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CONDITIONAL MEANS AND COVARIANCES OF NORMAL VARIABLES WITH SINGULAR COVARIANCE MATRIX

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Let \((\xi, \eta)\) be a partitioned zero mean normal random vector with covariance matrix \((A \ B')\) where \(\text{cov}(\xi) = A\) and \(\text{cov}(\eta) = C\), with \(\text{cov}()\) meaning the covariance matrix of the random vector enclosed. The usual way of getting the conditional mean and covariance matrix of \(\xi\), given that \(\eta = \beta\), is to divide the joint density of \(\xi\) and \(\eta\) by that of \(\eta\). The following is an alternative method which is more general, in that it does not require that \(\xi\) and \(\eta\) have a joint density, or even that \(\eta\) have a density.

This is the result we want to prove: The conditional mean and covariance of \(\xi\), given that \(\eta = \beta\), are

\[
E(\xi | \eta = \beta) = \beta C^+ B, \quad \text{cov}(\xi | \eta = \beta) = A - B'C^+B
\]

where \(C^+\) is the pseudoinverse of \(C\), that is, if \(C = TT^T\) with \(T r \times m\) of rank \(r\), then \(C^+ = T'(TT')^{-2}T\). If \(C^{-1}\) exists, then \(C^+ = C^{-1}\). The pseudoinverse of a symmetric matrix, although perhaps not under that name, is well known and has been used in statistics for some time. For a recent discussion and references, see [1].

Let \(E = T'(TT')^{-1}T = C^+ C = CC^+\) be the projector of the row space of \(C\). Note that \(CE = C\), and hence for any matrix \(F\) whose rows are in the row space of \(C\), \(FE = F\). In particular, \(B'\) in the covariance matrix above satisfies \(B'E = B'\), since the rows of \(B'\) are in the row space of \(C\). (The general covariance matrix
may be assumed to have the form $(S')'(S U) = (S'S U'S U')$, and the row space of $S'U$ lies in the row space of $U$, which has the same row space as $UU$.)

We will derive the formulas for conditional mean and covariance of $\zeta$, given $\eta = \beta$, by representing $\zeta$ in such a way that it is obvious what conditioning on $\eta$ means. We need only the fact that the sum of two normal random vectors is normal, and that if $\eta$ has covariance $C$, then $\eta M$ has covariance $M'C'M$. Let $\zeta$ be a zero mean normal random vector which is independent of $\eta$ and which has covariance $A - B'C^+B$. (This is a valid covariance matrix, for example, that of $\zeta - \eta C^+B$.) Then $\zeta = \zeta + \eta C^+B$ is our representation for $\zeta$, since the covariance matrix of $(\zeta, \eta) = (\zeta, \eta)(I_{C^+B} I)$

\[
\begin{pmatrix}
  I & B'C^+ & A - B'C^+B & 0 \\
  0 & I & 0 & C \\
  0 & 0 & I & C^+B \\
  0 & 0 & 0 & I
\end{pmatrix}
= \begin{pmatrix}
  A & B' \\
  B & C
\end{pmatrix}.
\]

Since $\zeta$ and $\eta$ are independent, it is obvious that the conditional mean of $\zeta + \eta C^+B$, given that $\eta = \beta$, is $\beta C^+B$, and that the conditional covariance is that of $\zeta$. Hence our general result: If $(\zeta, \eta)$ is a zero mean normal vector with $\text{cov}(\zeta, \eta) = \begin{pmatrix} A & B' \\ B & C \end{pmatrix}$, $\text{cov}(\zeta) = A$, and $\text{cov}(\eta) = C$, then the expected value and covariance of $\zeta$, given that $\eta = \beta$, are

$E(\zeta | \eta = \beta) = \beta C^+B$,
$\text{cov}(\zeta | \eta = \beta) = A - B'C^+B$

where $C^+$ is the pseudoinverse of $C$, that is, $C^+ = C^{-1}$ if $C^{-1}$ exists, otherwise, if $C = T'T$ with $T \in \mathbb{R}^{r \times m}$ of rank $r$, then $C^+ = T'(TT')^{-2}T$. 