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(6) OPTIMIZING THE DYNAMIC SHAPE

OF A VIBRATING BEAM

by

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CHAPTER I

INTRODUCTION

Engineers and scientists are constantly seeking the best approach to accomplish a task or to solve a problem. The "best way" is usually nothing more than the easiest way to attain the goal. Vibration problems that are encountered in every day conditions are gigantic if all the ramifications are applied to the situation. However a number of assumptions can reasonably be made such as assuming the beam or shaft under consideration has constant cross sectional area; the material is homogeneous; or the mass of the beam or shaft can be neglected.

Even after the problem has been "simplified" with reasonable assumptions, it sometimes remains a difficult problem to solve. For example, to increase the natural frequency, which part of the system should be changed to effect the desired increase? What corrections must be made if the cross sectional area is not constant along the beam? What are all the dampening factors and which can be reasonably neglected?

In many vibration problems an approximate solution must be accepted because an exact solution is very difficult if not impossible to attain. This paper will develop two approximate frequency-determining methods * which apply ~~will be applied~~ to the following problems;

p2.

* are developed

from 1

(a) Determining the optimum dynamic shape of a cantilever beam with a mass load at the free end, and

(b) Determining the necessary approach to accurately optimize the dynamic shape of a simply supported beam with a center mass load.

The dynamic shape of a beam is the actual physical form (rectangular, triangular, parabolic, etc.) of a beam (Figure 1). The <u>optimum</u> dynamic shape is the physical form of a beam, with or without a mass load, that will produce the highest natural frequency (Figure 2).

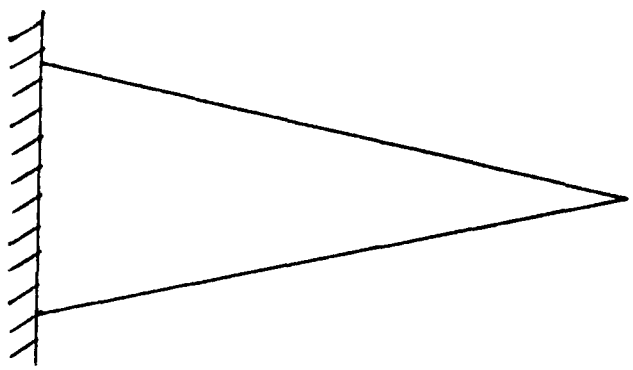

Figure 1.
Dynamic Shaped Cantilever Beam

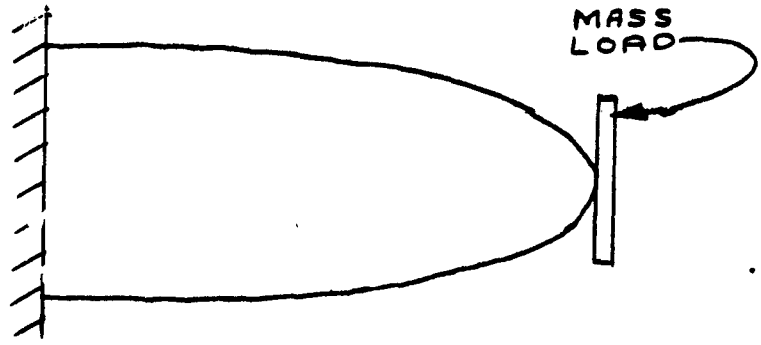

Figure 2.
Optimum Dynamic Shaped Cantilever Beam
With a Mass Load Equal to the Beam Mass

The two methods that will be developed are the Rayleigh principle and the Rayleigh-Ritz procedure. Each of these methods will be applied to the cantilever beam and the results will show that the Rayleigh principle is relatively easy to apply to a problem as compared to the Rayleigh-Ritz procedure. However, the frequency determined by the Rayleigh-Ritz procedure is more accurate and to accurately optimize the simply supported beam the Rayleigh-Ritz procedure will be applied.

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[1]Curreri, John R., Vibration Fixture Design, Copyright MB Electronics; 1961, Printed in the United States of America: p. 88.

CHAPTER II

RAYLEIGH PRINCIPLE

The Rayleigh method is a generalization of the "energy method"[2] which states that the potential energy at the extreme position (maximem amplitude of a vibrating body) is equal to the kinetic energy of the vibrating body in the neutral position. (The system is assumed to be conservative.) "Briefly, a shape is assumed for the first normal elastic curve; with this assumption the (maximum) potential and kinetic energies are calculated and are equated. Of course, if the exact shape had been taken as a basis for the calculation, the calculated frequency would be exactly correct also; for a shape differing somewhat from the exact curve a very useful and close approximation for the frequency is obtained."[3] In applying Rayleigh's method to any beam the change in potential energy of bending is given by

$$d(PE) = \frac{M \cdot d\phi}{2} \qquad \text{EQ. 2.1}$$

for any bending moment moving through a differential angular change. Equation 2.1 can be derived simply by considering an element dx under the influence of a bending moment M (Figure 3). The element is originally straight and is bent through an angle of $d\phi$ by the moment M. If

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[2]Den Hartog, J.P., Mechanical Vibrations, McGraw-Hill Book Co., Inc., New York, N.Y., 1956, p. 33.

[3]Ibid., p. 141.

the left end of the element is assumed to be fixed, the moment M at the right end turns through the angle $d\phi$. The work done by M on the beam is therefore $\frac{1}{2} M d\phi$, where the factor $\frac{1}{2}$ appears because both M and $d\phi$ are increasing from zero together.[4]

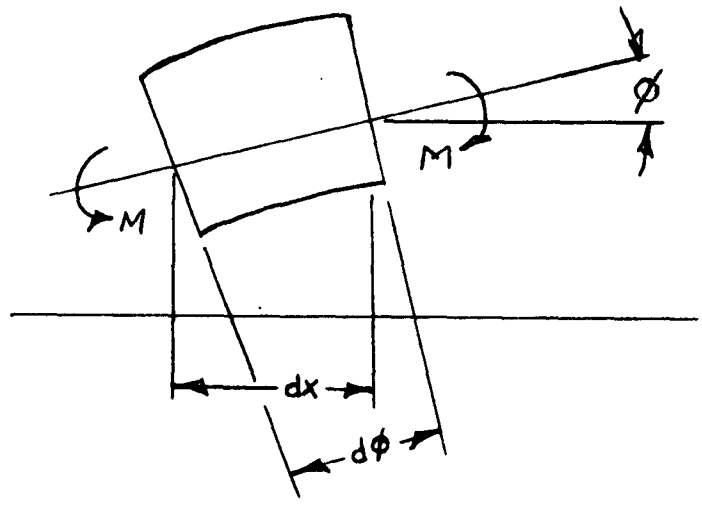

Figure 3.
Beam Element Under Influence of Bending Moment

The slope is given by $\tan \phi = \frac{dy}{dx} \cong \phi$ (for small angles) and the bending moment is $M = EI \frac{d^2y}{dx^2}$.

Substituting $d\phi$ ($d\phi = \frac{d^2y}{dx^2} dx$) and M in Equation 2.1 yields

$$d(PE) = \frac{1}{2} \left(EI \frac{d^2y}{dx^2} \right) \left(\frac{d^2y}{dx^2} dx \right)$$

$$PE = \int_0^{\ell} \frac{EI}{2} \left(\frac{d^2y}{dx^2} \right)^2 dx \qquad \text{EQ. 2.2}$$

---

[4] Ibid., p. 152

The change in kinetic energy is given by

$$d(KE) = \tfrac{1}{2}\,(dm)\,V^2 \qquad \text{EQ. 2.3}$$

V(Velocity) = Y (Deflection) · ω (Frequency)

dm (mass change along beam) = $\mu\left(\frac{\text{Mass}}{\text{Length}}\right)\cdot dx$

$$KE = \int_0^{\ell} \tfrac{1}{2}\,\frac{\mu\,\omega^2}{2}\; y^2\,dx\ . \qquad \text{EQ. 2.4}$$

If the beam has a mass load the kinetic energy is

$$KE = \int_0^{\ell} \tfrac{1}{2}\left(\frac{\mu\,\omega^2}{2}\right) y^2 dx + 1/2\ M_L\ (\text{Mass Load})\ \omega^2 Y_m{}^2 \qquad \text{EQ. 2.5}$$

where y_m is the beam deflection at the mass load.

Equating kinetic energy and potential energy the frequency becomes

$$\omega^2 = \frac{\int_0^{\ell} EI\left(\frac{d^2y}{dx^2}\right)^2 dx}{\int_0^{\ell} \mu\ y^2 dx + M_L\ y_m{}^2}\ . \qquad \text{EQ. 2.6}$$

Equation 2.6 is known as Rayleigh's equation and is relatively easy to apply if the shape of the deflection curve or normal function is known or approximated. If the normal function is not known the frequency obtained will be higher than the exact frequency for any one beam. If the area and inertia of the beam are not constant the normal function will be difficult to approximate with a single term as opposed to a series of terms.

CHAPTER III

RAYLEIGH-RITZ PROCEDURE

The Rayleigh-Ritz procedure is an extension of the Rayleigh principle and its application is very satisfactory for a beam whose area and inertia may not be constant along the beam. In the latter case the dynamic deflection curve is complicated and the normal function can best be described by a series of terms rather than a single term as used in the Rayleigh principle. Ritz defined the normal function by an infinite series of terms such as

$$y = a_1\phi_1(x) + a_2\phi_2(x) + a_3\phi_3(x) + a_4\phi_4(x) \quad \cdots\cdots$$

where every $\phi_i(x)$ satisfies the boundary conditions of the beam.

From Equation 2.6 the frequency is

$$\omega^2 = \frac{\int_0^{\ell} EI\left(\frac{d^2y}{dx^2}\right)^2 dx}{\int_0^{\ell} \mu\, y^2 dx + M_L\, Y_m^{\,2}} \qquad \text{EQ. 2.6}$$

The best coefficients (a_1, a_2, a_3, a_4, ... a_n) in the normal function, can be evaluated by minimizing the frequency $\frac{\partial\,\omega^2}{\partial\, a_n} = 0$.

$$\frac{\partial}{\partial a_n}\left(\frac{\int_0^{\ell} EI\left(\frac{d^2y}{dx^2}\right)^2 dx}{\int_0^{\ell} \mu\, y^2 dx + M_L\, Y_m^{\,2}}\right) = 0 \qquad \text{EQ. 3.1}$$

$$\left(\int_0^{\ell} \mu\, y^2 dx + M_L\, y_m^{\,2}\right)\cdot\frac{\partial}{\partial a_n}\int_0^{\ell} EI\left(\frac{d^2y}{dx^2}\right)^2 dx$$

$$-\int_0^{\ell} EI\left(\frac{d^2y}{dx^2}\right)^2 dx\cdot\frac{\partial}{\partial a_n}\left(\int_0^{\ell} \mu\, y^2 dx + M_L\, y_m^{\,2}\right) = 0. \qquad \text{EQ. 3.2}$$

Then substituting

$$\int_0^{\ell} \mu y^2 dx + M_L\, y_m^{\,2} = \frac{1}{\omega^2} \int_0^{\ell} EI\left(\frac{d^2y}{dx^2}\right)^2 dx \qquad \text{EQ. 2.6}$$

into Equation 3.2.

$$\frac{1}{\omega^2}\int_0^{\ell} EI\left(\frac{d^2y}{dx^2}\right)^2 dx \cdot \frac{\partial}{\partial a_n}\int_0^{\ell} EI\left(\frac{d^2y}{dx^2}\right)^2 dx$$

$$-\int_0^{\ell} EI\left(\frac{d^2y}{dx^2}\right)^2 dx \cdot \frac{\partial}{\partial a_n}\left(\int_0^{\ell} \mu\, y^2 dx + M_L\, y_m^{\,2}\right) = 0 \qquad \text{EQ. 3.3}$$

$$\frac{\partial}{\partial a_n}\left\{\int_0^{\ell} I\left(\frac{d^2y}{dx^2}\right)^2 dx - \frac{\omega^2}{E}\left(\int_0^{\ell} \mu y^2 dx + M_L\, y_m^{\,2}\right)\right\} = 0. \qquad \text{EQ. 3.4}$$

and let

$$s = \int_0^{\ell} I\left(\frac{d^2y}{dx^2}\right)^2 dx - \frac{\omega^2}{E}\left(\int_0^{\ell} \mu\, y^2 dx + M_L\, y_m^{\,2}\right) \qquad \text{EQ. 3.5}$$

The differential of "s" (Equation 3.5) with respect to each coefficient (a_1, a_2, a_3, a_4, ... a_n) will provide a set of homogeneous equations from which the frequency can be calculated. The accuracy of the Rayleigh-Ritz method depends on the number of terms used in the normal function. If all the terms were used the solution would converge to the exact frequency for any beam. Satisfactory results can be obtained, however, by using the first two terms for the systems under consideration.

CHAPTER IV

OPTIMIZING THE DYNAMIC SHAPE OF CANTILEVER BEAM WITH MASS LOAD AT FREE END

The mass load on the cantilever beam will be considered a point load (Figure 4).

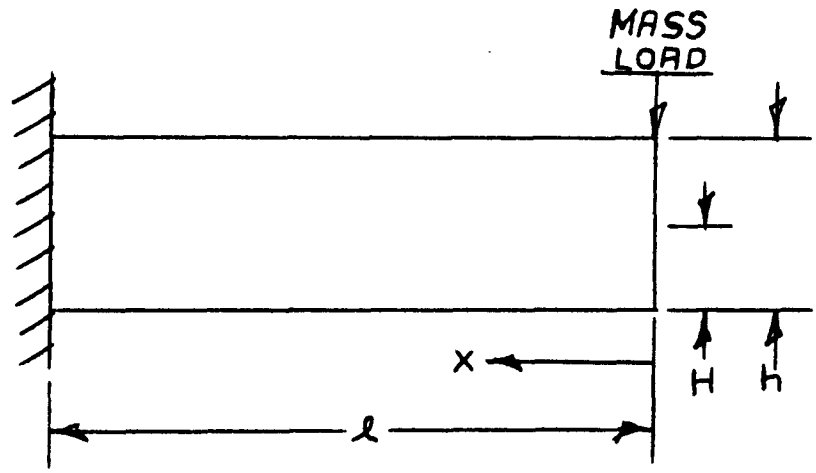

Figure 4.
Cantilever Beam with a Mass Load
at the Free End

The greatest deflection of a cantilever beam is contributed by the free end quarter of the beam. This can be simply proven with Dunkerley's formula.[5] Dunkerley's formula applies to systems with mass loads distributed along the system. Dunkerley's formula (as applied to a cantilever beam (Figure 5) states:

---

[5]Curreri, op. cit., p. 48.

$$\frac{1}{f^2} = \frac{1}{f_o^2} + \frac{1}{f_1^2} + \frac{1}{f_2^2} + \frac{1}{f_3^2} \cdots \frac{1}{f_n^2} \qquad \text{EQ. 4.1}$$

where f = approximate combined system frequency

f_o = natural frequency of beam alone

f_1 = natural frequency of weight 1 alone on massless beam

f_2 = natural frequency of weight 2 alone on massless beam

'

'

'

f_n = natural frequency of weight n alone on massless beam

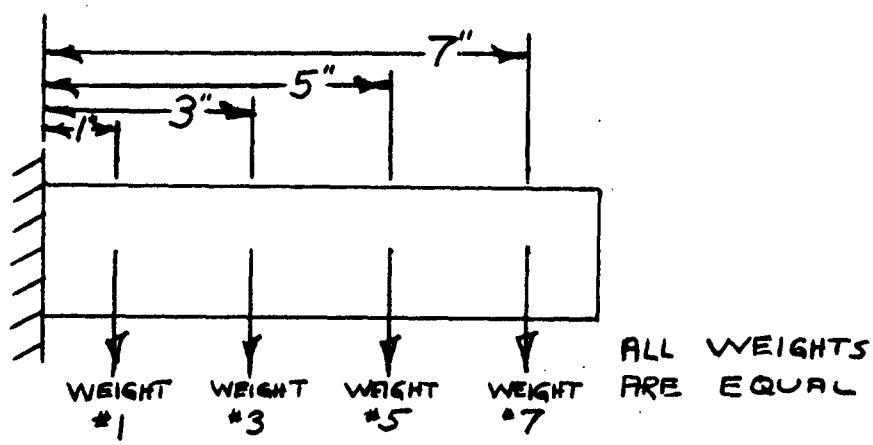

Figure 5.
Cantilever Beam Divided into Four Sections

Since the natural frequency of a single degree of freedom system can be written in terms of its static deflection ($f=3.12\sqrt{\frac{1}{y_{st}}}$), Equation 4.1 may be written as follows

$$y_{st} = y_o + y_1 + y_2 + y_3 + \cdots + y_n \, . \qquad \text{EQ. 4.2}$$

For the beam in Figure 5, $y_{st} = y_1 + y_3 + y_5 + y_7$ and $y_n = \frac{W_{(load)} \cdot L^3 \text{ (length to load)}}{3\ EI}$

$$y_{st} = \frac{W}{3\ EI}\left(1^3 + 3^3 + 5^3 + 7^3\right)$$

$$y_{st} = \frac{W}{8\ EI}\left(1 + 27 + 125 + 434\right). \qquad \text{EQ. 4.3}$$

Equation 4.3 shows that the end quarter (y_7) of the beam contributes to the greatest static deflection of the beam and is also the location of the least bending moment. From this it can be deduced that the removal of material at the end of the beam might reduce the mass effect more than the stiffness effect. This would increase the natural frequency because the static deflection (Equation 4.3) would be reduced with a resulting increase in frequency ($f = 3.12\sqrt{\frac{1}{y_{st}}}$).

The next step is to find the optimum dynamic shape for the beam which will give the highest fundamental frequency for any ratio of free end mass load to beam mass. As an example let the beam width be unity and the depth vary along the length of the beam (Figure 6). The half height of the beam will be given by

$$\frac{h}{2} = \left(\frac{H}{L^n}\right) x^n \qquad \text{EQ. 4.4}$$

where "n" defines the shape parameter as shown in Figure 6.[6]

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[6]Ibid., p. 86.

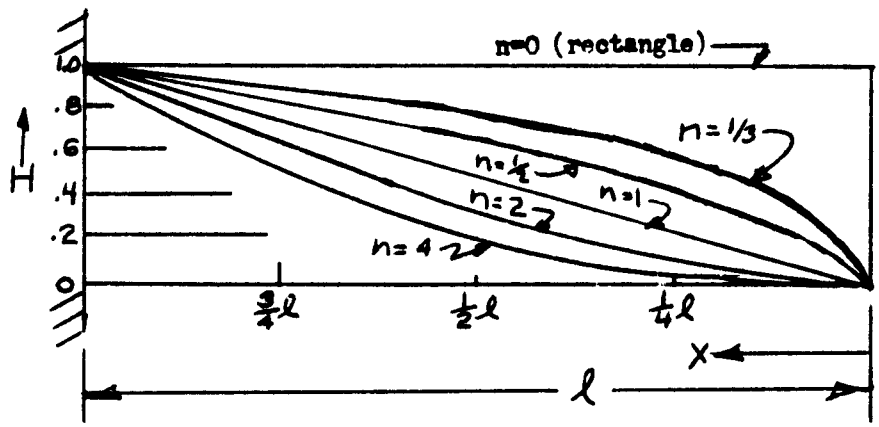

Figure 6.
Shape Parameters for a Cantilever Beam

The cross sectional area is given by

$$A = \left(\frac{2H}{\ell^n}\right) x^n \qquad \text{EQ. 4.5}$$

and the moment of inertia is

$$I = \frac{2}{3}\left(\frac{H\,x^n}{\ell^n}\right)^3 \qquad \text{EQ. 4.6}$$

With the above characteristics the potential energy (Equation 2.2) is given by

$$PE = \int_0^{\ell} \frac{E}{3}\left(\frac{Hx^n}{\ell^n}\right)^3 \cdot \left(\frac{d^2y}{dx^2}\right)^2 dx \qquad \text{EQ. 4.7}$$

and the kinetic energy is

$$KE = \int_0^{\ell} \frac{\mu\omega^2}{2}\, y^2 dx + \frac{M_L}{2}\, \omega^2\, y_m^{\,2} \qquad \text{EQ. 4.5}$$

where μ = mass/length = ρ (density) · A (area).

Let R = mass load/beam mass = $\frac{M_L}{M_B}$

$$\text{then } KE = \int_0^{\ell} \frac{1}{2}\, \omega^2\, \rho\, A\, y^2 dx + \frac{1}{2} R\, M_B\, \omega^2 y_m^{\,2}. \qquad \text{EQ. 4.8}$$

Since Beam mass = beam volume · density.

and Beam volume = $\int_0^{\ell} A \cdot dx$.

then $KE = \frac{1}{2}\ \omega^2 \rho \int_0^{\ell} A\ y^2 dx + \frac{1}{2}\ R\ \rho\ \omega^2 y_m{}^2 \int_0^{\ell} A \cdot dx$. EQ. 4.9

Substituting Equation 4.5 in Equation 4.9 the final form for the kinetic energy is

$$KE = \frac{\omega^2 \rho H}{\ell^n} \int_0^{\ell} x^n y^2\ dx + R\ y_m{}^2 \int_0^{\ell} x^n\ dx \qquad \text{EQ. 4.10}$$

Equating the potential energy (Equation 4.7) and the kinetic energy (Equation 4.10) the frequency equation for a dynamic shaped beam is

$$\omega^2 = \frac{E}{3 \cdot \rho} \left(\frac{H}{\ell^n}\right)^2 \left\{ \frac{\int_0^{\ell} (x^n)^3 \cdot \left(\frac{d^2y}{dx^2}\right)^2\ dx}{\int_0^{\ell} x^n y^2 dx + R\ y_m{}^2 \int_0^{\ell} x^n\ dx} \right\} \qquad \text{EQ. 4.11}$$

Equation 4.11 could be considered Rayleigh's frequency equation for a dynamic shaped beam.

RAYLEIGH APPLICATION

In applying the Rayleigh method to shaping a beam, a deflection curve must be assumed because the exact curve is not known for the shaped beam. By assuming the deflection curve to be $y = y_o\ (1 - \sin \frac{\pi x}{2\ell})$, where y_o is the deflection at the end of the beam the frequency (Equation 4.11) can be found for any combination of the shape parameter "n" and the ratio "R" (mass load to beam mass). Results of a number of calculations appear in Table I.

R ↓	Shape Parameter n: 0	½	1	2	3
0	$\frac{3.68H}{\ell^2}\sqrt{\frac{E}{3\cdot\rho}}$	$\frac{5.72H}{\ell^2}\sqrt{\frac{E}{3\cdot\rho}}$	$\frac{6.36H}{\ell^2}\sqrt{\frac{E}{3\cdot\rho}}$	$\frac{8.15H}{\ell^2}\sqrt{\frac{E}{3\cdot\rho}}$	$\frac{8.20H}{\ell^2}\sqrt{\frac{E}{3\cdot\rho}}$
1	$\frac{1.57H}{\ell^2}\sqrt{\frac{E}{3\cdot\rho}}$	$\frac{1.97H}{\ell^2}\sqrt{\frac{E}{3\cdot\rho}}$	$\frac{1.73H}{\ell^2}\sqrt{\frac{E}{3\cdot\rho}}$	$\frac{1.53H}{\ell^2}\sqrt{\frac{E}{3\cdot\rho}}$	$\frac{1.13H}{\ell^2}\sqrt{\frac{E}{3\cdot\rho}}$
10	$\frac{.543H}{\ell^2}\sqrt{\frac{E}{3\cdot\rho}}$	$\frac{.654H}{\ell^2}\sqrt{\frac{E}{3\cdot\rho}}$	$\frac{.567H}{\ell^2}\sqrt{\frac{E}{3\cdot\rho}}$	$\frac{.490H}{\ell^2}\sqrt{\frac{E}{3\cdot\rho}}$	$\frac{.366H}{\ell^2}\sqrt{\frac{E}{3\cdot\rho}}$

Table I

Frequency (Rayleigh Method) Versus Shape Parameter "n" and Mass Ratio "R"

The above approach for determining the fundamental frequency is only an approximation because the deflection curve (normal function) is not very accurate when the area and inertia are functions of "x" (length along the beam). The frequency from Table I for a rectangular beam (n=0) and no mass load (R=0) is $\frac{3.68\ H}{\ell^2}\frac{\sqrt{E}}{\sqrt{3\cdot p}}$ as compared to $\frac{3.52\ H}{\ell^2}\frac{\sqrt{E}}{\sqrt{3\cdot p}}$ which is the exact fundamental frequency for a rectangular beam with no mass load. This is a satisfactory solution for the specific case where the area and inertia are constant for the rectangular beam. As the shape parameter n" increases, the assumed deflection curve becomes less accurate and therefore the calculated frequency has a larger error.

RAYLEIGH-RITZ PROCEDURE

The Rayleigh-Ritz procedure is well suited for a beam whose area and inertia may not be constant along the beam and therefore the normal function could best be described by a series of terms. The normal function for this system is

$$y = a_1\left(1 - \frac{x}{\ell}\right)^2 + a_2\,\frac{x}{\ell}\left(1 - \frac{x}{\ell}\right)^2 + a_3\left(\frac{x}{\ell}\right)^2\cdot\left(1 - \frac{x}{\ell}\right)^2 \quad \cdots \quad a_n\left(\frac{x}{\ell}\right)^{n-1}\cdot\left(1-\frac{x}{\ell}\right)^2$$

which satisfies the boundary conditions for the cantilever beam.

Applying the "shaping" equation (Equation 4.5 and 4.6 to Equation 3.5) "s" is given by

$$s = \int_0^{\ell} \frac{2}{3}\left(\frac{Hx^n}{\ell^n}\right)^3\cdot\left(\frac{d^2y}{dx^2}\right)^2 dx \;-\; \frac{\omega^2}{E}\int_0^{\ell}\left(\frac{2Hx^n}{\ell^n}\right)\rho\, y^2 dx + M_L y_m^{\,2}$$

letting R = mass load/beam mass = $\frac{M_L}{M_B}$ and $M_B = \rho$(density) $\cdot\int_0^{\ell} A\cdot dx$(vol.)

then

$$s = \int_0^{\ell} \frac{2}{3}\left(\frac{Hx^n}{\ell^n}\right)^3\cdot\left(\frac{d^2y}{dx^2}\right)^2 dx \;-\; \frac{2\omega^2 H\rho}{E\,\ell^n}\left(\int_0^{\ell} x^n y^2 dx + Ry_m^{\,2}\int_0^{\ell} x^n dx\right)$$

and finally

$$s = \int_0^{\ell}\left(x^n\right)^3\cdot\left(\frac{d^2y}{dx^2}\right)^2 dx - \frac{3\rho\omega^2}{E}\left(\frac{\ell^n}{H}\right)^2\cdot\left(\int_0^{\ell} x^n y^2 dx + Ry_m^{\,2}\int_0^{\ell} x^n dx\right) \qquad \text{EQ. 4.12}$$

Applying the first two terms of the normal function

$$(y = a_1\left(1-\frac{x}{\ell}\right)^2 + a_2\left(\frac{x}{\ell}\right)\left(1-\frac{x}{\ell}\right)^2$$

to Equation 4.12 and performing the necessary operations as developed on page 8 the frequencies in Table II are calculated for bombinations of the shape parameter "n" and the mass ratio "R".[7]

---

[7]Ibid., p. 88.

R ↓	Shape Parameter n				
	0	½	1	2	3
0	$\frac{3.53H}{\ell^2}\sqrt{\frac{E}{3\cdot\rho}}$	$\frac{4.60H}{\ell^2}\sqrt{\frac{E}{3\rho}}$	$\frac{5.31H}{\ell^2}\sqrt{\frac{E}{3\cdot\rho}}$	$\frac{6.04H}{\ell^2}\sqrt{\frac{E}{3\cdot\rho}}$	$\frac{6.88H}{\ell^2}\sqrt{\frac{E}{3\cdot\rho}}$
1	$\frac{1.53H}{\ell^2}\sqrt{\frac{E}{3\cdot\rho}}$	$\frac{1.42H}{\ell^2}\sqrt{\frac{E}{3\cdot\rho}}$	$\frac{1.06H}{\ell^2}\sqrt{\frac{E}{3\cdot\rho}}$	$\frac{0.58H}{\ell^2}\sqrt{\frac{E}{3\cdot\rho}}$	$\frac{0.44H}{\ell^2}\sqrt{\frac{E}{3\cdot\rho}}$
10	$\frac{0.52H}{\ell^2}\sqrt{\frac{E}{3\cdot\rho}}$	$\frac{0.46H}{\ell^2}\sqrt{\frac{E}{3\cdot\rho}}$	$\frac{0.34H}{\ell^2}\sqrt{\frac{E}{3\cdot\rho}}$	$\frac{0.19H}{\ell^2}\sqrt{\frac{E}{3\cdot\rho}}$	$\frac{0.12H}{\ell^2}\sqrt{\frac{E}{3\rho}}$

Table II
Frequency (Rayleigh-Ritz Method) Versus Shape Parameter "n"
and Mass Ratio "R"

Figure 7 shows a plot of frequency coefficient versus shape parameter using the Rayleigh Principle and Rayleigh-Ritz procedure. The calculations by the Rayleigh-Ritz procedure are considered more accurate because the normal function (series of terms) is a closer approximation to the actual deflection curve for the shaped beam (area and inertia not constant).

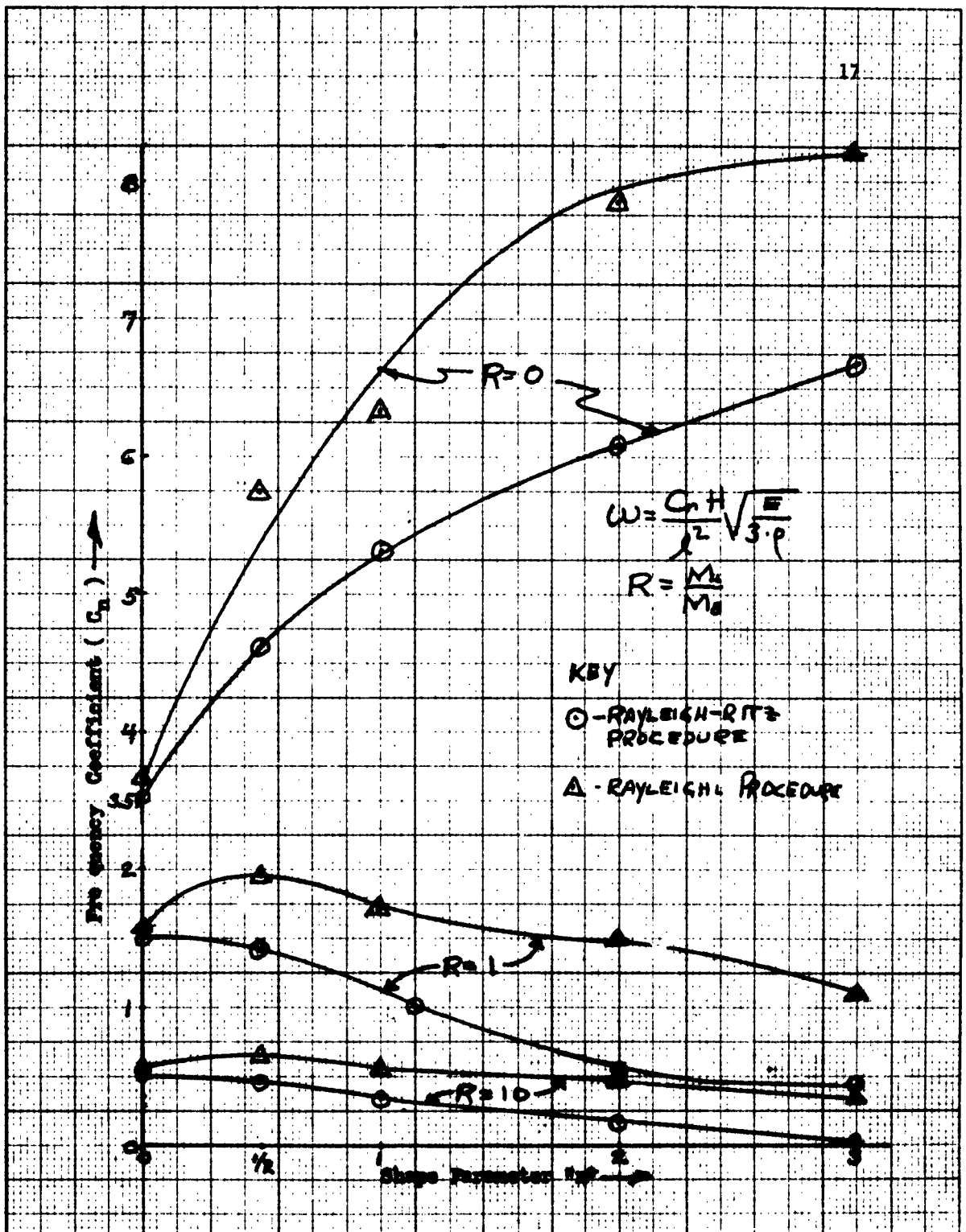

Figure 7.
Frequency Coefficient Versus Shape Parameter "H" Using Both Rayleigh and Rayleigh-Ritz Procedure.

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CHAPTER V

OPTIMIZING THE DYNAMIC SHAPE OF A SIMPLY SUPPORTED BEAM WITH A MASS LOAD AT THE CENTER OF THE BEAM

This Chapter will present the necessary approach to optimize the dynamic shape of a simply supported beam with a mass load (Figure 8).

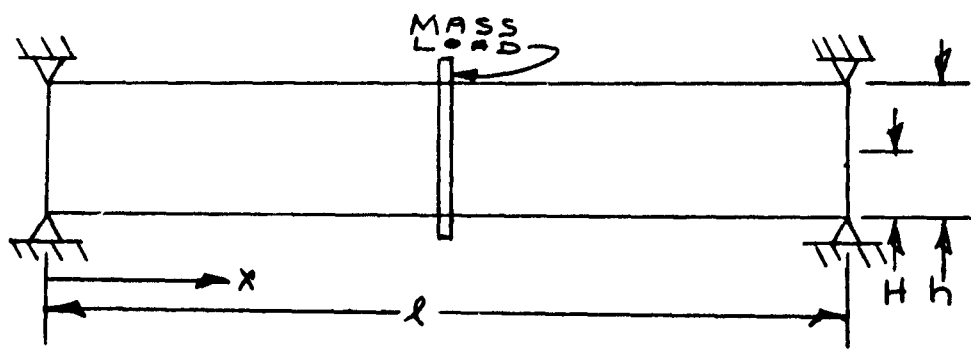

Figure 8.
Simply Supported Beam With a Center Mass Load

Based on the conclusions of the last chapter, the Rayleigh-Ritz procedure is best suited for determining the shape of this beam. The width of the beam in Figure 8 is again assumed to be unity and the depth is varied along the length of the beam. The half height of the beam is given by

$$\frac{h}{2} = \frac{H}{\left(\frac{\ell}{2}\right)^n}\left(\frac{\ell}{2} - x\right)^n \qquad 0 \leq x \leq \frac{\ell}{2} \qquad \text{EQ. 5.1A}$$

and

$$\frac{h}{2} = \frac{H}{\left(\frac{\ell}{2}\right)^n}\left(x - \frac{\ell}{2}\right)^n \qquad \frac{\ell}{2} \leq x \leq \ell \qquad \text{EQ. 5.1B}$$

where "n" defines the shape parameter as shown in Figure 9.

The cross sectional area is

$$A = \frac{2H}{\left(\frac{\ell}{2}\right)^n}\left(\frac{\ell}{2} - x\right)^n \qquad 0 \leq x \leq \frac{\ell}{2} \qquad \text{EQ. 5.2A}$$

and

$$A = \frac{2H}{\left(\frac{\ell}{2}\right)^n}\left(x - \frac{\ell}{2}\right)^n \qquad \frac{\ell}{2} \leq x \leq \ell \qquad \text{EQ. 5.2B}$$

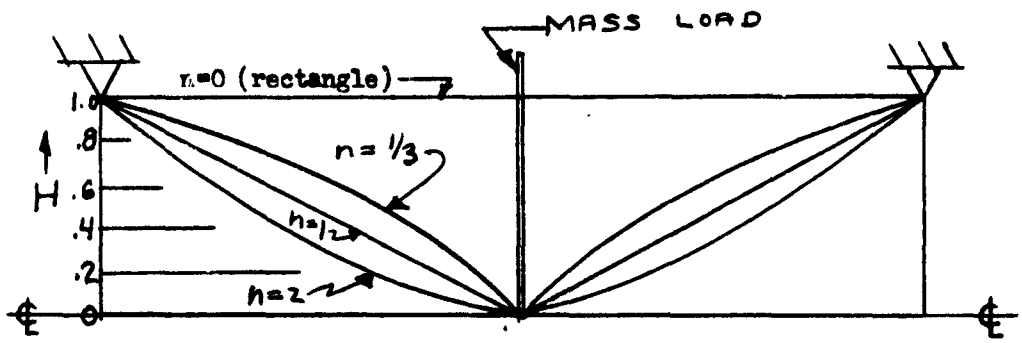

Figure 9.
Shape Parameter for Simply Supported Beam

The moment of inertia is

$$I = \frac{2}{3}\left(\frac{H\left(\frac{\ell}{2} - x\right)^n}{\left(\frac{\ell}{2}\right)^n}\right)^3 \qquad 0 \leq x \leq \frac{\ell}{2} \qquad \text{EQ. 5.3A}$$

and

$$I = \frac{2}{3}\left(\frac{H\left(x - \frac{\ell}{2}\right)^n}{\left(\frac{\ell}{2}\right)^n}\right)^3 \qquad \frac{\ell}{2} \leq x \leq \ell \qquad \text{EQ. 5.3B}$$

The frequency equation (4.11) when applied to the simply supported beam becomes

$$\omega^2 = \frac{E}{3\cdot\rho}\left(\frac{H}{\left(\frac{\ell}{2}\right)^n}\right)^2 \frac{\int_0^{\ell/2}\left(\frac{\ell}{2} - x\right)^{n^3}\cdot\left(\frac{d^2y}{dx^2}\right)^2 dx + \int_{\ell/2}^{\ell} x - \frac{\ell}{2}^{n^3}\left(\frac{d^2y}{dx^2}\right)^2 dx}{\int_0^{\ell/2}\left(\frac{\ell}{2} - x\right)^n y^2 dx + \int_{\ell/2}^{\ell}\left(x - \frac{\ell}{2}\right)^n y^2 dx}$$

$$+ Ry_m^2 \cdot \left(\int_0^{\ell/2}\left(\frac{\ell}{2} - x\right)^n dx + \int_{\ell/2}^{\ell}\left(x - \frac{\ell}{2}\right)^n dx\right) \qquad \text{EQ. 5.4}$$

Applying the Rayleigh-Ritz procedure to Equation 5.4 the "s" term as developed (Equation 3.5) is given by

$$s = \int_0^{\ell/2} \left(\frac{\ell}{2} - x\right)^{n^3} \cdot \left(\frac{d^2y}{dx^2}\right)^2 dx + \int_{\ell/2}^{\ell} \left(x - \frac{\ell}{2}\right)^{n^3} \cdot \left(\frac{d^2y}{dx^2}\right)^2 dx$$

$$- \frac{3\rho\omega^2}{E} \left(\frac{\ell}{2H}\right)^{n^2} \left\{ \int_0^{\ell/2} \left(\frac{\ell}{2} - x\right)^n y^2 dx + \int_{\ell/2}^{\ell} \left(x - \frac{\ell}{2}\right)^n y^2 dx \right.$$

$$\left. + R\, y_m^2 \left(\int_0^{\ell/2} \left(\frac{\ell}{2} - x\right)^n dx + \int_{\ell/2}^{\ell} \left(x - \frac{\ell}{2}\right)^n dx \right) \right\}. \qquad \text{EQ. 5.5}$$

Assuming the normal function to be

$$y = a_1 \sin \frac{\pi x}{\ell} + a_3 \sin \frac{3\pi x}{\ell} + a_5 \sin \frac{5\pi x}{\ell} + \cdots + a_n \sin \frac{n\pi x}{\ell}$$

and using only the first two terms. The frequency for a rectangular beam (n=0) with no load (R=0) is given by

$$s = \int_0^{\ell} \left(\frac{d^2y}{dx^2}\right)^2 dx - \frac{3\rho\omega^2}{E H^2} \int_0^{\ell} y^2 dx \qquad \text{EQ. 5.6}$$

Upon substituting the first two terms of the normal function into Equation 5.6

EQ. 5.7

$$s = \left(\frac{\pi}{\ell}\right)^4 \int_0^{\ell} \left(a_1^2 \sin^2 \frac{\pi x}{\ell} - 18 a_1 a_3 \sin \frac{\pi x}{\ell} \cdot \sin \frac{3\pi x}{\ell} + 81 a_3^2 \sin^2 \frac{3\pi x}{\ell}\right) dx$$

$$- \frac{3\rho\omega^2}{E H^2} \left\{ \int_0^{\ell} \left(a_1^2 \sin^2 \frac{\pi x}{\ell} + 2 a_1 a_3 \sin \frac{\pi x}{\ell} \cdot \sin \frac{3\pi x}{\ell} + a_3^2 \sin^2 \frac{3\pi x}{\ell}\right) dx \right\}$$

$$s = \frac{a_1^2 \pi^4}{2 \ell^3} + 81 \frac{a_3^2 \pi^4}{2 \ell^3} - \frac{3\rho\omega^2 \ell}{2 E H^2} \left(a_1^2 + a_3^2\right) \qquad \text{EQ. 5.8}$$

$$\frac{\partial s}{\partial a_1} = \left(\frac{\pi^4}{\ell^3}\right) a_1 - \left(\frac{3\rho\omega^2 \ell}{E H^2}\right) a_1 = 0 \qquad \text{EQ. 5.9}$$

$$\frac{\partial s}{\partial a_3} = \left(\frac{81\,\pi^4}{\ell^3}\right) a_3 - \left(\frac{3\rho\,\omega^2}{EH}\right) a_3 = 0 \quad , \qquad \text{EQ. 5.10}$$

Solving Equation 5.9 for frequency results in

$$\omega = \frac{\pi^2 H}{\ell^2}\sqrt{\frac{E}{3\cdot\rho}} \qquad \text{EQ. 5.11}$$

which is an exact answer. The solution to Equation 5.10 is the frequency of the third harmonic and it is also an exact answer.

By using Equation 5.5 in the same manner except using values other than n = 0 and R = 0 the frequencies can be found for any shape parameter "n" and any mass load ratio "R". By plotting the frequency coefficient versus the shape parameter the best shape (highest fundamental frequency) of the beam can be determined for a particular ratio of mass load to beam mass.

As mentioned in the introduction the exact answers to many vibration problems are difficult to obtain and approximate solution must be accepted rather than performing numerous numerical sets of calculations. For example; to find the frequency of the simply supported beam with a mass ratio "R" of 1 and a shape parameter "n" of 1, Equation 5.5 must be solved, then differentiated and the resulting homogeneous equations solved to determine the frequency. The solution will be very close to the exact solution <u>provided</u> no mistakes are made. Reviewing Equation 5.5 and substituting n=1 and R = 1 "s" becomes

$$s = \int_0^{\ell/2} \left(\frac{\ell}{2} - x\right)^3 \cdot \left(\frac{d^2y}{dx^2}\right)^2 dx + \int_{\ell/2}^{\ell} \left(x - \frac{\ell}{2}\right)^3 \cdot \left(\frac{d^2y}{dx^2}\right)^2 dx$$

$$- \frac{3\rho\,\omega^2 \ell^2}{4\,EH^2} \left\{ \int_0^{\ell/2} \left(\frac{\ell}{2} - x\right) y^2 dx + \int_{\ell/2}^{\ell} \left(x - \frac{\ell}{2}\right) y^2 dx \right.$$

$$\left. + y_m^2 \left(\int_0^{\ell/2} \left(\frac{\ell}{2} - x\right) dx + \int_{\ell/2}^{\ell} \left(x - \frac{\ell}{2}\right) dx \right) \right\} \qquad \text{EQ. 5.12}$$

performing the necessary multiplication and substituting the first two terms of the normal function, "s" equals

$$\left(\frac{\pi}{\ell}\right)^4 \int_0^{\ell/2} \left(\frac{\ell^3}{8} - \frac{3}{4}\ell^2 x + \frac{3}{2}\ell x^2 - x^3\right) \left(a_1^2 \sin^2 \frac{\pi x}{\ell} + 18\, a_1 \cdot a_3 \cdot \sin \frac{\pi x}{\ell} \cdot \sin \frac{3\pi x}{\ell} + 81\, a_3 \sin^2 \frac{3\pi x}{\ell}\right) dx + \left(\frac{\pi}{\ell}\right)^4 \int_{\ell/2}^{\ell} \left(x^3 - \frac{3}{2}\ell^2 x + \frac{3}{4}\ell x^2 - \frac{\ell^3}{8}\right) \left(a_1^2 \sin^2 \frac{\pi x}{\ell} + 18\, a_1\, a_3 \sin \frac{\pi x}{\ell} \cdot \sin \frac{3\pi x}{\ell} + 81\, a_3 \sin^2 \frac{3\pi x}{\ell}\right) dx$$

$$- \frac{3\rho\,\omega^2 \ell^2}{4\,EH^2} \left\{ \int_0^{\ell/2} \left(\frac{\ell}{2} - x\right) \left(a_1^2 \sin^2 \frac{\pi x}{\ell} + 2a_1\, a_2 \sin \frac{\pi x}{\ell} \cdot \sin \frac{3\pi x}{\ell} + a_3^2 \sin^2 \frac{3\pi x}{\ell}\right) dx \right.$$

$$+ \int_{\ell/2}^{\ell} \left(x - \frac{\ell}{2}\right) \left(a_1^2 \sin^2 \frac{\pi x}{\ell} + 2a_1 a_3 \sin \frac{\pi x}{\ell} \cdot \sin \frac{3\pi x}{\ell} + a_3^2 \sin^2 \frac{3\pi x}{\ell}\right) dx$$

$$+ \left(a_1^2 + 2a_1\, a_3 + a_3^2\right) \int_0^{\ell/2} \left(\frac{\ell}{2} - x\right) dx$$

$$\left. + \left(a_1^2 + 2a_1\, a_3 + a_3^2\right) \int_{\ell/2}^{\ell} \left(x - \frac{\ell}{2}\right) dx \right\} \qquad \text{EQ. 5.13}$$

The solution to Equation 5.12 is very long and will only give the frequency for a beam whose shape parameter is 1 and mass ratio is also 1. Sufficient combinations of "n" and "R" must be taken in the calculations so the frequency coefficient versus the shape parameter can be plotted. From this plot the optimum dynamic shape can be approximated.

CHAPTER VI

CONCLUSION

It ~~has been shown~~ that to accurately determine the optimum dynamic shape of a simply supported beam the process is very long and complicated. However it could be simplified by using the Rayleigh principle if a good deflection curve could be obtained. The accuracy of the Rayleigh principle depends on the normal function (shape of the deflection curve). It is difficult to approximate the normal function of a beam whose area and inertia are not constant with a single term. The normal function can best be approximated by a series of terms as was shown in the Rayleigh-Ritz procedure.

It was pointed out in the introduction that the accuracy of a solution required by an engineer might depend upon how much effort the engineer wants to expend in solving the problem. The accuracy required in shaping a simply supported beam is also a function of how much manpower can be applied to the problem.