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ASYMPTOTIC FORMS OF HERMITE POLYNOMIALS

by
H. SKOVGAARD

Technical Report 18
Prepared under contract Nonr-220(11)
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Department of Mathematics
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1. Introduction

The asymptotic behavior of Hermite polynomials, $H_n(z)$, as $n \to \infty$ has been investigated by several authors. The results previous to 1939, among which probably the best known are those of Plancherel and Rotach [8], are summarized in G. Szegö: *Orthogonal Polynomials* [10]. Some of the newer results are due to J. C. P. Miller [7], L. O. Heflinger [4] and M. Wyman. Since Hermite polynomials are special parabolic cylinder functions, attention should also be called to the results obtained in the complex plane by A. Erdélyi, M. Kennedy and J. L. McGregor [2] and by N. D. Kazarinoff [5].

In the present report Liouville's method of comparing two differential equations is used in the form in which it has been adapted to equations with a simple transition point by R. E. Langer, and later on further developed by T. M. Cherry and Erdélyi so as to obtain asymptotic solutions holding uniformly over an unbounded region of z. It is shown in section 2 that the comparison of the differential equation for Hermite polynomials with the Airy equation (2.7) furnishes a single asymptotic expression, (2.25), holding uniformly for all real $z \geq 0$ as $n \to \infty$; and this asymptotic expression is also valid for fixed n as $z \to \infty$. Because of the symmetry of Hermite polynomials it is of course sufficient to consider the interval $0 \leq z < \infty$.

From the representation thus obtained we derive in sections 3 and 4 simpler asymptotic forms in the oscillatory region, in the transition region and in the monotonic region, respectively. In the transition region two forms are derived, one of which is very simple but has only a narrow range of validity, while the other is more complicated, involves the Airy function, but is valid in a much larger neighborhood of the transition point. This latter formula is derived by the same technique as used by Erdélyi for the Laguerre polynomials, and its range of validity overlaps the oscillatory region and the monotonic region, in which the other two simplified forms hold.

In section 3 this has been carried through for $H_n(N^{1/2}x)$, $N = 2n + 1$, and in section 4 the corresponding results are deduced for $H_n[(2n)^{1/2}x]$. This has been done not only in view of the need for both sets of formulas, but also in order to see if both sets are equally good. In fact it turned out, that there is a difference regarding the error terms in the transition region, where the forms for $H_n(N^{1/2}x)$ are slightly more advantageous than the corresponding two forms for $H_n[(2n)^{1/2}x]$. Otherwise there is no difference either in error terms or in ranges of validity.

In connection with each formula comparison is made between the results obtained in this report and earlier known results.

I am indebted to Professor Erdélyi for suggesting this investigation and for much helpful advice during the preparation of this report. We are both indebted to Professor Wyman for showing us his unpublished results on Hermite Polynomials, which stimulated the present investigation.

2. An asymptotic representation of $H_n[(2n+1)^{1/2}x]$

For the Hermite polynomials $H_n(z)$, normalized as for instance in [3] and [10], we have

$$(2.1)\qquad H_n(z) = n! \sum_{m=0}^{[n/2]} \frac{(-1)^m (2z)^{n-2m}}{m!(n-2m)!} = (-1)^n e^{z^2} \frac{d^n}{dz^n} e^{-z^2}$$

and

$$(2.2)\qquad \exp(-z^2/2) H_n(z) = 2^{n/2} D_n(2^{1/2} z)$$

The parabolic cylinder function $D_n(2^{1/2}z)$, and hence $\exp(-z^2/2)H_n(z)$, is a solution of the differential equation

$$(2.3)\qquad \frac{d^2u}{dz^2} + (2n + 1 - z^2)u = 0;$$

another solution which is linearly independent of $D_n(2^{1/2}z)$ is $D_{-n-1}(2^{1/2}iz)$.

According to [3] we have for arbitrary fixed ν

$$(2.4)\qquad D_\nu(z) = \exp(-z^2/4) z^\nu \{1 + O(z^{-2})\}$$

$$\text{as}\quad z \to \infty,\quad |\arg z| < 3\pi/4.$$

In order to use [1, theorem 1], we set in (2.3)

$$(2.5)\qquad z = N^{\frac{1}{2}}x, \qquad N = 2n+1 = -i\nu, \qquad \arg \nu = \pi/2$$

and obtain

$$(2.6)\qquad \frac{d^2u}{dx^2} + \nu^2(x^2-1)u = 0,$$

which we shall study in the interval $I: 0 \leq x < \infty$. Since in this interval the equation has a transition point at $x = 1$, the comparison equation to be chosen will be

$$(2.7)\qquad \frac{d^2w}{dt^2} - tw = 0$$

This equation has the Airy functions

$$w_r = Ai(\omega^r t) \qquad\qquad r = 0, \pm 1,$$

as particular solutions; here $\omega = e^{2\pi i/3}$.

With a three times continuously differentiable function $\phi(x)$ satisfying $\phi'(x) \neq 0$ on I, we set

$$(2.8)\qquad t = -\nu^{2/3}\phi(x), \qquad u = [-\phi'(x)]^{\frac{1}{2}}Y(x)$$

and thereby transform (2.7) into

$$(2.9)\qquad \frac{d^2Y}{dx^2} - [\nu^2\phi\phi'^2 + \tfrac{1}{2}\{\phi, x\}]Y = 0$$

where the Schwarzian derivative $\{\phi, x\}$ is given by

$$(2.10)\qquad \{\phi, x\} = \frac{\phi'''}{\phi'} - \frac{3}{2}\left(\frac{\phi''}{\phi'}\right)^2$$

Thus we have

$$(2.11)\qquad Y_r(x) = (-\phi')^{-\frac{1}{2}}\, Ai(-\nu^{2/3}\omega^r\phi) \qquad\qquad r = 0, \pm 1,$$

as solutions to (2.9).

In order to achieve that the dominant terms for large ν in (2.6) and (2.9) coincide, we shall try to determine $\phi(x)$ so as to satisfy the equation

$$\phi\phi'^2 = x^2 - 1 \tag{2.12}$$

This equation has a real solution, which is negative in $0 \leq x < 1$, zero at $x = 1$ and positive for $x > 1$, and which is given explicitly by

$$\alpha(x) = \frac{2}{3}(-\phi)^{3/2} = \int_x^1 (1-t^2)^{1/2}\,dt = -\frac{1}{2}x(1-x^2)^{1/2} + \frac{1}{2}\cos^{-1}x \qquad x \leq 1 \tag{2.13}$$

$$\beta(x) = \frac{2}{3}\phi^{3/2} = \int_1^x (t^2-1)^{1/2}\,dt = \frac{1}{2}x(x^2-1)^{1/2} - \frac{1}{2}\cosh^{-1}x \qquad x \geq 1 \tag{2.14}$$

Here $\cos^{-1}x$ and $\cosh^{-1}x$ stand for the principal branches of the respective functions, and fractional powers of positive values are taken positive (everywhere in the present report).

It is easily verified that the function $\phi(x)$ so determined is three times continuously differentiable, and moreover that $\phi'(x) > 0$ in the whole interval I.

Next we write (2.6) in the form

$$\frac{d^2u}{dx^2} + [\nu^2\phi\phi'^2 + \tfrac{1}{2}\{\phi, x\}]u = F(x)\,u \tag{2.15}$$

where

$$F(x) = \tfrac{1}{2}\{\phi, x\} \tag{2.16}$$

and obtain an estimate of $F(x)$, holding uniformly in the interval I. By using (2.10) and (2.12) we find

$$F(x) = \frac{5}{16}\,\frac{x^2-1}{\phi^3} - \frac{3x^2+2}{4(x^2-1)^2} \qquad \text{if} \quad x \neq 1$$

and from (2.16), (2.10) and the properties of $\phi(x)$ it follows that $F(x)$ is continuous in I. From (2.14) we derive the asymptotic expansion

$$(2.17) \quad \beta(x) = \tfrac{1}{2}x^2 - \tfrac{1}{2}\ln(2x) - \tfrac{1}{4} + O(x^{-2}) \qquad \text{as} \quad x \to \infty$$

In fact, this is more accurate than we need at present, but (2.17) will be used in full at a later stage.

Since $\phi^3 = 9\beta^2/4$, it is then obvious that

$$F(x) = O(x^{-2}) \qquad \text{as} \quad x \to \infty;$$

so, since $F(x)$ is continuous in I, we may write

$$|(x+1)^2 F(x)| < A \qquad \text{in} \quad 0 \leq x < \infty,$$

where A is independent of x and ν.

Thus all the conditions in [1, theorem 1] will be satisfied, if we choose $\Lambda(\nu)$ as a constant and

$$(2.18) \quad g(t) = (1+t)^{-2}\,|1-t^2|^{-1/2}$$

Since

$$\int_x^\infty g(t)\,dt = O(x^{-2}) \qquad \text{as} \quad x \to \infty,$$

we have in the entire interval I

$$(2.19) \quad \int_x^\infty g(t)\,dt = O\left(\frac{1}{1+x^2}\right)$$

From the theorem referred to it then follows, that there exists a solution u_1 of (2.6) for which

$$(2.20) \quad u_1 = Y_1 + O\left[\frac{1}{\nu}\,\tilde{Y}_1 \int_x^\infty g(t)\,dt\right]$$

$$= Y_1 + \tilde{Y}_1\, O\left[\frac{1}{\nu(1+x^2)}\right]$$

holds uniformly for $\arg \nu = \frac{1}{2}\pi$, $0 \le x < \infty$. Here Y_1 is the function defined in (2.11) for $r = 1$. In the appraisal of the error, certain difficulties arise due to the fact that the zeros of Y_1 and u_1 do not coincide. In order to cope with these difficulties, a modification $\tilde{Y}_1$ of Y_1 is introduced. All zeros of $Ai\, z$ are negative: let a_0 be the largest of these, and set $\theta_1 = -\nu^{2/3}\omega\phi$. We may then take

$$\tilde{Y}_1 = \begin{cases} Y_1 & \text{if } \theta_1 > \frac{1}{2}a_0, \\ (-\phi')^{-1/2}\left[|Ai\,\theta_1|^2 + |Bi\,\theta_1|^2\right]^{1/2} & \text{if } \theta_1 \le \frac{1}{2}a_0 \end{cases} \tag{2.21}$$

From (2.3) and (2.5) it follows that u_1 may also be written

$$u_1 = c_1(n)\, D_n[(2N)^{1/2} x] + c_2(n)\, D_{-n-1}[(2N)^{1/2} ix] \tag{2.22}$$

and we proceed to determine c_1 and c_2 from a comparison of (2.20) with (2.22), keeping n fixed and making $x \to \infty$.

Since we know that

$$Ai\, z = \frac{\exp(-\frac{2}{3} z^{3/2})}{2\pi^{1/2} z^{1/4}}\{1 + O(z^{-3/2})\} \tag{2.23}$$

$$\text{as} \quad z \to \infty, \quad |\arg z| < \pi,$$

(see e. g. [6]), we find from (2.20), choosing for instance $\arg(-\phi')^{-1/2} = -\pi/2$,

$$u_1 = (-\phi')^{-1/2} Ai(-\nu^{2/3}\omega\phi)\{1 + O(x^{-2})\}$$

$$= \frac{\exp(i\nu\beta - 5\pi i/12)}{2\pi^{1/2}\nu^{1/6}(\phi\phi'^2)^{1/4}}\{1 + O(x^{-2})\}$$

Using (2.12) and (2.17) and substituting $\nu = iN$, we obtain

$$u_1 = e^{-\frac{1}{2}Nx^2} x^n \frac{\exp[\frac{1}{4}N(1 + \ln 4) - \frac{1}{2}n\pi i]}{2\pi^{1/2} N^{1/6}}\{1 + O(x^{-2})\} \tag{2.24}$$

On the right-hand side of (2.22) we use (2.4) and obtain, from the comparison with (2.24), that $c_2(n) = 0$ and

$$c_1(n) = (2N)^{-\frac{1}{2}n} \frac{\exp[\frac{1}{4}N(1+\ln 4) - \frac{1}{2}\pi i]}{2\pi^{1/2} N^{1/6}}$$

This result shows, in combination with (2.2), (2.11), (2.20) and (2.22), that

$$(2.25)\quad \boxed{\begin{aligned} &H_n(N^{\frac{1}{2}} x) \\ &= (2\pi)^{1/2} N^{n/2+1/6} \exp[\tfrac{1}{2}N(x^2 - \tfrac{1}{2})] \\ &\qquad \times (\phi')^{-1/2} Ai(N^{2/3}\phi)\{1 + O[n^{-1}(1+x^2)^{-1}]\} \end{aligned}}$$

where ϕ is given by (2.13) and (2.14). The formula (2.25) holds uniformly in $0 \leq x < \infty$ and for all n such that $n(1+x^2) \to \infty$, except that for $x < 1$ the error term must be modified as mentioned in connection with (2.20).

This result agrees with the asymptotic representations obtained in [2] for the parabolic cylinder functions in complex domains. It is also in accordance with the formula in [7, p. 69 cf. p. 74].

3. Specialized forms for $H_n(N^{\frac{1}{2}} x)$

In this section we shall derive from (2.25) simpler asymptotic forms for $H_n(N^{\frac{1}{2}}x)$ in each of the regions (i) $0 \leq x < 1$, (ii) a neighborhood of $x = 1$, and (iii) $1 < x < \infty$.

(i) $0 \leq x < 1$

In order to make use of the asymptotic form

$$(3.1)\quad Ai\, z = \pi^{-1/2} |z|^{-1/4} \{\cos(\tfrac{2}{3}|z|^{3/2} - \tfrac{1}{4}\pi) + O(|z|^{-3/2})\} \text{ as } z \to -\infty$$

we assume that $N^{2/3}\phi \to -\infty$ as $n \to \infty$. From (2.13) and (2.14) it follows that

$$(3.2)\qquad \phi(x) = 2^{1/3}(x-1)\{1 + O(x-1)\} \qquad \text{as} \quad x \to 1,$$

so that the assumption $N^{2/3}\phi \to -\infty$ is equivalent to

$$(3.3)\qquad n^{2/3}(1-x) \to \infty \qquad \text{as} \quad n \to \infty.$$

By (3.1) we now have

$$(\phi')^{-1/2} Ai(N^{2/3}\phi) = \frac{\cos(\frac{2}{3}N|\phi|^{3/2} - \frac{1}{4}\pi) + O(N^{-1}|\phi|^{-3/2})}{\pi^{1/2} N^{1/6} |\phi\phi'^2|^{1/4}}$$

which may be reduced to

$$(3.4)\qquad (\phi')^{-1/2} Ai(N^{2/3}\phi) = \frac{\cos(N\alpha(x) - \frac{1}{4}\pi) + O\{n^{-1}(1-x)^{-3/2}\}}{\pi^{1/2} N^{1/6}(1-x^2)^{1/4}}$$

by using (2.12), (2.13) and (3.2).

Since in this case $Ai(N^{2/3}\phi)$ and $Bi(N^{2/3}\phi)$ are of the same order of magnitude, the modified error term in (2.25) may be included in

$$O\{n^{-1}(1-x)^{-3/2}\},$$

and, using the above expression (3.4), we obtain from (2.25)

$$(3.5)\qquad \begin{aligned} H_n(N^{1/2}x) = {} & 2^{1/2} N^{n/2} \exp[\tfrac{1}{2}N(x^2 - \tfrac{1}{2})](1-x^2)^{-1/4} \\ & \times \{\cos[N\alpha(x) - \tfrac{1}{4}\pi] + O[n^{-1}(1-x)^{-3/2}]\} \end{aligned}$$

where $\alpha(x)$ is given by (2.13).

It is usual to set in the interval $0 \leq x < 1$

$$x = \cos\theta \qquad 0 < \theta \leq \tfrac{1}{2}\pi;$$

doing this one obtains from (3.5)

$$(3.6)\qquad \boxed{\begin{aligned} H_n(x) &= H_n(N^{1/2}\cos\theta) \\ &= 2^{1/2} N^{n/2} \exp[\tfrac{1}{4}N\cos 2\theta](\sin\theta)^{-1/2} \\ &\quad \times \{\cos[\tfrac{1}{4}N(2\theta - \sin 2\theta) - \tfrac{1}{4}\pi] + O(n^{-1}\theta^{-3})\} \end{aligned}}$$

This holds for $0 \le z < N^{1/2}$, as $n \to \infty$ and $n^{1/6}(N^{1/2} - z) \to \infty$, corresponding to $n\,\theta^3 \to \infty$.

This result agrees with the first two terms in the asymptotic expansion obtained by Plancherel and Rotach [8], and from the asymptotic expansions given by these authors it is seen that the error term in (3.6) can not be improved. Plancherel and Rotach, however, use another normalization of $H_n(z)$ and have, in our notation, the argument

$$(2n+2)^{1/2} \cos\theta$$

Numerical estimates of the error term are given by G. Sansone [9] for z bounded, and in the interval (i) by van Veen [11], who also uses the argument $(2n+2)^{1/2}\cos\theta$.

(ii) The neighborhood of $x = 1$

We first assume that $N^{2/3}\phi \to 0$ as $n \to \infty$, or by (3.2),

$$n^{2/3}(x-1) \to 0 \qquad \text{as} \quad n \to \infty, \tag{3.7}$$

Then, by (2.12), (3.2) and

$$Ai\ z = 3^{-2/3}[\Gamma(2/3)]^{-1} + O(z) \qquad \text{as} \quad z \to 0, \tag{3.8}$$

we find

$$(\phi')^{-1/2} Ai(N^{2/3}\phi) = \left(\frac{\phi}{x^2-1}\right)^{1/4} \{3^{-2/3}[\Gamma(2/3)]^{-1} + O[n^{2/3}(x-1)]\}$$

$$= 2^{-1/6}\, 3^{-2/3}\, [\Gamma(2/3)]^{-1} \{1 + O[n^{2/3}(x-1)]\}$$

Since the argument of the Airy function in this case tends to zero, we need not modify the error term in (2.25), so that we readily obtain

$$\boxed{\begin{aligned} &H_n(z) = H_n(N^{1/2} x) \\ &\quad = \frac{2^{1/4}\pi^{1/2}}{3^{2/3}\Gamma(2/3)} N^{n/2+1/6} \exp[\tfrac{1}{2}N(x^2 - \tfrac{1}{2})] \\ &\qquad \times \{1 + O[n^{2/3}(x-1)] + O(n^{-1})\} \end{aligned}} \tag{3.9}$$

This holds as $n \to \infty$ in a neighborhood of $z = N^{1/2}$ such that

$$n^{1/6}(z - N^{1/2}) \to 0$$

A formula with a much larger range of validity than (3.9), if more complicated, can be derived from (2.25) in the following way.

From (3.2) it follows that

$$N^{2/3}\,\phi = 2^{1/3}\,N^{2/3}\,(x-1) + O\,[n^{2/3}(x-1)^2] \qquad \text{as} \quad x \to 1 \tag{3.10}$$

We assume now that

$$n^{1/3}(x-1) \to 0 \qquad \text{as} \quad n \to \infty \tag{3.11}$$

instead of (3.7), and use for a while the abbreviation $t = 2^{1/3}N^{2/3}(x-1)$. Then, by use of the mean value theorem and (3.10), we obtain

$$Ai\,(N^{2/3}\,\phi) = Ai\;t + \widetilde{Ai}\,'(t)\;O\,[n^{2/3}(x-1)^2],$$

where $\widetilde{Ai}\,'(t)$ is a modification of $Ai\,'(t)$ analogous to the modification of Y_1 in (2.21). Taking $\widetilde{Ai}\;z$ in the similar sense, it can be verified that the following relation holds for all real z

$$Ai\,'(z) = \widetilde{Ai}(z)\;O(1 + |z|^{1/4}) \tag{3.12}$$

Thus

$$Ai\,(N^{2/3}\,\phi) = Ai\;t + \widetilde{Ai}\;t\{O\,[n^{2/3}(x-1)^2] + O\,(n\,|x-1|^{5/2})\}$$

As before, we have from (3.2)

$$(\phi')^{-1/2} = [(x^2-1)^{-1}\,\phi]^{1/4} = 2^{-1/6}\,\{1 + O\,(x-1)\}$$

By substitution in (2.25) we get the error terms

$$O\,[n^{2/3}(x-1)^2], \quad O\,(n\,|x-1|^{5/2}), \quad O\,(x-1) \quad \text{and} \quad O\,(n^{-1}),$$

the first one of which can be omitted since

$$O\,[n^{2/3}(x-1)^2] \le \max\{O\,(n\,|x-1|^{5/2}),\, O\,(x-1)\}$$

Hence

$$
\boxed{
\begin{aligned}
H_n(z) &= H_n(N^{1/2} x) \\
&= 2^{1/3}\, \pi^{1/2}\, N^{n/2+1/6} \exp[\tfrac{1}{2}N(x^2 - \tfrac{1}{2})] \\
&\qquad \times \{Ai\, t + \widetilde{Ai}\, t\,[O(n\,|x-1|^{3/2}) + O(x-1) + O(n^{-1})]\}
\end{aligned}
}
\tag{3.13}
$$

where $t = 2^{1/3} N^{2/3}(x-1)$. This holds as $n \to \infty$ in a neighborhood of $z = N^{1/2}$ such that $n^{-1/6}(z - N^{1/2}) \to 0$.

If, in particular, we take $z - N^{1/2} = O(n^{-1/6})$, then (3.13) yields the same result as given by Szegö [10, formula (8.22.[illegible]

(iii) $1 < x < \infty$

We assume that $N^{2/3}\phi \to \infty$, or

$$
n^{2/3}(x-1) \to \infty, \tag{3.14}
$$

which can be satisfied if one of the variables n or x becomes arbitrarily large. Then we find by use of (2.12) and (2.23)

$$
(\phi')^{-1/2} Ai(N^{2/3}\phi) = \frac{\exp(-\frac{2}{3}N\phi^{3/2})}{2\pi^{1/2} N^{1/6}(x^2-1)^{1/4}}\{1 + O(n^{-1}\phi^{-3/2})\} \tag{3.15}
$$

Since by (3.2) $\phi^{-3/2} = O[(x-1)^{-3/2}]$ as $x \to 1$, and by (2.14) and (2.17) $\phi^{-3/2} = O(x^{-2})$ as $x \to \infty$, it follows that the error term in (3.15) may be written as $O[n^{-1}x^{-1/2}(x-1)^{-3/2}]$ in the whole interval $1 < x < \infty$. Thus, we can neglect the error term in (2.25), when we substitute (3.15), and hence we have proved

$$
\begin{aligned}
H_n(N^{1/2}x) &= 2^{-1/2} N^{n/2} \exp\{\tfrac{1}{2}N[x^2 - \tfrac{1}{2} - 2\beta(x)]\}\,(x^2-1)^{-1/4} \\
&\qquad \times \{1 + O[n^{-1}x^{-1/2}(x-1)^{-3/2}]\}
\end{aligned}
\tag{3.16}
$$

where $\beta(x)$ is given by (2.14).

It is usual to set in $1 < x < \infty$

$$x = \cosh\theta \qquad 0 < \theta < \infty .$$

With this substitution (3.16) becomes

$$\boxed{\begin{aligned} H_n(z) &= H_n(N^{1/2}\cosh\theta) \\ &= N^{1/2 n}\exp[\tfrac{1}{4}N(2\theta + e^{-2\theta})](2\sinh\theta)^{-1/2} \\ &\quad\times\left\{1 + O\left(n^{-1}\sinh^{-3}\frac{2\theta}{3}\right)\right\} \end{aligned}} \tag{3.17}$$

This holds for $N^{1/2} < z < \infty$, as $n^{1/6}(z - N^{1/2}) \to \infty$, corresponding to $n\sinh^3(2\theta/3) \to \infty$. Hence (3.17) is valid both for fixed θ as $n \to \infty$ and for fixed n as $z \to \infty$.

The result obtained here agrees with [8] (cf. the remarks to (3.6)) and with the expansion due to L. Heflinger [4], who uses the same normalization as Plancherel and Rotach and in a smaller domain gives a numerical upper bound for the error term. It is seen from the results of these authors that the error term in (3.17) can not be improved in that part of the domain of validity, in which $\theta \to 0$, or $z - N^{1/2} = o(N^{1/2})$.

4. Asymptotic forms for $H_n[(2n)^{1/2}\xi]$

In order to obtain an asymptotic representation of $H_n[(2n)^{1/2}\xi]$, corresponding to (2.25), we might proceed in a similar manner as in section 2, studying instead of (2.6) the differential equation

$$\frac{d^2u}{d\xi^2} - 4n^2\left(\xi^2 - 1 - \frac{1}{2n}\right)u = 0 \tag{4.1}$$

This equation has $e^{-n\xi^2} H_n[(2n)^{1/2}\xi]$ as a solution, and the theorem in [1], which was used in section 2 can be extended to hold in the present case, although the new function ϕ will depend on n.

A much easier way, however, is to convert (2.25) by the transformation

$$x = \frac{\xi}{h}, \qquad \phi(x) = \frac{\Phi(\xi)}{h^{4/3}} \tag{4.2}$$

where $h = [(2n+1)/2n]^{½}$. Hence

$$\Phi(\xi) = h^{4/3}\,\phi\left(\frac{\xi}{h}\right) \tag{4.3}$$

and by (2.12) it follows that $\Phi(\xi)$ satisfies

$$\Phi\Phi'^2 = \xi^2 - h^2 \tag{4.4}$$

By substitution of (4.2) in (2.25), we can establish the following result

$$\boxed{\begin{aligned} &H_n[(2n)^{½}\,\xi] \\ &\quad = (2\pi)^{1/2}\,N^{n/4}\,(2n)^{-1/12}\exp[n\,\xi^2 - ¼N]\,(\Phi')^{-1/2} \\ &\quad\quad \times Ai[(2n)^{2/3}\,\Phi]\,\{1 + O[n^{-1}(1+\xi^2)^{-1}]\} \end{aligned}} \tag{4.5}$$

where Φ is given by (4.3), (2.13) and (2.14). This holds uniformly in $0 \le \xi < \infty$ and for all n such that $n(1+\xi^2) \to \infty$, except that for $\xi < h$ the error term must be modified as in (2.25).

A set of asymptotic forms for $H_n[(2n)^{½}\,\zeta]$ corresponding to the forms for $H_n(N^{½}x)$ developed in section 3 can now be derived from (4.5), although the calculations are more laborious in this case.

For the sake of convenience we note that $1 < h < 2$ for every n, and also

$$\begin{aligned} h &= \left(1 + \frac{1}{2n}\right)^{½} \\ &= 1 + ¼n^{-1} + O(n^{-2}) \qquad \text{as} \quad n \to \infty \end{aligned} \tag{4.6}$$

and

$$\xi - h = \xi - 1 + O(n^{-1}) \qquad \text{as} \quad n \to \infty \tag{4.7}$$

(i) $0 \le \xi < 1$

From (3.2) and (4.3) it follows that

$$(4.8) \qquad \Phi(\xi) = (2h)^{1/3}(\xi - h)\{1 + O(\xi - h)\} \qquad \text{as} \quad \frac{\xi}{h} \to 1$$

In order to use (3.1), let $(2n)^{2/3}\,\Phi \to -\infty$ as $n \to \infty$, or equivalently, as seen from (4.7) and (4.8)

$$(4.9) \qquad n^{2/3}(1 - \xi) \to \infty \qquad \text{as} \quad n \to \infty$$

Then, by (3.1), (4.3), (4.4) and (2.13)

$$(4.10) \quad (\Phi')^{-1/2}\,Ai[(2n)^{2/3}\,\Phi] = \frac{\cos[N\,\alpha(\xi/h) - \frac{1}{4}\pi] + O\{n^{-1}(1-\xi)^{-3/2}\}}{\pi^{1/2}(2n)^{1/6}\,(h^2 - \xi^2)^{1/4}}$$

and for the same reason as in section 3 case (i) the modified error term occurring in (4.5) may be included in $O\{n^{-1}(1 - \xi)^{-3/2}\}$.

Next, we introduce

$$\xi = \cos\sigma \qquad 0 < \sigma \le \tfrac{1}{2}\pi$$

and observe that (4.9) is equivalent to $n\sigma^3 \to \infty$ as $n \to \infty$. By (2.13)

$$\alpha\left(\frac{\xi}{h}\right) = -\frac{1}{2}\frac{\xi}{h}\left(1 - \frac{\xi^2}{h^2}\right)^{1/2} + \frac{1}{2}\cos^{-1}\frac{\xi}{h}$$

where, as $n \to \infty$,

$$-\frac{1}{2}\,\frac{\xi}{h}\left(1 - \frac{\xi^2}{h^2}\right)^{1/2} = -\frac{1}{2}\,h^{-2}\cos\sigma\left(\sin^2\sigma + \frac{1}{2n}\right)^{1/2}$$

$$= -\frac{n}{2N}[\sin 2\sigma + \tfrac{1}{2}n^{-1}\cot\sigma + O(n^{-2}\sigma^{-3})]$$

and furthermore, by Taylor's formula,

$$\frac{1}{2}\cos^{-1}\frac{\xi}{h} = \frac{1}{2}\sigma + \frac{1}{8}\,n^{-1}\cot\sigma + O(n^{-2}\sigma^{-3})$$

Hence

$$(4.11) \quad N\alpha\left(\frac{\xi}{h}\right) = \tfrac{1}{2}N\sigma - \tfrac{1}{4}n \sin 2\sigma + O(n^{-1}\sigma^{-3})$$

and in the denominator in (4.10)

$$(4.12) \quad (h^2 - \xi^2)^{\frac{1}{4}} = (\sin\sigma)^{\frac{1}{2}} \{1 + O(n^{-1}\sigma^{-2})\}$$

By substituting (4.11) and (4.12) into (4.10), and then (4.10) into (4.5), and using

$$(2n+1)^{n/2} = (2n)^{\frac{1}{2}n} e^{\frac{1}{2}} \{1 + O(n^{-1})\} \qquad \text{as} \quad n \to \infty$$

it is seen that

$$(4.13) \quad \boxed{\begin{aligned} H_n(z) &= H_n[(2n)^{\frac{1}{2}} \cos\sigma] \\ &= 2^{\frac{1}{2}} (2n)^{\frac{1}{2}n} \exp[\tfrac{1}{2}n \cos 2\sigma] (\sin\sigma)^{-\frac{1}{2}} \\ &\quad \times \{\cos[(n+\tfrac{1}{2})\sigma - \tfrac{1}{4}n \sin 2\sigma - \tfrac{1}{4}\pi] + O(n^{-1}\sigma^{-3})\} \end{aligned}}$$

This holds for $0 \leq z < (2n)^{\frac{1}{2}}$, as $n \to \infty$ and $n^{1/6}[(2n)^{1/2} - z] \to \infty$, corresponding to $n\sigma^3 \to \infty$.

This result is in agreement with the asymptotic representation obtained by Wyman by a contour integration method.

(ii) The neighborhood of $\xi = 1$

The formula analogous to (3.9) can be found most easily by introducing in (3.9) $x = h^{-1}\xi$. The assumption $n^{2/3}(x-1) \to 0$ as $n \to \infty$ corresponds to

$$n^{2/3}(\xi - 1) \to 0 \qquad \text{as} \quad n \to \infty$$

as seen by (4.7), and the error term $O[n^{2/3}(x-1)]$ may be replaced by $O[n^{2/3}(\xi-1)] + O(n^{-1/3})$. Hence

$$\boxed{\begin{aligned} H_n(z) &= H_n[(2n)^{1/2}\,\xi] \\ &= \frac{2^{1/3}\,\pi^{1/2}}{3^{2/3}\,\Gamma(2/3)} N^{n/2+1/6} \exp[n\,\xi^2 - \tfrac{1}{4}N] \\ &\quad \times \{1 + O[n^{2/3}(\xi-1)] + O(n^{-1/3})\} \end{aligned}} \tag{4.14}$$

which holds as $n \to \infty$ in a neighborhood of $z = (2n)^{1/2}$ such that

$$n^{1/6}[z - (2n)^{1/2}] \to 0 \,.$$

This result agrees with the results found by Wyman in the same neighborhood of $z = (2n)^{1/2}$.

Similarly, (3.13) may be converted as follows. The assumption (3.11) is equivalent to

$$n^{1/3}(\xi - 1) \to 0 \qquad \text{as} \quad n \to \infty \tag{4.15}$$

and

$$\begin{aligned} t &= 2^{1/3}\,N^{2/3}\left(\frac{\xi}{h} - 1\right) \\ &= 2n^{2/3}(\xi - 1) + O(n^{-1/3}) \end{aligned}$$

With the notation $r = 2n^{2/3}(\xi - 1)$, the mean value theorem and (3.12) show that

$$\begin{aligned} Ai\ t &= Ai\ r + \widetilde{Ai}\ r\ O\{(1 + |r|^{1/2})\,n^{-1/3}\} \\ &= Ai\ r + \widetilde{Ai}\ r\ \{O(n^{-1/3}) + O(|\xi - 1|^{1/2})\} \end{aligned}$$

and hence

$$\widetilde{Ai}\ t = \widetilde{Ai}\ r\,\{1 + O(n^{-1/3}) + O(|\xi - 1|^{1/2})\}$$

Lastly

$$\begin{aligned} O(n\,|x-1|^{5/2}) &= O(n\,|\xi-h|^{5/2}) \\ &= O(n\,|\xi-1+O(n^{-1})|^{5/2}) \\ &= O(n\,|\xi-1|^{5/2}) + O(n^{-3/2}) \end{aligned}$$

Thus

$$\boxed{\begin{aligned} H_n(z) &= H_n[(2n)^{1/2}\,\xi] \\ &= 2^{1/3}\,\pi^{1/2}\,N^{n/2+1/6}\exp[n\,\xi^2 - \tfrac{1}{4}N] \\ &\quad \times \{Ai\,r + \tilde{Ai}\,r\,[O(n\,|\xi-1|^{5/2}) + O(|\xi-1|^{1/2}) + O(n^{-1/3})]\} \end{aligned}} \tag{4.16}$$

where $r = 2n^{2/3}(\xi-1)$. This holds as $n \to \infty$ in a neighborhood of $z=(2n)^{1/2}$ such that $n^{-1/6}\,[z-(2n)^{1/2}] \to 0$.

It can be verified, that no better results than (4.14) and (4.16) could be obtained by working from (4.5), and following the same procedure as in the deduction of (3.9) and (3.13).

The results obtained by Wyman for $K_1 n^{-1/6} \leq |z-(2n)^{1/2}| \leq K_2 n^{-1/6}$ are included in (4.16).

(iii) $1 < \xi < \infty$

By means of the identity $h - 1 = [2n\,(h+1)]^{-1}$, it is seen that

$$n^{2/3}(\xi-h) = n^{2/3}(\xi-1) - \frac{1}{2n^{1/3}(h+1)}$$

This shows, in conjunction with (4.8), that the following assumptions are equivalent: $(2n)^{2/3}\,\Phi(\xi) \to \infty$, $n^{2/3}(\xi-h) \to \infty$ or

$$n^{2/3}(\xi-1) \to \infty \tag{4.17}$$

which does not necessarily imply that $n \to \infty$.

Proceeding now as in (iii) section 3, use of (4.3), (4.4) and (2.14) leads to

$$(4.18)\quad (\Phi')^{-1/2} Ai[(2n)^{2/3}\Phi]$$
$$= \frac{\exp[-N\beta(\xi/h)]}{2\pi^{1/2}(2n)^{1/6}(\xi^2-h^2)^{1/4}}\{1+O[n^{-1}\xi^{-1/2}(\xi-1)^{-3/2}]\}$$

holding in the interval $h < \xi < \infty$. Next, we introduce

$$\xi = \cosh\sigma \qquad\qquad 0 < \sigma < \infty$$

and observe that (4.17) is equivalent to $n\sigma^3 \to \infty$. Since this implies that $n \sinh^2\sigma \to \infty$ and $n \sinh\sigma \to \infty$, it follows by (2.14) that

$$\beta\left(\frac{\xi}{h}\right) = \frac{1}{2}\frac{\xi}{h}\left(\frac{\xi^2}{h^2}-1\right)^{1/2} - \frac{1}{2}\ln\left[\frac{\xi}{h}+\left(\frac{\xi^2}{h^2}-1\right)^{1/2}\right]$$

where

$$\frac{1}{2}\frac{\xi}{h}\left(\frac{\xi^2}{h^2}-1\right)^{1/2} = \tfrac{1}{2}h^{-2}\cosh\sigma\left(\sinh^2\sigma-\frac{1}{2n}\right)^{1/2}$$
$$= \frac{n}{2N}\left[\sinh 2\sigma - \tfrac{1}{2}n^{-1}\coth\sigma + O\left(\frac{\cosh\sigma}{n^2\sinh^3\sigma}\right)\right]$$

and

$$-\tfrac{1}{2}\ln\left[\frac{\xi}{h}+\left(\frac{\xi^2}{h^2}-1\right)^{1/2}\right]$$
$$= -\tfrac{1}{2}\ln\left[\cosh\sigma+\left(\sinh^2\sigma-\frac{1}{2n}\right)^{1/2}\right] + \tfrac{1}{2}\ln h$$
$$= -\tfrac{1}{2}\ln\left[e^{\sigma} - \tfrac{1}{4}n^{-1}(\sinh\sigma)^{-1} + O\left(\frac{1}{n^2\sinh^3\sigma}\right)\right] + \tfrac{1}{4}\ln\frac{N}{2n}$$
$$= -\tfrac{1}{2}\sigma + \frac{1}{8}n^{-1}e^{-\sigma}(\sinh\sigma)^{-1} + \tfrac{1}{4}\ln\frac{N}{2n} + O\left(\frac{e^{-\sigma}}{n^2\sinh^3\sigma}\right)$$

Observing that the error terms in both of these expressions may be replaced by $O[n^{-2}\sinh^{-3}(2\sigma/3)]$, we have

$$-N\beta\left(\frac{\xi}{h}\right) = -\tfrac{1}{2}n\sinh 2\sigma + \tfrac{1}{4}\coth\sigma + \tfrac{1}{2}N\sigma$$
$$-\tfrac{1}{4}e^{-\sigma}(\sinh\sigma)^{-1} - \tfrac{1}{4}N\ln\frac{N}{2n} + O\left(n^{-1}\sinh^{-3}\frac{2\sigma}{3}\right)$$

and hence

$$\text{(4.19)} \quad \exp\left[-N\beta\left(\frac{\xi}{h}\right)\right]$$
$$= \left(\frac{2n}{N}\right)^{\frac{1}{4}N} \exp[(n+\tfrac{1}{2})\sigma - \tfrac{1}{2}n\sinh 2\sigma + \tfrac{1}{4}]$$
$$\times\left\{1 + O\left(n^{-1}\sinh^{-3}\frac{2\sigma}{3}\right)\right\}$$

Furthermore, in the denominator of (4.18)

$$\text{(4.20)} \quad (\xi^2 - h^2)^{\frac{1}{4}} = (\sinh\sigma)^{\frac{1}{2}}\{1 + O(n^{-1}\sinh^{-3}\sigma)\}$$

Since the error terms in (4.5), (4.18) and (4.20) may be absorbed in the error term in (4.19), the following result is obtained by substituting (4.19) and (4.20) into (4.18), and then (4.18) in (4.5)

$$\text{(4.21)} \quad \boxed{\begin{aligned} H_n(z) &= H_n[(2n)^{\frac{1}{2}}\cosh\sigma] \\ &= (2n)^{\frac{1}{2}n}\exp[(n+\tfrac{1}{2})\sigma + \tfrac{1}{2}ne^{-2\sigma}](2\sinh\sigma)^{-\frac{1}{2}} \\ &\quad\times\left\{1 + O\left(n^{-1}\sinh^{-3}\frac{2\sigma}{3}\right)\right\}\end{aligned}}$$

This holds for $(2n)^{\frac{1}{2}} < z < \infty$ as $n^{1/6}[z - (2n)^{\frac{1}{2}}] \to \infty$, corresponding to $n\sigma^3 \to \infty$. Hence, in particular, (4.21) is valid both for fixed σ as $n \to \infty$ and for fixed n as $z \to \infty$.

There is agreement between (4.21) and the result found by Wyman in the same interval.

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