NOTICE: When government or other drawings, specifications or other data are used for any purpose other than in connection with a definitely related government procurement operation, the U. S. Government thereby incurs no responsibility, nor any obligation whatsoever; and the fact that the Government may have formulated, furnished, or in any way supplied the said drawings, specifications, or other data is not to be regarded by implication or otherwise as in any manner licensing the holder or any other person or corporation, or conveying any rights or permission to manufacture, use or sell any patented invention that may in any way be related thereto.
AN ISOPERIMETRIC EQUALITY; AND RELATED QUESTIONS

by

Robert Finn

TECHNICAL REPORT NO. 102

November 14, 1961

PREPARED UNDER CONTRACT Nonr-225(11)

(NR-041-086)

OFFICE OF NAVAL RESEARCH

Reproduction in Whole or in Part is Permitted for
any Purpose of the United States Government

APPLIED MATHEMATICS AND STATISTICS LABORATORIES

STANFORD UNIVERSITY

STANFORD, CALIFORNIA
AN ISOPERIMETRIC EQUALITY,
AND RELATED QUESTIONS

by

Robert Finn

Introduction.

Let $S$ be a piece of smooth two-dimensional surface, and let $\Gamma$ be a closed curve which bounds a simple region $\mathcal{Y}$ on $S$. Let $L(\Gamma)$ and $A(\mathcal{Y})$ be, respectively, the length and area of $\Gamma$ and $\mathcal{Y}$. A. Huber has proved in [1] the general inequality

$$\frac{L^2(\Gamma)}{A(\mathcal{Y})} \geq 4\pi - 2\mu^+$$

where $\mu^+$ is the surface integral of the Gaussian curvature $K$ of $S$, evaluated over that part of $\mathcal{Y}$ where $K > 0$.

Equality holds in (1) if and only if $K \equiv 0$ in $\mathcal{Y}$ and $\Gamma$ is a geodesic circle; thus the estimate is not a sharp one unless the Gaussian curvature vanishes identically on $S$.

One might suspect at first glance that (1) would be improved on replacing $\mu^+$ by $\mu$, the total curvature or curvatura integra of $\mathcal{Y}$. The inequality obtained in this way would, however, be incorrect. For example, we may choose for $\Gamma$ a unit circle on a plane, and then deform the plane interior to the concentric subcircle of radius $1/2$. This does not affect $\mu$, but the deformation can always be arranged to make the left side of (1) as small as desired. Neither can one assert in any sense the existence, in general, of curves for which $\frac{L^2}{A} \leq 4\pi - 2\mu$, since $\mu = \mu^+$ on any surface of non-negative curvature.

Nevertheless, the quantity $4\pi - 2\mu$ does in some cases occur in a natural way in connection with a relation of the form (1).
To make the idea clear, consider a circular cone of vertex half-angle $\alpha$, and a circle of radius $r$ on the cone. We compute easily,

$$L(r) = 2\pi r, \quad A(r) = \pi r^2 \csc \alpha, \quad \mu(r) = \mu = 2\pi(1 - \sin \alpha),$$

and hence

$$\frac{L^2(r)}{A(r)} = 4\pi - 2\mu.$$  This does not contradict Huber's theorem, as the surface is singular at the vertex. If the cone is smoothed out near this point, we obtain

$$L^2(r) = 4\pi - 2\mu + o(1)$$

as $r \to \infty$. Since this estimate is obviously independent of how the smoothing is accomplished, we see that a result of this type can be expected to hold also for surfaces on which the curvature is of mixed sign.

We shall show in this paper that if "circles" on the surface are defined with respect to global conformal parameters on $S$, then an estimate of the form (2) is true for every complete simply-connected open surface on which the curvature is absolutely integrable. If certain smoothness hypotheses are satisfied by $S$ at infinity, then these "circles" become, up to terms of negligible order, curves which are equidistant from a fixed point on $S$. In this sense, the image of a large circle in a plane on which $S$ is represented conformally will be a "circle" on $S$. As corollaries of the method, asymptotic estimates for the lengths of the images of circles and of radial lines in such a reference plane are given, depending only on the curvatura integra. Under some conditions, estimates above or below for the local stretching in the mapping can also be given. Also, new proofs are obtained, in the
case considered here, of theorems of Cohn-Vossen [2], Blanc and Fiala [3], and Huber [4].

The demonstrations have turned out to be considerably more intricate than I had at first anticipated, for the results can be obtained relatively easily when the curvature vanishes outside a compact subset of $S$. It seems, however, desirable to present the material in the generality imposed by the most natural geometrical assumptions.

It is hoped that the methods of this paper will find application also in other problems slated to the geometry of two-dimensional surfaces. One possibility would be to study again some of the problems considered by Huber in [4] with a view to providing new and perhaps simpler demonstrations. We have, however, not carried out this program.

I should like to thank my colleagues, Professors T. Frankel and C. Loewner, for many stimulating conversations. I am indebted particularly to Professor P. Malliavin for a suggestion which has led to a considerable improvement of my original results.

1. Preliminary remarks; notation and definitions:

By an abstract surface $S$ we shall mean an open Riemannian manifold whose metric is defined in terms of local parameters $t, n$ by a positive definite quadratic form

$$(3) \quad ds^2 = E(t, n) \, dt^2 + 2F(t, n) \, dtdn + G(t, n) \, dn^2 .$$

Such a surface is in particular a Riemann surface with angle defined locally by the given metric. We shall assume that $S$ is simply connected,
and in this case $S$ can be mapped conformally, in the metric (3), either onto the unit disc $|z| < 1$ (hyperbolic case) or else onto the entire $z$-plane (parabolic case). We write

$$ds^2 = e^{2u(x,y)} (dx^2 + dy^2) = e^{2u(z)} |dz|^2 = \lambda^2(z) |dz|^2 .$$

Here we have used $z$, as we shall throughout this paper, to denote both the complex variable $x + iy$ and the pair of numbers $(x, y)$.

From a knowledge of $E, F, G$ as functions of local parameters one can, by the Theorema Egregium, calculate the Gaussian curvature $K$ at each point of $S$. In terms of the conformal parameters $z$, this relation takes the form

$$K = -e^{-2u} \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = -e^{-2u} \Delta u .$$

Thus, the curvatura integra, or surface integral of the Gaussian curvature, over some region $\mathcal{G}$ on $S$, is expressed by

$$\mu(\mathcal{G}) = -\int \Delta u \, dx \, dy$$

the integration being taken over the corresponding region in the $z$-plane. This relation defines a mass distribution or measure $\mu(\mathcal{G})$ over $S$, such that the total mass $\mu(S) = \mu$ is the curvatura integra of $S$.

We shall suppose that the positive and negative mass associated with $S$ are individually finite, thus

$$T = \int_S |d\mu| = \int_\mathcal{G} |\Delta u| \, dx \, dy < \infty .$$
We shall consider surfaces $S$ which are complete in the sense of Hopf-Rinow [5], that is, every divergent path on $S$ has infinite length. A path on $S$ is said to be divergent if it is the topological image $p = p(t)$ of a half-open interval $0 \leq t < 1$, and if $p(t)$ lies outside any given compact set on $S$ for all $t$ sufficiently close to 1. Under our assumptions, we shall provide new proofs of the known facts, that $\mu \leq 2\kappa$ and $S$ is conformally parabolic.

In order to minimize technical complications, we assume throughout this paper that all entering functions are sufficiently regular to justify the operations which are performed. This will be guaranteed, for example, if $E, F,$ and $G$ are in the Hölder class $C^{2+\alpha}$ with respect to suitable parameters. Most of our results are however independent of smoothness considerations, and hence will be true for all surfaces whose curvature is defined and integrable, and for which $E, F, G,$ can be approximated by smooth functions such that the mappings converge, and such that the integral curvature over an arbitrary open set converges in $L_1$, to the given one. Such surfaces include, for example, polyhedra in which the curvature is concentrated at isolated singular points.

Throughout this paper the symbols $A$ and $C$ are used to denote constants, the value of which may change even within a given context. Thus, from $|z| < A(1 + |\xi|)$ we may conclude $|z| < A |\xi|$ for $|\xi| > 1$. The symbols $\epsilon$ and $\delta$ will be used in a similar sense, but will usually represent small quantities.

2. **A theorem on conformal type:**

The following result is contained in Huber [4, Theorem 15]. In the case required for this paper, it is possible to give a relatively simple
demonstration, which we present below.

**Theorem 2.1:** If an open simply-connected Riemann surface $S$ admits a complete conformal metric $e^{u(z)}|dz|$, and if $T < \infty$, then $S$ is parabolic.

**Proof:** If $S$ were not parabolic, it could be represented conformally on the unit disc $\Sigma : |z| < 1$. Corresponding to the subcircle $\Sigma_r: |z| < r$, we would have the representation, valid whenever $|z| < r$,

$$u(z) = -\int_{\Sigma_r} g(z; \xi) \, d\mu(\xi) + h_r(z)$$

where $\mu(\xi)$ is the associated measure, $g(z; \xi)$ is the Green's function for the unit disc, and $h_r(z)$ is harmonic. Since by assumption, $T = \int_{\Sigma} |d\mu(\xi)| < \infty$, the integrals converge, uniformly in any $\Sigma_{r_0}$, as $r \to 1$. Since $u(z)$ does not depend on $r$, we conclude that $h_r(z) \to h(z)$, again uniformly in any $\Sigma_{r_0}$, where $h(z)$ is harmonic.

Thus,

$$u(z) = -\int_{\Sigma} g(z; \xi) \, d\mu(\xi) + h(z).$$

This relation contradicts the assumed completeness of the metric.

For consider the analytic function

$$w = F(z) = \int_{\Sigma} e^{h(z)} \, \hat{h}(z) \, dz$$

defined in $\Sigma$, where $\hat{h}(z)$ is a harmonic function conjugate to $h(z)$.
in \( \Sigma \). \( F(z) \) maps the unit disc onto an unbranched Riemann surface over the \( w \)-plane, and takes \( z = 0 \) into the origin \( 0 \) in some sheet of the surface. Consider now all rays which originate at \( 0 \) in the \( w \)-plane. Not every such ray can be extended to infinity in the Riemann surface, for this would imply that the entire plane is a sheet of the surface, and the inverse mapping would determine a bounded analytic function over the plane. Hence at least one such ray \( \gamma_w \) meets the boundary at a finite point \( P \). Denote by \( \gamma_z \) the inverse image of \( \gamma_w \). Then,

\[
\int_{\gamma_z} e^{u(z)} |dz| = \int_{\gamma_z} e^{v(z)} e^{h(z)} |dz| = \int_{\gamma_w} e^{v[z(w)]} |dw|,
\]

where we have set \( v(z) = u(z) - h(z) \). We shall show that under the hypotheses, the last integral on the right is finite, thus establishing the existence on \( S \) of a divergent path of finite length. To do this, we study the function

\[
v(z) = -\int_{\Sigma} g(z; \xi) \, d\mu(\xi) = -\int_{\Sigma} \mathcal{G}(w; \omega) \, dm(\omega) = v[z(w)] = v(w)
\]

where \( \mathcal{G}(w; \omega) \) is the Green's function for the image \( \Sigma \) of \( \Sigma \), and \( dm(\omega) \) is the corresponding mass distribution over \( \Sigma \). We shall need two preliminary results, for which we postpone the demonstrations:

**Lemma 2.1:** There exists a finite constant \( C \), depending only on \( \Sigma \), on \( P \), and on \( \delta \), \((\delta \text{ small})\), such that whenever \( w \) and \( \omega \) have distances respectively less than \( \delta/2 \) and \( \delta \) from \( P \), then

\[
|\mathcal{G}(w; \omega)| < \frac{1}{2\pi} \left[ \log \frac{1}{r_{wU}} + C \right].
\]
Lemma 2.2: There exists a finite constant $C$, depending only on $\Sigma$, on $P$, and on $\delta$, such that whenever $|w - P| < 5/4$ and $|w - P| > \delta$, then $\left| \mathcal{J}(w; \omega) \right| < C$.

To complete the proof, choose $\delta < \frac{1}{2}$ and so small that the total mass $\int_{\Sigma_{\delta}} |dm(\omega)|$ interior to the circle $\Sigma_{\delta}$ of radius $\delta$ about $P$ is smaller than $2\kappa$. Denote the part of $\gamma_\omega$ which lies interior to $\Sigma_{\delta/2}$ by $\gamma$, and let $\alpha(M)$ be the measure of that set $E_M$ on $\gamma$, in which $|v(w)| > M$. We then have, by the above lemmas

$$M \alpha(M) < \int_{E_M} |v| |dw| < \frac{1}{2\kappa} \int_{\Sigma_{\delta}} \left\{ \int_{E_M} \left[ (C + \log \frac{1}{r_{\omega\omega}}) |dw| \right] \right\} |dm(\omega)|$$

$$+ CT \alpha(M)$$

with, as before, $T = \int_{\Sigma} |dm(\omega)| = \int_{\Sigma} |d\mu(\xi)|$. Now the integral in brackets on the right is clearly maximized when the set $E_M$ is an interval containing $\omega$ at its mid point. Thus,

$$\int_{E_M} (C + \log \frac{1}{r_{\omega\omega}}) |dw| \leq (1 + C) \alpha(M) + \alpha(M) \log \frac{1}{\alpha(M)}$$

so that, letting $\beta = \frac{1}{2\kappa} \int_{\Sigma_{\delta}} |dm(\omega)|$, ($\beta < 1$), we have

$$M < C + \beta \log \frac{1}{\alpha(M)}$$

for some (new) constant $C$. Hence

$$\alpha(M) < C e^{- \frac{1}{\beta M}}$$
ad $M \to \infty$. Therefore, we may write

$$\int_{\gamma} e^{V(w)} |dw| = \int e^{M} |d\alpha(M)| \leq C + \int^{\infty} e^{1 - \frac{1}{\beta} M} \, dM < \infty$$

and we conclude that the length of the path $\gamma_z$ is in the given metric finite.

It remains only to prove the Lemmas 2.1 and 2.2. We observe first that

$$\mathcal{H}(w; \omega) = \frac{1}{2\pi} \log \frac{1}{r_{\omega} w} + \alpha_1(w; \omega) + \alpha_2(w; \omega)$$

where

$$\alpha_1 = \begin{cases} \frac{1}{2\pi} \log \frac{1}{r} & \text{on } \gamma_1 \\ 0 & \text{on } \gamma_2 \end{cases}, \quad \alpha_2 = \begin{cases} \frac{1}{2\pi} \log \frac{1}{r} & \text{on } \gamma_2 \\ 0 & \text{on } \gamma_1 \end{cases}$$

$\gamma_1$ and $\gamma_2$ being the parts of the boundary of $\mathbb{T}$ lying, respectively, interior and exterior to $\Sigma_0$. Thus, since $\delta < 1/2$,

$$0 < \mathcal{H}(w; \omega) < \frac{1}{2\pi} \log \frac{1}{r_{\omega} w} + \alpha_2(w; \omega).$$

For fixed $w \in \Sigma_{\delta/2} \cap \mathbb{T}$, $\alpha_2(w; \omega)$ is harmonic and bounded in $\Sigma_0 \cap \mathbb{T}$, $|\alpha_2(w; \omega)| < A < \infty$ in this region. By the mean value theorem, the change in boundary data for $\alpha_2(w; \omega)$ as $w$ moves in $\Sigma_{\delta/2}$ does not exceed $\frac{\delta}{r_{\omega} - \delta/2}$, hence by the maximum principle for harmonic
functions, \(|\alpha_2(w; \omega)| < A + 2\) in \(\Sigma_\delta \cap \Pi\), uniformly for all \(w\) in \(\Sigma_{\delta/2} \cap \Pi\). This proves Lemma 2.1.

From the above discussion we see that 

\(|\alpha_2(w; \omega)| \leq A + 2\) if \(w \in \Sigma_{\delta/2} \cap \Pi\), and \(\omega\) lies on the boundary \(\Gamma_\delta\) of \(\Sigma_\delta\). Thus \(G(w; \omega)\) is uniformly bounded on \(\Gamma_\delta\). Since \(G(w; \omega) = 0\) on the boundary of \(\Pi\) and, for the given choice of \(w\), is harmonic in \((R - \Sigma_\delta) \cap \Pi\), \((R\) being the whole space), Lemma 2.2 follows similarly from the maximum principle. This completes the proof of Theorem 2.1.

3. Estimates of length and area:

We derive first estimates of these quantities from below, which are valid for arbitrary surfaces. Let the surface \(S\) be represented conformally on a plane region which includes the disc \(\Sigma_r: |z| \leq r\). Let \(\Gamma_r\) be the boundary of \(\Sigma_r\) and let \(L(r)\) and \(A(r)\) be, respectively, the length of the image \(\gamma_r\) of \(\Gamma_r\) on \(S\) and the area bounded by \(\gamma_r\). Let \(e^{u(z)}\) be the local length ratio in the mapping.

**Theorem 3.1:** Under the above hypotheses, there always holds

\[
L(r) \geq 2\pi e^{u_0} r \exp \left\{ -\frac{1}{2\pi} \int_0^r \frac{\mu(\Sigma_\rho)}{\rho} d\rho \right\}
\]

\[
A(r) \geq 2\pi e^{2u_0} \int_0^r \rho \exp \left\{ -\frac{1}{2\pi} \int_0^\rho \frac{\mu(\Sigma_\tau)}{\tau} d\tau \right\} d\rho
\]

where \(u_0 = u(0)\) and \(\mu(\Sigma_\rho)\) is the curvatura integra over \(\Sigma_\rho\).

Equality holds if and only if \(u(z)\) is a function only of \(r\).
Proof: Let \( \mu \) be the measure associated with \( u(z) \). From the relation \( \Delta u = -K e^{2u} \), we conclude

\[
\int_{\Sigma_\rho} d\mu = \mu(\Sigma_\rho) = -\int_{\Gamma_\rho} \frac{\partial u}{\partial n} \, ds = -\rho \frac{d}{d\rho} \int_{\Gamma_\rho} u \, ds
\]

where \( \mu(\Sigma_\rho) \) is the curvatura integra. Hence

\[
-\frac{1}{2\pi} \int_0^\rho \frac{\mu(\Sigma_\rho)}{\rho} \, d\rho = \frac{1}{2\pi} \int_{\Gamma_R} u \, ds - u_0
\]

\[
= \frac{1}{2\pi} \int_{\Gamma_R} (u - u_0) \, ds = \frac{1}{2\pi} \int_{\Gamma_R} \log e^{(u-u_0)} \, ds
\]

\[
\leq \log \frac{1}{2\pi} \int_{\Gamma_R} e^{(u-u_0)} \, ds
\]

from which the first inequality follows. \( \frac{3}{2} \)

The second relation is proved similarly. In fact, we write

\[
-\frac{1}{2\pi} \int_0^\rho \frac{\mu(\Sigma_\tau)}{\tau} \, d\tau = \frac{1}{2\pi} \int_{\Gamma_R} (u - u_0) \, ds = \frac{1}{2\pi} \int_{\Gamma_R} \log e^{2(u-u_0)} \, ds
\]

\[
\leq \frac{1}{2} \log \frac{1}{2\pi} \int_{\Gamma_R} e^{2(u-u_0)} \, ds
\]

Thus,

\[
e^{2u_0} \exp \left\{ -\frac{1}{\pi} \int_0^\rho \frac{\mu(\Sigma_\tau)}{\tau} \, d\tau \right\} \leq \frac{1}{2\pi} \int_{\Gamma_R} e^{2u} \, ds = \frac{1}{2\pi} \frac{dA(r)}{dr}
\]

from which the result follows on a further integration.
Corollary: Let \( S \) be simply connected and conformally parabolic, and suppose that \( \mu(\Sigma_r) \leq 2\pi \) for all sufficiently large \( r \). Then \( S \) has infinite area.

The proof is immediate from the second inequality of Theorem 3.1, for under the assumptions \( S \) can be mapped onto the entire finite plane.

Remark 1: If \( S \) is in addition complete, and if the curvature is absolutely integrable over \( S \), then \( \lim_{r \to \infty} \mu(\Sigma_r) \leq 2\pi \), cf., Huber [4, Theorem 10], also the corollary to Theorem 3.4 in this paper.

Remark 2: There exist complete surfaces \( S \) with finite area, which satisfy all the above hypotheses except the assumption \( \mu(\Sigma_r) \leq 2\pi \) for all large \( r \). An example is provided by the conformal metric
\[
ds = \frac{|dz|}{(|z| + 1) \log (|z| + 2)}
\]
over the \( z \)-plane. See Huber [4, p. 61].

If the surface \( S \) is assumed complete and of finite total curvature, the inequalities in Theorem 3.1 can, essentially, be replaced by equalities. This is the content of our next result.

Theorem 3.2: Let \( S \) be a complete, simply connected open surface, over which the curvature is absolutely integrable. Then \( S \) can be represented conformally on the whole \( z \)-plane, and in each such mapping the relation
\[
\mathcal{L}(r) = 2\pi e^{\frac{u_0 + o(1)}{r}} \exp \left\{ -\frac{1}{2\pi} \int_0^r \frac{\mu(\Sigma_p)}{\rho} d\rho \right\}
\]
is asymptotically satisfied as \( r \to \infty \). If in addition \( S \) has infinite
area (cf., the Corollary to Theorem 3.1), then

\[ A(r) = 2\pi e^{2u_0 + o(1)} \int_0^r \rho \exp \left\{ -\frac{1}{\pi} \int_0^\rho \frac{\mu(\Sigma_t)}{t} \, dt \right\} \, d\rho \]  

Theorem 3.2a: Let \( S \) be a complete, simply connected open surface for which \( \frac{1}{T} < \infty \) and \( -\infty < \mu < 2\pi \), \( \mu = \text{curvatura integra} \). Then \( S \) can be conformally represented on the whole \( z \)-plane and in each such mapping there holds asymptotically for large \( r \),

\[ A(r) = 2\pi e^{2u_0 + o(1)} \frac{2\pi r^2}{2\pi - \mu} \exp \left\{ -\frac{1}{\pi} \int_0^r \frac{\mu(\Sigma_t)}{\rho} \, d\rho \right\}. \]

Proof of Theorem 3.2: The first statement of the theorem follows from Theorem 2.1. In any representation on the \( z \)-plane, we then have for the logarithm of the local length ratio,

\[ u(z) = u(0) - \frac{1}{2\pi} \int_{\Sigma_r} \log |1 - \frac{z}{\xi}| \, d\mu(\xi) \]

\[ + h_r(z) - h_r(0) \]

for any \( r \), where \( h_r(z) \) is harmonic in \( \Sigma_r \). The integral converges absolutely as \( r \to \infty \), and we obtain as in the proof of Theorem 2.1,

\[ u(z) = u(0) - \frac{1}{2\pi} \int \log |1 - \frac{z}{\xi}| \, d\mu(\xi) + h(z) - h(0) \]

the integration being taken over the entire \( z \)-plane. We introduce
again the mapping

\[ w = F(z) = \int_{0}^{z} e^{h(\zeta) + i\tilde{h}(\zeta)} \, d\zeta, \]

and consider all rays on the unbranched Riemann surface defined by \( F(z) \), starting at the image of the origin. If there should be one such ray which cannot be extended to infinity, the method of proof of Theorem 2.1 shows that the inverse image of this segment is a divergent path of finite length, thus contradicting the assumed completeness.

We omit details. Therefore all segments extend to infinity, so that one branch of the Riemann surface necessarily covers the \( w \)-plane. But the inverse image of \( F(z) \) is by the monodromy theorem single valued. We conclude that \( F(z) = Az \) for a suitable constant \( A \), and therefore \( h(z) = \text{const.} \) Thus,

\[ u(z) = u(0) - \frac{1}{2\pi} \int \log |1 - \frac{z}{\xi}| \, d\mu(\xi) \]

for all \( z \). We base the proof of the theorem on this representation.

Let \( |z| = r \) and write

(7) \[ u(z) = u(0) + u_1(z) + u_2(z) \]

where

\[ u_1(z) = -\frac{1}{2\pi} \int_{\Sigma_{r/2}} \log \left| \frac{z - \xi}{\xi} \right| \, d\mu(\xi) \]

\[ u_2(z) = -\frac{1}{2\pi} \int_{\mathcal{E}_{r/2}} \log \left| \frac{z - \xi}{\xi} \right| \, d\mu(\xi) \]

\( \mathcal{E}_r \) being the exterior of \( \Sigma_r \).
In $\Sigma_{r/2}$, we have

$$\log \left| \frac{z - \xi}{z} \right| = \log \left| \frac{z}{\xi} \right| - \log \left| 1 - \frac{z}{\xi} \right|^{-1},$$

and

$$\int_{\Sigma_{r/2}} \log \left| 1 - \frac{z}{\xi} \right| |d\mu(\xi)| \leq T \log \frac{1}{1 - \eta} + \int_{\Sigma_{r/2}} \log \left| 1 - \frac{z}{\xi} \right| |d\mu(\xi)|$$

where

$$T = \int_{|\xi| < \infty} |d\mu(\xi)|, \quad \text{and} \quad 0 < \eta < 1/2.$$ 

In the last term on the right, the integrand is bounded, hence for a suitable constant $A$,

$$\int_{\Sigma_{r/2}} \log \left| 1 - \frac{z}{\xi} \right| |d\mu(\xi)| \leq T \log \frac{1}{1 - \eta} + A \int_{\Sigma_{\eta r}} |d\mu(\xi)|$$

If we choose $\eta = \eta(r)$ tending to zero but such that $\eta r \to \infty$, we see that

$$\int_{\Sigma_{\eta r}} \log \left| 1 - \frac{z}{\xi} \right| |d\mu(\xi)| = o(1)$$

as $r \to \infty$. Next, since $|\xi|^2$ is bounded for $|\xi| > \frac{1}{2}|z|$, 

$$\int_{\Sigma_{\eta r}} \log |\xi|^2 |d\mu(\xi)| = \int_{\Sigma_r} \log |\xi|^2 |d\mu(\xi)| + o(1)$$

$$= \int_0^r \frac{\mu(\Sigma_r)}{\rho} d\rho + o(1)$$
after an integration by parts. Thus

\[(8) \qquad u_1(z) = \frac{-1}{2\pi} \int_0^r \frac{\mu(S_\rho)}{\rho} \, d\rho + o(1).\]

Consider now \(\int_0^{2\pi} \left[ e^{u_2(z)} - 1 \right] \, d\theta\) for \(|z| = r\) and \(\theta = \arg z\).

Let \(\alpha(M)\) be the measure of the set \(E_M\) of \(\theta\) for which \(|u_2(z)| > M\).

Then

\[(9) \qquad M \, \alpha(M) < \int_{E_M} |u_2(z)| \, d\theta < \frac{1}{2\pi} \int_0^{2\pi} Q(\xi) \, |d\mu(\xi)|,\]

where

\[Q(\xi) = \left| \int_{E_M} \log |1 - \frac{z}{\xi}| \, d\theta \right|.\]

For \(|\xi| > \frac{1}{2} |z|\), \(|1 - \frac{z}{\xi}| < 3\), hence \(\log |1 - \frac{z}{\xi}|\) is bounded except at \(z = \xi\). It follows that \(Q(\xi)\) is maximized when \(E_M\) is an interval with its mid-point at \(\arg \xi\). For this configuration, we compute

\[Q(\xi) < A(1 + |\log \alpha(M)|) \, \alpha(M)\]

for some constant \(A\), and hence, from (9),

\[M < (1 + |\log \alpha(M)|) \, \epsilon(r)\]

where \(\epsilon(r) \to 0\) as \(r \to \infty\). We conclude.
\( \alpha(M) < A e^{\frac{-1}{\varepsilon(r)^M}} \)

for a suitable \( A \). But

\[
\int_0^{2\pi} |e^{u_2(z)} - 1| \, d\theta \leq \int_{M>0} |e^M - 1| \, d\alpha(M) + \int_{M>0} |e^{-M} - 1| \, d\alpha(M)
\]

from which there follows easily

\[
(10) \quad \int_0^{2\pi} |e^{u_2(z)} - 1| \, d\theta = o(1)
\]

as \( r \to \infty \).

We may now write, for \( |z| = r \), by (7), (8),

\[
\mathcal{L}(r) = r \int_0^{2\pi} e^{u(z)} \, d\theta = \text{re}^{\frac{1}{2}} \int_0^{2\pi} e^{u_1(z)} e^{u_2(z)} \, d\theta
\]

\[
= \text{re}^{\frac{1}{2}} + o(1) \exp \left( \frac{-1}{2\pi} \int_0^r \frac{\mu(\Sigma)}{\rho} \, dp \right) \int_{M > 0} e^{u_2(z)} \, d\theta
\]

\[
= 2\pi \text{re}^{\frac{1}{2}} + o(1) \exp \left( \frac{-1}{2\pi} \int_0^r \frac{\mu(\Sigma)}{\rho} \, dp \right)
\]

by (10). This proves (4).

To prove (5), observe that

\[
\frac{d}{dr} \mathcal{A}(r) = \int_r e^{2u(z)} \, ds.
\]
By \((7)\), \((8)\), and \((10)\), we find

\[
\frac{dQ(r)}{dr} = 2\pi e^{2u_0 + o(1)} r e^{\frac{1}{\pi} \int_0^r \frac{\mu(x)}{\rho^2} \, d\rho}
\]

from which \((5)\) follows by an integration.

**Proof of Theorem 3.2a:** By Theorem 2.1, \(S\) can be conformally represented on the whole \(z\)-plane. By the corollary to Theorem 3.1, \(S\) has infinite area. Hence by Theorem 3.2,

\[
\mathcal{A}(r) = 2\pi e^{2u_0 + o(1)} \int_0^r \rho F(\rho) \, d\rho
\]

where

\[
F(\rho) = \exp \left\{ -\frac{1}{\pi} \int_0^\rho \frac{\mu(x)}{\tau} \, d\tau \right\}.
\]

We have

\[
\int_0^r \rho F(\rho) \, d\rho = \frac{r^2}{2} F(r) + \frac{1}{2\pi} \int_0^r \rho^2 \frac{\mu(x)}{\rho} F(\rho) \, d\rho
\]

\[
= \frac{r^2}{2} F(r) + \frac{\mu}{2\pi} \int_0^r \rho F(\rho) \, d\rho + \frac{1}{2\pi} \int_0^r [\mu(\rho) - \mu] \rho F(\rho) \, d\rho
\]

Again using the fact that \(S\) has infinite area, we obtain

\[
[1 - \frac{\mu}{2\pi} + \epsilon(r)] \int_0^r \rho F(\rho) \, d\rho = \frac{r^2}{2} F(r)
\]

where \(\epsilon(r) \to 0\) as \(r \to \infty\), and from this \((6)\) follows.
From Theorems 3.2 and 3.2a we obtain immediately:

**Theorem 3.3:** Let \( S \) be a complete, simply connected open surface, for which \( T < \infty \) and \( -\infty < \mu < 2\pi \). Then \( S \) can be conformally represented on the whole \( \mathbb{z} \)-plane, and asymptotically for large \( r \), there holds

\[
\frac{\mathcal{L}^2(r)}{\mathcal{A}(r)} = 4\pi - 2\mu + o(1).
\]

We remark that it is easy to give examples of surfaces for which \( \mu(\Sigma_r) \rightarrow \mu < 2\pi \), for which \( T = \infty \), and for which (14) is incorrect.

**Theorem 3.3a:** Let \( S \) be a complete, simply connected open surface for which \( \mu = 2\pi \). Then if \( S \) has infinite area, there holds

\[
\frac{\mathcal{L}^2(r)}{\mathcal{A}(r)} = 4\pi - 2\mu + o(1).
\]

**Theorem 3.3b:** Let \( S \) be a complete, simply connected open surface for which \( \mu = 2\pi \). If exterior to a compact set on \( S \) the curvature is either non-positive or non-negative, then

\[
\frac{\mathcal{L}^2(r)}{\mathcal{A}(r)} = 4\pi - 2\mu + o(1).
\]

**Proof of Theorem 3.3a:** By Theorem 3.2,

\[
\mathcal{L}^2(r) = 4\pi^2 e^{2u_o + o(1)} r^2 p(r).
\]
where \( F(r) \) is defined by (12). By (13),
\[
  r^2 F(r) = \frac{1}{\pi} \int_0^r [\mu(\Sigma_\rho) - \mu] \rho F(\rho) \, d\rho .
\]

By assumption \( \mathcal{A}(r) = 2\pi e^{2u_0 + o(1)} \int_0^r \rho F(\rho) \, d\rho \) is unbounded. The result then follows from the fact that \( \mu(\Sigma_\rho) \to \mu. \)

Proof of Theorem 3.3b: If \( \mu = 2\pi \) and \( K \geq 0 \) at infinity, one sees from Theorem 3.1 that \( \mathcal{A} = \infty \), so that Theorem 3.3a applies. Suppose \( K \leq 0 \) at infinity. By Theorem 3.1,
\[
  \mathcal{A}(r) \geq 2\pi e^{2u_0} \int_0^r \rho \exp \left\{ - \frac{1}{\pi} \int_0^\rho \frac{\mu(\Sigma_\rho)}{\tau} \, d\tau \right\} \, d\rho
\]
\[
  = 2\pi e^{2u_0} \int_0^r \rho^{-1} H(\rho) \, d\rho
\]
defining \( H(\rho) \). We compute
\[
  \frac{dH}{d\rho} = 2\rho [1 - \frac{\mu(\Sigma_\rho)}{2\pi}] \exp \left\{ - \frac{1}{\pi} \int_0^r \frac{\mu(\Sigma_\rho)}{\rho} \, d\rho \right\}
\]
and hence, under the assumptions, \( \frac{dH}{d\rho} \leq 0 \) for all sufficiently large \( \rho \). We conclude \( H \to 0 \), since otherwise there would hold \( \mathcal{A}(r) \to \infty \); hence \( L^2(r) \to 0 \), qed.

We study now the image curves of the radial lines emanating from the origin in the \( z \)-plane on which the surface \( S \) is represented conformally. We shall give asymptotic estimates for the lengths \( L(r) \) of these curves as the radial segments tend to infinity.
Theorem 3.4: Let $S$ be a complete simply-connected open surface for which $T < \infty$. Then for any $\delta > 0$, there holds asymptotically for the length on $S$ of the image of a radial segment of length $r$ from the origin in the $z$-plane,

$$\frac{1 - \frac{\mu}{2\pi} - \delta}{r^2} < L(r) < \frac{1 - \frac{\mu}{2\pi} + \delta}{r^2}$$

(15)

Proof: We follow, essentially, the proof of Theorem 3.2. In the notation of that proof, we find

$$u_1(z) = -\frac{1}{2\pi} \int_0^r \frac{\mu(T \rho)}{\rho} \, d\rho + o(1)$$

(16)

$$= -\frac{\mu}{2\pi} \log r + \int_1^r \frac{[\mu - \mu(T \rho)]}{\rho} \, d\rho - \frac{1}{2\pi} \int_0^1 \frac{\mu(T \rho)}{\rho} \, d\rho + o(1)$$

$$= -\left(\frac{\mu}{2\pi} + o(1)\right) \log r .$$

The length $L(r)$ of the image on $S$ of a radial line is $L(r) = \int_0^r e^{u(z)} \, ds$, and we have, for large $r_0$,

$$e^{u_0} \int_{r_0}^r e^{u_0} (z) \, d\rho \leq \int_{r_0}^r e^{u(z)} \, ds \leq e^{u_0} \int_{r_0}^r e^{u(z)} \, d\rho$$

(17)

where $\delta^-$ and $\delta^+$ are any two numbers such that $\delta^- < -\frac{\mu}{2\pi} < \delta^+$. For any $\delta$, we have

$$\int_{r_0}^r e^{u(z)} \, d\rho = \int_{r_0}^r e^{u(z)} \, d\rho + \int_{r_0}^r [e^{u(z)} - 1] \, d\rho$$

(18)
and if \( r \leq 2r_0 \) we may write

\[
\left| \int_{r_0}^{r} \delta \left( u_2(z) - 1 \right) \, d\rho \right| < C \left( \int_{r_0}^{r} \left| e^{u_2(z)} - 1 \right| \, d\rho \right)
\]

Let \( E_M \) be the set on \([r_0, r]\) where \( |u_2(z)| > M \) and let \( \alpha(M) \) be its measure. Then

\[
\alpha(M) \cdot M < \int_{E_M} |u_2(z)| \, ds < \int_{E_M} |d\mu_\delta| \int_{E_M} \left| \log |1 - \frac{z}{\delta}| \right| \, ds \, z.
\]

The last integral is maximized for a \( \zeta \) such that \( \arg \zeta = \arg z \). We have, for \( \tau = |\frac{z}{\delta}| \) and \( E_M^* \), the image of \( E_M \),

\[
\int_{E_M} |\log |1 - \frac{z}{\delta}|| \, ds \leq |\zeta| \int_{E_M^*} |\log |1 - \tau|| \, d\tau.
\]

If \( r \leq 2r_0 \), then on the set \( E_M^* \) there holds always \( |\frac{z}{\delta}| = \tau \leq 4 \), and we obtain

\[
\int_{E_M^*} |\log |1 - \tau|| \, d\tau \leq A \alpha^*(M) \left[ 1 + \log \frac{1}{\alpha^*(M)} \right]
\]

\[
= A \frac{1}{|\zeta|} \alpha(M) \left[ 1 + \log \frac{1}{\alpha(M)} \right].
\]

Thus,

\[
\alpha(M) \cdot M < \epsilon(r) \cdot \alpha(M) \left[ 1 + \log \frac{2r_0}{\alpha(M)} \right]
\]

where \( \epsilon(r) \to 0 \) as \( r \to \infty \), from which

\[
\alpha(M) < C \text{ re} \frac{1}{\epsilon(r)^M}
\]

22
Hence, if \( r < 2r_0 \),

\[
\left| \int_{R^2_0} ^r \left[ e^{u_2(z)} - 1 \right] ds \right| \leq \left| \int_{M=0} [e^M - 1] d\alpha(M) \right| + \left| \int_{M=0} [e^{-M-1}] d\alpha(M) \right|
\]

\[
\leq C \cdot \epsilon(r) \cdot r
\]

for some constant \( C \). Hence in this case, (19) becomes

(20)\[
\left| \int_{R^2_0} ^r \rho^5 \left[ e^{u_2(z)} - 1 \right] d\phi \right| < C \cdot \epsilon(r_0) \cdot r_0^{1+\delta}
\]

Consider now an arbitrary \( r > r_0 \), and let \( n \) be the smallest integer for which \( 2^n r_0 > r \). Then \( r < 2^n r_0 \leq 2r \), and we find from (20)

\[
\left| \int_{R^2_0} ^r \rho^5 \left[ e^{u_2(z)} - 1 \right] d\phi \right| < C \cdot \epsilon(r_0) \cdot r_0^{1+\delta} \left[ 1 + 2^{1+\delta} + 2^n(1+\delta) \right]
\]

\[
= C \cdot \epsilon(r_0) \cdot r_0^{1+\delta} \cdot \frac{2^{(n+1)(1+\delta)} - 1}{2(1+\delta) - 1}
\]

(21)

\[
< C \cdot \epsilon(r_0) \begin{cases} 
    r_0^{1+\delta} & \text{if } \delta > -1 \\
    r_0^{1+\delta} & \text{if } \delta < -1 
\end{cases}
\]

Using this inequality, the theorem follows easily from (17) and (18).

**Remark:** The example of the complete conformal metric \( ds=\log(2+|z|)|dz| \) spread over the z-plane, shows that the constant \( \delta \) cannot in general be removed from the exponent in (15). In this case \( \mu = 0 \) and \( L(r) = r \log r + O(r) \).
Corollary: Under the hypotheses of Theorem 3.4, we have necessarily $\mu \leq 2\pi$.

For otherwise, $L(r)$ would by (15) be bounded on every radial segment, so that $S$ could not be complete.

Theorem 3.5: Let $S$ be as in Theorem 3.4, and let $\gamma$ be a divergent path on $S$. Let $\gamma_z$ be the inverse image of $\gamma$ under a representation of $S$ on the $z$-plane, and let $L_r(\gamma)$ be the length of that part of $\gamma$ such that $\gamma_z$ lies interior to a circle $\Sigma_r$ of radius $r$ about the origin. Then for any $\delta > 0$ there holds asymptotically

$$L_r(\gamma) \geq r^{1-\frac{\mu}{2\pi} - \delta}$$

as $r \to \infty$.

Comparing this result with Theorem 3.4, we see that the images of the radial lines in the $z$-plane behave asymptotically as approximations to geodesics on $S$.

Proof: Setting $|\xi| = \rho$, we have

$$L_r(\gamma) = \int_{\gamma_z \cap \Sigma_r} e^{u(\xi)} \, ds \geq \int_{\rho \leq r} e^{u(\xi)} \, d\rho .$$

This inequality will be strengthened if we omit all arcs of $\gamma_z$ on which values of $\rho$ are repeated; that is, if the maximum values of $\rho$ attained on $\gamma_z$ for all arc lengths $s < s_o$ is $\rho_o$, all arcs on $\gamma_z$ for which $s > s_o$, $\rho < \rho_o$ are to be omitted in the integration. In this case the integration is monotonic in $\rho$ and the estimates in the proof of Theorem 3.4 are easily seen to apply, so that for the length
L_r(y) of that part of \( \gamma_z \) for which \( \rho < r \) we obtain \( L_r(y) \geq r^{1-\frac{\mu}{2\pi}} \) by (15) for any \( \delta > 0 \), the stated result.

4. A geometrical assumption; sharpening of the above estimates:

In order to improve the estimate of Theorem 3.4, it is necessary to introduce a new assumption on the regularity of \( S \) at infinity. Such an assumption, if it is to be meaningful, must necessarily involve only quantities which can a-priori be determined in terms of the intrinsic geometry of the surface, and it should not depend on properties of the representations of \( S \) on the \( z \)-plane. The simplest such hypothesis which is available to us involves the rate of decay of the curvature as the point of evaluation moves to infinity on \( S \). To make this concept precise, we select a fixed point \( P \) of \( S \), and define the distance \( \sigma_Q \) from \( P \) to \( Q \) on \( S \) as the greatest lower bound of lengths of paths which join \( P \) to \( Q \) on \( S \). We shall assume in this section that there are fixed constants \( C \) and \( \delta > 0 \) such that uniformly for all \( Q \) on \( S \), \( |K| < C \sigma^{-2-\delta} \).

Under this hypothesis, we can sharpen our earlier estimates.

Theorem 4.1: Under the hypotheses of Theorem 3.4 and the additional hypothesis \( |K| < C \sigma^{-2-\delta} \), there holds

\[
L_r(r) = A r^{1-\frac{\mu}{2\pi}} [1 + O(r^{-\epsilon})]
\]

for some positive constants \( A \) and \( \epsilon \), whenever \( \mu < 2\pi \).
Remark: The assumption $|K| < C \sigma^{-2-\delta}$ cannot be deleted, and even an assumption $|K| < C(\sigma \log \sigma)^{-2}$ is not sufficient. This can be seen from the example (which we have already considered in another context) of the conformal metric $ds = \log(2 + |z|)|dz|$ spread over the $z$-plane. For the surface $S$ defined in this way, we have $T < \infty$, $\mu = 0$, $K = (r \log^2 r)^{-2} \sim (\sigma \log \sigma)^{-2}$, $L(r) = r \log r + O(r)$.

Proof of Theorem 4.1: As point $P$ we choose the image of the origin in the $z$-plane (we are at liberty to choose an arbitrary $P$ by adjusting the constant $C$) and we consider a radial segment from the origin of length $r$ in the $z$-plane, defining a path $PQ$ on $S$. Consider now a smooth path joining $P$ to $Q$ on $S$, whose length approximates the distance $\sigma Q$ to $Q$. Applying Theorem 5.5 to the inverse image of this path, we obtain, for given $\delta > 0$ and large $r$, $\sigma Q + \epsilon \geq r^{1 - \frac{\mu}{2\pi} - \delta}$ for any $\epsilon > 0$. Hence

$$\sigma Q \geq r^{1 - \frac{\mu}{2\pi} - \delta}$$

as $r \to \infty$.

By assumption, $|K(Q)| < \sigma^{-2-\delta} < r^{\mu-2-\delta}$ (not the same in all contexts) as $r \to \infty$. Now

$$\mu = \int_S d\mu = \mu(E_r) + \int_{\Sigma_r} Ke^{2u} \rho d\rho d\theta$$

where $E_r$ is the exterior of $\Sigma_r$, and
\[ |\int_{\mathbb{C}_r} K e^{2u} \rho d\rho d\theta| \leq \int_r^{\infty} \frac{\mu}{\rho} - 1 - 5 d\rho \int_0^{2\pi} e^{2u(z)} d\theta. \]

The circuit integral on the right equals \( \rho^{-1} \frac{dA(\rho)}{d\rho} \). All the hypotheses of Theorem 3.2 are satisfied, so we may write from (11)

\[ \int e^{2u(z)} d\theta = 2\pi e^{2u_0 + o(1)} - \frac{1}{\pi} \int_0^{\rho} \frac{\mu(\tau)}{\tau} d\tau, \]

\[ = 2\pi e^{2u_0 + o(1)} - \frac{\mu}{\rho} + o(1) \]

by (16). Thus,

\[ |\int_{\mathbb{C}_r} K e^{2u(z)} \rho d\rho d\theta| \leq 2\pi e^{2u_0} \int_r^{\infty} \rho^{-1 - 5} d\rho = \frac{2\pi}{6} e^{2u_0} r^{-5} \]

for some \( \delta > 0 \). We have proved:

\[ \mu(\Sigma_r) = \mu + o(r^{-5}) \]

This estimate may be used to replace the estimate (16) by the more precise relation

\[ u_1(z) = - \frac{\mu}{2\pi} \log r + A + o(r^{-5}) \]

as may be seen by repeating the estimate of \( u_1(z) \) in the proof of Theorem 3.2 for this case, and hence to replace (17) by

\[ \int_{\mathbb{C}_r} e^{u(z)} ds = e^A(1 + o(r^{-5})) \int_r^{\infty} \frac{\mu}{2\pi} e^{u_2(z)} d\rho. \]
The relations (18) and (21) then yield

\[ L(r) = e^{A} r^{\frac{1-\mu}{2\kappa}} [1 + O(r^{-\epsilon})] , \]

the stated result.

Similarly we may prove:

Theorem 4.2: Let \( S \) be an open, complete, simply connected surface for which \( T < \infty, \mu < 2\pi \), and let \( S \) be represented conformally on the \( z \)-plane. Then (cf. Theorems 3.2, 3.3)

\[ L(r) = 2\pi e^{u_0} r^{\frac{1-\mu}{2\kappa}} [1 + O(r^{-\epsilon})] \]

\[ A(r) = \frac{2\pi^2}{2\kappa - \mu} e^{2u_0} r^{\frac{2-\mu}{\kappa}} [1 + O(r^{-\epsilon})] , \]

\[ \frac{L^2(r)}{A(r)} = 4\pi - 2\mu + O(r^{-\epsilon}) \]

for some constant \( \epsilon > 0 \).

We omit details.

5. Asymptotic estimates for the length ratio:

We denote the local length ratio in the representation of \( S \) on a plane by \( \lambda(z) = e^{u(z)} \).

Theorem 5.1: Let \( S \) be an open, complete, simply-connected surface with finite total curvature \( \mu \), and suppose the region in which \( K > 0 \) has compact support on \( S \). Then in any conformal representation of \( S \)
on the $z$-plane, there is a constant $C$ such that $\lambda(z) \leq C r^{-\frac{\mu}{2\pi}}$ as $r \to \infty$. If the region is which $K < 0$ is compact on $S$, then $\lambda(z) \geq C r^{-\frac{\mu}{2\pi}}$ for some $C$.

Proof: We use again the decomposition (7). Suppose that in a given representation of $S$, $K < 0$ outside the circle $\Sigma_{r_0}$. By (8), if $r > r_0$,

$$u_1(z) = -\frac{1}{2\pi} \int_0^r \frac{\mu(\Sigma_\rho)}{\rho} d\rho + o(1)$$

$$= -\frac{1}{2\pi} \int_0^r \frac{\mu(\Sigma_\rho)}{\rho} d\rho - \frac{1}{2\pi} \int_{r_0}^r \frac{\mu}{\rho} d\rho + \frac{1}{2\pi} \int_{r_0}^r \frac{[\mu(\Sigma_\rho)]}{\rho} d\rho + o(1).$$

By assumption, the last integral on the right is non-positive, hence

$$\lambda_1(z) = e^{u_1(z)} \leq C r^{-\frac{\mu}{2\pi}}.$$ 

Consider now

$$u_2(z) = -\frac{1}{2\pi} \int_{E_{r/2}} \log \left| \frac{z - \xi}{\xi} \right| d\mu(\xi)$$

for $r > 2r_0$. We may neglect the integration over the region $|z - \xi| < |\xi|$, since in this region $d\mu(\xi) \leq 0$. But if $|z - \xi| > |\xi|$ and $\xi \in E_{r/2}$, then $1 < \left| \frac{z - \xi}{\xi} \right| < 3$. Thus

$$\lambda_2(z) = e^{u_2(z)} \leq e^{o(1)}$$

which proves the first part of the result. The corresponding inequality, when $K > 0$ outside $\Sigma_{r_0}$, is proved similarly.
Corollary: Under the hypotheses of Theorem 5.1, if $-\infty < \mu < 2\pi$, then $L(r) \leq C r^{1-\frac{\mu}{2\pi}}$, $L(r) \geq C r^{1-\frac{\mu}{2\pi}}$ respectively, in the two cases considered. If $\mu = 2\pi$, then $L(r) \leq C \log r$, $L(r) \geq C \log r$, respectively.

Remark: If the curvature has compact support on $S$, then one sees easily that $\lambda = r^{\frac{\mu}{2\pi}} [1 + O(r^{-1})]$. Estimates of this type cannot be expected, however, in a general case, even under assumptions of the type introduced in § 4. One may imagine, for example, a surface on which the curvature is concentrated at a sequence of points tending to infinity, the surface appearing in the neighborhood of each such point as the vertex of a cone. Such a surface can be constructed so that the curvature tends to zero at infinity as rapidly as desired, but $\lambda$ will nevertheless be singular at each point of the sequence. In order to obtain asymptotic estimates of the above type on $\lambda(z)$ it would be necessary to introduce a new postulate on the local smoothness of the curvature on $S$. 

30
FOOTNOTES

1. With reference to the ensuing discussion, cf. Osserman [6, Lemma 6].

2. Possible difficulties due to irregularity of the boundary can be avoided by a simple approximation procedure.

3. The last step in a consequence of a standard inequality between arithmetic and geometric means.

4. If \( \mu > 2\nu \) then \( S \) cannot be complete, cf. the corollary to Theorem 3.4.

5. The function \( \log |1 - \frac{z}{3}| \), considered as function of \( (w, \omega) \), has all the properties of the Green's function which entered in the proof of Theorem 2.1 except that it is singular at the image \( \omega_0 \) of \( \zeta = 0 \). However, one sees easily from the maximum principle that for \( w \) near \( P \), this function remains bounded on a fixed circle surrounding \( \omega_0 \). Hence the interior of this circle may be deleted from the image region and the proof of Theorem 2.1 repeated without further change.

6. This assumption assures a sufficient rate of decay so that the curvature is absolutely integrable on \( S \). An assumption \( |K| < C \sigma^{-2} \) would not suffice.
REFERENCES


<table>
<thead>
<tr>
<th>Stanford University Technical Report Distribution List</th>
</tr>
</thead>
<tbody>
<tr>
<td>Contract No. 225(11) (NR 041-066)</td>
</tr>
</tbody>
</table>

| Armed Services Technical Information Agency          | 10 |
| Arlington Hall Station, Arlington, Virginia          |
| Ass., Chief of Staff, G-4 for Research and Development U.S. Army Washington 25, D.C. | 1 |
| Ames Research Center, Moffett Field, California      | 1 |
| Attn: Technical Library                             |
| California Institute of Technology Institute Library | 1 |
| 1201 E. California Street, Pasadena 9, California   |
| Chief of Naval Operations Operations Evaluation Group OP 342 E Washington 25, D.C. | 1 |
| Chief, Bureau of Ordnance Department of the Navy Washington 25, D.C. | 1 |
| Attn: Repl Retel                                      |
| Chief, Bureau of Aeronautics Department of the Navy Washington 25, D.C. | 1 |
| Chief, Bureau of Ships Ass., Chief for Research and Development Department of the Navy Washington 25, D.C. | 1 |
| Chairman Research and Development Board The Pentagon Washington 25, D.C. | 1 |
| Commander, U.S.N.O.T.S. Pasadena Annex 3202 E. Foothill Boulevard Pasadena 9, California Attn: Technical Library | 1 |
| Commander, U.S.N.O.T.S. Moffett Field, California Attn: Technical Library | 1 |
| Commander General U.S. Proving Ground Aberdeen, Maryland | 1 |
| Commanding Officer Office of Naval Research Branch Office 3000 Geary Street San Francisco 9, California | 1 |
| Commanding Officer Office of Naval Research Branch Office 466 Broadway New York 13, New York | 1 |
| Commanding Officer Office of Naval Research Branch Office 1530 E. Green Street Pasadena 1, California | 1 |
| Commanding Officer Branch Office Office of Naval Research Navy No. 100 Fleet Post Office New York, New York | 2 |
| Commanding Officer Ballistic Research Laboratory U.S. Proving Ground Aberdeen, Maryland Attn: Dr. H. Kent | 1 |
| Commanding Officer Naval Ordnance Laboratory White Oak Silver Spring 19, Maryland Attn: Technical Library | 1 |
| Computation Division Directorate of Management Analysis Control of the Air Force Headquarters USAF Washington 25, D.C. | 1 |
| Director, David Taylor Model Basin Washington 25, D.C. Attn: Hydrodynamics Laboratory Technical Library | 1 |
| Director, National Bureau of Standards Department of Commerce Washington 25, D.C. Attn: National Hydraulics Lab. | 1 |
| Director, Pennsylvania State School of Engineering Ordnance Research Laboratory State College, Pennsylvania | 1 |
| Department of Mathematics Library University of Illinois Urbana, Illinois | 1 |
| Department of Statistics University of California Berkeley 4, California | 1 |
| Diamond Ordnance Factory Laboratory Department of Defense Washington 25, D.C. Attn: Dr. W. K. Saunders | 1 |
| Engineering Library University of California Los Angeles 29, California | 1 |
| Engineering Societies Library 25 W. 39th Street New York, New York | 1 |
| Fisk University Attn: Library Nashville, Tennessee | 1 |
| Headquarters USAF Director of Research and Development Washington 25, D.C. | 1 |
| Hydrodynamics Laboratory California Institute of Technology 1201 E. California Street Pasadena 4, California Attn: Executive Committee | 1 |
| Harvard University Technical Report Collection Pierce Hall, Room 503A Cambridge 38, Massachusetts | 1 |
| John Gorham Library Chicago 1, Illinois | 1 |
| Library B-230 Massachusetts Institute of Technology Lincoln Laboratory Lexington 73, Massachusetts | 1 |
| Library Princeton University Princeton, New Jersey Attn: Verna E. Bayles | 1 |
| Library Sorbonne Institute of Oceanography La Jolla, California | 1 |
| Louisiana State University Attn: Library University Station Baton Rouge 3, Louisiana | 1 |
| Los Angeles Engineering Field Office Air Res. and Development Command 5504 Hollywood Blvd, Los Angeles 28, California Attn: Capt. N. E. Nelson | 1 |
| Massachusetts Institute of Technology Attn: Library Cambridge 39, Massachusetts | 1 |
| Mathematical Reviews 190 Hope Street Providence 9, Rhode Island | 1 |
| Mathematics Department University of Colorado Boulder, Colorado | 1 |
| Mathematics Department University of Pennsylvania Philadelphia 4, Pennsylvania | 1 |
| Mathematics Library Rec. Bldg., Purdue University W. Lafayette, Indiana | 1 |
| Mathematics Library Syracuse University Syracuse 3D, New York | 1 |
| Mathematics Library Rockefeller Foundation Building, N. Y. | 1 |
| Mathematics Library Applied Mathematics Group, N. Y. | 1 |
| National Bureau of Standards Library Room 301, New York Building Washington 25, D.C. | 1 |
| New York Public Library Acquisitions Branch 575 Avenue at 42nd Street New York 38, New York | 1 |

May 1961
<table>
<thead>
<tr>
<th>Name</th>
<th>Address</th>
<th>City</th>
<th>State/Country</th>
</tr>
</thead>
<tbody>
<tr>
<td>Professor L. Krieger</td>
<td>Institute of Mathematical Sciences</td>
<td>New York 3, New York</td>
<td></td>
</tr>
<tr>
<td>Professor J. Nitsche</td>
<td>Department of Mathematics</td>
<td>Minneapolis 14, Minnesota</td>
<td></td>
</tr>
<tr>
<td>Professor C. D. Oides</td>
<td>Department of Mathematics</td>
<td>Los Altos, California</td>
<td></td>
</tr>
<tr>
<td>Professor Charles H. Pagan</td>
<td>Department of Electrical Engineering</td>
<td>Pasadena, California</td>
<td></td>
</tr>
<tr>
<td>Mr. J. D. Pierson</td>
<td>The Martin Company</td>
<td>Baltimore 3, Maryland</td>
<td></td>
</tr>
<tr>
<td>Professor M. S. Pelosi</td>
<td>Division of Engineering</td>
<td>Pasadena 4, California</td>
<td></td>
</tr>
<tr>
<td>Professor W. Prager</td>
<td>Division of Applied Mathematics</td>
<td>Providence 13, Rhode Island</td>
<td></td>
</tr>
<tr>
<td>Dr. Milton Rose</td>
<td>Applied Mathematics Division</td>
<td>Upton, Long Island</td>
<td></td>
</tr>
<tr>
<td>Professor P. C. Rosenblum</td>
<td>Department of Mathematics</td>
<td>University of Minnesota</td>
<td></td>
</tr>
<tr>
<td>Professor A. E. Ross</td>
<td>Department of Mathematics</td>
<td>Notre Dame, Indiana</td>
<td></td>
</tr>
<tr>
<td>Dr. H. Rouse</td>
<td>Iowa Institute of Hydraulic Research</td>
<td>Iowa City, Iowa</td>
<td></td>
</tr>
<tr>
<td>Dr. C. Saltzer</td>
<td>Department of Mathematics</td>
<td>Cleveland 6, Ohio</td>
<td></td>
</tr>
<tr>
<td>Professor W. Sears</td>
<td>Graduate School of Aeronautical Engineering</td>
<td>Illano, New York</td>
<td></td>
</tr>
<tr>
<td>Professor Laurie Snell</td>
<td>Department of Mathematics</td>
<td>Hanover, New Hampshire</td>
<td></td>
</tr>
<tr>
<td>Professor D. C. Spencer</td>
<td>Fine Hall</td>
<td>Princeton, New Jersey</td>
<td></td>
</tr>
<tr>
<td>Professor J. J. Stoker</td>
<td>Institute of Mathematical Sciences</td>
<td>New York 3, New York</td>
<td></td>
</tr>
<tr>
<td>Professor C. A. Truesdell</td>
<td>901 N. College Avenue</td>
<td>Bloomington, Indiana</td>
<td></td>
</tr>
<tr>
<td>Professor J. L. Ullman</td>
<td>Mathematics Department</td>
<td>University of Michigan</td>
<td>Ann Arbor, Michigan</td>
</tr>
<tr>
<td>Professor S. E. Warschewski</td>
<td>Institute of Technology</td>
<td>Minneapolis 14, Minnesota</td>
<td></td>
</tr>
<tr>
<td>Professor A. Weinstein</td>
<td>Institute for Fluid Dynamics and Applied Mathematics</td>
<td>University of Maryland</td>
<td>College Park, Maryland</td>
</tr>
<tr>
<td>Dr. J. D. Wilkes</td>
<td>Director, Naval Analysis Group</td>
<td>Washington 25, D. C.</td>
<td></td>
</tr>
<tr>
<td>Professor A. Zygmund</td>
<td>Mathematics Department</td>
<td>University of Chicago</td>
<td>Chicago 37, Illinois</td>
</tr>
<tr>
<td>Mr. R. Vasudevan</td>
<td>Research Assistant</td>
<td>School of Science and Engineering</td>
<td>La Jolla, California</td>
</tr>
<tr>
<td>Distribution via ONR London</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Commanding Officer</td>
<td>Branch Office</td>
<td>New York, New York</td>
<td></td>
</tr>
<tr>
<td>Dr. R. F. Arent</td>
<td>U.S. Naval Weapons Laboratory</td>
<td>Dahlgren, Virginia</td>
<td></td>
</tr>
<tr>
<td>Professor H. C. Kranzer</td>
<td>Department of Mathematics</td>
<td>Berkeley, California</td>
<td></td>
</tr>
<tr>
<td>Professor J. B. Milas</td>
<td>Department of Mathematics</td>
<td>U.S. Naval Ordnance Laboratory</td>
<td>White Oak, Silver Spring, Maryland</td>
</tr>
<tr>
<td>Professor M. A. Vasudevan</td>
<td>Mathematics Department</td>
<td>Department of Mathematics</td>
<td>Massachusetts</td>
</tr>
<tr>
<td>Professor R. B. Ogashall</td>
<td>The Library, Physics Department</td>
<td>Tromsø, NORWAY</td>
<td></td>
</tr>
<tr>
<td>Mrs. R. S. Olds</td>
<td>Norwegian Institute of Technology</td>
<td>Trondheim, NORWAY</td>
<td></td>
</tr>
<tr>
<td>Other Foreign Address</td>
<td>Hydrodynamics Laboratory</td>
<td>Ottawa, CANADA</td>
<td></td>
</tr>
<tr>
<td>Contract No. 22511</td>
<td>Office of Naval Research Branch Office</td>
<td>Boston 10, Massachusetts</td>
<td></td>
</tr>
<tr>
<td>Professor W. Wilcox</td>
<td>Department of Naval Research</td>
<td>New York University</td>
<td></td>
</tr>
<tr>
<td>Professor J. B. Albas</td>
<td>Department of Electrical Engineering</td>
<td>New York University</td>
<td></td>
</tr>
<tr>
<td>Professor C. H. Wilcox</td>
<td>Department of Mathematics</td>
<td>Providence, Rhode Island</td>
<td></td>
</tr>
<tr>
<td>Professor N. Levinson</td>
<td>Mathematics Department</td>
<td>Massachusetts Institute of Technology</td>
<td>Cambridge, Massachusetts</td>
</tr>
<tr>
<td>Professor L. Greenberg</td>
<td>Department of Mathematics</td>
<td>California Institute of Technology</td>
<td>Pasadena, California</td>
</tr>
<tr>
<td>Additional copies for project, leader and assistants and reserve for future requirements</td>
<td></td>
<td></td>
<td>50</td>
</tr>
</tbody>
</table>