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Thomas J. Watson Research Center, Yorktown Heights

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ON ABSTRACT DUAL LINEAR PROGRAMS *

by

A. J. Hoffman
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1. Introduction

This note examines the duality theorem of linear programming in the context of a general algebraic setting. It is well known that, when the constants and variables of primal and dual programs are real numbers (or any ordered field), then (i) any value of the function to be maximized does not exceed any value of the function to be minimized; and (ii) max = min. Property (i) is a triviality, and property (ii) depends on the hyperplane separation theorem [3], the simplex method [2] or some other argument [4]. All of the arguments used to prove (ii), however, seem to depend on the properties of a field. The proof of (i), however, does not; in fact, its triviality will persist in the abstract setting described below in Section 2. We then formulate some questions, which it is the main purpose of this note to advertise. That these questions have some interest will be illustrated in Section 3, where the duality theorem will be shown to hold in some unusual surroundings.

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2. Abstract Formulation of Linear Programming Duality

We shall be concerned only with that portion of the duality theorem which considers properties (i) and (ii) mentioned in the introduction.

We assume that we deal with a set S which contains all the constants and all possible values of our variables. In addition, S admits the operations of addition (under which S is a commutative semi-group); multiplication (under which S is a semi-group); and multiplication is distributive with respect to addition. Furthermore, S is partially ordered under a relation "$\leqq$" satisfying $a \leqq b$ implies $x + a \leqq x + b$ for all $x \in S$. Finally, S admits a subset $P \subset S$ such that $a \leqq b$, $x \in P$ implies $xa \leqq xb$ and $ax \leqq bx$.

We now formulate two dual linear programs; $A = (a_{ij})$ is an m by n matrix; $b = (b_1, \ldots, b_m)$ is a vector with m components; $c = (c_1, \ldots, c_n)$ is a vector with n components; all entries in S.

Problem 1: Choose n elements $x_1, \ldots, x_n$ of S so that

$$\sum_j a_{ij} x_j \leqq b_i \qquad (i=1, \ldots, m) \tag{2.1}$$

$$x_j \in P \qquad (j=1, \ldots, n) \tag{2.2}$$

in order to maximize

$$\sum_j c_j x_j . \tag{2.3}$$

The meaning of (2.3) is that we seek elements $x_1^0, \ldots, x_n^0$ which satisfy (2.1) and (2.2) such that, if $x_1, \ldots, x_n$ are any elements satisfying (2.1) and (2.2), we have

$$\sum_j c_j x_j \leqq \sum_j c_j x_j^0 \tag{2.4}$$

Problem 2: Choose m elements $y_1, \ldots, y_m$ satisfying

$$\sum_i y_i a_{ij} \geqq c_j \qquad (j = 1, \ldots, n) \tag{2.5}$$

$$y_i \in P \qquad (i = 1, \ldots, m) \tag{2.6}$$

in order to minimize

$$\sum_i y_i b_i \,. \tag{2.7}$$

Remarks analogous to (2.4) explain the meaning to be attached to (2.7).

Before proving property (i), let us note that

$$\begin{aligned} &a_i \leqq b_i \qquad i = 1, \ldots, k \\ &\text{implies } \sum_i a_i \leqq \sum b_i \end{aligned} \tag{2.8}$$

To prove (2.8), it is clearly sufficient, by induction, to prove it in the case $k = 2$. But $a_1 \leqq b_1$ implies $a_1 + a_2 \leqq b_1 + a_2$. Also, $a_2 \leqq b_2$ implies $b_1 + a_2 \leqq b_1 + b_2$. Hence $a_1 + a_2 \leqq b_1 + b_2$, by the transitivity of partial ordering.

To prove property (i), let $x_1, \ldots, x_n$ satisfy (2.1) and

(2.2), $y_1, \ldots, y_m$ satisfy (2.5) and (2.6), and one sees that the usual proof applies. For, consider

$$\sum_j \sum_i y_i a_{ij} x_j = \sum_i \sum_j y_i a_{ij} x_j . \tag{2.9}$$

The right-hand side of (2.9) is

$$\sum_i y_i \left(\sum_j a_{ij} x_j\right) .$$

Since $\sum_j a_{ij} x_j \leqq b_i$, we have

$$y_i \sum_j a_{ij} x_j \leqq y_i b_i ,$$

since $y_i \in P$; and

$$\sum_i y_i \left(\sum_j a_{ij} x_j\right) \leqq \sum_i y_i b_i ,$$

by (2.8). Similarly, the left side of (2.9) is

$$\sum_j \left(\sum_i y_i a_{ij}\right) x_j \geqq \sum_j c_j x_j .$$

By the transitivity of partial ordering,

$$\sum_j c_j x_j \leqq \sum_i y_i b_i ,$$

which is property (i).

We now pose the following problems:

A. Find all (some) sets S satisfying the postulates such that, if (2.1), (2.2), (2.5), (2.6) have solutions, then the maximum of (2.3) and the minimum of (2.7) exist and are equal - i.e., duality holds.

Two examples of such sets S will be given in the next section.

B. If S is a set satisfying the postulates, for which duality fails, find all matrices A with the property: if $b_1, \ldots, b_m$ and $c_1, \ldots, c_n$ are taken so that (2.1), (2.2), (2.5) and (2.6) have solutions, then duality holds for this matrix A.

As an example of problem B, let S be the set of integers, P the nonnegative integers, multiplication, addition and "$\leq$" have the usual meanings. Then duality does not hold in general. The class of matrices A for which it does hold are the totally unimodular matrices [1], [5].

3. Examples of Sets S For Which von Neumann Duality Holds

Example 1: Let U be a set, S any algebra of subsets of S (denote the complement of a by $\bar{a}$, interpret multiplication and addition as intersection and union respectively, "$\leq$" means "$\subset$", and P = S).

Theorem 3.1: In Example 1, duality holds.

Proof: Observe that (2.1) and (2.2) always have solutions; trivially, we can set $x_j = \phi$ for every j. Also, (2.5) and (2.6) have solutions if and only if $\sum_i a_{ij} \geq c_j$ for every j, which we shall assume. It is now straightforward to show that

$$x_j = \prod_i (\overline{a_{ij}} + b_i) \quad , \qquad (j = 1, \ldots, n)$$

and

$$y_i = \sum_j c_j (\overline{b}_i + a_{ij} \prod_k (\overline{a}_{kj} + b_k a_{kj})) \qquad (i = 1, \ldots, m)$$

verify (2.1), (2.2), (2.5), (2.6) and the equality of (2.3) and (2.7).

Example 2: Let S be the set of positive fractions. We shall say that a/b if (b/a) is an integer. Let multiplication in S be ordinary multiplication, addition in S be (g.c.d.), "$\leqq$" mean "|", P = S.

Example 3: Let S be the set of all integers. Multiplication in S is ordinary addition, addition in S is min (i.e., a + b = min (a, b)), "$\leqq$" in S is the ordinary inequality, P = S.

Theorem 3.2: In Examples 2 and 3, duality holds.

Proof: We first remark that, by considering the exponents of each prime number present in each fraction, we see that duality for Example 2 will follow from Example 3, which we now treat.

Clearly (2.1), (2.2), (2.5) and (2.6) have solutions. In Problem 1, we seek $\{x_j\}$ in order to maximize

$$\min_j \{c_j + x_j\} , \tag{3.3}$$

where

$$\min_j \{a_{ij} + x_j\} \leqq b_i \qquad i = 1, \ldots, m. \tag{3.4}$$

Let j (i) be any mapping of $\{1, \ldots, m\}$ into $\{1, \ldots, n\}$,

and let

$$(3.5) \qquad \bar{x}_k = \left\{ \begin{array}{l} \infty \text{ if } k \neq j(i) \text{ for any } i \\ \min\ (b_i - a_{ik}) \text{ over all } i \text{ such that } k = j(i) \end{array} \right\}.$$

Clearly, $\{\bar{x}_k\}$ satisfy (3.4), and (3.3) becomes

$$\min_k \left\{ \begin{array}{l} \infty \text{ if } k \neq j(i) \text{ for any } i \\ \min\ (c_k + b_i - a_{ik}), \text{ over all } i \text{ such that } k = j(i) \end{array} \right\}.$$

Another way of stating this value of (2.3) is as follows: the mapping $j(i)$ picks out certain entries in the matrix $(c_j + b_i - a_{ij})$, and (3.3) is the least of those entries. In particular, we may select a mapping $j(i)$ so that

$$c_{j(i)} + b_i - a_{ij(i)} = \max_j\ c_j + b_j - a_{ij}\,, \qquad (i = 1, \ldots, m).$$

Thus, we can obtain a value for (3.3) which is the minimum of the row maxima of $(c_j + b_i - a_{ij})$.

For Problem 2,

$y_i = \max_j\ \{c_j - a_{ij}\}$ satisfies (2.5) and (2.6), and (2.7) becomes $\min_i \max_j \{b_i + c_j - a_{ij}\}$. This is the same as the solution we found for Problem 1, proving the theorem.

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