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A Note on Generating Chi Random Numbers

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Introduction

Marsaglia [1] has given a simple method for generating exponential random numbers on a digital computer. We present a similar method for generating random numbers with the chi distribution. Such random numbers may be used to generate normal random numbers.

I. The chi distribution (of rank two) $F$ is

$$F(a) = \begin{cases} 0, & a \leq 0, \\ 1 - e^{-a^2/2}, & 0 < a. \end{cases}$$

Let $x$ be a random variable with the distribution $F$. Let

$$G_c(a) = \text{Prob}(x \leq a| x \leq c) \text{ where } 0 < c.$$ Then for $0 \leq a \leq c$,

$$G_c(a) = \frac{(1 - e^{-a^2/2})(1 - e^{-c^2/2})}{(1 - e^{-c^2/2})}$$

$$= 1 - \sum_{k=1}^{\infty} q_k (1 - a^2/c^2)^k,$$

where

$$q_k = (c^2/2)^k/[k!(e^{c^2/2} - 1)].$$

Let

$$H_c(a) = \begin{cases} 1 - e^{-(a^2-c^2)/2}, & c < a, \\ 0, & a \leq c. \end{cases}$$

Then

$$F(a) = (1 - e^{-c^2/2})G_c(a) + e^{-c^2/2}H_c(a).$$

Thus a random number with the distribution $F$ may be generated as follows. Generate a uniform random number $u$, i.e., a random number uniformly distributed on $(0,1)$. If $u < (1 - e^{-c^2/2})$, 


generate a random number with the distribution $G_c$; otherwise generate one with distribution $H_c$.

A random number $y$ with the distribution $H_c$ may be generated by setting $y = \sqrt{\lambda r + c^2}$, where $r$ is a random number with the exponential distribution.

A random number $x$ with the distribution $G_c$ can be generated by setting

$$x = c \cdot \min[\max(u_1, u_2), \ldots, \max(u_{2z-1}, u_{2z})],$$

where the $u_i$ are independent uniform random numbers, and $z$ is a random integer taking on the value $k$ with probability $q_k$. This fact is easily verified by noting that the distribution of $x$ is just the series (1).

For a binary computer the best choice for $c$ is $c = 2$. On the IBM 7090 computer the average time to generate a chi random number $x$ by this method is 112 cycles. (A cycle is 2.14 microseconds on this computer). This assumes that the exponential random numbers are generated by the method given in [1].

If $x$ is generated by setting $x = \sqrt{2r}$, the average time is 165 cycles.

II. To generate normal random numbers we make use of the following well-known fact. Let $(\alpha, \beta)$ be the rectangular coordinates of a random point uniformly distributed on the unit circle. Then if $x$ is a chi random number $y = \alpha x$ and $z = \beta x$ are independent standard normal random numbers.
The following methods for generating such a pair \((\alpha, \beta)\) are well known.

**Method 1.** Test independent pairs of uniform numbers \((u, v)\) until a pair is found which satisfies \(u^2 + v^2 \leq 1\). Then set \(\alpha = u / \sqrt{u^2 + v^2}\) and \(\beta = v / \sqrt{u^2 + v^2}\).

**Method 2.** Test independent pairs \((u, v)\) until a pair is found which satisfies \(u^2 + v^2 \leq 1\). Then set \(\alpha = 2uv / (u^2 + v^2)\) and \(\beta = (v^2 - u^2) / (u^2 + v^2)\).

To generate normal random numbers we can use the following procedure. Generate a chi random number \(x\). If \(x < c\), use method 2 to generate \((\alpha, \beta)\). If \(c \leq x\) use method 1 to generate \((\alpha, \beta)\). Note that we can decide if \(c \leq x\) before we set \(x = \sqrt{c^2 + 2r}\). Therefore this square root operation can be combined with that used to generate \((\alpha, \beta)\). In effect when \(c \leq x\), we compute a pair of normal random numbers \(y\) and \(z\) by

\[
y = x\alpha = u[(2r + c^2) / (u^2 + v^2)]^{1/2}
\]

and

\[
z = x\beta = v[(2r + c^2) / (u^2 + v^2)]^{1/2}.
\]

This procedure takes 156 cycles to generate one normal random number on the 7090.
REFERENCES