of the identities (4.2), with $K_j = (\nabla \times E)_j + i\omega\mu_{jk} H_k$, gives

$$\int_\Omega \{E_j(\nabla \times \Psi)_j - (\nabla \times E)_j \Psi_j\} dx = 0 \quad \text{for all} \quad \Psi \in L_2(\nabla x; \Omega) \ ;$$

i.e., $E \in L_2^0(\nabla x; \Omega)$. Thus E, H defines a strict solution.

(The implication "$\Longleftarrow$") Multiply the first of Maxwell's equations (1.2) by $\Phi \in L_2^0(\nabla x; \Omega)$ and integrate over Ω . Then the term

$$\int_\Omega (\nabla \times H)_j \Phi_j dx = \int_\Omega H_j(\nabla \times \Phi)_j dx$$

because $\Phi \in L_2^0(\nabla x; \Omega)$ and $H \in L_2(\nabla x; \Omega)$. This gives the first of identities (4.2) and the second follows by a similar argument.

Maxwell's equations (1.2) determine a linear operator Λ on the Hilbert space $L_2(\Omega) \times L_2(\Omega)$, with domain

$$D(\Lambda) = L_2^0(\nabla x; \Omega) \times L_2(\nabla x; \Omega) \ ,$$

defined by $\Lambda(A, B) = (J, K)$ where

$$(4.3) \qquad \left\{ \begin{array}{l} J_j = (\nabla \times B)_j - (i\omega\epsilon_{jk} + \sigma_{jk}) A_k \\ \\ K_j = (\nabla \times A)_j + i\omega\mu_{jk} B_k \end{array} \right\} \quad (A, B) \in D(\Lambda) \ .$$

Corollary 1 implies that Λ is one-to-one on $D(\Lambda)$, and that Λ^{-1} is bounded: $\|A, B\| \leq C_1 \|\Lambda(A, B)\|$, where $\|A, B\|^2 = \|A\|^2 + \|B\|^2$ is the norm in $L_2(\Omega) \times L_2(\Omega)$.

Theorem 2 is equivalent to the assertion that $R(\Lambda)$, the range of Λ, is equal to $L_2(\Omega) \times L_2(\Omega)$. This will be established by showing that (i) $R(\Lambda)$ is dense in $L_2(\Omega) \times L_2(\Omega)$ and (ii) $R(\Lambda)$ is closed in $L_2(\Omega) \times L_2(\Omega)$. The first assertion is called

<u>Theorem 3.</u> Under the hypotheses of Theorem 2,

$$\overline{R(\Lambda)} = L_2(\Omega) \times L_2(\Omega) \quad ,$$

where the bar denotes closure in $L_2(\Omega) \times L_2(\Omega)$.

<u>Proof.</u> $\overline{R(\Lambda)}$ is a closed linear subspace and

$$L_2(\Omega) \times L_2(\Omega) = \overline{R(\Lambda)} \oplus N \quad ,$$

by a standard theorem on Hilbert space [2, p. 25] . Hence, it is sufficient to show that if $E, H \in R(\Lambda)$ is orthogonal to $R(\Lambda)$ in $L_2(\Omega) \times L_2(\Omega)$ then $E = H = 0$. Now $E, H \perp R(\Lambda)$ means

$$\int_\Omega (E_j \bar{J}_j + H_j \bar{K}_j)\, dx = 0 \quad \text{for all } A \in L_2^0(\nabla x; \Omega),\ B \in L_2(\nabla x; \Omega) \ .$$

Combine this identity with Maxwell's equations (4.3), and take first

$$A = \bar{\Phi} \quad \text{with} \quad \Phi \in L_2^0(\nabla x; \Omega),\ B = 0 \quad ,$$

and second

$$A = 0, \quad B = \bar{\Psi} \quad \text{with} \quad \Psi \in L_2(\nabla x; \Omega) \ .$$

This gives the pair of identities

$$\int_\Omega \{E_j(i\omega\epsilon_{jk} - \sigma_{jk})\,\Phi_k + H_j(\nabla \times \Phi)_j\}\,dx = 0 \ ,$$

$$\int_\Omega \{E_j(\nabla \times \Psi)_j - i\omega\mu_{jk} H_j\,\Psi_k\}\,dx = 0 \ ,$$

for all $\Phi \in L_2^0(\nabla x; \Omega)$, $\Psi \in L_2(\nabla x; \Omega)$. These identities state that E, H is a solution with finite energy of a modified diffraction problem (the adjoint problem), with $-\omega$ for ω, $\epsilon_{kj}, \mu_{kj}, \sigma_{kj}$ for $\epsilon_{jk}, \mu_{jk}, \sigma_{jk}$, and source fields $J = K = 0$. But, the conditions for the validity of Theorem 1 clearly imply the same conditions for the modified problem. Thus, E, H is a strict solution of the modified problem (by Lemma 3) with source fields $J = K = 0$ and therefore $E = H = 0$ by Theorem 1. This proves Theorem 3.

<u>Proof of Theorem 2</u>. Let $J \in L_2(\Omega)$, $K \in L_2(\Omega)$. Then, by Theorem 3, there exist sequences of fields $E^n \in L_2^0(\nabla x; \Omega)$, $H^n \in L_2(\nabla x; \Omega)$ whose source fields J^n, K^n converge to J, K in $L_2(\Omega) \times L_2(\Omega)$. Applying Corollary 2 with $E = E^n$, $H = H^n$, $E' = E^m$, $H' = H^m$ gives

$$\|E^n - E^m\|^2 + \|H^n - H^m\|^2 \leq C_1(\|J^n - J^m\|^2 + \|K^n - K^m\|^2) \ .$$

It follows that $\{E^n\}$ and $\{H^n\}$ define Cauchy sequences in $L_2(\Omega)$. Hence, limit fields

$$E = \lim_{n\to\infty} E^n, \quad H = \lim_{n\to\infty} H^n \quad (\text{in } L_2(\Omega))$$

exist because $L_2(\Omega)$ is complete. Now, E^n and H^n define strict solutions with source fields J^n, K^n. Hence, by Lemma 3, E^n and H^n define solutions with finite energy; i.e., identities (4.2) hold with E^n for E, H^n for H, J^n for J and K^n for K. Making $n \to \infty$ in these identities gives the same identities for the limit fields, since all the fields converge in $L_2(\Omega)$. Thus E, H is a solution with finite energy, and therefore a strict solution (by Lemma 3) having the prescribed source fields J, K which proves Theorem 2.

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