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RESEARCH MEMORANDUM

DIFFERENTIAL GAMES III:
THE BASIC PRINCIPLES OF
THE SOLUTION PROCESS

Rufus Isaacs

RM-1411

December 1954

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SUMMARY

This report is a continuation of RM-1399, Differential Games II: The Definition and Formulation.

Its essential contents are: the differential equation technique for solving differential games (at least "in the small") and the Verification Theorem, a device which enables one to prove that the answers obtained are the correct solutions.

TABLE OF CONTENTS

		Page
1.	The Nature of a Solution	1
2.	The Main Equation	3
3.	Semipermeable Surfaces	6
4.	The Verification Theorem	8
5.	The Path Equations	12
6.	The Retrogression Principle	15
7.	Solutions in the Small	16

DIFFERENTIAL GAMES III: THE BASIC PRINCIPLES OF THE SOLUTION PROCESS

by

Rufus Isaacs

(This work is a continuation of RM-1391, which we will refer to as II.)

1. The Nature of a Solution

When we analyse a differential game we shall be interested in three goals: the optimal tactics of each player and the value of the game. We shall say the game has a solution when all three items exist; collectively they are the solution. We will have solved the game when either we have found a solution or, in case none exists, when we have become as enlightened as the situation permits.

The term optimal tactics used above refers, as already discussed in II, to such $\phi(x)$ and $\psi(x)$ as may appear in all K-strategies which are ε-strategies for all sufficiently small positive ε. One must bear in mind that the utility of K-strategies is for proofs; for practical purposes we are concerned with the optimal $\phi(x)$, $\psi(x)$ (henceforth denoted by $\bar{\phi}(x)$, $\bar{\psi}(x)$) themselves. We are not far wrong if we conceive of them directly as optimal strategies and in the future we will often speak in such terms.

We recall that the data of a game include a particular starting point in $\mathcal{E}$ and that we have used the term "game" rather freely for what should be a family of games. When we speak of the solution of a game for a certain subset $\mathcal{E}'$ of $\mathcal{E}$, we will refer to all games with starting points in $\mathcal{E}'$.

The process of solving a game splits into two phases. It generally turns out that the region $\mathcal{E}$ is to be divided into a number of parts separated by surfaces which we shall later call singular surfaces.* In each part the solution will be smooth; that is, V will be of class C_1 and the Φ and Ψ will be continuous functions of x. On the singular surfaces (or singular manifolds of lower dimension) various kinds of special behaviour will take place. There are a number of possibilities. The whole subject of singular surfaces is difficult and sometimes their study is the essence of the problem. They shall be dealt with later.

We shall use the term in the small to refer to the "smooth" parts of the solution found between the singular surfaces. The problem of identifying the singular surfaces and assemblying the "smooth" parts into the total solution will be described by the phrase in the large.

We shall see that the technique in the small is one of differential equations. This phase of the problem plays a part in full solution somewhat analogous to that played by the Euler equations in the calculus of variations. In fact, for some one person games the approaches become identical. There is nothing radically new in the concept of a singular surface. They appear in some classical problems, but inconspicuously. Motives for emphasizing them appear only when we introduce the second player

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* There may also be singular manifolds of dimension $< n-1$; such of course cannot separate the components of $\mathcal{E}$.

and look at matters game theoretically.*

It is not possible to make a categorical statement as to which of the phases is the more important. There are examples where the solutions in the small are extremely simple (for example, the paths may be straight lines traversed at constant speed) while the assortment of singular surfaces is abundant and involved. On the other hand, there may be few or no singular surfaces, but we may encounter involved differential equations.

2. The Main Equation

We suppose that the value of a differential game exists. It will depend on the starting point x and we denote it by V(x). We shall show that V(x) satisfies a first order partial differential equation, to be called the main equation, whenever V(x) is of class C_1.**

For games with integral payoff the main equation is

$$\min_{\phi} \max_{\psi} \left[\sum_j V_{x_j} f_j(x,\phi,\psi) + G(x,\phi,\psi) \right] = 0 \tag{1}$$

and, for games with terminal payoff,

$$\min_{\phi} \max_{\psi} \sum_j V_{x_j} f_j(x,\phi,\psi) = 0 \quad . \tag{2}$$

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* Example 2 of the following chapter will illustrate this point.

** That is, V has all its first partials and they are continuous.

From the minimax assumption, it makes no difference if the min and max in (1) and (2) are reversed.

We shall give two proofs. The first will follow immediately. It is frankly heuristic. But it is instructive. For example, it is possible to solve differential games approximately by discrete methods. The first proof will point the way.

We utilize what we have called the tenet of transition. The germ of the idea is that we are dealing with a family of games based on different starting points. Let us consider an interval of time in midplay. At its commencement the path has reached some definite point of $\mathcal{E}$. We consider all possible x which may be reached at the end of the interval for all possible navigational choices by both players. We suppose that, for each endpoint, the game beginning there has already been solved; in other words, V is known there. Then the payoff resulting from each choice ϕ, ψ <u>during the interval</u> will be known and the navigation variables are to be so chosen as to render it minimax. When we let the duration of the interval approach zero, the result yields a differential equation.

We couch the above reasoning formally. First let the payoff be integral. Let V be known at x in $\mathcal{E}$ which has been reached in a play at time t. A short time later – let us candidly label the interim dt – the play has progressed to the (variable) point x'. Then

$$(3) \qquad \mathcal{P} = \text{payoff at } x = \int_t^{t+dt} G(x,\phi,\psi)dt + V(x')$$

= (in the language of primitive calculus*)

$$\left[G(x,\phi,\psi) + \sum_j V_{x_j} f_j(x,\phi,\psi) \right] dt + V(x) \quad .$$

The ϕ and ψ in the last line are, of course, their values at x. To minimax $\mathcal{P}$, they must be chosen to minimax the bracket. (Their values will then be those of an optimal tactic at x.) But when this is done, the payoff will be the value; thus, to make the equation balance, the bracket must be zero. We have (1).

When the payoff is terminal, almost the same reasoning applies. In this case optimal play during the interval means that V will be unchanged during it; we can suppress the integral in (3) and proceed as before.

This proof instructs us as to a distinction between games with the two kinds of payoff. Where the optimal path of x is differentiable, along it

$$(4) \qquad \frac{dV}{dt} = \dot{V} = \sum_j V_{x_j} f_j(x,\phi,\psi) \quad .$$

By examination of the bracket above we learn:

<u>When the payoff is integral the paths of optimal play penetrate the surfaces on which V is constant;</u>** <u>when it is</u>

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* The reader speaking a more sophisticated tongue (mean value theorems, etc.) can readily translate our idea into terms of greater rigor.

** except, of course, when G is 0 during an interval

integral, the paths remain throughout the play in the same surface of constant V.

The second proof of the main equation depends on a new concept.

3. Semipermeable Surfaces

We take it that each small portion of the surfaces under discussion separate the neighboring space. As orientation is germane to our purpose, we distinguish the two directions in which the surface may be penetrated, calling them the P- and E-directions. The "side" of the surface reached after penetration in the P-[E-] direction will be called the P-[E-] side. We take a point x on a so oriented surface and visualize the full vectogram at x. We will say the surface is semipermeable at x when the following is true:

There is at least one value $\bar{\phi}$ of ϕ such that if $\phi = \bar{\phi}$, no vector in ψ-vectogram penetrates the surface in the E-direction. Similarly, there is a $\bar{\psi}$* which prevents penetration in P-direction.

A surface having this property at each point will be called semipermeable.

Suppose the f_i are all separable. Then we observe that if $\phi = \bar{\phi}$ and $\psi \neq \bar{\psi}$, there will be strict penetration in the P-direction. It is this fact that causes, in the separable case, optimal

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* $\bar{\phi}[\bar{\psi}]$ will be used interchangeably to denote a value of $\phi[\psi]$ with the described property or the set of all such values.

strategies to be best strategies.

We have already seen that we can transform a game with integral payoff into one with terminal payoff. Consider an instance of the latter which we supposed solved, and for which $V(x)$ has at least two values.

Any surface which separates the parts of $\mathcal{E}$ where $V > c$ and $V < c$ (c, any constant) must be semipermeable with V decreasing as the surface is crossed in the P-direction. For if, at some point x of the surface, there were no ϕ, P could not prevent E from pulling x into the side with the larger V. Similarly, there is a ψ.

The critical reader who demands more precise reasoning than this will see in the next section how it may be supplied. There the reasoning is based on K-strategies which we have accepted as the sole thoroughly rational bulwark of the theory.

Now suppose in a certain subregion of $\mathcal{E}$, V is of class C_1. Then the surfaces on which V is constant will be semipermeable. The vector grad $V = \left\{V_{x_i}\right\}$ is normal to such surfaces. Whether a moving point penetrates the surface in one direction or the other or not at all depends on the sign of its velocity component along this vector. That is, in the case where V is of class C_1, the semipermeability condition for the surfaces of constant V is

$$\min_{\phi} \max_{\psi} \sum_{i} V_{x_i} f_i(x,\phi,\psi) = 0 \quad .$$

But this is the main equation.

Suppose we had begun with an integral payoff case and performed the transformation in II §3. We see that if we compare the games emanating from two points which differ only in their values of x_{n+1} (let d be this difference) the games will be identical except that there is a difference d between pairs of corresponding payoffs. Thus $V_{x_{n+1}} = 1$. Also $f_{n+1} = G$. Putting these values into (2), we obtain (1).

4. The Verificiation Theorem

We do not purport to give an existence theorem for differential games. Our interest lies in solving problems. In the sequel we shall explain methods and exhibit examples. But what we do need is a technique for showing that the results of our methods actually are solutions. Such is provided by the subject theorem. It is actually no more than a sedulous application of the semi-permeable surface concept of the last section. Its advantages are, as the reader will later see, that it verifies our solution methods almost automatically. In fact, it may be quite possible to construct an existence theorem from the solution technique and the present theorem for a properly delineated class of games.

Let us suppose that we have found an alleged solution of a certain game with terminal payoff. We suppose first that it is a game of degree, reserving the other case for later. By this we mean H is a continuous function on $\mathcal{C}$. Let V(x) be the purported value. There may be a subdivision of $\mathcal{E}$ by singular surfaces. Experience has shown that the only types of such surfaces on which V fails to be differentiable are ones that are never

crossed by the optimal paths.* We will suppose this to be the case here and we shall either remove any such (alleged) singular surfaces from $\mathcal{E}$, or think of them as "slits" so that on the "cut" $\mathcal{E}$ we may expect V to be of class C_1. There may also be a part of $\mathcal{E}$ with the property that the alleged optimal paths beginning there do not terminate. If so, we remove it from $\mathcal{E}$. What is left of $\mathcal{E}$ after both these kinds of surgery we will call $\mathcal{E}'$. Generally it fulfills the hypothesis of the

Verfication Theorem. If $V(x)$ is of class C_1 in $\mathcal{E}'$, if it satisfies(2) and equals H on $\mathcal{C}$, then $V(x)$ is the value of the game, provided the assumptions made in passing from $\mathcal{E}$ to $\mathcal{E}'$ are correct. The optimal tactics consist for those classes of functions of x such at each position they provide the min and max in (2).

Proof: We can, as we have seen in II, without changing the nature of the game, arrange that all velocity vectors in each full vectogram have at most unit length. We do so.

Let us select a tactic $\Phi(x)$ for P such that for each x, Φ is minimizing in (2). We let E play any K-strategy; let $\psi(x)$ be its tactic. Play starts from x^0. Given an $\varepsilon > 0$, we are going to complete P's K-strategy by constructing a σ_ι. We shall speak as if P were to play indefinitely; of course, we need but curtail our scheme when $\mathcal{C}$ is reached.

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* Example 3 of II is an instance. Although there are infinitely many singular surfaces, $V = -x_1$.

Divide time into intervals $I_m (m \leq t < m + 1, m = 0, 1, \cdots)$. In I_m, x cannot be further than m + 1 from the starting position x^o. We can take it, that for x so bounded, each V_{x_j} and f_j is uniformly continuous in x, the uniformity holding for all ϕ, ψ. The same is true of $\sum_j V_{x_j} f_j = Q(x, \phi, \psi)$. From the bound on speed, we can complete σ_t by subdividing I_m into I_{mp} such that, during each, the change in $Q < \frac{\varepsilon}{2^{m+1}}$. The I_{mp} may be further subdivided into I_{mpq} by the t'_k of σ'_t. In each I_{mpq} ϕ and ψ are constant ($= \phi_{mp}$ and ψ_{mpq}) and so $\frac{dV^*}{dt} = Q$. Let x' be the position at the start of I_{mp} and x be a point reached during I_{mpq}. Then

$$|Q(x, \phi_{mp}, \psi_{mpq}) - Q(x', \phi_{mp}, \psi_{mpq})| < \frac{\varepsilon}{2^{m+1}}$$

and, as ϕ_{mp} satisfies (2) at the start of I_{mp},

$$\max_{\psi} Q(x', \phi_{mp}, \psi) = 0$$

we have

$$\frac{dV}{dt} \text{ during } I_{mpq} = Q(x, \phi_{mp}, \psi_{mpq}) < \frac{\varepsilon}{2^{m+1}} .$$

As this inequality holds through I_m, the gain there of $V < \frac{\varepsilon}{2^{m+1}}$ and so the total gain $< \varepsilon$. But when C is reached, V = H = payoff and so the latter $< V(x^o) + \varepsilon$.

Likewise we can construct a σ'_t for E insuring a payoff $> V(x^o) - \varepsilon$. Thus $V(x^o)$ is the value.

Above we have tacitly assumed the game will terminate. Whether this is true is a question of a game of degree. Suppose that the answer assures us of termination; it generally happens that it is occasioned by the optimal tactics.

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* forward time derivative

Now let us consider the singular surfaces slashed from $\mathcal{E}$ in the construction of $\mathcal{E}'$. If no velocity $f(x,\Phi,\Psi)$ leads x onto such a surface from sufficiently proximate points.then the assumption of the surface being uncrossed during optimal play is corroborated.

Suppose besides a part $\mathcal{E}''$, in which it was alleged that optimal play did not terminate, was removed from $\mathcal{E}$. It may be that neither player can invoke termination from $\mathcal{E}''$. In that event, $\mathcal{E}''$ might well have been discarded at the outset; it has hardly a legitimate claim to belong to $\mathcal{E}$ in the first place. More usual – this being a game whose essence is conflict – one player will strive for termination; the other will oppose it. The boundary between $\mathcal{E}'$ and $\mathcal{E}''$ will be semipermeable; the details fit into a discussion of this subject still to come.

We now turn to games of kind. We already know that our objective here is to divide $\mathcal{E}$ into discrete parts corresponding to the discrete payoffs. We can generally consider $\mathcal{E}''$ as one of these parts. This is certainly true if we decree a stop rule (see II, section 3); the case where we don't will be treated in a subsequent chapter.

The salient point is that the boundaries of this subdivision are appropriately oriented semipermeable surfaces.

<u>Verification Theorem</u> (Games of Kind). If semipermeable surfaces with continuously turning tangent hyperplanes exist in $\mathcal{E}$ such that they meet $\mathcal{C}$ at the loci where H changes values and they are correctly oriented (that is, the P-side corresponds to a payoff

lower than that of the F-side), then, if these surfaces separate $\mathcal{E}$, the value of the game will be constant in each component and equal to the of H on $\mathcal{C}$ therein. We assume that for each component of $\mathcal{E}$, except possibly $\mathcal{E}''$, the game will terminate.

Proof: If S is one of these surface we can imbed it in a family of semipermeable surfaces whose union is a thin layer containing S. We define a function $V_1(x)$ over the layer constant on each surface, of class C_1, and decreasing in the P-direction. This construction amounts to a problem in partial differential equations and its solution is known to exist. If the reader demands an explicit construction we can easily infer one from our solution techniques to be explained later.

We reason about this layer just the way we did about $\mathcal{E}'$ in the preceding proof. We find, say, that if x is on the P-side of S, say at a "distance" $\mathcal{E}$ from S, by playing a suitable K-strategy P can keep x from crossing S.

5. The Path Equations

We work with integral payoffs, the other species being handled by suppressing G.

Being confronted with a particular problem let us write (2) and ascertain the maximizing ψ and minimizing ϕ as functions of the x_i and V_i*. Let them be

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* We will henceforth write V_i in place of V_{x_i}. V_x will stand for the vector $\{V_i\}$ (= grad V).

(5) $\Phi = \Phi(x, V_x)$, $\Psi = \Psi(x, V_x)$ *

If there is a choice involved in (5), select Φ, Ψ to be continuous functions of their 2n arguments insofar as it is possible. (The last clause generally acquires a definite meaning in a particular example.) Substitute (5) into (2):

(6) $$\sum_i V_i(x) f_i(x, \Phi, \Psi) + G(x, \Phi, \Psi) = 0 \quad .$$

We have now a true partial differential equation for V in (6). It also shall be referred to as the main equation.

We differentiate (6) with respect to each x_k now thinking of V as a function of x. Doing so in accordance with the rules of elementary calculus, we examine the components as they arise. First we have

(7) $$\sum_i V_{x_i x_k} f_i$$

which can also be written

(8) $$\sum_i (V_k)_{x_i} \dot{x}_i = \dot{V}_k$$

that is, the time derivative of V_k over a direct optimal path. Next we have

(9) $$\sum_i V_i f_{ik} + G_k$$

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* We are not really overworking these symbols. When V becomes known as a function of x and so substituted in (5), it is clear that we will have our old Φ, Ψ.

where $f_{ik}(x,u,v) = \frac{\partial f_i}{\partial x_k}(x,u,v)$ and $G_k = \frac{\partial G}{\partial x_k}$. Then we encounter

$$(10) \qquad \sum_{j=1}^{\varrho} \frac{\partial}{\partial \phi_j} \left(\sum_i V_i f_i + G\right) \frac{d\Phi_j}{dx_k}$$

Each ϕ_j is supposed subject to constant bounds such as (10) of II. The minimizing Φ_j occurs either interior to the constraining interval or at an endpoint. If the former, the $\frac{\partial}{\partial \phi_j}$ ( ) term of (10) is 0 because of the minimizing property of Φ; if the latter, the $\frac{\partial \Phi_j}{\partial x_k}$ is 0 as Φ_j is constant. In either case then (10) vanishes. The same is true of the remaining terms devolving on the ψ_i. We conclude

$$(11) \qquad \dot{V}_k = -\sum_i V_i \quad , f_{ik}(x,\Phi(x,V_x), \Psi(x,V_x)) + G_k(x,\Phi(x,V_x), \Psi(x,V_x)).$$

Rewriting the K.E.*, slightly specialized,

$$(12) \qquad \dot{x}_k = f_k(x,\Phi(x,V_x), \Psi(x,V_x)) \quad .$$

This set (11), (12) of 2n ordinary differential equations in the 2n unknowns x_k, V_k shall be called the <u>path equations</u>. It is not necessary to presuppose the existence of second partials of V such as appear in (7) and (8). Actually (11), (12) are the characteristic equations of (6) (slightly special in that the terms (10) are nullified). Solutions of (6) can be built from integrals of (11), (12) in the standard manner,** a procedure which we shall

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* We shall use such obvious abbreviations henceforth.

** See, for example, Courant-Hilbert, <u>Methoden der Mathematische Physik</u>, T.II.

shortly adapt to our purposes.

6. The Retrogression Principle

When solving a game we reverse time; we start at $\mathcal{C}$ and work backwards into $\mathcal{E}$. The motivation can be easily understood if the game is quantized.

Let us replace the K.E. by approximating difference equations. The exact manner is not critical for our present ends; we'll settle for any reasonable discrete facsimile. Consider starting x so near $\mathcal{C}$ that it can be reached in one move by each player. What we have here is a one move discrete game; the navigation variables are chosen as the optimal strategies for it. This settles the value of the game in a certain thin part $\mathcal{E}_1$ of $\mathcal{E}$ bordering $\mathcal{C}$. Next we similarly investigate the starting points from which $\mathcal{E}_1$ can be reached after one move each by the players. Inasmuch as values at the end of the composite move are known, we again can formulate matters as a one-move game. (What we are doing here is applying the tenet of transition; compare the first proof of the main equation in Section 2.) Thus the value becomes known in a second layer $\mathcal{E}_2$ bordering $\mathcal{E}_1$. We proceed thus, filling $\mathcal{E}$.

The value of the game is thus determined by a chain of causes and consequences that proceeds counterchronologically from $\mathcal{C}$.

Accordingly, we let $\tau = -t$ and use the symbol $\overset{\circ}{x}$ for $\frac{dx}{d\tau}$, so that $\overset{\circ}{x} = -\dot{x}$. For reference, we rewrite the P.E.:

(13) $$\overset{\circ}{V}_k = \sum_i V_i f_{ik}(x,\hat{\phi},\hat{\psi}) + G_k(x,\hat{\phi},\hat{\psi})^*$$

(14) $$\overset{\circ}{x}_k = -f_k(x,\hat{\phi},\hat{\psi}) \quad .$$

7. The Solution in the Small

The surface $\mathcal{C}$ furnishes a natural seat of initial conditions for these retrogressive equations. But an important detail calls for attention first.

Consider a position very near $\mathcal{C}$. One or the other player may be able to force or deter an imminent termination despite any opposition from his opponent. Let $\nu = (\nu_1,\ldots,\nu_n)$ be vector normal to $\mathcal{C}$ from point x on $\mathcal{C}$ and extending into $\mathcal{E}$. If, say,

(15) $$\min_{\phi} \max_{\psi} \sum_i \nu_i f_i(x,\phi,\psi) > 0$$

then E can prevent immediate termination from a position sufficiently near x. If (15) holds with the inequality reversed, P can compel immediate termination.

There is the question of whether a player will benefit from the exercise of such power. Sometimes the answer is obvious. We cite the case of termination time payoff; clearly E will defer termination whenever he can. But in other instances E may see that avoidance of the frying pan now will only lead to the fire later. We leave the intricacies of such questions to individual

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* For terminal payoff games, we merely suppress the G.

cases. But we must crystallize one concept.

The above situation can imply that only a certain part of $\mathcal{C}$ will be effective under optimal play. We call it the <u>useable part</u>. It is the residue of $\mathcal{C}$ when we remove the portions where one player can – and profitably can – forestall termination from nearby points.

Now we formulate the initial conditions for (13), (14). Let

$$x_i = h_i(s_1,\ldots,s_{n-1}) \tag{16}$$

be a parametric representation of $\mathcal{C}$. On $\mathcal{C}$, V is known. Let it be V(s). If the payoff is terminal, V(s) = H; if integral, V(s) = 0. We need to know the values of the V_i on $\mathcal{C}$. We have

$$\frac{\partial}{\partial s_\ell} V(s) = \sum_i V_i \frac{\partial h_i}{\partial s_\ell} \qquad (\ell=1,\ldots,n-1) \tag{17}$$

which are n–1 equations for the n unknowns V_i. The remaining equation is the main equation (6). We need only solve this system for the U.P. of $\mathcal{C}$. Sometimes a double solution will appear. The reason is that there is nothing in our analysis to distinguish the two sides of $\mathcal{C}$. It is not hard to do so by other means and then discard the solution not pertaining to $\mathcal{E}$.

Thus the values of x_i and V_i will be known on the U.P. of $\mathcal{C}$. They are to be employed as initial values in integrating (13), (14). The solutions furnish (reversed) paths $x_i(\tau,s_1,\ldots,s_{n-1})$ extending from the points s of $\mathcal{C}$ into $\mathcal{E}$. Should they fill $\mathcal{E}$ univalently (exactly one path through each point) the game is virtually solved. The elements of the solution can be obtained

by routine calculation and the verification theorem will show it is correct.

If $\mathcal{E}$ is not so filled, we can still expect to obtain a solution in part of $\mathcal{E}$. It will be the first step of the solution in the large. As the later portions of the solution are constructed, we may find ourselves repeating the whole procedure with new surfaces playing the role here allotted to $\mathcal{C}$.

Although we wish to segregate the in-the-large phases to later paragraphs, it will be instructive to mention one simple type of singular surface now. Suppose the functions Φ, Ψ of (5) are not continuous; but, for example, Φ has a simple discontinuity when a certain function $u(x,V_x)$ changes sign. On $\mathcal{C}$ we will know the sign of u and can construct the paths accordingly. From these solutions of (13) and (14) we will know the value of u along a path. Let us say that, for each s (that is, path) there occurs a value of τ where u ceases to have the favorable sign, and all such points together constitute a surface $\mathcal{T}$ in $\mathcal{E}$. Thus our solution construction is halted at $\mathcal{T}$.

But $\mathcal{T}$ may well be a <u>transition surface</u>, that is one crossed by the optimal paths but on which at least one of the optimal tactics is discontinuous.* To find out, we need but proceed with the construction. We use $\mathcal{T}$ as a new seat of initial conditions and on its far side solve the path equations anew. These initial values of x_i, V_{x_i} are of course obtained from the earlier paths, but we utilize for Φ the new value ascertained from the changed sign of u.

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* For an instance see Example 3 of II.

Observe that V remains class C_1 across $\mathcal{T}$.

The next chapter will consist of various examples that can be handled with our present state of knowledge.

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