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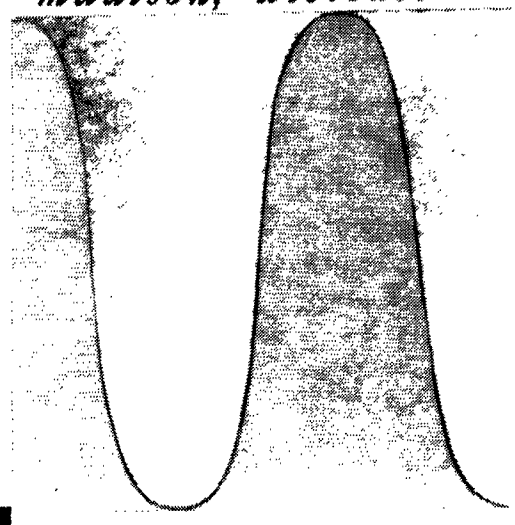

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IDEMPOTENTS IN GROUP ALGEBRAS

Walter Rudin

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ABSTRACT

It is known that the idempotent elements in the complex group algebra of any commutative group have some rather striking properties. The present note exhibits simple examples of idempotents on non-commutative groups which fail to have these properties.

IDEMPOTENTS IN GROUP ALGEBRAS

Walter Rudin

I. INTRODUCTION. If G is a group, its group algebra $L^1(G)$ consists of all complex functions f on G for which the norm

$$\|f\| = \sum_{x \in G} |f(x)| \tag{1}$$

is finite; addition is pointwise, and multiplication is defined by convolution:

$$(f * g)(x) = \sum_{y \in G} f(y)\, g(y^{-1}x) \quad . \tag{2}$$

Any $f \in L^1(G)$ for which

$$f * f = f \tag{3}$$

will be called an <u>idempotent on</u> G .

The <u>support</u> of a complex function f on G is the set of all $x \in G$ at which $f(x) \neq 0$. The <u>support group</u> of f is the smallest subgroup of G which contains the support of f .

By methods involving Fourier transforms and the Pontryagin duality theory, the idempotents on abelian groups are completely known [2; p. 199]. (For non-discrete locally compact abelian groups, the classification of the idempotent

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measures was completed by P. J. Cohen [1].) Let us draw attention to the following facts, of which (A) and (D) are probably the most striking:

(A) If f is an idempotent on an abelian group G, <u>then the support group of</u> f <u>is finite</u>.

(B) Idempotents on abelian groups are <u>self-adjoint</u> (i.e., $f(x^{-1})$ is the complex conjugate of $f(x)$).

(C) On a finite abelian group there are only <u>finitely many</u> idempotents (namely 2^n if the group has n elements). On a countable abelian group there are <u>at most countably many</u> idempotents.

(D) If f is an idempotent on an abelian group and if $\|f\| > 1$, then $\|f\| \geq \frac{1}{2}\sqrt{5}$ [3; p. 72]. (Note that there are no idempotents f with $\|f\| < 1$, except $f = 0$.)

<u>It is the purpose of the present note to show that each of the above statements becomes false if the word "abelian" is omitted</u>.

II. Consider a set E which contains the integers and the three symbols α, β, γ, let

$$(4) \qquad \begin{cases} a = (\alpha\beta\gamma) \\ b = (\beta\gamma)(\ldots -2\ -1\ 0\ 1\ 2 \ldots) \end{cases}$$

be permutations of E, and let G be the group generated by a and b. The relations

$$a^3 = 1, \quad b^{2k-1} a = a^2 b^{2k-1} \tag{5}$$

hold for all integers k, and G consists of the distinct elements

$$a^n b^k \qquad (n = 0, 1, 2; \ k = 0, \pm 1, \pm 2, \ldots) . \tag{6}$$

Setting $\omega = \exp\{2\pi i/3\}$, define

$$f_0(a^n b^k) = \begin{cases} \frac{1}{3}\omega^n & \text{if } k = 0 , \\ 0 & \text{if } k \neq 0 , \end{cases} \tag{7}$$

and

$$f_j(x) = f_0(xb^{-j}) \qquad (x \in G; \ j = 0, \pm 1, \pm 2, \ldots) . \tag{8}$$

I claim that

$$f_0 * f_j = f_j \quad \text{and} \quad f_{2m-1} * f_j = 0 \tag{9}$$

for all integers j and m. Indeed,

$$(f_0 * f_j)(a^n b^j) = \sum_{r=0}^{2} f_0(a^{n-r}) f_j(a^r b^j)$$

$$= \frac{1}{9} \sum_{r=0}^{2} \omega^{n-r} \cdot \omega^r = f_j(a^n b^j) ,$$

whereas (5) shows that

$$(f_{2m-1} * f_j)(a^n b^{2m-1+j}) = \sum_{r=0}^{2} f_{2m-1}(a^{n-r} b^{2m-1}) f_j(a^{-r} b^j)$$

$$= \frac{1}{9} \sum_{r=0}^{2} \omega^{n-r} \omega^{-r} = 0$$

If now c_m are complex numbers such that $\sum_{-\infty}^{\infty} |c_m| < \infty$, and if

$$f = f_0 + \sum_{-\infty}^{\infty} c_m f_{2m-1} \tag{10}$$

then

$$\|f\| = 1 + \sum_{-\infty}^{\infty} |c_m| < \infty \quad , \tag{11}$$

and the equations (9) show that $f * f = f$.

Taking infinitely many $c_m \neq 0$, we thus obtain <u>idempotents on</u> G <u>with infinite support</u> (and, a fortiori, with infinite support group). The example $f = f_0 + f_1$ shows that <u>there exist idempotents on</u> G <u>with finite support but infinite support group.</u> Equation (11) shows that <u>every number</u> ≥ 1 <u>is the norm of some idempotent on</u> G . Unless all c_m are 0, <u>the idempotents</u> (10) <u>are not self-adjoint</u>.

III. I have not succeeded in proving the existence of self-adjoint idempotents with infinite support, but it is easy to give examples in which the support group is infinite.

Put

$$(12)\qquad \begin{cases} a = (\alpha\beta\gamma)(12)(34)(56)\ldots , \\ b = (\alpha\beta\gamma)(23)(45)(67)\ldots . \end{cases}$$

Then ab has infinite order, so that the group G generated by a and b is infinite. The relations $a^2 = b^2$, $a^6 = b^6 = 1$ hold. Define $g_1(a^n) = \frac{1}{6}$, $g_1 = 0$ elsewhere; $g_2(b^n) = \frac{1}{6}\exp\{\frac{n\pi i}{3}\}$, $g_2 = 0$ elsewhere. Then

$$(13)\qquad g_1 * g_1 = g_1,\ g_2 * g_2 = g_2,\ g_1 * g_2 = g_2 * g_1 = 0 .$$

Hence $g = g_1 + g_2$ is an idempotent on G whose support S is finite. Since $a \in S$ and $b \in S$, G is the support group of g; and since g_1 and g_2 are self-adjoint, so is g.

IV. Even on a <u>finite</u> group there can be uncountably many idempotents, both self-adjoint and non-self-adjoint. To see this, let G be the non-cyclic group of order 6, with generators a and b. The relations $a^3 = b^2 = 1$, $ba = a^2 b$ hold. If p, q, r are complex numbers, subject to

$$(14)\qquad p^2 + pq + q^2 = \frac{1}{12} - r^2 ,$$

and if

$$(15)\qquad \begin{cases} f(1) = \frac{1}{3}, \quad f(a) = -\frac{1}{6} + ir, \quad f(a^2) = -\frac{1}{6} - ir, \\ \\ f(b) = p + q, \quad f(ab) = -p, \quad f(a^2 b) = -q, \end{cases}$$

explicit computation shows that $f * f = f$. If r is real and $12 r^2 < 1$, then p and q can be taken real in (14), and the resulting idempotents f are self-adjoint. If r is not real, f is not self-adjoint.

V. We conclude with a positive result:

<u>Theorem</u>. <u>If</u> f <u>is an idempotent on</u> G <u>and if</u> $\|f\| = 1$, <u>then the support of</u> f <u>is a finite subgroup</u> H <u>of</u> G, <u>and</u>

$$(16)\qquad f(xy) = |H|\, f(x)\, f(y) \qquad (x, y \in H) .$$

Here $|H|$ denotes the number of elements of H . We sketch the proof. Let S be the support of f , let $m = \max |f(x)|$ $(x \in G)$, and let H be the set of all $x \in G$ at which $|f(x)| = m$. Clearly H is finite. For $x \in H$, we have

$$(17)\qquad \Big|\sum_y f(y)\, f(y^{-1} x)\Big| = m .$$

Since $\|f\| = 1$, (17) is only possible if $y^{-1} x \in H$ for every $y \in S$, i.e., if $S^{-1} H \subset H$. Since $H \subset S$, it follows that H is a group, and then that $S = H$.

Also, $|f(x)| = |H|^{-1}$ on H. The equation $f(x) = \sum f(y)\, f(y^{-1}x)$ then forces the arguments of $f(y)\, f(y^{-1}x)$ to be equal to the argument of $f(x)$, for all $x, y \in H$, and this gives (16).

Since non-negative idempotents have norm 1 or 0, the above theorem characterizes them as well.

Finally, observe that (16) implies that $f(xy) = f(yx)$ for all $x, y \in G$. In other words, <u>all idempotents of norm</u> 1 <u>lie in the center of the group algebra</u>. It would be interesting to know whether statement (A) of the Introduction is true for all central idempotents.

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