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CERTAIN PROBLEMS OF THE THEORY OF
STATISTICAL LINEARIZATION AND ITS APPLICATION

By

I. Ye. Kazakov

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# CERTAIN PROBLEMS OF THE THEORY OF STATISTICAL LINEARIZATION AND ITS APPLICATION

I. Ye. Kazakov

## Summary

We will examine the general aspects of the theory of statistical linearization based on the approximation of an arbitrary nonlinear transformation of a random function by an equivalent linear operator. Applications of this theory to an analysis of the accuracy of nonlinear automatic control systems are given. Practical methods of determining the equivalent operator for a nonlinear transformation of random functions are recommended.

## Introduction

A theoretical probability analysis of the accuracy of nonlinear dynamic systems in the presence of random disturbances is a very complex problem. As a consequence of this, of considerable practical importance for the investigation and designed of nonlinear control systems is the approximate method of statistical linearization of nonlinear transformations which has been developed and is widely used now

[1, 2]. This method is reasonably simple in a practical application since the principal calculations are performed by using the well-developed linear theory of transformation of random functions. This is the main practical virtue of the approximate method.

However, the method of statistical linearization of nonlinear transformations was developed for nonlinear inertia-free elements and is inapplicable to more complex nonlinear transformations. In addition it does not take into account the spectrum change of the random functions at the output of the inertia-free element, which is of interest in certain cases of the calculation of dynamic systems.

The elimination of these shortcomings is possible by the generalization of the existing method of statistical linearization based on the use of the general theory of approximation of random functions [3, 4]. In this report we have developed a series of statistical linearization of nonlinear operators of a general form on the basis of a quadratic approximation of the transformed random function. The application of this theory to analysis of closed nonlinear continuous and discontinuous automatic control systems is examined.

## 1. General Aspects of the Theory of Statistical Linearization of Nonlinear Transformation

Let the random function I(t) be related with the random function X(t) by a nonlinear transformation of the general form

$$Y(t) = f[X(\tau), t]. \tag{1.1}$$

The function f is an arbitrary nonlinear function characterizing the nonlinear transformation realizable over the function X(t) for obtaining the function Y(t). In particular this can be a static characteristic of an inertia-free nonlinear element without a lag and with

a lag or a solution of a nonlinear equation of arbitrary form

$$P\left(t, \frac{d}{dt}\right)Y = F(t, Y, Y', \ldots, Y^{(i)}, X, \ldots, X^{(k)}). \tag{1.2}$$

where F is a nonlinear function; $P\left(t, \frac{d}{dt}\right)$ is a polynomial relative to $\frac{d}{dt}$ with variable coefficients.

In the general case the form of the function f describing the behavior of the output value of some dynamic systems can be unknown. In such a case the dynamic system should be defined physically as a complex of apparatus and the form of the function f relating the output value Y(t) with the input value X(t) should be determined experimentally.

We will approximate the nonlinear transformation (1.1) by a linear dependence between the random functions selected on the basis of a certain criterion of the best approximation of the random function Y(t) by a linear operator applied to the random function X(t). Let us represent the random function X(t) and Y(t) as

$$X(t) = m_x(t) + X^0(t); \tag{1.3}$$

$$Y(t) = m_y(t) + Y^0(t), \tag{1.4}$$

where $m_x(t)$, $m_y(t)$ are the mathematical expectations of the corresponding random functions including the regular components; $X^0(t)$ and $Y^0(t)$ are centered random functions.

We will represent the approximating function U(t) as

$$U(t) = m_u(t) + U^0(t). \tag{1.5}$$

The mathematical expectation $m_u(t)$ and the random component $U^0(t)$ of the function U(t) will be represented in the form

$$m_u(t) = k_0 \int_T W(t, \tau) m_x(\tau) d\tau; \tag{1.6}$$

$$U^0(t) = k_1 \int_T W(t, \tau) X^0(\tau) d\tau, \tag{1.7}$$

where $k_0$ and $k_1$ are the statistical coefficients of amplification; $W(t, \tau)$ is the weight function of the linear system realizing the optimal approximation; T is the region of the change of the variables t and $\tau$ or the interval of the observation of the function $X(t)$.

For practical purposes we can consider two criteria of the linear approximation of random functions. The first criterion consists of fulfilling the condition of equality of the mathematical expectations and the correlation function of the true and approximate random function. The second criterion consists of fulfilling the minimum condition of the mean square value of the difference of the true and approximate random functions.

By using the first criterion we derived the following equation for determining $K_0$, $K_1$, $W(t, \tau)$:

$$m_y(t) = k_0(t) \int_T W(t, \tau) m_x(\tau) d\tau; \tag{1.8}$$

$$K_y(s, t) = k_1(s) k_1(t) \int_T \int_T W(s, \tau) W(t, \tau') K_x(\tau, \tau') d\tau d\tau', \tag{1.9}$$

where $K(s,t)$, $K_x(\tau, \tau')$ are the correlation functions of the random functions of Y and X respectively.

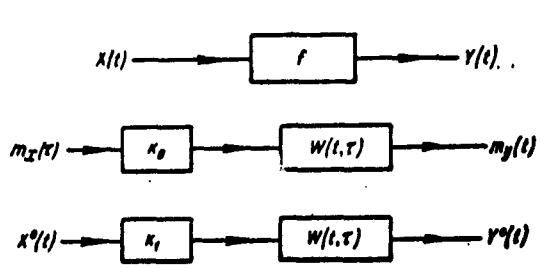


Fig. 1. Equivalent linear transformation.

By using the criterion of the minimum of the root mean square

error at any $s \in T$

$$M\{|Y(s) - U(s)|^2\} = \min,$$

we will obtain for determining the optimal linear approximate operator the equation [4]

$$K_{yx}(s, t) - k_1 \int_T K_x(\tau, t) W(s, \tau) d\tau + m_x(t) \lambda(s) = 0, \tag{1.10}$$

where

$$\lambda(s) = m_y(s) - k_0 \int W(s, \tau) m_x(\tau) d\tau. \tag{1.11}$$

Therefore the nonlinear transformation of the general form (1.1) of the random function X(t) is replaced by two linear operations over the mathematical expectation and random components, as shown in Fig. 1.

For a complete determination of the equivalent linear transformation we must indicate a method of calculating the coefficients $k_0$ and $k_1$ as well as a method for finding the weight function W(s, t).

When approximating with respect to the first criterion we will determine $k_0$ from the condition (1.8)

$$k_0(t) = \frac{m_y(t)}{\int_T W(t, \tau) m_x(\tau) dt}. \tag{1.12}$$

We will take coefficient $k_1$ equal to

$$k_1(t) = \left[\frac{K_y(t, t)}{K_x(t, t)}\right]^{\frac{1}{2}}. \tag{1.13}$$

The weight W(s, t) should be determined from the quadratic integral equation (1.9).

When approximating according to the second criterion we have one equation (1.10) for determining the three characteristics $k_0$, $k_1$, $W(t, \tau)$; therefore, it can also be dealt with by selecting two coefficients $k_0$ and $k_1$. We will determine coefficient $k_0$ in the same

manner as in the first case, by expression (1.12). The coefficient $k_1$ will be determined in the following manner:

$$k_1(t) = \frac{K_{yx}(t, t)}{K_x(t, t)}. \tag{1.14}$$

When determining the coefficient $k_0$ on the basis of equality (1.12), condition (1.11) becomes the following:

$$\lambda(s) \equiv 0. \tag{1.15}$$

On the strength of equality (1.15), equation (1.10) for determining the weight function $W(s, t)$ takes the form

$$\int_t K_x(\tau, t) W(s, \tau) d\tau = \frac{1}{k_1(s)} K_{yx}(s, t). \tag{1.16}$$

Therefore, the determination of the approximate linear operator reduces to a solution of equation (1.16) and a calculation of the coefficients $k_0$ and $k_1$ by formulas (1.12) and (1.14) at given $K_{yx}(s, t)$, $K_x(\tau, t)$, $m_y(s)$, $m_x(\tau)$.

Of practical importance is the case where the interval of observation of the random function $X(t)$ is a semi-infinite interval $-\infty < t < T$. Then equation (1.16) when $s \subset T$ takes the form

$$\int_{-\infty}^{s} K_x(\tau, t) W(s, \tau) d\tau = \frac{1}{k_1(s)} K_{yx}(s, t). \tag{1.17}$$

## 2. Equivalent Linear Transformations for a Nonlinear Circuit

As was shown above the finding of an equivalent linear transformation lies in calculating the coefficients $k_0$ and $k_1$ and in a solution of equation (1.9) relative to $W(s, t)$ when approximating with respect to the first criterion or equation (1.16) when approximating according to the second criterion.

When approximating according to the first criterion a direct solution of equation (1.9) in the general case is difficult. However, in the special case where the interval of the change of variables is semi-infinite and the functions X(t) and Y(t) are stationary, the equivalent weight function on the basis of (1.9) can be determined by the following expression:

$$W(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \Phi(i\omega) e^{i\omega t} d\omega, \tag{2.1}$$

where

$$|\Phi(i\omega)|^2 = \frac{1}{k_1^2} \frac{G_y(\omega)}{G_x(\omega)}; \tag{2.2}$$

$G_y(\omega)$ is the spectral density of the random function Y(t);

$G_x(\omega)$ is the spectral density of the random function X(t).

The determination of the weight function W(s, t) by the second criterion in the general case based on equation (1.16) at given functions $K_{yx}(s, t)$ and $K_x(\tau, t)$ is possible by using the integral canonical concepts of random function [5]. In the special case of practical importance, where the function X(t) is given for a semi-infinite interval $-\infty < t < T$ in the form of an integral canonical presentation

$$X^0(t) = \int_{-\infty}^{t} Z(\lambda) u(t, \lambda) d\lambda, \tag{2.3}$$

where $Z(\lambda)$ is white noise with a correlation function equal to

$$K_z(\lambda, \mu) = G(\lambda) \delta(\lambda - \mu), \tag{2.4}$$

and $u(t, \lambda)$ is the weight function of a certain linear system transforming white noise $Z(\lambda)$ to the random function $X^0(t)$, the solution of equation (1.17) takes the form [5]:

$$W(s, t) = \int_{-\infty}^{t} \frac{u^-(\lambda, t)}{G(\lambda)} d\lambda \int_{-\infty}^{t} \frac{1}{k_1} K_{yx}(s, \tau) u^-(\lambda, \tau) d\tau, \tag{2.5}$$

where $u^{-}(\lambda, t)$ is the weight function of the reverse system transforming the random function X to white noise Z; $G(\lambda)$ is the density of the dispersion of white noise Z.

In an even more particular case, if $m_x$ = const, the function $X^{o}(t)$ is stationary and function $Y^{o}(t)$ is stationarily related with $X^{o}(t)$, then, as shown in [3] and [5], the solution of equation (1.17) takes the form

$$W(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \Phi(i\omega) e^{i\omega t} d\omega, \tag{2.6}$$

where

$$\Phi(i\omega) = \frac{1}{k_1 2\pi \Psi(i\omega)} \int_{0}^{\infty} e^{-i\omega t} dt \int_{-\infty}^{\infty} \frac{G_{yx}(\omega)}{\Psi^{*}(i\omega)} e^{i\omega t} dt; \tag{2.7}$$

$G_x(\omega) = \Psi(i\omega)\ \Psi^{*}(i\omega)$ is the spectral density of the random process $X^{\Psi}(t)$; $G_{yx}(\omega)$ is the mutual spectral density. Here the coefficients $k_0$ and $k_1$ are determined by the following formulas:

$$k_1 = \frac{K_{yx}(0)}{K_x(0)}, \quad k_0 = \frac{m_y}{m_y} \frac{1}{\int_0^{\infty} W(\tau)\, d\tau}. \tag{2.8}$$

## 3. Equivalent Linear Transformation for Nonlinear Inertia-Free Element

The characteristic of a nonlinear inertia-free element is the simplest nonlinear function. This form of nonlinear transformation has been studied in detail in works on statistical linearization. Such an elementary nonlinear transformation can be approximately replaced by an inertia-free linear transformation with with amplification coefficients (transmission factor) $k_0$ and $k_1$ respectively for the mathematical expectation and the random component. This corresponds to the condition $W(t, \tau) = \delta(t - \tau)$ in formulas (1.6) and (1.7).

However, to solve certain problems we need a more accurate approximation of the nonlinear inertia-free transformation based on the above-mentioned theory.

Of practical importance is the case of a stationary input signal. Then with a stationary characteristic of the nonlinear element on the basis of the first criterion, the coefficients $k_0$ and $k_1$ are determined by the expression

$$k_1 = \left[\frac{K_y(0)}{K_x(0)}\right]^{\frac{1}{2}}; \tag{3.1}$$

$$k_0 = \frac{m_y}{m_x}\frac{1}{\Phi(0)}. \tag{3.2}$$

The frequency characteristic and the weight function of the equivalent linear operator are determined by formulas (2.2) and (2.1).

To use formula (2.1) we must know how to calculate the function $G_y(\omega)$ for the nonlinear element. This last operation can be performed fairly simply by assuming a two-dimensional normal distribution law of the random function X(t) at the input to the nonlinear element.*

$$\varphi(x_1, x_2) = \frac{1}{2\pi\sigma_x^2\sqrt{1-k_x^2(\tau)}}\times$$
$$\times\exp\left\{-\frac{(x_1-m_x)^2+(x_2-m_x)^2-2k_x(\tau)(x_1-m_x)(x_2-m_x)}{2\sigma_x^2[1-k_y^2(\tau)]}\right\}. \tag{3.3}$$

Keeping in mind the known expansion of the function of the normal distribution into a series

$$\frac{1}{2\pi\sqrt{1-k^2(\tau)}}\exp\left\{-\frac{x^2+y^2-2kxy}{2[1-k^2]}\right\} = \frac{1}{2\pi}\sum_{n=0}^{\infty}\frac{k^n(\tau)}{n!}\frac{d^n e^{-\frac{x^2}{2}}}{dx^n}\frac{d^n e^{-\frac{y^2}{2}}}{dy^n}, \tag{3.4}$$

we can obtain reasonably simply the expression for the spectral density $G_y(\omega)$ in the form

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* In the general case of arbitrary distribution we must use an orthagonal expansion of this law into a series based on the normal distribution.

$$G_y(\omega) = \sum_{n=0}^{\infty} \frac{A_n^2}{n!} \frac{1}{2\pi\sigma_x^2} \int_{-\infty}^{\infty} k_x^n(\tau) e^{-i\omega\tau} d\tau, \tag{3.5}$$

where

$$A_n = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(\sigma_x x + m_x) \frac{d^n e^{-\frac{x^2}{2}}}{dx^n} dx,$$

$$k_x(\tau) = \frac{1}{\sigma_x^2} \int_{-\infty}^{\infty} G_x(\omega) e^{i\omega\tau} d\omega,$$

and f is the characteristic of the nonlinear element.

When applying the second method for determining the equivalent linear operator in the stationary case to the inertia-free nonlinear element we must use formulas (2.6), (2.7), and (2.8). The calculations by these formulas require a preliminary determination of the function $G_{xy}(\omega)$ for which we need the coupling moment of the functions Y(t) and X(t). The coupling moment can be determined by calculations with the given two-dimensional law of the distribution of random function X(t) or experimentally.

It is necessary to note that in the normal two-dimensional distribution law of the function at the input to a symmetrical inertia-free nonlinear element, as Booton showed [1], the coupling moment of the input and output random functions is proportional to the correlation function of the input random function. This means that the equivalent linear operator in this case is inertia-free and can be characterized by two coefficients: $k_0$ and $k_1$.

The equivalent linear transformation for the inertia-free nonlinear element for the first criterion has recently been investigated by a number of authors. From these investigations and from an analysis of the results of the calculations by formulas (2.7)-(2.8) we can make only preliminary conclusions that approximation of inertia-free

nonlinear transformation with respect to the first criterion more accurately reproduces the first two probability moments of random functions in complex circuits containing the given nonlinear element.

## 4. A Practical Method for Determining the Equivalent Linear Transformation for Nonlinear Circuits

As follows from what has been stated above, the equivalent linear transformation for a nonlinear circuit can be determined if the probability characteristic of the random function at the input and output of the circuit are given. Here the equivalent linear transformation will depend on the form of the nonlinear circuit and on the probability characteristics at the random function at the input.

It follows from physical considerations of the transformation of the random function by a nonlinear circuit that the equivalent nonlinear transformation little depends on the form of the correlation function at the input. These considerations permit us to determine the equivalent linear transformation of a nonlinear circuit in a certain form of the random function at the input to the nonlinear circuit having a correlation function of the $\delta$-function type. Stationary white noise X(t) has such a correlation function.

$$K(\tau, t) = 2\pi G_0 \delta(\tau - t): \tag{4.1}$$

where $G_0$ is the spectral density of white noise, $2\pi G_0$ is its dispersion density. In this case, keeping in mind that the dispersion of white noise is infinite, we will replace $K_x(t, t)$ in formula (1.14) by the corresponding dispersion density $2\pi G_0$. Then we obtain

$$k_1(s) = \frac{K_{xy}(s, s)}{2\pi G_0}. \tag{4.2}$$

Substituting expressions (4.1) and (4.2) into equation (1.17), we will reduce it to the form

$$W(s, t) = \frac{K_{yx}(s, t)}{K_{yx}(s, s)}. \tag{4.3}$$

Introducing the designation

$$k_{yx}(s, t) = \frac{1}{K_{yx}(s, s)} K_{yx}(s, t) \tag{4.4}$$

for the normalized correlation function and showing explicitly the dependence of W(s, t) on the probability characteristics $m_x$ and $\delta_x = \sqrt{2\pi G_0}$ of the input disturbance of the nonlinear circuit, we can rewrite formula (4.3) as

$$W(s, t, m_x, \sigma_x) = k_{yx}(s, t). \tag{4.5}$$

The weight function W found depends on $m_x$, $\delta_x$ and on the form of the nonlinear circuit. The function $k_{yx}(s, t)$ at different values of $m_x$ and $\delta_x$ for this circuit can be obtained experimentally upon the effect on the input of the circuit of white noise with variable characteristics $m_x$ and $\delta_x$.

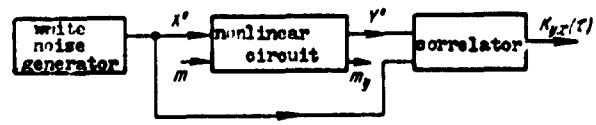


Fig. 2. Experimental determination of the equivalent weight function of a stationary nonlinear circuit.

If the nonlinear circuit is stationary and the mathematical expectation of its input disturbance is constant, then formula (4.5) will yield a stationary weight function

$$W(\tau, m_x, \sigma_x) = k_{yx}(\tau). \tag{4.6}$$

Such a weight function corresponds to the frequency characteristic

$$\Phi(i\omega, m_x, \sigma_x) = \int_0^\infty W(\tau, m_x, \sigma_x) e^{-i\omega\tau} d\tau. \tag{4.7}$$

This formula can be obtained from expressing (2.7) if we take into account that in the case under consideration $\Psi(i\omega) = \Psi^*(i\omega) = \sqrt{G_0}$,

$$k_1 = \frac{K_{yx}(0)}{2\pi G_0} \text{ и } K_{yx}(\tau) = \int_{-\infty}^{\infty} G_{yx}(\omega) e^{i\omega\tau} d\omega.$$

Let us rewrite expression (2.7) in the form

$$\Phi(i\omega, m_x, \sigma_x) = \int_0^\infty e^{-i\omega t} dx \int_{-\infty}^{\infty} g_{yx}(\omega, m_x, \sigma_x) e^{i\omega t} d\omega, \tag{4.8}$$

where

$$g_{yx}(\omega, m_x, \sigma_x) = \frac{G_{yx}(\omega, m_x, \sigma_x)}{K_{yx}(0)}.$$

Therefore, the equivalent linear transformation for a nonlinear circuit can be calculated beforehand depending on the magnitude of the mathematical expectation and the standard deviation of the random function at the input.

A practical method of determining the equivalent linear transformation of a nonlinear stationary circuit is the experimental determination of the relation of the input and output random function $K_{yx}(\tau)$ according to the scheme in Fig. 2 and subsequent calculation of the coefficient $k_0$, $k_1$ and the transfer function $\Phi(p, m, \sigma)$. These calculations are performed by the following formulas:

$$k_1(m, \sigma) = \frac{K_{yx}(0, m, \sigma)}{\sigma^2}; \tag{4.9}$$

$$W(\tau, m, \sigma) = \frac{1}{\sigma^2} K_{yx}(\tau, m, \sigma); \tag{4.10}$$

$$\Phi(i\omega, m, \sigma) = \int_0^\infty W(\tau, m, \sigma) e^{-i\omega\tau} d\tau; \tag{4.11}$$

$$k_0(m, \sigma) = \frac{m_y(m, \sigma)}{m} \frac{1}{\int_0^\infty W(\tau, m, \sigma) d\tau}. \tag{4.12}$$

Example. Let us determine the equivalent linear transformation with respect to the second criterion for a nonlinear circuit, the equation of which has the form

$$(Tp+1)Y = L\,\mathrm{sign}\,[X-Y].$$

When $m_x = m = \text{const}$ and $X^0(t)$ which is stationary white noise having a level of spectral density $G_0$, let the value $m_y$ and the function $K_{yx}(\tau, m, \sigma)$, where $\sigma^2 = 2\pi G_0$, are experimentally determined or calculated. The function $K_{yx}(\tau, m, \sigma)$ for the circuit considered can be approximated by the expression

$$K_{yx}(\tau, m, \sigma) = A(m, \sigma)\, e^{-B(m, \sigma)|\tau|}.$$

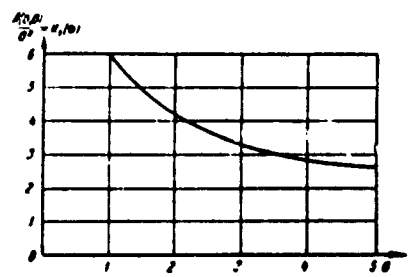

Fig. 3. Graphs of the coefficient $k_1(\sigma)$ for the nonlinear circuit.

From the above formulas (4.9)-(4.12) we will determine successfully the coefficient $k_0$, $k_1$ and the transfer function $\Phi(p, m, \sigma)$

$$k_1(m, \sigma) = \frac{A(m, \sigma)}{\sigma^2}; \quad W(\tau, m, \sigma) = e^{-B(m, \sigma)\tau};$$

$$k_0(m, \sigma) = B(m, \sigma)\frac{m_y(m, \sigma)}{m}; \quad \Phi(p, m, \sigma) = \frac{1}{p + B(m, \sigma)}.$$

Graphs or tables can be compiled for coefficients $A(m, \sigma)$ and $B(m, \sigma)$ as a numerical example let us examine the case: $L = 1$, $T = 0.1$, $m = 0$.

In this case $m_y = 0$. The value of the coefficient $B(0, \sigma)$ coincides with $k_1(\sigma)$, the coefficient $A(0,\sigma) = \sigma^2 k_1(\sigma)$. The graph in Fig. 3 shows the dependence of the coefficient $k_1(\sigma)$ for this case.

## 5. The Use of the Theory of Statistical Linearization for a Theoretical Probability Analysis of Nonlinear Closed Stationary Systems

Let us now examine a closed nonlinear dynamic system, which is shown in Fig. 4. We will write the equations of this system in the form

$$L(p)Z = X - Y; \; Y = f(Z), \tag{5.1}$$

where f is the operator of the nonlinear circuit, L(p) is the linear differential operator. Using statistical linearization of the nonlinear circuit, we will derive two systems of equations. The first system for determining the mathematical expectations of the functions is

$$L(p)m_z = m_x - m_y; \; \Psi(p, m_z, \sigma_z)m_y = k_0 m_z, \tag{5.2}$$

where $\Psi(p, m_z, \sigma_z) = \frac{1}{\Phi(p, m_z, \sigma_z)}$ is the equivalent linear operator of the nonlinear circuit; $k_0(m_z, \sigma_z)$ is the statistical coefficient of amplification of the nonlinear circuit with respect to the mathematical expectation.

To determine the random components of the functions we will derive the system of equations

$$L(p)Z^\circ = X^\circ - Y^\circ, \; \Psi(p, m_z, \sigma_z)Y^\circ = k_1 Z^\circ, \tag{5.3}$$

where $k_1(m_z, \sigma_z)$ is the statistical coefficient of amplification of the circuit with respect to the random component.

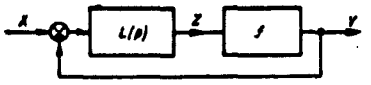


Fig. 4. Block diagram of a nonlinear closed circuit.

The linear theory of the transformation of random functions is fully applicable upon integration of systems of equations (5.2) and (5.3). The solution of these equations must be done by the method of successive approximation with the use of the graphs of the coefficient $k_0(m, \sigma)$, $k_1(m, \sigma)$, of the analytical expressions for the operator $\Psi(p, m, \sigma)$, and the graphs of its coefficients.

The theory of statistical linearization is formally extended to linear systems of discontinuous automatic control by replacement of the appropriate continuous type of differential equations by equations in finite differences.

## Conclusions

An approximate theoretical probability analysis of nonlinear dynamic systems based on the theory of statistical linearization of nonlinear operators opens broad opportunities in the investigation of nonlinear continuous and discontinuous automatic control systems for which the first two probability moments of random functions are sufficient characteristics.

Two criteria of the approximation of a random function were examined: 1) the criterion of the equality of the first two probability moments and 2) the criterion of the minimum of the mean square value of the difference of functions. When approximating the nonlinear operators the second criterion is more simple in a practical sense.

Of practical value is the approximate method of determining the equivalent linear operator with respect to the second criterion based on an input signal in the form of a stationary random white noise.

A combination of the principle of statistical linearization with experimental methods of deriving the dynamic characteristic of a nonlinear circuit enables us to investigate real components whose equations are unknown. Here the experimental determination of the dynamic characteristic of the nonlinear circuit consists of calculating the mutual correlation function of the output and input upon a random signal at the input in the form of a stationary white noise. A normalized mutual correlation function coincides with the weight function of an equivalent linear operator.

A further development of the theory of statistical linearization of nonlinear operators should apparently be carried out in the following directions: 1) an investigation of the accuracy of the method of statistical linearization of nonlinear operators and the obtainment of the fundamental and practical estimates of the accuracy of the calculation; 2) an investigation of the limit of the application of the method of statistical linearization for open and closed dynamic systems; 3) a study of typical cases of applying the principle of statistical linearization of nonlinear circuits and the accumulation of factual material on the equivalent characteristics (compilation of tables, graphs, and algorithms of the calculation of equivalent characteristics).

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## DISCUSSION

### Questions

I. A. Bol'shakov (USSR). Of considerable interest are closed systems whose nonlinearity lies in an element separating the error signal. If the characteristic of this element is at infinity or simply equals zero, then a stationary distribution of the error of tracing does not exist and the method of statistical linearization is deprived of a physical meaning. However, in one of the recent works of investigator Kazakov, the method of linearization was applied to this type of nonlinearity. Does the author of the report consider this legal?

J. M. Milsem (Canada). I would like to ask Mr. Kazakov how valid is his methods for dynamic nonlinearities in the presence of a closed control circuit which now by definition are nonlinear? As we are all in agreement, the distribution of the probability density can considerably differ from gaussian which, by assumption, is at the input of the circuit.

In addition does the author have any comments relative to an analytical forecast of the distortion of the distributions of the probability density in these nonlinear closed circuits?

## Statements

R. L. Stratonovich (USSR). The problem of statistical linearization is very important for an analysis of statistical processes enclosed automatic control systems when selecting their optimal parameters. At the same time if the calculation of the correlation functions needed for linearization is a problem of the same degree of difficulty as the initial problem (which must be solved by linearization), then the method to a considerable extent is assured. The generalization of the method proposed by Kazakov for the case of inertia transformation requires verification from this point of view.

Heretofore insufficient attention has been devoted to the limits of applicability of the method of statistical linearization. I will attempt to enumerate the conditions of the applicability of the method and the possible sources of errors corresponding to them.

1. It is necessary that the higher moments (third, fourth, and others) be disregarded for the problem under consideration.

2. It is necessary that a distribution law of a known form be at the input of the nonlinear element in spite of the presence of feedback. I. Ye. Kazakov used the gaussian law of distribution, whereas in nonlinear systems with a large amplification factor yielding a small error the gaussian law does not take place. This is shown in particular in the first and second examples of Barret's report. Fortunately the gaussian law is not necessary for the application of the linearization method in the generalized form. For example, in Barret's examples linearization can be fulfilled by means of the distribution laws that were found. Hence we see that linearization is an effective method in combination with other methods which serve to determine the probability density. Here it is necessary mainly to

mention such a vigorous method of analysis of inertia systems as the apparatus of the Markow processes.

3. The following shortcoming has heretofore been characteristic of proposed theories of linearization. They have not taken into account the fundamental difference of nonlinear transformations from linear transformations which is manifested in the nonlinear case by the inequality

$$K_{yy} K_{xx} - K_{xy}^2 > 0 \tag{1}$$

(whereas in linear transformations this magnitude equals zero). This effect can be called "losses of correlation" (in contrast to "losses of information" which does not occur). As a consequence of this inequality upon replacing the nonlinear transformation $y = f(x)$ by the linear transformation $y' = \alpha + \beta x$ it is impossible to satisfy with two coefficients the three equalities:

$$\overline{y'} = \overline{y};\ K_{y'x} = K_{yx};\ K_{y'y'} = K_{yy};\quad (K_{ab} = \overline{ab} - \bar{a}\bar{b}). \tag{2}$$

Two variations of the method arise. In the first $\alpha$ and $\beta$ are determined from the equality $\overline{y} = \overline{y}'$; $K_{yy} = K_{y'y'}$, and in the second they are determined by means of equalities $\overline{y} = \overline{y}'$; $K_{yx} = K_{y'x}$. The third equality remains unsatisfied. This defect can be eliminated. We will introduce A, the operator destroying the correlation, and we will find the nonlinear transformation in the form

$$y' = \alpha + \beta x + \gamma A x \tag{3}$$

(in order not to develop specially a rule with action with operator A, it is convenient (in the stationary phase) to take it in the form of a shift operator $e^{-pT}$ for a certain indeterminably large time interval T). Then the three magnitudes can satisfy all three equalities (2)

and we obtain:

$$\alpha = \bar{y} - (\beta + \gamma)\bar{x}; \quad \beta = \frac{K_{yx}}{K_{xx}}; \quad \gamma = \frac{\sqrt{K_{yy}K_{xx} - K_{xy}^2}}{K_{xx}}. \tag{4}$$

We will apply this method to example 2 of Barret's report. The control system is described by the equation

$$\ddot{x} + a\dot{x} = -f(x) + \xi. \tag{5}$$

We will set $\overline{\xi} = 0$; $\overline{\xi\xi_\tau} = N\delta(\tau)$; $f(-x) = -f(x)$. In this case the stationary distribution (Boltzmann distribution) has the form

$$W_{st}(x) = Ce^{-\frac{2u(x)}{N}}; \quad \left(u(x) = -\int f(x)\,dx; \quad C^{-1} = \int e^{-\frac{2u(x)}{N}}\,dx\right).$$

Therefore, the magnitude entering into (4) equal

$$\begin{gathered} \bar{y} = 0; \quad K_{yx} = C\int f(x)\,xe^{-\frac{2u(x)}{N}}\,dx; \\ K_{yy} = C\int f^2(x)\,e^{-\frac{2u(x)}{N}}\,dx. \end{gathered} \tag{6}$$

Substituting (3) and (5) and solving this equation, we find the spectral density of fluctuating of the x-coordinate

$$S_x(\omega) = \frac{N}{\omega^4 + (a^2 - 2\beta)\,\omega^2 + \beta^2 + \gamma^2}; \tag{7}$$

If we use the imperfect method, then with the first variant the term $-2\beta$ would be absent in the demoninator of expression (7) and with the second variant, the term $\gamma^2$.

4. When performing linearization, strictly speaking it is necessary to satisfy inequality (2) in which the magnitude x, y pertain to one instant of time, and the corresponding equalities for different instants of time are:

$$\overline{y'(t)} = \overline{y(t)}; \quad K_{y'x}(t, t') = K_{yx}(t, t'); \quad K_{y'y'}(t, t') = K_{yy}(t, t') \tag{8}$$

or in the stationary case

$$\bar{y}' = \bar{y}; \quad K_{y'x}(\tau) = K_{xy}(\tau); \quad K_{y'y'}(\tau) = K_{yy}(\tau); \ (\tau = t_2 - t_1).$$

Then even in the case of the inertia-free nonlinear transformation $y = f(x)$, approximation (3) will be replaced by the inertia transformation

$$y'(t) = \alpha(t) + \int \beta(t, t')x(t')dt' + A\int \gamma(t, t')x(t')dt', \tag{9}$$

where $\beta$, $\gamma$ satisfy equations:

$$\begin{gathered} \int \beta(t, s)K_{xx}(s', t')ds = K_{yx}(t, t'); \\ \iint [\beta(t, s)\beta(t', s') + \gamma(t, s)\gamma(t', s')][K_{xx}(s, s')ds\,ds' = K_{yy}(t, t'). \end{gathered} \tag{10}$$

An increase in the accuracy corresponding to a change to a complete system (8) is associated, however, with an increase of difficulties in the calculation. The main difficulty is to find the correlation functions $K_{xx}$, $K_{xy}$, $K_{yy}$ of a closed system. It is necessary to note that in example (7) linearization facilitates determination of the unknown correlation function of the x-coordinates and, consequently, the variable y. As for equations (10), in the stationary case they can by the method of the Fourier transform. Since we are dealing with a calculating method, we need not observe the condition of the physical realization of transformation (9).

J. F. Barret (England). I wish to cite the work of Dr. A. T. Fuller where the problem of statistical linearization for this specific example is examined.

S. S. L. Chan (USA). I was greatly impressed by the comment of Mr. Stratonovich concerning the loss of correlation in nonlinear systems. However, I have certain doubts. While the output magnitude y is as a whole determined by the input noise $\xi$, heretofore y and $\xi$ have

been correlated in the strict sense of this word. The loss of correlation sooner occurs because of the method of determining correlation functions of the second order than because of an actual loss of correlation between two signals.

Another point of view of this problem is associated apparently with the use of gaussian-type signals for which the joint distribution is fully determined by correlation functions of the second order, which is invalid for random signals of another type. Correlation functions cannot represent signals in nonlinear systems like they represent signals in linear systems. In nonlinear systems even with an input signal with a gaussian distribution, the distribution of the output magnitude is not gaussian.

A. A. Pervozvanskiy (USSR). The problem of the limits of the applicability of the statistical linearization method has two sides: 1) a revelation of the possibilities of the method assuming that the distribution laws differ from normal; 2) determination of a class system for which the method is effectively applicable.

The second question concerns the need to establish the difference between the distribution laws at the output of a circuit consisting of inertia-free nonlinearity and a linear filter. An estimate of the difference with respect to higher moments is difficult.

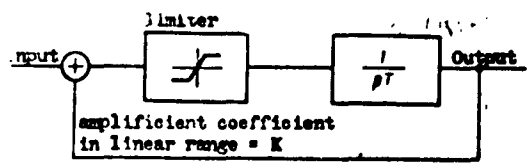
It is expeditious to use the extreme property of entropy of a normal process established by Shannon. Somewhat generalizing it we can assert that the normal process has the highest entropy per degree of freedom among all processes having an identical determinant composed of correlation function.

A calculation of the entropy at the output of the aforementioned circuit is not too complex. The expression for entropy of a normal process with a given correlation determinant was given by Shannon.

A comparison of these expressions permits us to answer the problem concerning the degree of normalization. Such an application of the methods of the information theory can be of substantial interest.

J. K. Lubbock (England). I have two comments relative to this report.

1. The author asserts that when using the second criterion for an inertia-free nonlinear element in a stationary system at the output of which the signal has its gaussian distribution, the equivalent linear operator becomes inertia-free; this result was proved by Nuttall [1] for a more general class of input processes which he called a separable class. This is valid also for an even more general semi-separable class* of processes [2].




2. A method of approximation of such a kind is extremely valuable for analysis and synthesis of nonlinear systems, in particular nonlinear closed systems, but one should be very careful when using it since such cases can be encountered where the error of approximation can not be disregarded. I don't know, did the author investigate the order of error which should be expected in any specific practical situation. In one of his early works, Booton [3] cited an example of a closed system, shown in the figure, which proved to be unsatisfactory because the error of the system was about 10 times greater than the

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* It is not clear in what respect these classes pertain to seperable random processes determined in mathematics. See for example J. L. Dub, probability processes. ILL, M., 1956. (Editors comment).

magnitude of the input signal. In such an unreal case the input signal of the nonlinear limiting has almost a gaussian distribution (if the signal at the input of the system is gaussian), but if amplification of the limiter increases or the time constant of the integrator decreases so that it operates more satisfactorily as a repeator, then the method of linearization leads to an equivalent system which is a very poor approximation of the initial nonlinear system.

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V. S. Pugachuv (USSR). The method of statistical linearization is in essence an approximate method. Therefore, it can be used only for an approximate physical analysis of nonlinear systems. This method in principle is inapplicable for an exact solution of such problems as, for example, determination of the distribution laws of parameters of state of a nonlinear circuit. It should be considered from this point of view.

The idea of the method of statistical linearization is fully analogous to the idea of the method of harmonic linearization or harmonic balance in the deterministic theory of nonlinear systems. Both these methods are convenient practical instruments for an approximate calculation of nonlinear systems. This explains the extensive practical use of the method of harmonic linearization, the ever growing

interest in the method of statistical linearization, and the great number of works devoted to this method and its application which have appeared recently in various countries. The report of Kazakov gave a further generalization of this method which took into account the distortion of the correlation function upon propagation of random functions through nonlinear elements.

From a practical point of view it is important that when using the general method of statistical linearization described by Kazakov, we can perform statistical linearization of nonlinear elements directly from the experimental data.

It should be noted that the improvements in the method of statistical linearization proposed here in the report of R. A. Stratonovich did not yield anything new in comparison with the general method of statistical linearization proposed in the report of Kazakov, since the formula for the output signal of the approximate linear element taken by Stratonovich is, evidently, a spectral case of the more general formula contained in Kazakov's report.

I. Ye. Kazakov answers questions asked of him.

Question. Is it possible to determine the law of the distribution of a random process at the input to a nonlinear element in a closed system by using the method of statistical linearization?

Answer. By using the method of statistical linearization it is impossible in principle to determine the distribution law at the input to a nonlinear element in a closed circuit. For this purpose we need a more complete and accurate method.

Question. Can the method of statistical linearization be used for nonlinear elements having characteristics of a type of assymptotic curve?

Answer. In principle the equivalent statistical coefficient of amplification can be determined for any nonlinear dependences. These statistical coefficients of amplification have a statistical sense.

Question. In what cases can we use the usual method of statistical linearization when examining inertia-free nonlinearities in a closed system and in what cases is it necessary to use the generalized theory presented in the report?

Answer. When estimating the stability of a closed circuit having nonlinear elements it is necessary to use the more complete statistical characteristics of nonlinear elements which take into account the changes of the spectrum upon propagation of a signal through the nonlinear element.

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