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INVARIANT IMBEDDING AND RAREFIED GAS DYNAMICS

J. Aroesty, R. Bellman, R. Kalaba and S. Ueno

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PREFACE

This Memorandum is concerned with the application of invariant imbedding to the study of rarefied gas dynamics. A new formulation of the problem of linearized Couette flow is presented using ideas which were first developed in astrophysics and neutron-transport theory. This study should be of interest to specialists in the fields of aerodynamics and heat transfer who are concerned with the calculation of flow fields in the upper atmosphere, as well as to astrophysicists and nuclear physicists.

SUMMARY

This Memorandum applies the techniques of invariant imbedding to the study of rarefied gas flows. The problem of linearized Couette flow is investigated, and it is shown how the assumption of the Krook scattering model results in a formulation which is similar to that obtained in radiative transfer for conservative isotropic scattering in a plane-parallel atmosphere.

By a simple enumeration of physical processes, the nonlinear integral-differential equation governing the reflection function is obtained, and a suitable transformation is shown to render this equation amenable to numerical computation.

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INVARIANT IMBEDDING AND RAREFIED GAS DYNAMICS

1. INTRODUCTION

The use of invariance principles and invariant imbedding techniques, originating in the work of Ambarzumian and Chandrasekhar, has provided new insights and computational solutions to various problems in the fields of radiative transfer and neutron transport theory [1]. Let us indicate how these same techniques can be applied to representative problems in the kinetic theory of gases. The possibility of this is suggested by the essential similarity of all particle processes: the behavior of neutrons or photons, or the structureless spherically symmetric molecule of the classical kinetic theory. For the present, we shall restrict our area of interest to linearized scattering models, finite collision frequencies, and relatively simple geometries. We leave to future study the application of invariance techniques to nonlinear scattering models and to cases in which the Boltzmann gain and loss operators are only conditionally convergent. A classic problem, that of plane shear flow between two infinite flat plates, will be used to present the ideas of invariant imbedding in the context of rarefied gas dynamics.

2. KINETIC-THEORY APPROACH

Various kinetic-theory approaches to the problem of plane shear flow are described in some detail by Willis [2]. The physical picture is the following.

An infinite plane wall moves with a small constant velocity V_w in its plane at a distance d from a fixed wall. Both walls are maintained at the same temperature, and may consist of either similar or dissimilar materials. It is desired to find the gas velocity and the stress at the moving wall as a function of the degree of rarefaction, which is measured by an appropriate Knudsen number. In the body of the gas, we shall use the single relaxation time model of Krook [3] in its linear version. Linearization is made possible by requiring that the Mach number of the moving wall be much less than unity, and then using standard perturbation techniques.

We define the following quantities:

$$(2.1) \qquad n = \int\int_{-\infty}^{+\infty}\int f d\vec{v}, \quad \tau = \delta n_0 \sqrt{h_0}\, d,$$

$$\bar{c} = \sqrt{\bar{u}^2 + \bar{v}^2 + \bar{w}^2},$$

$$n\bar{V} = \int\int_{-\infty}^{+\infty}\int f\bar{v} d\vec{v}, \quad u,v,w = \sqrt{h}\,\bar{u}, \sqrt{h}\,\bar{v}, \sqrt{h}\,\bar{w},$$

$$\frac{3n}{2h} = \int\int_{-\infty}^{+\infty}\int f(\bar{v} - \bar{V})^2\, d\vec{v}, \quad y = \frac{x}{d}.$$

The nonlinear Krook [3] equation for this one-dimensional geometry can be written

$$(2.2) \qquad \frac{\bar{u}df}{dx} = -\,\delta n[f - n(\frac{h}{\pi})^{3/2}\exp(-\,h(\bar{c} - \bar{V})^2)].$$

The linearized version of this equation, obtained by setting $\sqrt{h}\,\bar{V}_w \ll 1$,* (where V_w is the velocity of the moving wall) is

$$(2.3) \qquad \frac{\bar{u}df}{dx} = -\,\delta n[f - n(\frac{h}{\pi})^{3/2}\exp(-\,h\bar{c}^2)(1 + 2h\bar{v}\,\bar{V})].$$

If $f = n_0(\frac{h_0}{\pi})^{3/2}\exp(-\,h_0\bar{c}^2) + f_1$, where $f_1 = n_0(\frac{h_0}{\pi})^{3/2}\exp(-\,h_0\bar{c}^2)v\Phi(u,x)$, then the equation for the perturbation distribution function may be written

$$(2.4) \qquad \frac{udf_1}{dy} = \tau\left[-\,f_1 + \frac{2e^{-c^2}}{\pi^{3/2}}\,v\int\int_{-\infty}^{+\infty}\int f_1\,v\,dudvdw\right].$$

The boundary conditions corresponding to purely diffuse reflection from the wall are

(2.5) (a) Fixed plate: $f_1^+(x = 0) = 0$,

(b) Moving plate: $f_1^-(x = d) = n_0(\frac{h_0}{\pi})^{3/2}\,e^{-c^2}\cdot 2\,v\,M_w$, where $M_w = \sqrt{h}\,V_w$.

It is possible to develop an equation for the function which is closely related to the radiative transport equation for isotropic scattering in a plane slab, i.e.,

$$(2.6) \qquad \frac{ud\Phi}{dy} = \tau\left[-\,\Phi + \frac{1}{\sqrt{\pi}}\int_{-\infty}^{+\infty} e^{-u^2}\,\Phi\,du\right].$$

The diffuse boundary conditions associated with this equation are

$$(2.7) \qquad y = 0,\ \ \Phi^+ = 0,\ \ y = 1,\ \ \Phi^- = 2\sqrt{h}\,V_w.$$

---

*This is equivalent to requiring that the Mach number be everywhere small. $\sqrt{h}\,V_w$ is proportional to the Mah number of the wall.

Willis has obtained a numerical solution to equation (2.6) by recasting it in a form similar to the Milne integral equation of radiative transfer. In addition, he has shown that straightforward Neumann iteration is suitable only for the very rarefied case, corresponding to $\tau < 1$. Since each subsequent calculation in the Neumann iteration scheme corresponds to higher-order collisions, it is not surprising that it is rather slowly convergent as the collision frequency is increased. This is quite similar to the state of affairs in radiative transfer, where it is known that a calculation based on successive absorption and scattering processes is ill suited for computation as slab thickness is increased, or conversely, as the optical mean free path is decreased.

3. INVARIANT IMBEDDING APPROACH

The invariant-imbedding approach permits us to concentrate our attention on the values of quantities at the boundaries. This is particularly relevant to problems in heat transfer and aerodynamics, where values of certain molecular fluxes at the boundary are related to such macroscopic observables as energy transport and momentum transport to the walls. Once values have been obtained for the distribution function at the boundary, however, it is possible to calculate values of the distribution function within the body of the gas by relatively simple techniques.

The introduction of other boundary conditions besides the presently chosen diffuse re-emission is postponed so that we may focus on the essentials of the technique. In addition, it should be noted that our choice of one moving wall and one fixed wall is somewhat different from the usual Couette flow problem, where the two walls move in opposite directions. This choice was made because it obviated the need for the calculation of transmission functions. The method of linearization and the diffuse boundary conditions at the fixed wall result in the fixed wall acting as a "sink" for perturbing molecules, in the sense that it emits only particles which possess the equilibrium distribution function.

Consider Fig. 1. A reflection function $\rho(d,u,u_0',v,v_0',w,w_0')$ is defined which relates incoming molecules at d to outgoing molecules at d:

$$(3.1)\quad f_1^+(u,v,w) = \int_0^{+\infty}\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} \rho(d,u,u_0',v,v_0',w,w_0') \cdot f_1^-(u_0',v_0',w_0')\,du_0'dv_0'dw_0'.$$

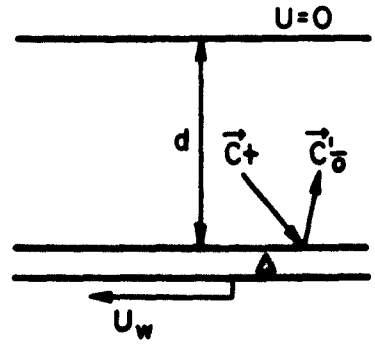

Fig. 1

Linearity of the transport equation permits us to consider a Dirac δ-function incident distribution centered about u_0, v_0, w_0. Thus $\rho(d,u,u_0,v,v_0,w,w_0)$ is the outgoing distribution function of particles at d due to an incident monochromatic beam at d with velocity u_0, v_0, w_0. We observe that the sink-like character of the fixed diffuse wall guarantees that all outgoing perturbing particles at the moving wall are due solely to the disturbance generated by it. Simple particle-counting techniques are utilized to relate the reflection function $\rho(d,u,u_0,v,v_0,w,w_0)$ at d to the reflection function $\rho(d + \Delta,u,u_0,v,v_0,w,w_0)$ at $d + \Delta$. We recognize that the original disturbing beam of molecules at $d + \Delta$ is modified by interactions within Δ, and the resulting incoming distribution function at d undergoes the same sort of processes at d as the original beam underwent at $d + \Delta$. A new problem is initiated at d, which differs from the problem initiated at $d + \Delta$ by the fact that the incident distribution is continuous rather than monochromatic.

We introduce the mean free path as the unit of length, and use $\tau = \delta\, n_0 \sqrt{h_0}\, d$ rather than d for the spacing between plates.

The spirit of the technique is exhibited in the simple particle-counting approach. To terms of order $(\Delta)^2$, $\rho(\tau + \Delta,u,u_0,v,v_0,w,w_0)$ is equal to the results

of the following processes:*

(a) The incoming monochromatic beam is diminished by collisions in Δ, reflected at τ, and then diminished again by collisions in Δ.

(b) The incoming monochromatic beam is scattered within Δ from u_0, v_0, w_0 to u, v, w.

(c) The incoming beam is reflected at τ, and then scattered into u, v, w within Δ.

(d) The incoming beam is scattered within Δ, then reflected at τ into u, v, w.

(e) The incoming beam is reflected at τ, scattered in Δ, and then reflected again at τ.

Taking account of these interactions, we have

$$(3.2)\quad \rho(\tau+\Delta, u, u_0, v, v_0, w, w_0)$$

$$= [1 - \frac{\Delta}{u}]\rho(\tau, u, u_0, v, v_0, w, w_0)[1 - \frac{\Delta}{u_0}] + \frac{2v}{(\pi)^{3/2}u} e^{-c^2} v_0 \Delta$$

$$+ 2e^{-c^2} \frac{v\Delta}{(\pi)^{3/2}u} \int_0^{\infty} \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} \rho(\tau, \dot{u}, u_0, \dot{v}, v_0, \dot{w}, w_0) \dot{v} d\dot{u} d\dot{v} d\dot{w} +$$

---

*A consequence of the linearized Krook scattering model is that $\Delta\tau f_1(u,v,w)/u$ is the number of particles which are scattered out of the element of velocity space centered about u, v, w, and

$$\Delta\tau \frac{2e^{-c^2} v}{(\pi)^{3/2}u} \int\int_{-\infty}^{+\infty}\int f_1 v_1 du_1 dv_1 dw_1$$

is the number of particles scattered into this element, all in an infinitesimal thickness Δ.

$$+ \frac{2}{(\pi)^{3/2}} \Delta \int_0^{\infty} \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} e^{-C'^2} \frac{v_0 v'}{\dot{u}} \rho(\tau,u,\dot{u},v,\dot{v},w,\dot{w}) d\dot{u} d\dot{v} d\dot{w}$$

$$+ \frac{2}{(\pi)^{3/2}} \Delta \int_0^{\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \rho(\tau,u,\bar{u},v,\bar{v},w,\bar{w}) e^{-\bar{C}^2} \bar{v} d\bar{u} d\bar{v} d\bar{w}$$

$$\cdot \int_0^{\infty} \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} \rho(\tau,\dot{u},u_0,\dot{v},v_0,\dot{w},w_0) \dot{v} d\dot{u} d\dot{v} d\dot{w}.$$

The requirements of Sec. 2 suggest the transformation

$$(3.3) \qquad \rho(\tau,u,u_0,v,v_0,w,w_0) = \frac{2}{\pi^{3/2}} e^{-C^2} \frac{vv_0}{u} S(\tau,u,u_0).$$

Letting $\Delta \to 0$, we obtain the following symmetric form for the one-dimensional reflection function $S(d,u,u_0)$:

$$(3.4) \quad \frac{dS}{d\tau}(\tau,u,u_0) + \left(\frac{1}{u} + \frac{1}{u_0}\right) S(\tau,u,u_0)$$

$$= 1 + \frac{1}{\sqrt{\pi}} \int_0^{\infty} e^{-u_0^2} S(\tau,u,u_0) \frac{du_0}{u_0} + \frac{1}{\sqrt{\pi}} \int_0^{\infty} e^{-u^2} S(\tau,u,u_0) \frac{du}{u}$$

$$+ \frac{1}{\pi} \int_0^{\infty} e^{-u^2} S(\tau,\dot{u},u_0) \frac{d\dot{u}}{\dot{u}} \int_0^{\infty} e^{-\bar{u}_0^2} S(\tau,u,\bar{u}_0) \frac{d\bar{u}_0}{\bar{u}_0}.$$

The initial condition associated with (3.4) is $S(0,u,u_0) = 0$, resulting from the observation that $\tau = 0$ corresponds to the free-molecule limit when there are no collisions within d to reflect molecules from u_0 into u.

The last equation closely resembles the radiative-transport equation for isotropic scattering in a plane

slab that Chandrasekhar also obtains by invariance principles [4]. A significant difference, however, between radiative transport and kinetic theory is that in kinetic theory, incoming distribution functions are not generally of a delta-function character, so that it is necessary to integrate over all incoming velocities in order to obtain the distribution function at the wall.

The computational solution of (3.4) has been readily obtained using techniques which were originally developed for problems in radiative transfer. In forthcoming publications we shall discuss this, consider the inclusion of nondiffuse particle-surface interaction at the moving wall, and present asymptotic observations which are relevant to the problems of near-free-molecule flow and slip flow.

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