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ON COMPACTIFICATION OF METRIC SPACES

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Technical (Final) Report
Contract No. 62558-3315
February 1963
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Technical (Final) Report
Contract No. 62558–3313
February 1963

The research reported in this document has been sponsored by the U.S. Navy through Office of Naval Research
Let \( f: X \to X^* \) be a homeomorphism of a metric separable space \( X \) into a compact metric space \( X^* \), such that \( f(X) = X^* \). The pair \((f, X^*)\) is then called a metric compactification of \( X \). If \( X \) is an absolute \( G_{\delta} \)-space \((F_\sigma\text{-space})\) (i.e. a \( G_{\delta} \) set \((F_\sigma\text{-set})\) in some compact space), then \( X \) is said to be of the first kind (cf. [6]) if there exists a compactification \((f, X^*)\) of \( X \) such that \( X = \bigcap_{i=1}^{\infty} G_i \), where \( G_i \) are sets open in \( X^* \) and \( \dim(\text{Fr}(G_i)) < \dim X \), \( i = 1, 2, \ldots \) \((\text{Fr}(G_i)\) - being the boundary of \( G_i \) and \( \dim X \) - the dimension of \( X \)). An absolute \( G_{\delta} \)-space, \((F_\sigma\text{-space})\) which is not of the first kind is said to be of the second kind. In the present study spaces \( X \) which are both absolute \( F_\sigma \) and absolute \( G_{\delta} \)-spaces of the second kind are constructed for any positive finite dimension, a problem related to one of A. Lelek in [11] is solved and a sufficient condition on \( X \) is given, under which \( \dim(X^* - f(X)) \geq 1 \) for any compactification \((f, X^*)\) of \( X \). It is noted also, that an analogous condition assures \( \dim(X^* - f(X)) \geq n \).
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I N T R O D U C T I O N

Let $f: X \to X^*$ be a homeomorphism of a metric separable space $X$ into a compact metric space $X^*$ such that $f(X) = X^*$. The pair $(f, X^*)$ is then called a metric compactification of the metric space $X$, it is known\(^1\) that for each metric separable space $X$ there exists a homeomorphism $f: X \to J^N$ of $X$ into the Hilbert cube $J^N$. Thus denoting $X^* = f(X)$ (the closure of $f(X)$ in $J^N$) we obtain a compactification $(f, X^*)$ of $X$. It can be shown\(^3\) that there always exists a compactification $(f, X^*)$ such that $\dim X^* \leq \dim X$ where $\dim X$ denotes the dimension of $X$ in the sense of Menger-Urysohn\(^8\). What can be said about the dimension $\dim (X^* - f(X))$ of the set $X^* - f(X)$ is considered in the present study. This question is closely related to some results obtained by B. Knaster in [6] and A. Lelek in [11]\(^4\).

I. SOME COMPACTIFICATIONS OF METRIC SPACES

I.1. Let $X$ be a given topological space. Let $X^* = X \cup \{x^*\}$, where $x^* \not\in X$ is an additional point, and let us define the topology in $X^*$ by taking as open sets all sets open in $X$ and all subsets $U$ of $X^*$, such that $X^* - U$ is a closed compact subset of $X$. Then, the theorem of Alexndroff states:

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1) S. [8], p. 119, Theorem 1.
2) S. [4], p. 65, Theorem V, 6. Also [9], p. 72.
3) S. [4], p. 10 and 24. Also [8], p. 162.
4) I learned recently that some problems considered in the present study have been solved by Lelek in an entirely different way. (not published).
(1) The space $X^*$ is a compact topological space and $X^*$ is a Hausdorff space if and only if $X$ is a locally compact Hausdorff space.

The space $X^*$ is called the one-point compactification of the space $X$.

A topological embedding is usually allowed rather than insist that $X$ actually be a subset of $X^*$.

Thus by a compactification of a space $X$ a pair $(f, X^*)$ is understood, such that $f : X \to X^*$ is a homeomorphism of $X$ into a compact space $X^*$ and $\overline{f(X)} = X^*$ (i.e., the image $f(X)$ of $X$ is dense in $X^*$).

In this sense the one-point compactification of a non-compact space $X$ is a pair $(i, X^*)$ where $i : X \to X^*$ is the identity mapping and $\overline{i(X)} = X^* = X \cup \{x^*\}$.

Another compactification of a topological space $X$ is the Stone-Čech compactification $(\varepsilon, \beta(X))$.

This compactification is defined as follows:

Let us take the set $F(X)$ of all continuous functions $f : X \to J$ mapping $X$ into the interval $J = [0, 1]$ and the product $J^{F(X)}$ with the Tychonoff topology. Let us define the mapping $\varepsilon : X \to J^{F(X)}$ by correlating each point $x \in X$ the point $\varepsilon(x)$ whose $f$-th coordinate is $f(x)$, for each $f \in F(X)$.

The mapping $\varepsilon(x)$ is a continuous mapping of $X$ into $J^{F(X)}$, and in the case when $X$ is a completely regular $T_1$-space it turns out to be a homeomorphism. In this case we define $\beta(X)$ by $\beta(X) = \overline{\varepsilon(X)}$ and the pair $(\varepsilon, \beta(X))$ is called the Stone-Čech compactification of $X$.

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5) S. [5], p. 150, also [3], p. 73.
6) S. [5], p. 152. For properties of the Stone-Čech compactification, see also [2] and [13].
7) S. [5], p. 153.
Let us note that:

(2) If \((e, \beta(X))\) is the Stone-\(\check{C}\)ech compactification of a completely regular \(T_1\)-space \(X\) and \(f: X \rightarrow Y\) is a continuous mapping of \(X\) into a compact Hausdorff space \(Y\), then \(f[e^{-1}(x)]\) has a continuous extension on \(\beta(X)\) into \(Y\).\(^7\)

Numerous other compactifications are constructed for various purposes. One of the, used in the dimension theory, is the Wallman compactification \((\Phi, w(X))\). It turns out to be topologically equivalent to the Stone-Cech compactification, if \(w(X)\) is a Hausdorff space.\(^8\)

1.2. Considering the one-point compactification \((i, X^*)\) of a metric space, we note that the space \(X^*\) is generally not a metric space. For instance, if \(X\) is a metric space which is not locally compact, then by (1) \(X^*\) cannot be a metric space (since every metric space is a Hausdorff space). Thus if we seek for a given metric space \(X\), a compactification \((i, X^*)\), where \(X^*\) is also a metric space, we generally cannot achieve this, by merely adding a single point and should provide for the set \(X^* - i(X)\) to contain more than one point.

In the present study we confine ourselves to metric compactifications \((i, X^*)\) of metric separable spaces \(X\) only. This means the assumption that \(X\) is a separable metric space and \(X^*\) a metric space. As already noted, the one-point compactification is generally not a metric compactification. Let us show that an analogous statement holds for the Stone-\(\check{C}\)ech compactification \((e, \beta(X))\). This will be

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7) S. [5], p. 153.
8) Ibidem, p. 168. For properties of the Wallman compactification, [15].
shown by the following

**Theorem 1.** If $X$ is a non compact metric space and $(e, \beta(X))$ the Stone-Čech compactification of $X$, then $\beta(X)$ is not a metric space.

**Proof.** Suppose, to the contrary, that $\beta(X)$ is a metric space. Let $e(X)$ be the image of $X$ in $\beta(X)$.

Since $X$ is not compact, there exists a sequence $A = \{a_n\}_{n=1}^{\infty}$ of points $a_n \in X$ which does not contain any convergent subsequence. Consider the points $e(a_n) = b_n$. Since $\beta(X)$ is compact and metric, the sequence $\{b_n\}_{n=1}^{\infty}$ contains a convergent subsequence $\{b'_n\} \subseteq \{b_n\}$. Let $b'_n = b \in \beta(X)$ and consider the points $a'_n = e^{-1}(b'_n)$. By $A' = \{a'_n\} \subseteq A$ the sequence $A'$ does not contain any convergent subsequence. Therefore $A'$ is a closed subset of $X$. Let us define the real function $f: A' \to [0,1]$ by $f(a'_n) = \begin{cases} 0 & \text{for } n = 2k \\ 1 & \text{for } n = 2k-1 \end{cases}$

Since $A'$ does not contain any convergent subsequence, the function $f: A' \to [0,1]$ is continuous; and since $A'$ is a closed subset of the metric space $X$, we can, using Tietze’s extension theorem, extend this function, to a continuous function $f: X \to J$ (the extended function is denoted also by $f$).

By (2), the function $fe^{-1}$ has then a continuous extension $\bar{f}$ on the whole of $\beta(X)$. But since $\bar{f}(b'_n) = fe^{-1}(b'_n) = f(a'_n) = \begin{cases} 0 & \text{for } n = 2k \\ 1 & \text{for } n = 2k-1 \end{cases}$ and $b'_n \to b$ the function $\bar{f}$ cannot be continuous at the point $b$. This contradiction shows that $\beta(X)$ is not a metric space.

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9) S. [8], p. 117.
Remark 1. Since, as noted at the end of Section 1.1, the Wallman compactification \((\Phi, w(X))\) is in case of Hausdorff space \(w(X)\) topologically equivalent to that of Stone–Čech it follows by Theorem 1 that if \(X\) is a non-compact metric space, then the space \(w(X)\) is not a metric space.

II. PROBLEMS ON COMPACTIFICATIONS

II.1. The results of Section 1 indicate that metric compactifications of metric spaces are generally neither the Stone–Čech nor the one-point compactification. Now, since for metric compactifications the set \(X^* = f(X)\) generally contains more than one point, there arises a problem of finding the structure of this set for some classes of metric spaces \(X\). For example the following questions can be put:

(a) Is it always possible to find a compactification \((f, X^*)\) of \(X\) such that \(X^* - f(X)\) would be countable?
(b) Is it always possible to find a compactification \((f, X^*)\) such that \(\dim [X^* - f(X)] < \dim X\)?

Regarding question (a), it is known that each space which does not contain a subset dense in itself, has a compactification \((f, X^*)\) such that \(X^* - f(X)\) is countable. On the other hand, it is easily seen that for each compactification of the set \(X\) of rational numbers the set \(X^* - f(X)\) is uncountable.

Indeed, since \(f : X \rightarrow X^*\) is a homeomorphism, each point of \(f(X)\) is a limit point and therefore \(X^*\) is perfect. Hence \(X^*\) is uncountable.

10) S. [7], p. 194, IV.
11) S. [3], p. 98.
Regarding (b), it is known, that for each space $X$, there exists a compactification $(f, X^*)$ such that $\dim X^* = \dim X$ and thus $\dim [X^* - f(X)] \leq \dim X$. Easy examples show that in many cases this weak inequality $\leq$ can be replaced the strong $\prec$. It suffices, for example to take any $n$-dimensional cube $J^n; n=1,2,...$ and any point $p \in J^n$. The set $X - J^n - (p)$ can be compactified by adding this single point. We then have $X^* - J^n$ and $\dim [X^* - f(X)] = \dim (p) = 0 < \dim X$, where $f = i$ is the identity mapping. On the other hand, it is not always possible to achieve the strong inequality $\dim (X^* - f(X)) \prec \dim X$. Indeed, for a 0-dimensional space $X$, $\dim (X^* - f(X)) \prec \dim X = 0$ means that $X^* - f(X)$ is empty and hence $X$ is compact. It follows that for a 0-dimensional non compact space $X$ this strong inequality is impossible. The problem of finding examples of $n$-dimensional spaces $X, n > 0$ of a simple topological structure for which $\dim [X^* - f(X)] \prec \dim X$ does not hold for any compactification $(f, X^*)$ of $X$ is more complicated. More precisely, this problem may be formulated as follows:

(c) Let $X$ be a given $n$-dimensional space and $k \leq n$ an integer. Under what conditions on $X$ shall we have $\dim [X^* - f(X)] \geq k$ for each compactification $(f, X^*)$ of $X$?

II.2. B. Knaster discovered in [6] that there exist two kinds of absolute $G_\delta$-spaces (also called $G_{\delta\varepsilon}$-spaces in compact spaces or topologically complete spaces). Their definition is:

An absolute $G_\delta$-space is said to be of the first kind, if there exists a compactification $(f, X^*)$ such that $f(X) = \bigcap_{i=1}^m G_i$ and $\dim \{f(x) | x \in G_i\} < \dim X$, where $G_i, i = 1,2,...$ are sets open in $X^*$ and
Fr(G_i) denotes the boundary of G_i in X^o. An absolute Gδ-space is said to be of the second kind if it is not of the first kind.

It was shown by Lelek \(^{12}\) that

(3) An absolute Gδ-space of finite dimension is of the first kind, if and only if there exists a compactification (f, X^o) of X such that \(\dim (X^o - f(X)) < \dim X\).

Now, it was shown in [6] that the Cartesian product N x J, where N is the set of irrational numbers in the interval J = [0,1], is an absolute Gδ-space of the second kind. It was further proved in [11], that if Z is any compact space with \(\dim Z = n > 0\), then the space \(X = N \times Z\) is an absolute Gδ-space of the second kind. These results provide a solution of problem (c) for \(n-k\) in the class of finite dimensional absolute Gδ-spaces. The sequel will i.a. include a solution of the following problems:

(a₁) Does there exist, for any positive finite dimension \(n = 1,2,\ldots\), a finite dimensional space X, which is both an absolute Fσ and Gδ-space of the second kind?

(a₂) Is it true that each absolute Gδ-space X of the second kind, having a positive finite dimension, n, contains a topological image of a set of the form N x Z, where N is the set of irrational numbers of the interval J = [0,1] and \(\dim Z = \dim X\)?

(a₃) Problem (c), for the case \(k = 1\)

12) S. [11], p. 31, Theorem 1.
and finally

(a₄) Construction of a weakly infinite dimensional absolute F and G₁-space of the first kind, such

that for each compactification (f, X*) there is dim (X* - f(X)) = ∞.  

Before proceeding with a solution of problems (a₁) - (a₄), we quote in the next section some facts

on coverings.

III. COVERINGS

By covering of a space Y, a family G = {G₁} of sets G₁ is understood such that Y = U G₁. If

G₁ are open (closed) sets the covering is called open (closed). If the diameters δ(G₁) of all G₁ are

< ε, G is called an ε-covering and if G is finite - a finite covering.

dₜ(Y) denotes the infimum of all numbers ε > 0 such that there exists a finite open ε-covering

of Y satisfying

(4) G₁ ∩ G₂ ∩ ... ∩ Gₙ = 0, for any set of n + 1 indices i₁ < i₂ < ... < iₙ (i.e., such that

the intersection of any n + 1 different sets Gᵢ is empty).

It is known that for finite coverings of a space Y the existence of an open ε-covering satis-

fying (4) is equivalent to that of a closed ε-covering satisfying (4), and that for a compact space Y,

13 A space is called weakly infinite-dimensional if it is a union of a sequence of finite dimensional

spaces Xₖ, with dim Xₖ → ∞, for k → ∞.
\[ \dim Y \leq n \text{ if and only if } d_{n+1}(Y) = 0. \]

Let us now prove a property of the Lebesgue number \( \lambda \) of a finite covering.

Let \( F_0, F_1, \ldots, F_m \) be a finite family of closed subsets of a compact space \( Z \).

Then there exists a number \( \lambda > 0 \) (the Lebesgue number of the family \( \{ F_0, F_1, \ldots, F_m \} \)) such that if a point \( p \in Z \) is at distance \( \leq \lambda \) from all the sets \( F_{k_0}, F_{k_1}, \ldots, F_{k_n} \), these sets have a non-empty intersection.

**Proof.** Suppose the contrary. Then there exists a sequence of points \( p_0, p_1, \ldots, p_n \in Z \),

\( n = 0, 1, 2, \ldots \) and families \( S_0 = (F_{k_0}, F_{k_1}, \ldots, F_{k_n}), \ldots, S_j = (F_{k_0}, \ldots, F_{k_j}), \ldots \) of sets such that the point \( p_j \) is at distance \( \leq \frac{1}{j+1} \) from all the sets \( F_{k_i} \) of the family \( S_j \), but \( \bigcup_{0}^{n} F_{k_i} = \emptyset \).

Since the number of different families \( S_j, j = 0, 1, \ldots \) constructed from a given finite family of sets \( \{ F_{k_i} \}_{k=0}^{m} \) is finite, some family — say \( S_a \) — must appear in the sequence \( \{ S_j \}_{j=0}^{\infty} \) an infinite number of times. Thus there exists a subsequence \( \{ p_{n}^a \} \subset \{ p_{n} \} \) such that \( p_{n}^a \) is at distance \( \leq \frac{1}{n+1} \) from all the sets \( F_{k_0}, \ldots, F_{k_n} \) of \( S_a \). Since \( Z \) is compact, the sequence \( \{ p_{n}^a \} \) contains a convergent subsequence to some point \( p \in Z \). Denoting this subsequence by \( \{ p_{n}^a \} \), we have \( p_{n}^a \to p \in Z \).

Now, by \( \rho(p_{n}^a, F_{k_i}) \leq \frac{1}{n+1} \) for \( i = 0, 1, \ldots, n_a \) and every \( n = 0, 1, \ldots \) and by \( p_{n}^a \to p \) we have \( \rho(p, F_{k_i}) = 0 \). Since \( F_{k_i} \) are closed sets, it follows that \( p \notin F_{k_i}^c \), \( i = 0, 1, \ldots, n_a \) which is incom-
compatible with the fact that \( \bigwedge_{k=1}^{n} F_k = 0 \) (by the definition of \( S_j \)).

It follows by (5) that

(6) If \( Y \) is a closed subset of a compact space \( Z \) and \( Y \subseteq \bigcup_{k=1}^{n} F_k \), where \( F_k \) are closed sets such that any different \( n+1 \) of them have an empty intersection; then, replacing each \( F_k \) by its \( \epsilon \)-neighborhood \(^{16}\) \( G_k = S(F_k, \epsilon) \) (in \( Z \)) with \( 2 \epsilon < \lambda \) we get an open (in \( Z \)) covering \( G = \{ G_k \} \) of the set \( Y \), such that for the family \( \{ G_k \} \) of closures of \( G_k \), any \( n+1 \) different sets \( G_k \) have also an empty intersection \(^{17}\).

Another consequence of (5) is;

(7) If the closed sets \( F_1, F_2, \ldots, F_m \) in a compact space \( Z \) have an empty intersection:

\[ \bigwedge_{k=1}^{m} F_k = 0, \]

then, there exists a number \( \epsilon > 0 \) such that no set of diameter \( \leq \epsilon \) has a non-empty intersection with each of the sets \( F_1, F_2, \ldots, F_m \).

Indeed, it suffices to take \( \epsilon = \frac{\lambda}{2} \) and to apply (5).

We shall now give some properties of coverings of simplexes.

Let \( \sigma^s = (p_0, \ldots, p_s) \) be a closed \( s \)-dimensional simplex with vertices \( p_0, p_1, \ldots, p_s \) in the Euclidean \( s \)-dimensional space \( E^s \) and let \( f: \sigma^s \to Z \) be a homeomorphism of \( \sigma^s \) into a space \( Z \).

Let \( \sigma^{s-1,1} \) denote the \((s-1)\) dimensional closed face of \( \sigma^s \) opposite to the vertex \( p_1 \in \sigma^s \), i.e.

\(^{16}\) An \( \epsilon \)-neighborhood of a set \( F \) is by definition the union over all \( p \in F \) of the sets \( S_p = \{ x; \rho(p, x) < \epsilon; \ x \in Z \} \).

\(^{17}\) For a proof of (6) see also [14], p. 414, Lemma 2 and [10], p. 257.
Then \( r^n \) is a curvilinear simplex with vertices \( q_i = f(p_i) \) and \( (s-1) \)-dimensional faces \( r^{s-1,1}, \)
i = 0,1, \ldots, s. Since \( f \) is a homeomorphism and \( \bigcap_{i=0}^{s} \sigma^{s-1,1} = 0 \), we have that \( \bigcap_{i=0}^{s} r^{s-1,1} = 0 \). Thus applying (7) with \( m = s \) to the closed sets \( F_i = r^{s-1,1} \), we find that there exists a number \( \epsilon > 0 \) such that no set with diameter \( \leq \epsilon \) intersects each of the faces \( r^{s-1,1} \).

Let now \( \epsilon > 0 \) be this number and let us show that

(8) Let \( \epsilon > 0 \) be a number such that no set with diameter \( \leq \epsilon \) intersects each face \( r^{s-1,1} \). Let further \( r^n = \bigcup_{k=0}^{s} F_k \), where \( F_k \) are closed sets with diameters \( \delta(F_k) \leq \epsilon \), \( k = 0,1, \ldots, m \). Then some \( s+1 \) sets \( F_{k_0}, \ldots, F_{k_s} \) have a non empty intersection.

Since \( \delta(F_k) \leq \epsilon \), no \( F_k \) containing a vertex \( q_i \) of \( r^n \) intersects the face \( r^{s-1,1} \) opposite to \( q_i \). Since \( f \) is one-to-one, no set \( f^{-1}(F_k) \) containing a vertex \( p_j \) of \( \sigma^n \) intersects the face \( \sigma^{s-1,1} \) opposite to \( p_j \). Now, the sets \( f^{-1}(F_k), k = 0,1, \ldots, m \) cover the simplex \( \sigma^n \) and are closed, since \( f \) is continuous. Thus applying the same procedure as in the proof of [2,24] in [1], p. 194 we obtain that some \( s+1 \) sets \( f^{-1}(F_k), j = 0,1, \ldots, s \) have a non empty intersection. Hence also the sets \( F_{k_j}, j = 0,1, \ldots, s \) have a non empty intersection.
IV. THE SOLUTION OF PROBLEMS FORMULATED IN II

IV. 1. An n-dimensional absolute $F_\sigma$ and $G_\delta$-space $X$ and its properties.

Let $\sigma^n = (p_0, p_1, \ldots, p_n)$ be the n-dimensional closed simplex in the n-dimensional Euclidean space $\mathbb{E}^n$ with vertices $p_0 = (0,0,\ldots,0)$ and $p_i = (0,\ldots,0,1,0,\ldots,0)$, $i = 1, 2, \ldots, n$ (i.e., $p_i$ is the point in $\mathbb{E}^n$ whose $i$-th coordinate is 1 and all other coordinates are 0). Let $A = \{a_j\}$ $j = 1, 2, \ldots$ be the sequence of points of the form $a_j = \frac{1}{j}$, $j = 1, 2, \ldots$ on the real axes $\mathbb{E}^1$ and let $a_0 = 0 \in \mathbb{E}^1$.

Denote by $\text{Fr}(\sigma^n) = \bigcup_{i=0}^{n} \sigma^{n-1,i}$ the boundary of the simplex $\sigma^n$.

Define

(9) $X = (A \times \sigma^n) \cup [(a_0) \times \text{Fr}(\sigma^n)]$

We have $X \subset \mathbb{E}^{n+1}$ and the closure $\overline{X}$ of $X$ in $\mathbb{E}^{n+1}$ is $\overline{X} = (A \times \sigma^n) \cup [(a_0) \times \sigma^n] = [A \cup (a_0)] \times \sigma^n$.

Since $\overline{X}$ is a compact subset of $\mathbb{E}^{n+1}$ (as a product of two compact spaces $A \cup (a_0)$ and $\sigma^n$), $\overline{X}$ is a compact space, and since $X$ can be written as a union $[(a_0) \times \text{Fr}(\sigma^n)] \cup \bigcup_{i=1}^{\infty} [(a_j) \times \sigma^n]$ of a countable number of compact sets, it follows that $X$ is an absolute $F_\sigma$ space.

On the other hand the set $\overline{X} - X$ equals the interior of the simplex $(a_0) \times \sigma^n$. Since this interior is a union of compact sets, the set $\overline{X} - X$ is an $F_\sigma$ set and therefore $X$ is a $G_\delta$-set in $\overline{X}$. It follows that

(b) The set $X$ defined in (9) is both an absolute $F_\sigma$ and $G_\delta$-space. Evidently, $\dim X = n$. 

We shall now show that

\( b^*_i \) For each compactification \((f, X^*)\) of \( X \) there is \( \dim [X^* - f(X)] \geq \dim X = n \).

Indeed, suppose to the contrary that \( \dim [X^* - f(X)] \leq n - 1 < \dim X \) and take the sets

\[ r_j^n = f([a_j] \times \sigma^n), \quad r_j^{n-1,i} = f([a_j] \times \sigma^{n-1,i}) \quad i = 0, 1, \ldots, n, \quad j = 0, 1, \ldots \]

By \( a_j \to a_0 \) for \( j \to \infty \), we have that for every \( i = 0, 1, \ldots, n \), dist \( \{([a_j] \times \sigma^{n-1,i}), ([a_0] \times \sigma^{n-1,i})\} \to 0 \) if \( j \to \infty \), where

\[ \text{dist}(A, B) = \max \{\sup_{x \in A} d(x, B), \sup_{x \in B} d(A, x)\} \] is the distance of the sets \( A \) and \( B \) in the sense of Hausdorff \(^{18}\). Since \( f: X \to X^* \) is a homeomorphism, and \([A \cup \{a_0\}] \times \text{Fr}(\sigma^n)\) is compact it follows that

\[ \text{dist}(r_j^{n-1,i}, r_0^{n-1,i}) \to 0 \quad \text{for} \quad j \to \infty \quad \text{and each} \quad i = 0, 1, \ldots, n. \]

Now the space \( X^* \) being compact, there exists a subsequence \( \{j\} \) of \( \{j\} \) such that the sequence of sets \( \{r_j^n\} \) converges to a continuum \( C \subseteq X^* \) \(^{19}\). Writing \( j \) instead of \( j' \), we have

\[ \text{dist}(r_j^n, C) \to 0 \quad \text{for} \quad j \to \infty. \]

Since \( f \) is one-to-one, it follows that \( C \cap \bigcup_{j=1}^{\infty} r_j^n = \emptyset \), and since the set \( \bigcup_{j=0}^{\infty} r_j^{n-1,i} \) is an \( (n-1) \)-dimensional compact subsets of \( C \), we have by the assumption \( \dim [X^* - f(X)] \leq n - 1 \) and Corollary 1, in \([4]\), p. 32, that \( \dim C \leq n - 1 \). Thus by the definition of \( d_n(Y) \) (Cf. section III) we obtain \( d_n(C) = 0 \). Hence, by \( (6) \), there exists for every \( \varepsilon > 0 \) an \( \varepsilon \)-covering of \( C \) by sets \( G_k \) open in \( X^* \), \( k = 0, 1, \ldots, m \) such that

\[ \bar{G}_k \cap \bar{G}_{k_1} \cap \ldots \cap \bar{G}_{k_n} = 0 \quad \text{for any set of subscripts} \quad k_0 < k_1 < \ldots < k_n. \]

\(^{18}\) S. [8], p. 106

\(^{19}\) S. [9], p. 110. Also [16], p. 11.
Now, since $\bigcap_{i=0}^{n-1} r_i - 0$, we can by (7), choose for this covering an $\varepsilon$ so small that no $G_k$ intersects each set $r_i^{n-1}$, i.e., $r_0^{n-1}$. Hence by (10) no set $G_k$ intersects all the faces $r_i^{n-1}$, i.e., $r_0^{n-1}$, for sufficiently large $j$. Let $G = \bigcup_{k=0}^{m} G_k$. By $C \subseteq G$ and dist $(r_j, C) - 0$ for $j \to \infty$ there exists a $j_0$ such that $r_j^{n} \subseteq G$ for $j \geq j_0$. Fixing any $j \geq j_0$, we find that the sets $F_k = r_j^{n} \cap G_k$, for $k = 0, 1, \ldots, m$ satisfy the assumptions of (8) with $n$ replaced by $n$, and $r$ by $r_j$. Hence by (8) some $n+\alpha$ sets $F_k \ldots \ldots F_{k_n}$, and therefore also the sets $G_k \ldots \ldots G_{k_n}$ have a nonempty intersection, which is incompatible with (11). Thus (b') is proved.

By (b'), (b1') and (3) we obtain

**Theorem 2.** The set $X$ defined in (9) is both an absolute $F_{\sigma}$ and $G_\delta$-space of the second kind and of dimension $n$.

This theorem gives an answer to problem (a1).

IV. 2. On a problem of A. Lelek.

The following problem p.313 in [11], p. 34 was formulated by Lelek.

Does there exist, for each absolute $G_\delta$-space $X$ of the second kind with finite, positive dimension, a compact space $Z$ with positive dimension, such that $X$ contains a topological image of the set $N \times Z$ ($N$ being the set of irrational numbers of the interval $J = (0, 1)$)?

A negative answer to this question was given in [12]. Now it is easily seen that a negative
answer to problem \((a_2)\) posed in section II contains as a special case, a negative answer to that of Lelek. (It suffices to take in \((a_2)\) \(n = \dim X = 1\)). We now proceed to prove that the answer to \((a_2)\) is negative.

Indeed, let \(X\) be the space defined in (9). We shall show that there does not exist a space \(Z\) with \(\dim Z = \dim X = n\) such that \(N \times Z\) has a topological image in \(X\).

Suppose, to the contrary, that such a space \(Z\) exists and let \(h : N \times Z \to X\) be a homeomorphism of \(N \times Z\) into \(X\). Fix a point \(\xi \in N\). Then the \(n\)-dimensional space \((\xi) \times Z\) has a topological image in \(X\). Now \(X\) being a countable union of compact disjoint sets \((a_j) \times \sigma^n\) and \((a_j) \times \text{Fr}(\sigma^n)\), \(j = 1, 2, \ldots\)

and \((\xi) \times Z\) being \(n\)-dimensional, it follows that \(h((\xi) \times Z)\) has an \(n\)-dimensional intersection with some set \((a_j(\xi)) \times \sigma^n^{19})\). This intersection, as \(n\)-dimensional subset of \(\sigma^n\), contains an open subset of \((a_j(\xi)) \times \sigma^n^{20})\). Since \(h\) is one-to-one, the sets \(h((\xi) \times Z)\) and \(h((\xi') \times Z)\) are disjoint for \(\xi \neq \xi', \xi, \xi' \in N\) and since \(N\) is uncountable, we get an uncountable family of disjoint open sets contained in \(X\), which is impossible.

IV. 3. A theorem on compactification.

We shall now prove a theorem with help of which it will be possible to construct for any \(n = 1, \ldots, \mathcal{M}_0\), a \(n\)-dimensional space \(X\) which is not locally compact at a single point and such that for each

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19) This is a consequence of the Sum Theorem for Dimension \(n\), Cf. [4], p. 30.
20) This follows easily from Theorem IV, 3 in [4], p. 44.
compactification \((f, X^*)\) of \(X\) there is \(\dim (X^* - f(X)) \geq 1\).

**Theorem 3.** Suppose that the space \(X\) contains a sequence \(\{C_i\}_{i=1,2,...}\) of continua \(C_i\) and a point \(p\) such that

(c₁) the sets \(C_i\) are closed and open in the union \(\bigcup_{i=1}^{\infty} C_i\) and disjoint \(C_i \cap C_j = 0\) for \(i \neq j\)

(c₂) there exists a number \(\delta > 0\), such that for each \(i = 1,2,...\), the diameters \(\delta(C_i) \geq \delta\)

and

(c₃) \(\bigcup_{i=1}^{\infty} C_i - \bigcup_{i=1}^{\infty} C_i = (p)\).

Then \(X\) is not locally compact at the point \(p\), and for each compactification \((f, X^*)\) of \(X\) there

is \(\dim (X^* - f(X)) \geq 1\).

**Proof.** Let \(U_p\) be an arbitrary neighborhood containing the point \(p\). We have to show that the closure \(\bar{U}_p\) is not compact. By (c₃) there exists a sequence of points \(p_i \in \bigcup_{i=1}^{\infty} C_i\) such that \(p_i \to p\) for \(i \to \infty\) and such that the sequence \(\{p_i\}_{i=1,2,...}\) has only a finite number of points in common with each \(C_i\).

Thus we may assume, that for each \(i = 1,2,...\) there is \(p_i \in C_i\). Let \(S = S(p,r)\) be a spherical neighborhood of \(p\) with radius \(r < \delta/2\) contained in \(U_p\). By \(p_i \to p\), the sets \(C_i \cap S\) are not empty for \(i\) sufficiently large and since \(C_i\) are connected, we get, by (c₃), that for these \(i\) there is \(C_i \cap \text{Fr}(S) \neq 0\), where \(\text{Fr}(S) = \{q; \rho(p,q) = r, q \in X\}\) is the boundary of \(S\). Choose from each such set \(C_i \cap \text{Fr}(S)\) a point \(q_i\) and consider the sequence \(\{q_i\}\). Since \(\bar{S} \subseteq \bar{U}_p\), we have \(\{q_i\} \subseteq \bar{U}_p\) and since \(q_i \in \text{Fr}(S)\), there is \(\rho(q_i,p) = r > 0\). Now, by \(q_i \in C_i\) for \(i\) sufficiently large, (c₁) and (c₃),...
any convergent subsequence of \( \{q_i\} \) tends to \( p \), which is impossible by \( \rho(q_i, p) = r > 0 \). Thus \( \bar{U}_p \) is not compact. It remains to show that if \((f, X^*)\) is any compactification of \( X \), then \( \dim[X^* - f(X)] \geq 1 \).

For this purpose let us consider the sets \( X_1 = \bigcup_{i=1}^\infty C_i \cup (p) \) and \( f(X_1) \). The closure \( \overline{f(X_1)} = X_1^* \subset X^* \) is a compactification of \( X_1 \). Let \( y \) be any point of \( X_1^* - f(X_1) \). Then the point \( y \neq f(x) \). Indeed, if there would exist a point \( x \in X \) such that \( y = f(x) \) then there would be \( x \neq X_1 \), since \( f \) is one-to-one.

Now by \( y \in f(X_1) \) there exists a sequence of points \( x_n \in X_1 \) such that \( f(x_n) \to y \). Thus by the continuity of \( f^{-1} \) it should be \( x_n \to x \in X_1 \). But by \((c_2)\) the set \( X_1 \) is closed in \( X \), and since \( x_n \in X_1 \) it follows that \( x \in X_1 \). This contradiction shows that \( y \neq f(x) \). Thus

\[(12) \quad [X_1^* - f(X_1)] \cap f(X) = [\overline{X_1} - f(X_1)] \cap f(X) = 0\]

Let us take further \( r < \frac{\delta}{2} \) and construct (analogously with the first part of the proof) points \( p_i \to p \), \( p_i \in C_i \) and \( q_i \in C_i \), such that \( \rho(p, q_i) = r > 0 \) for \( i \) sufficiently large. Since \( X_1^* = f(X_1) \) is compact and \( f(C_i) \subset X_1^* \) we can choose a subsequence of the sequence \( f(C_i) \) of continua converging to some continuum \( C \). Denoting the subscripts of this subsequence by \( i \) we have therefore that \( \text{dist}(f(C_i), C) \to 0 \) for \( i \to \infty \).

Now, by \( p_i \to p \), \( p_i \in C_i \), it follows that \( C \) contains the point \( f(p) \). If \( C \) would reduce to this point \( f(p) \), then by \( q_i \in C_i \) there would be \( f(q_i) \to f(p) \) and since \( f^{-1} \) is continuous there would also be \( q_i \to p \), in contradiction to \( \rho(p, q_i) = r > 0 \). It follows that \( C \) contains at least two points, and since it is a continuum we have \( \dim C \geq 1 \). Therefore \( \dim(C - (f(p))) \geq 1 \).

21) S. [9], p. 110.
Now, by (cI) we have $C \cap (C_i) = 0$ for each $i = 1, 2, \ldots$. Therefore by $X_i^o < X^*$ and (12) it follows that $\dim [X^* - f(X)] \geq 1$. Theorem 3 is proved.

Remark 2. In a quite analogous way one could prove that

if the space $X$ contains topologically the set defined by (9) and

$A \times \sigma^a - A \times \sigma^a = (a_0) \times Fr(\sigma^a)$, then for each compactification $(f, X^*)$ of $X$ there is

$\dim [X^* - f(X)] \geq n$. (For $n = 2$, see Fig. 3).

Example 1. Let $X = (a_0) \cup \bigcup_{j=1}^\infty (a_j) \times J$ where $a_0 = 0$ and $a_j = \frac{1}{2^{j-1}}$, $j = 1, 2, \ldots$ are real numbers on the real axes and $J = (0, 1)$ (See Fig. 1). This 1-dimensional space $X$ is not locally compact at the single point $a_0 = 0$, and by Theorem 3 $\dim [X^* - f(X)] \geq 1$ for any compactification $(f, X^*)$ of $X$. It is also easily seen that $X$ is an absolute $F_o$ and $G_{\delta}$-space and thus, by (3) and $\dim X = 1$, we obtain that $X$ is an absolute $F_o$ and $G_{\delta}$-space of the second kind.

Fig. 1
Example 2. Let $n = 2, 3, \ldots$ and let $X = (J^n - X_1) \cup \{0\}$, where $X_1 = \{x; x = (x_1, x_2, \ldots, x_n)\}$, $x_i = 0$, $0 \leq x_i \leq 1$, for $i = 2, 3, \ldots, n$} and $0 = (0, 0, \ldots, 0)$ (if $n = \aleph_0$, $J^n$ is the Hilbert cube).

It is clear that $\dim X = n$, and that $X$ is not locally compact at the single point $0 = (0, 0, \ldots, 0)$. It is also easy to construct a sequence $C_i$ of continua in $X$, such that the assumptions of Theorem 3 be satisfied for the point $p = (0, 0, \ldots, 0)$. Hence $\dim \{X^n - f(X)\} \geq 1$ for any compactification $(f, X^n)$ of $X$ (for $n = 3$, see Fig. 2).
According to Remark 2, for each compactification \((f, X^*)\) of this full cube \(X\) excluding the interior of the square \(OABC\) (but including \(OA, AB, BC\) and \(CO\)) \(\dim \left[ X^* - f(X) \right] \geq 2\).

Fig. 3.

IV.4. A weakly infinite-dimensional absolute \(F_\sigma\) and \(G_\delta\)-space

As stated in (3), a finite dimensional absolute \(G_\delta\)-space \(X\) is of the first kind if and only if
there exists a compactification \((f, X^*)\) of \(X\) such that \(\dim (X^* - f(X)) < \dim X\).

We shall now show that the above condition is not necessary for infinite dimensional spaces. More precisely, we shall construct an absolute \(F_\sigma\) and \(G_\delta\)-space of the first kind which is weakly infinite-dimensional and such that for each compactification \((f, X^*)\) of \(X\), there is \(\dim [X^* - f(X)] = \infty\). Let us take, for fixed \(n\), the set of points \(x_m = \frac{1}{2^n} + \frac{1}{2^n}, \ m = n + 1, n + 2, \ldots\) on the real axes, and let \(A_n = \bigcup_{m=n+1}^{\infty} (x_m, m)\). Define \(X_n = (A_n \times \sigma^n) \cup \left(\frac{1}{2^n} \times \text{Fr} (\sigma^n)\right)\) where \(\sigma^n\) is an \(n\)-dimensional closed simplex with diameter \(\delta (\sigma^n) = \frac{1}{2^n}\), and \(\text{Fr} (\sigma^n)\) is the boundary of \(\sigma^n\). The set \(X\) is then defined by

\[(13) \quad X = \bigcup_{n=1}^{\infty} X_n\]

The set \(X\) can be considered as a subset of the Hilbert cube \(I^\infty\), and the closure \(\overline{X}\) equals

\[\overline{X} = \bigcup_{n=1}^{\infty} X_n \cup \left[ \bigcup_{n=1}^{\infty} \left(\frac{1}{2^n} \times \text{Int} (\sigma^n)\right) \right] \cup \{0\},\]

where \(\text{Int} (\sigma^n) = \sigma^n - \text{Fr} (\sigma^n)\) and \(0 = (0,0,\ldots)\) is the point all whose coordinates are zero. It is also easily seen that \(\overline{X}\) may be written in the form

\[\bigcup_{n=1}^{\infty} \overline{X}_n \cup \{0\}, \quad \overline{X}_n = (A_n \cup \left(\frac{1}{2^n}\right)) \times \sigma^n.\]

Since \(\overline{X}\) is a compact space and \(X\) is a countable union of compact sets, we find that \(X\) is an absolute \(F_\sigma\)-space. Further, we can write each set

\[\left(\frac{1}{2^n}\right) \times \text{Int} (\sigma^n)\]

as a union \(\bigcup_{i=1}^{\infty} F^n_i\) of compact sets \(F^n_i, i = 1, 2, \ldots\). Thus \(\overline{X} - X = \bigcup_{n=1}^{\infty} \bigcup_{i=1}^{\infty} F^n_i \cup \{0\}\) is an \(F_\sigma\)-set and thus \(X\) is an absolute \(G_\delta\)-space. Moreover, the sets \(\overline{X} - \left[ \bigcup_{n=1}^{\infty} \bigcup_{i=1}^{\infty} F^n_i \cup \{0\}\right] = G_\delta\)

are open in \(\overline{X}\), \(\dim [\text{Fr} (G_\delta)] < \infty\) and \(\bigcap\ G_\delta = X\). Hence, \(X\) is an absolute \(F_\sigma\) and \(G_\delta\)-space of the first kind. By the definition of \(X\), it follows that \(X\) is a weakly infinite-dimensional space i.e.
dim $X = \infty$.

We shall now show that for each compactification $(f, X^\circ)$ of $X$ there is $\dim \left[ X^\circ - f(X) \right] = \infty$. For this purpose, let us note that the set $X_\circ$ is homeomorphic with the space defined in (9), and hence by $(b'_1)$ there is $\dim \left[ X_\circ^\circ - f(X_\circ) \right] \geq \dim X_\circ - n$ for each compactification $(f, X_\circ^\circ)$ of $X_\circ$. Now it is easily seen that

$$\text{(14) } f(X_\circ) \cap f(X - X_\circ) = \emptyset$$

where $f(X_\circ)$ is the closure of $f(X_\circ)$ in $X^\circ$.

Indeed, suppose to the contrary that the set in (14) is not empty and let $y \in f(X_\circ) \cap f(X - X_\circ)$. We have $f(X - X_\circ) = \bigcup_{k \neq n} f(X_k)$. Then $y = f(x)$ where $x \in X_k$ for some $k \neq n$. Since $y \in f(X_\circ)$, there exists a sequence $\{y_i\}_{i=1}^{\infty}$ such that $y_i \to y$ and $y_i = f(x_i)$ with $x_i \in X_k$. Since $f$ is continuous, it follows by $y = f(x)$ that $x_i \to x$. This is impossible, since $x_i \in X_k$, $x \not\in X_k$ and $X_k$ is a closed (also open) set in $X$.

Now $X_\circ^\circ = f(X_\circ)$ is a compactification of $X_\circ$ and therefore, by $\dim \left[ X_\circ^\circ - f(X_\circ) \right] \geq \dim X_\circ - n$ and (14), we have that $\dim \left[ X^\circ - f(X) \right] \geq n$. Since $n$ is arbitrary, it follows that $\dim \left[ X^\circ - f(X) \right] = \infty$.

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22) For weakly infinite-dimensional spaces $X$, $\dim X = \omega$ is sometimes written instead of $\dim X = \infty$. 
REFERENCES


ON COMPACTIFICATION OF METRIC SPACES

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ABSTRACT: Let \( f: X \rightarrow X' \) be a homeomorphism of a metric separable space \( X \) into a compact metric space \( X' \), such that \( f(X) = X' \). The pair \((f, X')\) is then called a metric compactification of \( X \). If \( X \) is an absolute \( G_\delta\)-space \((F_\sigma\text{-space})\) (i.e., \( G_\delta \) set \((F_\sigma\text{-set})\) in some compact space), then \( X \) is said to be of the first kind (cf. [6]) if there exists a compactification \((f, X')\) of \( X \) such that \( X = \bigsqcup G_i \), where \( G_i \) are sets open in \( X' \) and \( \dim \left( \text{Fr}(G_i) \right) < \dim X \), \( i = 1, 2, \ldots \) \((\text{Fr}(G_i)\) being the boundary of \( G_i \) and \( \dim X \) - the dimension of \( X \)). An absolute \( G_\delta\)-space, \((F_\sigma\text{-space})\) which is not of the first kind is said to be of the second kind. In the present study spaces \( X \) which are both absolute \( F_\sigma \) and absolute \( G_\delta\)-spaces.
of the second kind are constructed for any positive finite dimension, a problem related to one of A. Lelek in [11] is solved and a sufficient condition on \( X \) is given, under which \( \dim (X^e - f(X)) \geq 1 \) for any compactification \((i, X^e)\) of \( X \). It is noted also, that an analogous condition assures \( \dim (X^e - f(X)) \geq n \).