NOTICE: When government or other drawings, specifications or other data are used for any purpose other than in connection with a definitely related government procurement operation, the U. S. Government thereby incurs no responsibility, nor any obligation whatsoever; and the fact that the Government may have formulated, furnished, or in any way supplied the said drawings, specifications, or other data is not to be regarded by implication or otherwise as in any manner licensing the holder or any other person or corporation, or conveying any rights or permission to manufacture, use or sell any patented invention that may in any way be related thereto.
TRANSLATION

SOLVING THE BOUNDARY LAYER PROBLEM

BY

V. Ya. Shkadov

FOREIGN TECHNOLOGY DIVISION

AIR FORCE SYSTEMS COMMAND

WRIGHT-PATTERSON AIR FORCE BASE
OHIO
SOLVING THE BOUNDARY LAYER PROBLEM

By: V. Ya. Shkadov

English Pages: 6


UNEDITED ROUGH DRAFT TRANSLATION
SOLVING THE BOUNDARY LAYER PROBLEM

V. Ya. Shkadov

The movement of an incompressible viscous fluid in a boundary layer is described in Kotschin et al. [1] by the equation for the stream function \( \psi (x, y) \)

\[
\frac{\partial^2 \psi}{\partial y^2} - \frac{\partial \psi}{\partial x} \frac{\partial^2 \psi}{\partial y^2} = U \frac{dU}{dx} + v \frac{\partial^2 \psi}{\partial y^2}
\]  

(1)

with the boundary conditions

\[
\psi = 0, \quad \frac{\partial \psi}{\partial y} = 0 \quad \text{when} \quad y = 0, \quad \frac{\partial \psi}{\partial y} \rightarrow U \quad \text{when} \quad y \rightarrow \infty
\]  

(2)

If the velocity of the fluid \( U(x) \) on the external boundary of the boundary layer is given in the form of a power series \( U(x) = a_0 + a_1 x + a_2 x^2 + \ldots \), the solution for \( \psi (x, y) \) may also be found in the form of a series with respect to \( x \), the coefficients of which must be found by numerical integration.

At present Curle [2] has carried calculations out to the \( x^{11} \) term for bodies with blunted fronts and symmetrical with respect to the onflowing stream. When using this solution one must limit himself to the first few terms of the \( U(x) \) expansion, but this is not always sufficient, especially for asymmetrical bodies. In these cases one
must use a solution which cannot represent \( U(x) \) as a rapidly converging series.

The author himself [3] has pointed out the possibility of producing a solution to the boundary layer equations definable by the dimensionless combinations \( U' x / U, U'' x^2 / U, U''' x^3 / U, \ldots \), in which the primes indicate differentiation with respect to \( x \).

We will introduce a new independent variable \( \eta = \eta(x,y) \) and the unknown function \( f(x, \eta) \) so that

\[
\psi(x, y) = \sqrt{V x z}(x, \eta) \quad (\eta = \sqrt{V x z})
\]

For \( f(x, \eta) \) is derived the equation

\[
\frac{\partial f}{\partial \eta} + \frac{1}{2} \frac{\partial f}{\partial \eta^2} \frac{U' x}{U} \left[ 1 - \left( \frac{\partial f}{\partial \eta} \right)^2 + \frac{1}{2} \frac{\partial^2 f}{\partial \eta^2} \right] = x \left( \frac{\partial f}{\partial \eta} \frac{\partial f}{\partial x \partial \eta} - \frac{\partial f}{\partial x} \frac{\partial f}{\partial \eta} \right)
\]

with the bounding conditions

\[
f = 0, \quad \frac{\partial f}{\partial \eta} = 0 \text{ when } \eta = 0, \quad \frac{\partial f}{\partial \eta} \to 1 \text{ when } \eta \to \infty
\]

Let us examine the equation

\[
\frac{\partial f}{\partial \eta} + \frac{1}{2} \frac{\partial f}{\partial \eta^2} \frac{U' x}{U} \left[ 1 - \left( \frac{\partial f}{\partial \eta} \right)^2 + \frac{1}{2} \frac{\partial^2 f}{\partial \eta^2} \right] = \sum_{i=1}^{\infty} \left( p_i - p_i p_{i+1} + p_{i+1} \right) \left( \frac{\partial f}{\partial \eta} \frac{\partial f}{\partial \eta p_i} - \frac{\partial f}{\partial x} \frac{\partial f}{\partial \eta p_i} \right)
\]

in which the function depends on \( \eta, p_1, p_2, \ldots \)

We will look for a solution to this equation which satisfies the boundary conditions (4). If, in the space of the variables \( p_1, p_2, \ldots \), function \( U(x) \) parametrically gives a curve with the relationships \( p_1 = U' x / U, p_2 = U'' x^2 / U, \) it is easy to see that along this curve

\[
x \frac{d p_i}{d x} = p_i - p_i p_{i+1} + p_{i+1} \quad (i=1,2,\ldots)
\]

therefore the right half of Eq. (5) represents

\[
x \left( \frac{\partial f}{\partial \eta} \frac{\partial f}{\partial x \partial \eta} - \frac{\partial f}{\partial x} \frac{\partial f}{\partial \eta} \right)
\]
and Eq. (5) coincides with (3).

Consequently, in this special case function \( f(\eta, P_1, P_2, \ldots) \) found by the solution of Eq. (5) with boundary conditions (4) will describe the flow in the boundary layer.

Two solutions for Eq. (5) may be produced: expansion into a series with respect to \( P_1, P_2, \ldots \) and into one with respect to \( P_1 - 1, P_2, \ldots \). The first solution corresponds to boundary layer flow beginning at the sharp edge on which \( P_1 = 0, P_2 = 0, \ldots \), while the second describe a boundary layer beginning at a critical point at which \( P_1 = 1, P_2 = 0, \ldots \).

The coefficients of these series are functions of \( \eta \) and are found by numerical integration of ordinary differential equations. The necessary calculations were done on a "Strela" computer. They show that the coefficients of the series rapidly diminish as \( \eta \) grows, therefore in what follows the terms containing \( P_4, P_5, \ldots \) are considered too small and are dropped. For bodies with a sharp forward edge

\[ f = f_0 + 8f_1P_1 + 8'(f_1P_1^2 + f_2P_2) + 8''(f_2P_2^2 + f_3P_3 + f_4P_4) + \cdots \]  

Substituting \( f \) into Eq. (5) and setting expressions with various combinations of \( P_1, P_2, \) and \( P_3 \) equal to zero we may derive equations for \( f_{1k} \).

Because, however, numerical integration necessitates awkward right sides and consequent storage of much information, \( f_{1k} \) was computed by a different method. Let

\[ U(x) = 1 + ax + bx^2 + cx^3 \]  

It can be found that

\[ \begin{align*}
ax &= \frac{1}{d} \left( P_1 - \frac{1}{2} P_2 + \frac{1}{6} P_3 \right), \\
bx^2 &= \frac{1}{d} \left( \frac{1}{2} P_1 - \frac{1}{2} P_3 \right), \\
px^3 &= \frac{1}{d} \left( \frac{1}{6} P_3 \right), \\
d &= 1 - P_1 + \frac{1}{2} P_3 - \frac{1}{6} P_5.
\end{align*} \]
By expanding into a series with respect to \( x \) we will find the solution to Eq. (3) with the boundary conditions (4) in the form

\[
f = f_0 + s_1 x + (a^2 h_1 + b h_2 + c^2) x^2 + \cdots
\]  

(8)

The functions \( f_0 (\eta), g_1 (\eta), h_1 (\eta), \ldots \) satisfy the equations

\[
\begin{align*}
 f''_0 + \frac{1}{2} f s_0^* &= 0 \\
 g''_1 + \frac{1}{2} h_1^* s_1^* - h_0 s_1' + \frac{3}{2} f s_0^* &= -1 + h_0^2 - \frac{1}{2} f s_0^* \\
 h''_1 + \frac{1}{2} f h_1^* - f s_1' + \frac{5}{2} f h_0^* &= - \frac{3}{2} s_1^* + 2 h_0 s_1' - \frac{1}{2} (h_0^* s_1^* + h_1^* s_0^*) + 1 - f s_0^* + \frac{1}{2} f s_0^*
\end{align*}
\]

(9)

with the boundary conditions

\[
\begin{align*}
 f_0 &= 0, \quad f_0' = 0, \quad g_1 = 0, \quad g_1' = 0, \ldots \quad \text{when} \ \eta = 0 \\
 f_1' &= 1, \quad g_1' = 0, \quad h_1' = 0, \ldots \quad \text{when} \ \eta \to \infty
\end{align*}
\]

(10)

If we assume that \( p_1, p_2, \text{and} \ p_3 \) are small enough, it is possible to expand \( ax, bx^2, \) and \( cx^2 \) into series of the form (6) by using (7). Substituting expressions for them into (8), collecting the terms with the same \( p_1, p_2, p_3 \) combinations, and comparing with (6) we finally find \( f_{1k} \). For shearing stress at a wall \( \tau = \mu \frac{\partial^2 \psi}{\partial y^2} \) when \( y = 0 \) we have

\[
\begin{align*}
\tau &= \left( \frac{\partial \psi}{\partial x} \right)_{y=0} - \frac{1}{x} \\
&= f''_{\infty} (0) + 8 p_3 f'_{\infty} (0) + 8 (p_1 f'_{\infty} + p_1 f'_{\infty}) (0) + \\
&\quad + 8 p_2 f'_{\infty} (0) + p_3 p_3 f'_{\infty} (0) + p_1 p_1 f'_{\infty} (0) + 8 (p_2 f''_{\infty} + p_2 p_2 f'_{\infty}) (0) + \\
&\quad + p_3 p_3 f''_{\infty} (0) + p_1 p_1 f''_{\infty} (0) + 8 (p_2 p_3 f''_{\infty} + p_2 p_3 f''_{\infty}) (0) + \\
&\quad + p_2 p_3 p_3 f''_{\infty} (0) + p_2 p_3 p_3 f''_{\infty} (0)
\end{align*}
\]

(11)

where the second derivatives of \( f_{1k} \) have the following values:

\[
\begin{align*}
f'_{\infty} (0) &= 0.33206, & f'_{10} (0) &= 0.11550, & f''_{\infty} (0) &= -0.03129 \\
f'_{20} (0) &= -0.00216, & f'_{30} (0) &= 0.00224, & f''_{20} (0) &= 0.00122 \\
f'_{20} (0) &= 0.00008, & f'_{30} (0) &= 0.00079, & f''_{20} (0) &= -0.00074 \\
f'_{20} (0) &= -0.00003, & f'_{30} (0) &= -0.00002, & f''_{20} (0) &= 0.00023 \\
f'_{20} (0) &= 0.00003, & f'_{30} (0) &= 0.00002, & f''_{20} (0) &= 0.00002
\end{align*}
\]
The calculations from formula (11) coincide well with the results shown in Görtler and Witting [4] and Terrill [5]. To exemplify, breakaway of the boundary layer $U = 1 - x^2$ takes place when $x = 0.64$, closely matching the value of $x = 0.67$ [5].

In order to examine the boundary layer on bodies with blunted foreparts we will introduce

$$x_i = \frac{U_x}{U} - 1, \quad x_2 = \frac{U_x x^2}{U} - 3 \left( \frac{U_x x}{U} - 1 \right), \quad x_3 = \frac{U_x x^3}{U} - 6 \frac{U_x x^2}{U} + 15 \left( \frac{U_x x}{U} - 1 \right) \tag{12}$$

Let $U = x \left( 1 + ax^2 + bx^4 + cx^6 \right)$; then

$$ax^4 = \frac{1}{D} \left( \frac{1}{2} x_1 - \frac{1}{4} x_2 + \frac{1}{16} x_3 \right), \quad bx^4 = \frac{1}{D} \left( \frac{1}{8} x_1 - \frac{1}{32} x_2 \right) \tag{13}$$

$$cx^4 = \frac{1}{D} \left( \frac{1}{48} x_2 \right), \quad D = 1 - \frac{1}{2} x_1 + \frac{1}{16} x_2 - \frac{1}{48} x_3$$

Using the solutions found by expanding with respect to $x$ (Curle [2]), we will by the above-described method obtain

$$f = F_{w0} + x_1 F_{w1} + x_1^2 F_{w2} + x_1^3 F_{w3} + \ldots \tag{14}$$

and for shearing stress at a wall

$$\tau = \frac{\sqrt{U_x}}{2} \left( -\frac{1}{2} F_{w0}(0) + x_1 F_{w1}(0) + x_1^2 F_{w2}(0) + x_1^3 F_{w3}(0) + \ldots \right)$$

$$+ x_1^2 F_{v0}(0) + x_1^2 x_2 F_{v1}(0) + x_1^2 x_2^2 F_{v2}(0) + x_1^2 x_2^3 F_{v3}(0) + \ldots$$

$$+ x_1^2 x_2^2 F_{v4}(0) + x_1^2 x_2^2 F_{v5}(0) + x_1^2 x_2^2 F_{v6}(0)$$

Here

$$F_{w0}(0) = 1.233588, \quad F_{w1}(0) = 0.52445, \quad F_{w2}(0) = -0.06739$$
$$F_{v0}(0) = -0.01731, \quad F_{v1}(0) = 0.01058, \quad F_{v2}(0) = 0.06758$$
$$F_{v3}(0) = 0.00111, \quad F_{v4}(0) = -0.00229, \quad F_{v5}(0) = -0.00070$$
$$F_{v6}(0) = -0.000058, \quad F_{v7}(0) = 0.00014$$
$$F_{v8}(0) = 0.000008$$
By using (15) the shear stress and point of breakaway are calculated for various cases described in Curle [2] and Terrill [6]; in doing so good coincidence is ascertained. By way of example we cite below values of the magnitudes

\[ \tau = \left( \frac{U}{y^2} \right)^2, \]

calculated for \( U = U_0 (x - x^3 + 0.0789 x^5) \) according to formula (15) and also borrowed from Curle [2].

<table>
<thead>
<tr>
<th>( x )</th>
<th>0.2</th>
<th>0.4</th>
<th>0.56</th>
<th>0.64</th>
<th>0.69</th>
<th>0.75</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \tau )</td>
<td>1.189</td>
<td>1.07</td>
<td>0.715</td>
<td>0.378</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>( T )</td>
<td>1.189</td>
<td>1.07</td>
<td>0.712</td>
<td>0.569</td>
<td>0</td>
<td>0</td>
</tr>
</tbody>
</table>

Received January 24, 1962

REFERENCES


# DISTRIBUTION LIST

<table>
<thead>
<tr>
<th>DEPARTMENT OF DEFENSE</th>
<th>Nr. Copies</th>
<th>MAJOR AIR COMMANDS</th>
<th>Nr. Copies</th>
</tr>
</thead>
<tbody>
<tr>
<td>DISTRIBUTION LIST</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>DEPARTMENT OF DEFENSE</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>HEADQUARTERS USAF</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>AFCIN-3D2</td>
<td>1</td>
<td>AFSC</td>
<td></td>
</tr>
<tr>
<td>ARL (ARB)</td>
<td>1</td>
<td>SCFDD</td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td>DDC</td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td>TDBTL</td>
<td></td>
</tr>
<tr>
<td>OTHER AGENCIES</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>CIA</td>
<td>1</td>
<td>AEDC (AEY)</td>
<td></td>
</tr>
<tr>
<td>NSA</td>
<td>6</td>
<td>SSD (SSF)</td>
<td></td>
</tr>
<tr>
<td>DIA</td>
<td>9</td>
<td>AFPTC (FTY)</td>
<td></td>
</tr>
<tr>
<td>AID</td>
<td>2</td>
<td>AFSWC (SWP)</td>
<td></td>
</tr>
<tr>
<td>OTS</td>
<td>2</td>
<td>ASD (ASYIM)</td>
<td></td>
</tr>
<tr>
<td>AEC</td>
<td>2</td>
<td></td>
<td></td>
</tr>
<tr>
<td>FWS</td>
<td>1</td>
<td></td>
<td></td>
</tr>
<tr>
<td>NASA</td>
<td>1</td>
<td></td>
<td></td>
</tr>
<tr>
<td>ARMY (FSTC)</td>
<td>3</td>
<td></td>
<td></td>
</tr>
<tr>
<td>NAVY</td>
<td>3</td>
<td></td>
<td></td>
</tr>
<tr>
<td>NAFEC</td>
<td>1</td>
<td></td>
<td></td>
</tr>
<tr>
<td>RAND</td>
<td>1</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

FTD-TT-63-283/1+2+4