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BISTATIC CROSS SECTIONS OF CYLINDRICAL WIRES

J. T. deBettencourt

PICKARD & BURNS, INC.
103 Fourth Avenue
Waltham 54, Massachusetts

Subsidiary Of
GORHAM CORPORATION

Scientific Report No. 1
Contract No. AF30(602)-2412

Prepared For
ROME AIR DEVELOPMENT CENTER
Air Research and Development Command
United States Air Force
Griffiss Air Force Base
New York

F&B Pub. No. 735A

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FOREWORD

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Patricia Concannon developed the digital computer program and obtained and plotted the data for the figures.
ABSTRACT

Bistatic scattering cross sections of a thin, perfectly conducting cylinder are developed using the variational method. This method was used by Tal \(^1\) for obtaining monostatic (back scatter) radar cross sections. Resulting expressions are developed for wires of 3/2 \(\lambda_o\) length or less, where \(\lambda_o\) is the free space wavelength. The wires are assumed to be in air or vacuo.

The resulting calculations apply to the case where the incident electric vector lies in the incident plane and the receiving antenna is polarized parallel to the scattering plane.

Curves are plotted for \(\sigma_{MS}/\lambda_o^2\) vs \(\theta_o\) where \(\sigma_{MS}\) is the monostatic scattering cross section in the same units as \(\lambda_o^2\), for various values of \(2h/\lambda_o\) in the range \(0.5 \leq 2h/\lambda_o \leq 1.5\) for a thickness parameter \(\Omega = 2 \log_e 2h/a = 10\) and 13.6. The wire length and radius are \(2h\) and \(a\), respectively. \(\theta_o\) is the angle of incidence (and back scattering) with respect to the wire. Results for monostatic cross sections agree with those of others for their values of \(\Omega\) and \(2h/\lambda_o\).

For bistatic cross sections \(\sigma_{BS}/\lambda_o^2\) curves are given for \(\sigma_{BS}/\lambda_o^2\) vs \(\theta\) for \(\Omega = 10\), in the range \(0.5 \leq 2h/\lambda_o \leq 1.5\), with \(\theta_o\) as the parameter. \(\theta\) is the scattering or reflection angle with respect to the wire.

The curves for bistatic scattering show aspect and frequency sensitivity. Double humps or lobes are generally asymmetric in \(\theta\) for given values of \(\theta_o\) and particularly for wire lengths greater than 0.75 \(\lambda_o\).
Thus "forward scattering" gives larger cross sections than back scattering under certain combinations of $\theta_0$ and $2h/\lambda_0$.

If the incident wave from a transmitter is polarized such that $\psi_T$ is the polarizing angle of the incident electric vector with respect to the plane of incidence, the curves can be used by multiplying $\sigma/\lambda_0^2$ by $\cos^2 \psi_T$. If, in addition, the receiver is polarized at an angle $\psi_R$ with respect to the scattering or reflection plane, the values of $\sigma/\lambda_0^2$ should be multiplied by $(\cos^2 \psi_T \cos^2 \psi_R)$.

Application of the curves of $\sigma/\lambda_0^2$ to determination of total transmission losses in a bistatic system are discussed, with a sample calculation.
# TABLE OF CONTENTS

<table>
<thead>
<tr>
<th>Section</th>
<th>Page</th>
</tr>
</thead>
<tbody>
<tr>
<td>I. INTRODUCTION</td>
<td>1</td>
</tr>
<tr>
<td>II. MONOSTATIC (RADAR) CROSS SECTIONS - VARIATIONAL EXPRESSIONS OF TAI</td>
<td>4</td>
</tr>
<tr>
<td>III. BISTATIC SCATTERING CROSS SECTIONS</td>
<td>20</td>
</tr>
<tr>
<td>IV. THE CURRENT DISTRIBUTION $I(z)$</td>
<td>32</td>
</tr>
<tr>
<td>V. APPLICATION VIA TRANSMISSION EQUATION TO TRANSMISSION LOSSES</td>
<td>45</td>
</tr>
<tr>
<td>VI. REFERENCES</td>
<td>49</td>
</tr>
<tr>
<td>APPENDIX I. EVALUATION OF INTEGRALS USED IN SCATTERING COEFFICIENTS</td>
<td></td>
</tr>
<tr>
<td>I-I. INTRODUCTION</td>
<td>50</td>
</tr>
<tr>
<td>I-II. THE INTEGRAL $g_c$</td>
<td>51</td>
</tr>
<tr>
<td>I-III. THE INTEGRAL $g_b$</td>
<td>52</td>
</tr>
<tr>
<td>I-IV. THE $\gamma_c$ INTEGRAL</td>
<td>53</td>
</tr>
<tr>
<td>I-V. THE $\gamma_b$ INTEGRAL</td>
<td>54</td>
</tr>
</tbody>
</table>
### LIST OF ILLUSTRATIONS

<table>
<thead>
<tr>
<th>Figure</th>
<th>Description</th>
<th>Page</th>
</tr>
</thead>
<tbody>
<tr>
<td>1.</td>
<td>Geometry of Incident Electric Wave Vector and Wire Scatterer</td>
<td>5</td>
</tr>
<tr>
<td>2.</td>
<td>Angular Distribution of Back Scatter Cross Section</td>
<td>9</td>
</tr>
<tr>
<td>3.</td>
<td>Angular Distribution of Back Scatter Cross Section</td>
<td>10</td>
</tr>
<tr>
<td>4.</td>
<td>Angular Distribution of Back Scatter Cross Section</td>
<td>11</td>
</tr>
<tr>
<td>5.</td>
<td>Angular Distribution of Back Scatter Cross Section</td>
<td>12</td>
</tr>
<tr>
<td>6.</td>
<td>Angular Distribution of Back Scatter Cross Section</td>
<td>13</td>
</tr>
<tr>
<td>7.</td>
<td>Angular Distribution of Back Scatter Cross Section</td>
<td>14</td>
</tr>
<tr>
<td>8.</td>
<td>Angular Distribution of Back Scatter Cross Section</td>
<td>15</td>
</tr>
<tr>
<td>9.</td>
<td>Angular Distribution of Back Scatter Cross Section</td>
<td>16</td>
</tr>
<tr>
<td>10.</td>
<td>Angular Distribution of Back Scatter Cross Section</td>
<td>17</td>
</tr>
<tr>
<td>11.</td>
<td>Polar Plot of Back Scatter $\frac{\theta}{\lambda_o Z}$</td>
<td>18</td>
</tr>
<tr>
<td>12.</td>
<td>Geometry of Bistatic Scattering ($\mathbf{E}_{inc}$ Lies in Plane of Incidence)</td>
<td>21</td>
</tr>
<tr>
<td>13.</td>
<td>Bistatic Scattering Cross Section Vs Scattering Angle</td>
<td>25</td>
</tr>
<tr>
<td>14.</td>
<td>Bistatic Scattering Cross Section Vs Scattering Angle</td>
<td>26</td>
</tr>
<tr>
<td>15.</td>
<td>Bistatic Scattering Cross Section Vs Scattering Angle</td>
<td>27</td>
</tr>
<tr>
<td>16.</td>
<td>Bistatic Scattering Cross Section Vs Scattering Angle</td>
<td>28</td>
</tr>
<tr>
<td>17.</td>
<td>Bistatic Scattering Cross Section Vs Scattering Angle</td>
<td>29</td>
</tr>
<tr>
<td>18.</td>
<td>Distribution of Current $I =</td>
<td>I</td>
</tr>
<tr>
<td>19.</td>
<td>Distribution of Current $I =</td>
<td>I</td>
</tr>
<tr>
<td>20a.</td>
<td>Distribution of Current $I =</td>
<td>I</td>
</tr>
<tr>
<td>20b.</td>
<td>Distribution of Current $I =</td>
<td>I</td>
</tr>
<tr>
<td>Figure</td>
<td>Description</td>
<td>Page</td>
</tr>
<tr>
<td>--------</td>
<td>-------------</td>
<td>------</td>
</tr>
<tr>
<td>21a.</td>
<td>Distribution of Current $I =</td>
<td>I</td>
</tr>
<tr>
<td>21b.</td>
<td>Distribution of Current $I =</td>
<td>I</td>
</tr>
<tr>
<td>22.</td>
<td>Distribution of &quot;Forced Component of Current&quot; $I_F =</td>
<td>I_F</td>
</tr>
<tr>
<td>23.</td>
<td>Distribution of &quot;Forced Component of Current&quot; $I_F =</td>
<td>I_F</td>
</tr>
<tr>
<td>24.</td>
<td>Distribution of &quot;Resonant Component of Current&quot; $I_R =</td>
<td>I_R</td>
</tr>
<tr>
<td>25.</td>
<td>Distribution of &quot;Resonant Component of Current&quot; $I_R =</td>
<td>I_R</td>
</tr>
</tbody>
</table>
BISTATIC CROSS SECTIONS OF CYLINDRICAL WIRES

I. INTRODUCTION

For a certain radio echo propagation problem it was required to investigate optimum locations of a transmitter and receiver with respect to a scattering source assumed to be a cylindrical parasitic wire of length less than a couple of wavelengths. Of particular interest was the case of separated transmitter and receiver, giving rise to bistatic scattering, as opposed to the usual radar back scatter, or monostatic radar, case. The problem thus involves the determination of the scattering cross section of the wire, its aspect sensitivity and frequency dependence.

For radar or monostatic back scattering, several analytic methods have been employed to determine the back scatter cross section. These include the e.m.f. method, iteration method, and use of the variational principle. The methods and results are reviewed in texts by King, Mentzer and King and Wu, with references to the original work. For convenience we shall use the variational principle as applied by Tai for back scatter cross sections, but then extend the development to bistatic arrangements.

The assumptions regarding the scattering model used here for a free space wavelength \( \lambda_0 \), are that the cylindrical wire of length \( 2h \) and radius \( a \) is not too thick and can be described by a thickness parameter \( \Theta = 2 \log_e 2h/a \), that \( 2h/\lambda_0 \leq 1.5 \) and that the wire has infinite
conductivity. At a later time the effect of finite conductivity can be determined by assuming that such effect is ascribable to an internal impedance-per-unit-length term included in the expressions for current, as was done by Cassedy and Fainberg for back scatter cross sections. The impedance per unit length $z_i$ depends upon conductivity and dielectric constant of the wire.

In Section II the expressions of Tai using the variational principle are reviewed. (Certain typographical errors in his text are mentioned, the details being delineated in the Appendix.) Computations using the corrected expressions were made for monostatic radar cross sections for $\Omega = 10$ and 13.6. Curves of $\sigma / \lambda_o^2$ vs $\theta_o$ agree with those of others for given values of $2h/\lambda_o$. Additional values of cross sections are also plotted for other values of $2h/\lambda_o$.

In Section III expressions are developed for bistatic cross sections, $\sigma_{BS}$. Values of $\sigma_{BS}$ in square wavelengths are computed for $\Omega = 10$. Curves are plotted showing the variation of $\sigma_{BS}/\lambda_o^2$ vs scatter (or reflection) angle $\theta$ for various incident angles $\theta_o$. The polarizing angle $\psi_T$ of the incident field from a transmitter is assumed zero, i.e., the incident field lies in the plane containing the wire and direction of wave propagation and reflection. Values are given for $2h/\lambda_o = 0.5, 0.75, 1.0, 1.25$ and 1.5. Where $\theta = \theta_o$ the curves become those for the back scattering case of Section II.

Section IV contains the development of expressions for the current distributions. Curves are presented in normalized form for the
case of $2h = \lambda_0 / 2$ and $2h = \lambda_0$ in an attempt to understand the lobe phenomena in the scattered field, by the "forced" and "resonant" current contributions to total current.

The use of these curves can be extended to those cases when the incident electric field and the receiver antenna polarization are not parallel to the incident or scattering planes. If $\psi_T$ is the angle of incident electric vector direction (polarization) with respect to the incident plane, the curves for $\sigma / \lambda_0^2$ should be multiplied by $\cos^2 \psi_T$. If, additionally, the receiver antenna electric polarization is described by an angle $\psi_R$ with respect to the scattering plane, then the values of $\sigma / \lambda_0^2$ should be multiplied by $(\cos^2 \psi_T \cos^2 \psi_R)$.

A discussion of the application of the curves of $\sigma / \lambda_0^2$ is contained in Section V, with reference to total transmission losses for bistatic scattering as determined from the transmission equation.
II. MONOSTATIC (RADAR) CROSS SECTIONS -
VARIATIONAL EXPRESSIONS OF TAI

Consider a wave incident upon a cylindrical wire of length 2h, radius a, such that \( \bar{\eta} = 2 \log_e 2h/a \) and the geometry of the situation is as in Figure 1. The angle of incidence is \( \theta_0 \) measured between the wire-axis (Z-axis) and the direction of incidence. The incident electric vector \( E_i \) is a complex vector denoted by \( \tilde{E}_i \) (tilde represents complex quantity, bar represents vector quantity). This vector is polarized at an angle \( \psi \) between its direction in space and the plane containing the direction of incidence and wire axis.

For the radar backscattering case, the scattered field component of interest is that scattered back along the direction \( \theta_0 \) towards the transmitter and received by a receiver located coincident with the transmitter. It is assumed that the wire is in the far-field of the transmitter and that the receiver antenna is in the far-field of that scattered by the wire. Hence the only components of interest are those perpendicular to the direction of propagation, or \( \theta_0 \) components in Figure 1.

Let \( E_i \) be the magnitude \( |\tilde{E}_i| \), i.e. peak value in time. The current induced in the wire is then proportional to \( E_i \cos \psi \sin \theta_0 \). The development from here follows Tai(1). The first-order trial function for the distribution of the current \( I_z \) is assumed to be of the form:

\[
I(z) = I_0 \left[ f_c(z) + A f_s(z) \right]
\] (1)
FIGURE 1. GEOMETRY OF INCIDENT ELECTRIC WAVE VECTOR AND WIRE SCATTERER
where
\[
\begin{align*}
    f_c(z) &= \cos \beta_o z \cos (\beta_o h \cos \theta_o) - \cos \beta_o h \cos(\beta_o z \cos \theta_o) \\
    f_s(z) &= \sin \beta_o z \sin (\beta_o h \cos \theta_o) - \sin \beta_o h \sin(\beta_o z \cos \theta_o)
\end{align*}
\]
with
\[
\beta_o = 2\pi / \lambda_o
\]
and where \(I_0\) and \(A\) are constants. (Tai evaluates \(A\) and we shall find it useful to evaluate \(I_0\) to obtain "forced" and "resonant" components of \(I(z)\) distribution.)

Tai (1) obtains the back scattering cross section for the monostatic radar case as

\[
\sigma_{MS}(\theta_o, \psi) = \frac{4\pi \cos^4 \psi}{\beta_o^2} \left| S \right|^2 = \frac{\lambda_o^2}{\pi} \cos^4 \psi \left| S \right|^2.
\]  \(3\)

(This is Tai's \(\sigma(\theta, \psi)\) in his equation (10), and we use \(\beta_o\) for his \(k\).)

The quantity \(S\) of Tai's is complex and may be considered as a "scattering coefficient", being independent of \(\psi\).

By averaging over all values of \(\psi\), Tai obtains an average cross section

\[
\sigma_{MS}(\theta_o) = \langle \sigma_{MS}(\theta_o, \psi) \rangle = \frac{3}{8\pi} \lambda_o^2 \left| S \right|^2. \tag{4}
\]

(This is Tai's \(\sigma(\theta)\) in his equation (12); we are interested in (3) rather than (4).)

When \(\psi = 0\)

\[
\sigma_{MS}(\theta_o, \psi = 0) = \frac{\lambda_o^2}{\pi} \left| S \right|^2. \tag{5}
\]
The constant $A$ in (1) is given by Tal as

$$A = \frac{g_s Y_c}{g_c Y_s} \quad (6)$$

and $S$ becomes

$$S = \frac{g_c}{Y_c} + \frac{g_s}{Y_s} \quad (7)$$

The quantity $g_c$ is real, $g_s$ imaginary, and $Y_c$ and $Y_s$ are complex integrals:

$$g_c = \beta_0 \sin \theta_0 \int_{-h}^{h} f_c(z) e^{j \beta_0 z \cos \theta_0} dz \quad (8a)$$

$$g_s = \beta_0 \sin \theta_0 \int_{-h}^{h} f_s(z) e^{j \beta_0 z \cos \theta_0} dz \quad (8b)$$

$$Y_c = j \int_{-h}^{h} \int_{-h}^{h} f_c(z) f_c(z') K(z-z') dz' dz \quad (8c)$$

$$Y_s = j \int_{-h}^{h} \int_{-h}^{h} f_s(z) f_s(z') K(z-z') dz' dz \quad (8d)$$

with the kernel $K(z-z')$ being given by

$$K(z-z') = \beta_0 (1 + \frac{1}{\beta_0^2} \frac{\partial^2}{\partial z^2}) e^{-j \beta_0 R} \frac{R}{R} \quad (9)$$

$$R = \left[ (z-z')^2 + a^2 \right]^{1/2}$$

The evaluation of the integrals (8) is given in Appendix A.

Typographical errors in the printed version of Tal's paper occur in his evaluation of (8b), (8c) and (8d), i.e. in his equations (Appendix) (3)
in which a "cos x" should be "sin x", his equation (6) should contain a term "cos^2 \Omega xL (4x)" and his equation (7) should contain a term "sin^2 \Omega xL (4x)". The symbols are explained in Appendix A.

With these corrections, values of the average cross section, given by (4), were computed using an IBM 1620 computer for 2h/a = 900, \( \Omega = 13.6 \), and 2h/\( \lambda o \) = 0.5, 1.5 and 2.0. (The results compare almost exactly with Tai's curves Figures 2, 3, and 4, so that his results are satisfactory but the printed equations need the corrections aforementioned.)

Curves for the average cross section (i.e. Tai's \( \sigma (\theta) / \lambda^2 \))

\[ \sigma_{MS}^2(\theta_0) / \lambda^2 \] vs \( \theta_0 \) for \( \Omega = 13.6 \) (2h/a = 900) are shown in Figures 2, 3, and 4 for 2h/\( \lambda o \) = 0.5, 0.75 and 1.5 respectively; Figures 2 and 4 being the same as those of Tai (1), King (2), King and Wu (3) and Mentzer (3).

For a thicker antenna for which 2h/a = 150, \( \Omega = 10 \) similar curves are given in Figures 5 through 10 for 2h/\( \lambda o \) = 0.5, 0.75, 0.80, 0.90, 1.0, and 1.25 respectively. A polar plot for 2h/\( \lambda o \) = 1.5 is shown in Figure 11 for \( \Omega = 13.6 \) corresponding to Figure 4.

The small integral values chosen (for \( \Omega = 10 \)) between 2h/\( \lambda o \) = 0.75 and 1.25 were an attempt to depict the "double hump" development shown previously at 2h/\( \lambda o \) = 0.10. With 2h/\( \lambda o \), a single broad peak is noted. At 2h/\( \lambda o \) = 0.80 a minor double hump begins to appear, being better developed at 0.90. The two peaks are well developed in the case 2h/\( \lambda o \) = 1.0, but a minor lobe appears at broadside (\( \theta o = 90^\circ \)) incidence in Figure 9. Assymetry appears in the major lobes at 2h/\( \lambda o \) = 1.25 in Figure 10. The lobes appear to separate further as...
Figure 2. Angular distribution of backscatter cross-section.

- $\frac{2h}{\lambda_0} = 5$
- $\Omega = 13.6$ (2h, 900)

Graph showing the angular distribution with 0° on the x-axis and θ on the y-axis (in degrees).
\[ \frac{2h}{\lambda_0} = 0.75 \]
\[ \Omega = 13.6 \quad \left( \frac{2h}{a} = 900 \right) \]

**Figure 3. Angular Distribution of Back Scatter Cross-Section**
Figure 4. Angular distribution of back scatter cross-section.
\[ \frac{2h}{\lambda_0} = 0.75 \]
\[ \Omega = 10.0 \ (\frac{2h}{a} = 150) \]

FIGURE 6. ANGULAR DISTRIBUTION OF BACK SCATTER CROSS-SECTION
\[ \frac{2h}{\lambda_0} = 0.80 \]

\[ \Omega = 10.0 \quad (\frac{2h}{a} = 150) \]

**FIGURE 7.** ANGULAR DISTRIBUTION OF BACK SCATTER CROSS-SECTION
\[ \frac{2h}{\lambda_0} = 0.90 \]
\[ \Omega = 10.0 \quad (\frac{2h}{a} = 150) \]

**FIGURE 8.** ANGULAR DISTRIBUTION OF BACK SCATTER CROSS-SECTION
\begin{align*}
\frac{2h}{\lambda_0} &= 1.0 \\
\Omega &= 10.0 \quad (\frac{2h}{a} = 150)
\end{align*}

\textbf{FIGURE 9. ANGULAR DISTRIBUTION OF BACK SCATTER CROSS-SECTION}
$\frac{2h}{\lambda_o} = 1.25$

$\Omega = 10.0$  \hspace{1cm} $\frac{2h}{a} = 150$

$\theta_0$ (IN DEGREES)

\[ \frac{\sigma(\theta_0)}{\lambda_o} \]

**FIGURE 10.** ANGULAR DISTRIBUTION OF BACK SCATTER CROSS-SECTION
$2h/\lambda_0$ increases from 0.75 to 1.25. Apparently the "forced" term and "resonant" term in the current are playing relatively different roles, as mentioned by King (2). A center-driven antenna $2h/\lambda_0 = 1$ has one major lobe, a parasitic scatterer has two lobes as in Figure 9.

The plots given in the above figures are by way of checking the expressions we use with those presented by others plus additional values of cross sections for different values of $\Omega$. The development of the double-lobes is also depicted. These curves show angular response of average cross sections for perfectly conducting parasitic scatterers, $2h/\lambda_0$ lying between 0.5 and 2.0.

We shall now develop bistatic cross sections for the case $\psi = 0$ for perfectly conducting wire scatterers.
III. BISTATIC SCATTERING CROSS SECTIONS

For bistatic scattering, the transmitter (causing the incident field $E_{\text{inc}}$) and the receiver (detecting the scattered field $E_{\text{sc}}$) are separated in location. Referring to Figure 12 we shall consider the specialized case where the incident field $E_{\text{inc}}$ lies in the plane of incidence containing the wire (Z-axis) and the direction of incidence (polarizing angle $\psi = 0$).

The incident field induces a current $I(z)$; the scattered field $E_{\text{sc}}$ vector lies in the scattering plane containing the direction of the current $I(z)$ (assumed to be the same as the wire or Z-axis) and the direction of propagation of the scattered wave.

$I(z)$ is due to $E_i$, the amplitude of the incident field and is given by (1). This expression contains the constant $A$ which is given by (6). We need to evaluate $I_o$ in (1). This is done as follows. We start with Tai's expression for the scattering coefficient $S$, and ultimately given by (7).

It is

$$S = \frac{n_0 \beta_0^2 \sin \theta_0}{4\pi E_i \cos \psi} \int_{-h}^{h} l(z) e^{i \beta_0 z \cos \theta_0} dz$$

and upon substituting (2)

$$S = 30 \beta_0 I_o \frac{g_c + Ag_s}{E_i \cos \psi}$$

Putting in the values of $S$ from (7) and $A$ from (6)

$$I_o = \frac{1}{30 \beta_0} \frac{g_c}{\gamma_c} E_i \cos \psi.$$
FIGURE 12. GEOMETRY OF BISTATIC SCATTERING ($E_{\text{inc}}$ LIES IN PLANE OF INCIDENCE)
Now the $\theta$-component of the scattered field $E_{sc}$ in the direction $\theta$, in the far-field at a distance $R_o$ due to $l(z)$, is

$$E_{sc} = -j30\beta_o \sin \theta \frac{-j\beta_o R_o}{R_o} I_0 \int_{-h}^{h} (f_c + A f_s) e^{j\beta_o z \cos \theta} dz$$  \hspace{1cm} (13)$$

where $f_c$ and $f_s$ are functions of the incident angle $\theta_i$, not the scattering angle $\theta$.

Let us write

$$F = \frac{1}{\beta_o \sin \theta} (G_c + AG_s) = F_c + AF_s$$  \hspace{1cm} (14)$$

where $G_c$ and $G_s$ are integrals similar to $g_c$ and $g_s$, integrals of Tai but are functions of both $\theta_i$ and $\theta$ rather than $\theta_i$ alone, i.e.

$$G_c = \beta_o \sin \theta F_c \hspace{1cm} G_s = \beta_o \sin \theta F_s$$  \hspace{1cm} (15)$$

$$F_c = \int_{-h}^{h} f_c e^{j\beta_o z \cos \theta} dz \hspace{1cm} F_s = \int_{-h}^{h} f_s e^{j\beta_o z \cos \theta} dz$$

noting that $A$ is not a function of $z$.

We now define a bistatic scattering coefficient $\sigma_{BS}$ for $\psi = 0$ in a fashion similar to that for the radar monostatic case, or

$$\sigma_{BS} (\psi = 0) = 4 \pi R_o^2 \left| \frac{E_{sc}}{E_i} \right|^2 \bigg| \psi = 0$$

$$= \frac{\lambda_o}{\psi} \left| S_1 \right|^2$$

where the bistatic scattering coefficient $\left| S_1 \right|$ becomes

$$\left| S_1 \right| = \left| \frac{g_c}{\gamma_c} (G_c + AG_s) \right| = \left| \frac{g_c g_{sc}}{\gamma_c} + \frac{g_s g_s}{\gamma_s} \right|$$  \hspace{1cm} (17)$$
When \( \theta = \theta_o \), i.e. back scattering (radar) \( G_c = \hat{g}_c \), \( G_s = \hat{g}_s \) and
\[
\left| S_1 \right| = \left| S \right| \text{ in (17)}.
\]
The \( G \)-functions are obtained from evaluating the \( F \)-integrals in (6).

We use some abbreviations (which differ a bit from those of Tai). These are
\[
\begin{align*}
q &= \beta_o \cos \theta^o, & x &= \beta o h, & \ell &= \beta o h \cos \theta = x \cos \theta \\
q_o &= \beta o \cos \theta^o, & \ell_o &= \beta o h \cos \theta = x \cos \theta_o
\end{align*}
\]
\quad (18)

Then
\[
\begin{align*}
&f_c = \cos \ell_o \cos \beta o z - \cos x \cos \ell o z \\
&f_s = \sin \ell_o \sin \beta o z - \sin z \sin \ell o z
\end{align*}
\]
\quad (19)

and
\[
\begin{align*}
F_c &= \cos \ell_o I_1 - \cos x I_2 \\
F_s &= \sin \ell_o I_3 - \sin x I_4
\end{align*}
\]
\quad (20)

The values of the various \( I_n \) integrals above are
\[
\begin{align*}
I_1 &= \int_{-h}^{h} \cos \beta o z e^{i q z} dz \\
I_2 &= \int_{-h}^{h} \cos q o z e^{i q z} dz \\
I_3 &= \int_{-h}^{h} \sin \beta o z e^{i q z} dz \\
I_4 &= \int_{-h}^{h} \sin q o z e^{i q z} dz
\end{align*}
\]
\quad (21)
Making use of Pierce's Table of Integrals (Nos. 414 and 415) the I-
integrals may be evaluated.

\[ I_1 = \frac{2}{\beta_0 \sin^2 \theta} \left( \sin x \cos \theta - \cos \theta \cos x \sin \theta \right) \]  
(22a)

\[ I_2 = \frac{2}{\beta_0 (\cos^2 \theta - \cos^2 \theta)} \left( \cos \theta \sin \beta_0 \cos \theta \cos \theta \cos \theta \cos \theta \sin \theta \right) \]  
(22b)

\[ I_3 = \frac{j}{\beta_0 \sin^2 \theta} \left( \cos \theta \sin x \cos \theta - \cos x \sin \theta \right) \]  
(22c)

\[ I_4 = \frac{j}{\beta_0 (\cos^2 \theta - \cos^2 \theta)} \left( \cos \theta \sin \theta \cos \theta \cos \theta \cos \theta \sin \theta \right) \]  
(22d)

The integrals in (22) are thus functions of \( \theta \), \( \theta_0 \) and \( \beta_0 h \), but not the thickness parameter \( \xi \). Evaluating (22) for a set of values of \( \theta \), \( \theta_0 \), and \( x \), the results are then substituted in (20) to obtain \( F_C \) and \( F_S \), and ultimately \( G_C \) and \( G_S \) in (15). [In a computation routine, the values of (22) are multiplied by \( \beta_0 \sin \theta \), yielding immediately useful values for the \( G_C \) and \( G_S \) integrals.]

A set of values of \( \theta \), \( \theta_0 \), and \( x \), were calculated and plotted as a function of the scattering angle \( \theta \) for various values of \( \theta_0 \) as a parameter. The thickness parameter \( \xi \) was chosen as 10 corresponding to \( 2h/a = 150 \).

The figures correspond to increasing values of \( 2h/\lambda_0 \) as follows:

- Figure 13: \( 2h = \frac{\lambda_0}{2} \), \( x = \beta_0 h = 0.5\pi \)
- Figure 14: \( 2h = \frac{3}{4} \lambda_0 \), \( x = 0.75\pi \)
- Figure 15: \( 2h = \lambda_0 \), \( x = 1.0\pi \)
The curves of $\sigma_{BS}$ are normalized in square wavelengths and apply for $\psi = 0$. (These are not the average values over all values of $\psi$ used in back scatter cross sections of the previous Section.)

Of interest is the asymmetry of $\sigma_{BS}/\lambda_o^2 (\psi = 0)$ with $\theta$ at certain lengths and incident angles. For $2h = \lambda_o/2$, the curves are reasonably symmetrical about $\theta = 90^\circ$ (broadside). However, those for $2h = 0.75$ begin to show the asymmetry, in that decreasing $\theta$ below $90^\circ$ towards smaller values shifts the maximum at values of $\theta$ at angles correspondingly less than $90^\circ$. A fuller development is shown in Figure 15 for $2h/\lambda_o = 1.0$. Consider here the curve for $\theta_o = 50^\circ$. A lesser maximum occurs in the polar diagram at $\theta = 50^\circ$ than at $\theta = 123^\circ$.

When $\theta_o = 130^\circ$ $(180^\circ - 50^\circ)$, the larger maximum occurs at $57^\circ$ $(180^\circ - 123^\circ)$ than at $\theta = 130^\circ$ $(180^\circ - 50^\circ)$. It appears that the asymmetrical curve for $\theta_o = K$ has as a mirror image with varying $\theta$ the curve for $\theta_o = 180^\circ - K$.

Thus for certain angles, the cross section is larger in the "forward scatter" region than for "back scatter." Consider Figure 15 again. If $\theta_o = 50^\circ$ and $\theta = 50^\circ$ (back scatter), $\sigma_{BS}/\lambda_o^2 = 0.66$; but at $\theta = 123^\circ$, $\sigma_{BS}/\lambda_o^2 = 1.06$, a ratio of $1.61$ or $2$ db. This does not always hold true; for example, at $\theta_o = 90^\circ$ there is only one maximum and reception at angles $\theta$ greater or less than $90^\circ$ gives smaller values of $\sigma_{BS}/\lambda_o^2$. 

Figure 16
$$2h = \frac{5}{4} \lambda_o \quad x = 1.25 \pi$$

Figure 17
$$2h = \frac{3}{2} \lambda_o \quad x = 1.5 \pi$$
Values of $\sigma_{BS}/\lambda_o^2$ for monostatic (radar) and bistatic scattering are shown below for a few values of $\theta$, $2h/\lambda_o = 1.0$, $\Omega = 10$.

**TABLE 1.** $\sigma_{BS}/\lambda_o^2$ at $\theta = \theta_0$, $\theta = \theta_1$ (1st Max.) and $\theta = \theta_2$ (2nd Max.) and Ratios to That for Back Scattering ($\theta = \theta_0$)

<table>
<thead>
<tr>
<th>$\theta$</th>
<th>$\theta = 30^\circ$</th>
<th>$\theta = 40^\circ$</th>
<th>$\theta = 50^\circ$</th>
<th>$\theta = 60^\circ$</th>
<th>$\theta = 70^\circ$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\theta_0$</td>
<td>30°</td>
<td>1 (0)</td>
<td>.46</td>
<td>1 (0)</td>
<td>.66</td>
</tr>
<tr>
<td>$\theta_1$</td>
<td>.95 (2.9)</td>
<td>52°</td>
<td>.56</td>
<td>1.08 (.3)</td>
<td>.66</td>
</tr>
<tr>
<td>$\theta_2$</td>
<td>2.37 (3.7)</td>
<td>124°</td>
<td>.78</td>
<td>1.70 (2.3)</td>
<td>1.06 *</td>
</tr>
</tbody>
</table>

There appears to be an advantage for reception other than at back scatter angles for double hump polar diagrams. The advantage may be at angles closer to the wire than $\theta_0$ (viz. $\theta_0 = 70^\circ$, with $\theta = \theta_1 < 80^\circ$, there is a 2.9 db advantage over that at $\theta = \theta_0 = 70^\circ$); the greater advantage is usually at the second maximum $\theta = \theta_2$ as $\theta$ increases from 0 to 180°.

In the case of $2h/\lambda_o = 0.75$ (Figure 14), exhibiting a single lobe, the advantage is only slight for scattering on the maxima vs back scattering until very small values of $\theta_0$ are employed.
IV. THE CURRENT DISTRIBUTION $I(z)$

A clue to the behavior of multi-lobed scattering polar diagrams, particularly for $2h > \lambda_o/2$, can be found in an examination of the current distributions.

The current distribution, assumed axial, is given by (1)

$$ I(z) = I_0 \left[ f_c(z) + Af_s(z) \right] $$

where $f_c(z)$ and $f_s(z)$ are given by (2).

The constant $A$ is given by equation (6) as

$$ A = \frac{g_s Y_c}{g_c Y_s} $$

with the integrals $g_s$, $g_c$, $Y_c$, and $Y_s$ defined by (a), (b), (c) and (d) and evaluated in the Appendix. The constant $I_0$ is given by (12) as

$$ I_0 = \frac{1}{30 \beta_o} \frac{g_c}{Y_c} E_i \cos \psi $$

where $\beta_o = 2 \pi / \lambda_o$, $E_i$ is the amplitude of the incident wave and $\psi$ the polarization angle (Figure 1).

Substituting (12) and (6) in (1), we may obtain a normalized current distribution which we call $I$, viz.

$$ I = \frac{I(z)}{30 \beta_o} \frac{g_c}{E_i \cos \psi} = \frac{g_c}{Y_c} \left[ f_c(z) + Af_s(z) \right] $$

$$ = \frac{g_c}{Y_c} f_c(z) + \frac{g_s}{Y_s} f_s(z) $$

which we may write in polar form

$$ I = |I| e^{i \theta} $$
Computations of $I$ and $\theta_0$ were made for $\Omega = 10$ and for $2h = \frac{\lambda_0}{2}$ ($x = \pi/2$) and $2h = \lambda_0(x = \pi)$, to illustrate the behavior of $I$.

The variations of $|I|$ and $\theta_0$ vs $z/h$ for $2h = \lambda_0/2$ ($x = \pi/2$) are shown in Figures 18 and 19 respectively. The variations of $|I|$ for $2h = \lambda_0(x = \pi)$ vs $z/h$ are shown in Figures 20(a) and (b) and the phase angle $\theta_0$ in Figures 21(a) and (b).

Regarding the case $2h = \lambda_0/2$, a single maximum occurs in $|I|$, Figure 18, for all values of $\theta_0$. The phase angle variations, all in the fourth quadrant, indicate a single peak in the scattered field with asymmetry due to $\theta_0$.

For $2h = \lambda_0(x = \pi)$, there are principally two maxima in $|I|$, and these are asymmetrical. The phase angles shift from the third quadrant $-1 \leq z/h < 0$ to the first quadrant for $z/h > 0$. When $\theta_o = 90^\circ$ (broadside incidence), the single broad peak with a constant $\theta_0$ indicates a small single peak in the scattered field.

The current distribution (23) can be considered due to a "forced" component and a "resonant" component, as suggested by King (2) (Chapter IV). The "forced" terms contain trigonometric terms in $\beta_0 \cos \theta_0 x$, whence they depend upon the phase of the electric field along the wire. The other "resonant" terms depend upon $\beta_0 x$ only.

We write

$$I = I_F + I_R$$

where

$$I_F = - \left[ \frac{\delta_c}{Y_c} \cos x \cos (x \cos \theta_0 \frac{z}{h}) + \frac{\mu_s}{Y_s} \sin x \sin (x \cos \theta_0 \frac{z}{h}) \right]$$

(25)
FIGURE 18. DISTRIBUTION OF CURRENT \( I=|I|e^{j\phi} \)
\[ \theta_I \text{ vs } \frac{z}{h} \]

\[ 2h = 0.5 \]
\[ \lambda_0 = 5.0 \]
\[ \Omega = 10.0 \]

**Figure 19. Distribution of Current**

\[ I = |I|e^{j\theta_I} \]
Figure 20a. Distribution of current $I = |I| e^{i\theta I}$
Figure 20b. Distribution of current $I = |I| e^{j\theta I}$
Figure 21a. Distribution of current \( I = |I| e^{i\theta I} \)
\[ I_R = \left[ \frac{\varepsilon_0}{\gamma_c} \cos (x \cos \theta_o) \cos \left( x \frac{z}{h} \right) + \frac{\varepsilon_0}{\gamma_s} \sin (x \cos \theta_o) \sin \left( x \frac{z}{h} \right) \right] \] (26)

where

\[ x \frac{z}{h} = \beta_o h \frac{z}{h} = \beta_o z \]

\[ x \cos \theta_o \frac{z}{h} = \beta_o z \cos \theta_o. \]

We find that at broadside incidence, \( \theta_o = 90^\circ \), the contribution of \( I_R \) to the total current is 100\% (\( I_F = 0 \)) when \( 2h = \lambda_o / 2 \) (\( x = \pi / 2 \)) but is 50\% when \( 2h = \lambda_o (x = \pi ) \) and \( z/h = 0 \).

Plots of \( I_F = I_F e^{j \theta F} \) and \( I_R = I_R e^{j \theta R} \) are shown, for \( 2h = \lambda_o / 2 \), in Figures 22 and 23 for \( I_F \) and Figures 24 and 25 for \( I_R \). These curves illustrate the relative effects of \( I_F \) and \( I_R \) by comparison with the total current \( I \) (Figures 18 and 19).
\[ I_F = |I_F| e^{i\theta_F} \]

**Figure 22. Distribution of "Forced Component of Current"**

\[
\begin{align*}
2h &= 0.5 \\
\lambda_0 &= 10.0 \\
\Omega &= 10.0
\end{align*}
\]
\[ I_F = |I_F| e^{i\theta_F} \]

**Figure 23. Distribution of "Forced Component of Current"**

\[ \theta_F \text{ vs } \frac{z}{h} \]

\[ \frac{2h}{\lambda_0} = 0.5 \]

\[ \Omega = 10.0 \]
FIGURE 25. DISTRIBUTION OF "RESONANT COMPONENT OF CURRENT"

\[ I_R = |I_R| e^{j\theta_R} \]
V. APPLICATION VIA TRANSMISSION EQUATION TO TRANSMISSION LOSSES

For bistatic cross section model measurements or in a system, an estimate of required receiver sensitivity can be obtained or an estimate of cross section calculated from the transmission equation.

A transmitter with average power $P_T$ feeds a loss-less antenna of gain $G_T$ located a distance $R_T$ from the wire. A receiver is located a distance $R_R$ from the wire and has a loss-less antenna of gain $G_R$. The gains $G$ takes into account the polar diagrams of the antennas and are the values of gain in the direction of the wire. It is assumed that electric vector polarization of the transmitter is parallel to the incident plane, so that the incident polarization angle $\psi_T = 0$. Similarly, it is assumed that receiver polarization is parallel to the scattering or reflection plane containing the wire so that $\psi_R = 0$. The appropriate value of cross section to be used is $\sigma_{BS}(\theta_o, \theta)$ of the previous Section.

The received power will then be

$$P_R = \frac{\frac{P_T}{4\pi R_T^2}}{G_T} \frac{\sigma}{4\pi R_R^2} \frac{G_R}{4\pi} \frac{\lambda_o^2}{4\pi}$$ (27)

assuming that the load impedance of the receiver input is matched to the receiver antenna impedance. Writing losses as quantities such that losses in decibels will be positive, the total system loss is

$$L_T = \frac{P_T}{P_R} = \left(\frac{4\pi R_T^2}{\lambda_o}\right) \left(\frac{4\pi R_R^2}{\lambda_o}\right) \frac{1}{(\sigma/\lambda_o)^2} \frac{1}{4\pi G_T G_R}$$ (28a)

$$= (L_o)_T (L_o)_R L_{sc} \frac{1}{G_T G_R}$$ (28b)
Equations (28a and 28b) are written in the form to allow use of the curves for $\sigma/\lambda_o^2$, where the free space losses

$$\begin{align*}
(L_o)_T &= \left(\frac{4\pi R_T}{\lambda_o}\right)^2 \\
(L_o)_R &= \left(\frac{4\pi R}{\lambda_o}\right)^2
\end{align*}$$

and the "scattering loss" $L_{sc}$ is

$$L_{sc} = \frac{1}{(4\pi\sigma/\lambda_o^2)}$$

In decibels

$$L_T(db) = L_{oT}(db) + L_{oR}(db) + L_{sc}(db) - G_T(db) - G_R(db)$$

where the free space losses, in decibels, are given by

$$L_o(db) = 36.56 + 20 \log_{10} R(miles) + 20 \log_{10} f(mc/s)$$

$$= -23.43 + 20 \log_{10} R(miles) + 20 \log_{10} f(kc/s)$$

$$= -27.56 + 20 \log_{10} R(meters) + 20 \log_{10} f(mc/s)$$

$$= -87.56 + 20 \log_{10} R(meters) + 20 \log_{10} f(kc/s)$$

and the "scattering loss" in decibels is

$$L_{sc}(db) = -10 \log_{10} \left(\frac{4\pi\sigma}{\lambda_o^2}\right)$$

The gains $G_T(db)$ and $G_R(db)$ are the antenna gains in decibels in the direction of the wire.
Putting (33) and (32) into (31) and using distances R in miles and frequencies in kc/s, (31) becomes

\[
L_T(\text{db}) = -57.85 + 20 \log_{10} R_T(\text{miles}) + 20 \log_{10} R_R(\text{miles})
\]

\[
+ 40 \log_{10} f(\text{kc}) - G_T(\text{db}) - G_R(\text{db}) - 10 \log_{10} \frac{\sigma}{\lambda_o^2}
\]

As a special case, if \(R_T = R_R = R\) and \(G_T = G_R = G\)

\[
L_T(\text{db}) = -57.85 + 40 \log_{10} R(\text{miles}) + 40 \log_{10} f(\text{kc}) - 2G(\text{db}) - 10 \log_{10} \frac{\sigma}{\lambda_o^2}
\]

From (34), if distances, frequencies, antenna gains and \(L_{sc}\) are known total loss \(L_T\) can be computed which will give the required transmitter power for a given receiver power, by virtue of (28a), i.e.

\[
P_T(\text{dbw}) = P_R(\text{dbw}) + L_T(\text{db})
\]  

Conversely, if \(P_T\) and \(P_R\) are known so that \(L_T\) is known, and other parameters are known, then \(L_{sc}(\text{db})\) can be computed, viz (equation (34))

\[
L_{sc}(\text{db}) = -10 \log_{10} \left(\frac{4 \pi \epsilon_0}{\lambda_o^2}\right) = L_T(\text{db}) + 46.86 - 20 \log_{10} R_T(\text{miles})
\]

\[
- 20 \log_{10} R_R(\text{miles}) - 40 \log_{10} f(\text{kc}) + G_T(\text{db}) + G_R(\text{db})
\]

Finally, the above has assumed that polarization angles \(\psi_T = 0\) and \(\psi_R\). For non-zero values of \(\psi\), the received power in (27) will decrease, and (27) must be multiplied by \(\cos^2 \psi_T \cos^2 \psi_R\). The total losses in (28) will increase by \(1/(\cos^2 \psi_T \cos^2 \psi_R)\), or the loss \(L_T(\text{db})\) in (31) et seq.

will increase by a polarization loss \(L_p(\text{db})\) given by

\[
L_p(\text{db}) = -20 \left[ \log \cos \psi_T + \log \cos \psi_R \right]
\]
Consequently (31) is modified to take account of $L_P$ and

$$L_T(db) = L_{oT}(db) + L_{oR}(db) + L_{sc}(db) + L_p(db) - G_R(db) - G_T(db) \tag{38}$$

where $L_o(db)$ is given by (32), $L_{sc}(db)$ by (33) and $L_p(db)$ by (37).

As an example consider

\[ f = 100 \text{ kc} \quad R_T = R_R = 500 \text{ miles} \quad \psi_T = 60^\circ = \psi_R \]
\[ \frac{2h}{\lambda_o} = 1.0 \quad \theta_o = 60^\circ \]
\[ \Omega = 10.0 \quad \theta = 123^\circ \]
\[ G_T = G_R = 2.15 \text{ db} \]

From (32) $L_{oT}(db) = L_{oR}(db) = -23.43 + 53.9^\circ + 40.00 = 60.55 \text{ db}$. From Figure 15, for $\theta_o = 60^\circ$, $\theta = 123^\circ$, $\sigma / \lambda_o^2 = 1.14$ and $L_{sc} = -11.56 \text{ db}$.

From (37) $L_p = -20 (-6.0206) = 12.04 \text{ db}$. Hence, using (38)

$$L_T(db) = 121.10 - 11.56 + 12.04 - 4.30 = 117.28 \text{ db}$$

Hence a 10-watt transmitter ($P_T = 10 \text{ dbw}$) would produce a received signal (matched receiver) of $10 - 117.28 = -107.28 \text{ dbw}$ or $1.871 \times 10^{-11} \text{ watts}$.

For a receiver resistance of 50 ohms the input voltage would be $-107.28 + 16.29 = -90.29 \text{ db}$ with respect to one volt or 29.71 db above one microvolt or about 30.6 microvolts.
VI. REFERENCES


APPENDIX A

EVALUATION OF INTEGRALS USED IN SCATTERING COEFFICIENTS

A-I. INTRODUCTION

The purpose of this Appendix is to correct certain typographical errors in the evaluation of certain integrals employed in the variational method of Tai\(^{(1)}\). We shall employ here the notation of Tai. Let

\[ k = \frac{2 \pi}{\lambda} \quad \theta = \text{angle of incidence onto wire} \]
\[ x = \frac{k \ell}{2} = \frac{\pi \ell}{\lambda} \quad \ell = \text{half length of wire} \]
\[ q = \cos \theta \quad \lambda = \text{free space wavelength} \]

A-II. THE INTEGRAL \( g_c \)

\[ g_c = k \sin \theta \int_{\ell}^{f_c(z)} e^{jkz \cos \theta} dz \]  \( (I-1) \)

Using simple trigonometric identities, it can be shown that

\[ g_c = \frac{4q \cos^2 x - (1 + q^2) \sin 2q \cos x - 2q(1-q^2) x \cos x}{2q(1-q^2)^{1/2}} \]  \( (I-2) \)

This is identical with Appendix A(2) of Tai\(^{(1)}\).
A-III. THE INTEGRAL \( g_{a} \)

\[
g_{a} = k \sin \theta \int_{-\ell}^{\ell} f_{a}(z) e^{jkz} \cos \theta \, dz \quad (1-3)
\]

\[
= -j \frac{4q \sin^2 qz \cos z - (1+q^2) \sin 2qz \sin x + 2q(1-q^2) x \sin x}{2q(1-q^2)^{1/2}} \quad (1-4)
\]

This differs from Appendix A(3) of Tai\(^{(1)}\) in the very last term of the numerator, where we find "sin x" instead of his "cos x."

In the above

\[
f_{c}(z) = \cos qk \ell \cos kx - \cos qk \ell \cos k \ell \quad (1-5)
\]

\[
f_{a}(z) = \sin qk \ell \sin kx - \sin qk \ell \sin k \ell
\]

with

\[
I_{z} = I_{o} \left[ f_{c}(z) + Af_{a}(z) \right] \quad (1-6)
\]

is the trial current distribution, \( A \) and \( I_{o} \) being constants.
A-IV. THE Ψ \text{C} INTEGRAL

\[ \Psi_c = \frac{j}{k} \int_{-\ell}^{\ell} \int_{-\ell}^{\ell} f_c(z) f_c(z') K(z-z') \, dz \, dz' \]  

(I-7)

where the kernel is

\[ K(z-z') = k \left( 1 + \frac{1}{k^2} \frac{\partial^2}{\partial z^2} + \frac{e^{-jkR}}{R} \right) \]

and

\[ R = \left[ (z-z')^2 + a^2 \right]^{1/2} \]

Storer* has shown how integrals of this type may be evaluated by changing variable. The result is

\[ \Psi_c = \cos^2 x \left[ -1 + \cos 2x \cos 2q + q \sin 2x \sin 2q + j (\sin 2x \cos 2q - q \cos 2x \sin 2q) \right] - \frac{1}{2} \left[ \frac{1+q^2}{q^2} \cos^2 x \cos 2q + \sin 2x \sin 2q \right] \]

\[ \left[ L(1+q) 2x - L(1-q) 2x \right] + j \left\{ \cos^2 x \left[ \frac{1+q^2}{q^2} \sin 2q + (1-q^2) x \right] \right. 
\]

\[ - \cos^2 qx \sin 2x \right\} \left\{ \log e 4 + \Omega - L(1+q) 2x - L(1-q) 2x \right\} \]

\[ + \cos^2 qx L(4x) \]  

(I-8)

A-V. THE $\gamma_s$ INTEGRAL

We have

$$\gamma_s = \int_{-L}^{L} f_s(z) f_s(z') K(z-z') \, dz \, dz'$$

and in a fashion like that for $\gamma_c$, this integral becomes

$$\gamma_s = \sin^2 x \left[ -1 + \cos 2x \cos 2qx + q \sin 2x \sin 2qx \right] + \frac{1}{2} \left[ \frac{1+q^2}{q^2} \sin^2 x \cos 2qx \right.$$  

$$\left. \sin 2x \sin 2qx \right] \begin{bmatrix} L(1+q) 2x - L(1-q) 2x \end{bmatrix} + j \left\{ -\sin^2 x \left[ \frac{1+q^2}{q} \sin 2qx -(1-q^2) x \right] + \sin^2 qx \sin 2x \right\}$$

$$\left\{ \log_q 4 + \Omega - L(1+q) 2x - L(1-q) 2x \right\} + \sin^2 qx \sin^2 x \right\}$$

(1-10)

In the $\gamma$-integrals above, the evaluations assume that the wire is thin, i.e. $\ell >> a$. The $L$-functions are tabulated functions, given by sine and cosine integrals.

$$L(1 \pm q) \, \partial p = \int_0^{(1 \pm q) \partial p} \frac{1-\pm q}{u} \, du + \sin (1 \pm q) \, \partial p + \sin (1 \pm q) \, \partial p$$

(1-11)

The $\gamma_c$-integrals (1-8) differ from that of Appendix A(6) of [ref1] in that a term $\cos^2 qx \, L(4x)$ was omitted from the latter. Similarly in the $\gamma_s$-integral (1-10), a term $\sin^2 qx \sin 4x$ was omitted from [ref1]'s expression in his Appendix A(7).

With the above corrections, the expressions in the text for $\gamma_s$, $\gamma_c$, and $\gamma_s$ were used with the evaluations of (1-2), (1-4), (1-8) and (1-10).