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A Superposition Property of Angle Modulation

by
J. W. Goodman

Technical Report No. 710-1
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PREPARED UNDER
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SUMMARY

A superposition property of angle modulation is derived for a class of modulating waveforms.
I. A SUPERPOSITION PROPERTY OF ANGLE MODULATION

It is well known that amplitude modulation is a linear operation in the sense that the sidebands resulting from a sum of modulations may be found by linear superposition of the sidebands caused by each individual modulation. Angle modulation, however, is a highly non-linear operation, with significant sideband energy arising from the interaction of the various modulation components. Thus, the type of linear superposition of sidebands which holds for amplitude modulation does not hold for angle modulation. This report shows that a different type of linear superposition does hold for an important class of angle modulating waveforms, namely those containing at least one periodic component.

Consider the waveform

\[ y(t) = \cos(\omega_c t + P(t)) \]  

where \( \omega_c \) is a fixed angular frequency, and \( P(t) \) is a periodic angle modulation of period \( T \). Since \( P(t) \) is periodic, \( y(t) \) has a discrete frequency spectrum which may be written

\[ y(t) = \sum_n d_n \cos \left( \omega_c t + \frac{2\pi n}{T} + \varphi_n \right) \]  

A second modulation is now added, giving

\[ Y(t) = \cos(\omega_c t + P(t) + f(t)) \]

where \( f(t) \) may be random or causal. We now find an expression for \( Y(t) \) that demonstrates explicitly the effect of \( f(t) \) on the discrete spectrum of \( y(t) \).

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*Angle modulation refers to either phase modulation or frequency modulation in its equivalent phase modulating form.*
Begin by rewriting (3) as

\[ Y(t) = \text{Re} \{ e^{j(\omega_c t + P(t) + f(t))} \} \]

\[ = \text{Re} \{ e^{j(\omega_c t + f(t))} e^{jP(t)} \} \quad (4) \]

The periodic function \( e^{jP(t)} \) may now be expanded in a complex Fourier series,

\[ e^{jP(t)} = \sum_n c_n e^{j\frac{2\pi n}{T} t} \quad (5) \]

where the \( c_n \) are given by

\[ c_n = \frac{1}{T} \int_0^T e^{j[\omega_c + \frac{2\pi n}{T}] t} \, dt \quad (6) \]

Substitution of the series in (4) and term-by-term multiplication by the first exponential gives

\[ Y(t) = \text{Re} \left\{ \sum_n c_n e^{j(\omega_c + \frac{2\pi n}{T}) t} \right\} \quad (7) \]

Taking the real part as indicated,

\[ Y(t) = \sum_n R_n \cos [(\omega_c + \frac{2\pi n}{T}) t + f(t)] - I_n \sin [(\omega_c + \frac{2\pi n}{T}) t + f(t)] \quad (8) \]

where

\[ R_n \triangleq \text{Re} \{ c_n \} \]

\[ I_n \triangleq \text{Im} \{ c_n \} \quad (9) \]
Use of the trigonometric identity

\[
A \cos \theta + B \sin \theta = \sqrt{A^2 + B^2} \cos \left[ \theta + \tan^{-1} \left( \frac{B}{A} \right) \right]
\]  

(10)

gives

\[
Y(t) = \sum_n \sqrt{R_n^2 + I_n^2} \cos \left[ \left( \omega_c + \frac{2\pi n}{T} \right) t + \tan^{-1} \left( \frac{I_n}{R_n} \right) \right] + f(t)
\]  

(11)

The definitions

\[
d_n = \sqrt{R_n^2 + I_n^2}
\]

\[
\beta_n = \tan^{-1} \left( \frac{I_n}{R_n} \right)
\]  

(12)

lead to the desired result.

\[
Y(t) = \sum_n d_n \cos \left[ \left( \omega_c + \frac{2\pi n}{T} \right) t + \beta_n + f(t) \right]
\]  

(13)

Note that when \( f(t) \) is identically zero \((10)\) reduces to \((2)\), thereby ensuring that the \( d_n \) and \( \beta_n \) are the same in both equations. Equation (13) was previously known to be approximately true,\(^1\)\(^2\) but the above proof demonstrates that it is, indeed, exact.

Thus, the periodic modulation \( P(t) \) may be considered to divide the carrier into a number of subcarriers separated in frequency by an amount equal to the fundamental frequency of \( P(t) \). The added modulation \( f(t) \) then angle modulates each of these subcarriers identically. When \( f(t) \)

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is causal, the sidebands of the original carrier are a linear superposition of the sidebands of the various subcarriers, providing the contributions at any one frequency may be taken into account. When the integrated power spectrum of \( Y(t) \) may be found by simple linear addition of the power spectra of the various subcarriers.