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Robert F. Sellars, Capt., USN
Commander (Acting)

Wm. B. McLean, Ph.D.
Technical Director

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COMPRESSIBILITY EFFECTS DURING WATER ENTRY

By

Paul Dergarabedian

Underwater Ordnance Department

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FOREWORD

As the speed of air-to-water missiles increases, the water impact decelerations which these missiles experience at water entry become increasingly important to structural design. For example, at low speeds of even a few hundred feet per second, decelerations of several hundred gravitational units are common. At these entry speeds, and increasingly more so as the speeds increase, the compressibility of water has important influence on the peak impact loads and decelerations.

The research reported here considers the effect of water compressibility in the fundamental case of vertical impact of round-nosed missiles, by applying mathematical techniques previously used only in supersonic aerodynamics.

This work has been supported jointly by the Office of Naval Research (Project TOWER: Treatise on Water-Entry Research) and this Station under Local Project 701, from 1950 to 1954.

This report is transmitted for information only. It does not represent the official views or final judgment of this Station. It presents information released at the working level that is still subject to modification and withdrawal.

D. W. STEEL, Acting Head
Research Division

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Underwater Ordnance Department

ABSTRACT

Compressibility effects during the initial impact phase of air-to-water missiles are presented in the form of a review of existing theoretical and experimental work conducted at the U. S. Naval Ordnance Test Station during the last few years.

The theoretical section of the report presents three different approaches to the problem of finding the pressures which act upon the missile when it strikes the water with a velocity much lower than the speed of sound in water, taking into account the compressibility of the water.

The experimental section of the report presents results obtained in measuring the impact pressure of missiles or spheres striking the water surface.

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INTRODUCTION

Compressibility effects during the first instants of water-entry impact of an air-to-water missile are of interest both from a structural and a ballistic viewpoint. During the initial impact, the effects of compressibility are important, whereas during the latter phases of entry the incompressible theory is adequate for most problems. It should be emphasized that the amount of experimental data concerning the compressible phenomenon is very limited, so that the treatment of compressible effects is essentially a theoretical one at the present time.

Extremely high pressures can be developed on the nose of a missile at water impact. If a flat, rigid disk is considered as striking a water surface normally, the instantaneous pressure due to the compression wave set up in the water is $p = \rho cV$ (Appendix A) where ρ is the density of the water, c is the velocity of sound in the water, and V is the normal velocity of the disk. If this problem is considered from the incompressible viewpoint, the pressure at impact has an infinite value. In the case of a sphere striking a water surface normally, there is a small area over which such a shock will exist for a finite interval during which the intersection of the sphere with the water surface expands faster than the velocity of sound in water. For low entry velocities, $V \ll c$, the time during which the rate of growth of the wetted-surface radius is supersonic is very small. Hence for most cases of water entry, the impact or shock phase occurs during a very short interval, usually of the order of a few microseconds. This factor alone makes experimental work very difficult to perform since instruments with very rapid response are required in order to measure events occurring during a short interval. In addition, there are indications of air trapped between the missile and the water which acts as a cushion in its effects on the shock phase. Since the pressure plugs in the missile have a finite diameter, any pressure measurement indicates an averaged value. The phenomenon of trapped air may also delay and reduce the peak pressures.

The structural effect of the shock phase is of interest in connection with the deformation of the nose of the missile, but it is not well known. The structural effect is also of theoretical interest since it combines hydrodynamic and elasticity theory in the solution of the problem.

As one part of the over-all problem of water entry shock, the impact phase is important since it is the first to occur and the subsequent behavior of the missile is dependent on what happens during this phase. The fact that the compressibility effects are more pronounced during this phase means that only theoretical and experimental work of a fundamental nature can reveal their relative importance.

The incompressible theory of water entry of blunt bodies has been extensively developed by Shiffman and Spencer (Ref. 1) and by others. Based on this theory, the pressure distribution on a sphere entering water vertically has been determined at the Naval Ordnance Test Station (NOTS), and the theoretical results have been checked experimentally through pressure measurements with piezoelectric gages at the stagnation point of a 12-inch diameter sphere.

Incompressible theory, however, predicts an infinite pressure at the stagnation point of a blunt body when it comes in contact with the water surface. Actually, it is expected that during the impact phase, pressures of the order of the shock or piston-impulsive pressure ρcV are experienced for a few microseconds. Since a theoretical prediction of the pressures during this phase requires that account be taken of the compressibility of the water, the formulation of this problem and several solutions will be considered. The problem was first stated by L. Trilling of this Station, who based his formulation on the theory of weak waves and obtained a solution to the two-space dimensional problem after making several assumptions. The problem was later taken up at NOTS by R. H. Owens, who reformulated the problem using the more general Reisz method, but made no attempt at a detailed solution. The vertical impact of a sphere on the water surface was considered at NOTS by R. H. Korkegi, using the retarded potential solution as suggested by F. E. Marble, consultant to this Station, from which actual pressure distributions were obtained.

TRILLING'S SOLUTION FOR TWO-SPACE DIMENSIONS

Since this report deals with the first contact of a striking body with a plane water surface where the compressibility of the water is to be considered, the problem resolves itself into one concerning the propagation of a pressure wave of finite amplitude into the water. This involves considerations in terms of the dynamics of a compressible liquid.

When a body strikes a water surface at a velocity $\bar{V}$ where $|\bar{V}|/c \ll 1$ (c is the velocity of sound in water), the theory of weak

waves (Ref. 2) may be applied to compute its motion. The flow is irrotational and isentropic, and the velocity potential satisfies the wave equation up to terms of the order of V^2/c^2. The boundary conditions are applied on the undisturbed free surface, and the displacement of the body is neglected. The effect of splash is ignored.

The velocity potential φ satisfies the equation

$$\varphi_{tt} - c^2 \nabla^2 \varphi = 0 \tag{1}$$

The velocity $\overline{V}$ and the pressure p are determined by

$$\overline{V} = \nabla \varphi \quad , \quad p = -\rho \varphi_t \tag{2}$$

Just before impact, the fluid is at rest or

$$\varphi(0, x, y, z) = 0 \tag{3}$$

Since the differential equation and the boundary conditions have been linearized, the solutions for the vertical and horizontal components of motion may be found separately. In this report only the vertical component will be considered.

The boundary condition for the vertical component is

$$\begin{aligned} \varphi(t, x, 0, z) &= 0 \qquad \text{on } S' \\ \varphi_y(t, x, 0, z) &= V \qquad \text{on } S \end{aligned} \tag{4}$$

where S is the wetted surface of the body and S' is the horizontal free surface of the liquid. The problem (Eq. 1 - 4) is similar to that of a lifting three-dimensional wing in a steady supersonic stream for the case of two-space dimensions (Ref. 3, p. 73), and the methods of supersonic airfoil theory may be adapted to solve it.

When a two-dimensional body strikes a plane water surface at a constant velocity $\overline{V}$, its wetted surface $|x| < x_e(t)$, grows at a rate proportional to the slope of the body section since $dx_e/dy = \dot{x}_e/v$. The potential function satisfies the equation

$$\varphi_{tt} - c^2 (\varphi_{xx} + \varphi_{yy}) = 0 \tag{5}$$

with the initial condition

$$\varphi(0, x, y) = 0$$

The vertical component of motion satisfies the boundary conditions

$$\varphi(t,x,0)=0 \quad , \qquad x > x_e(t)$$

$$\varphi_y(t,x,0)=V \quad , \qquad x < x_e(t)$$

Figure 1 shows the growth of the strip $x_e(t)$ in the plane y = 0.

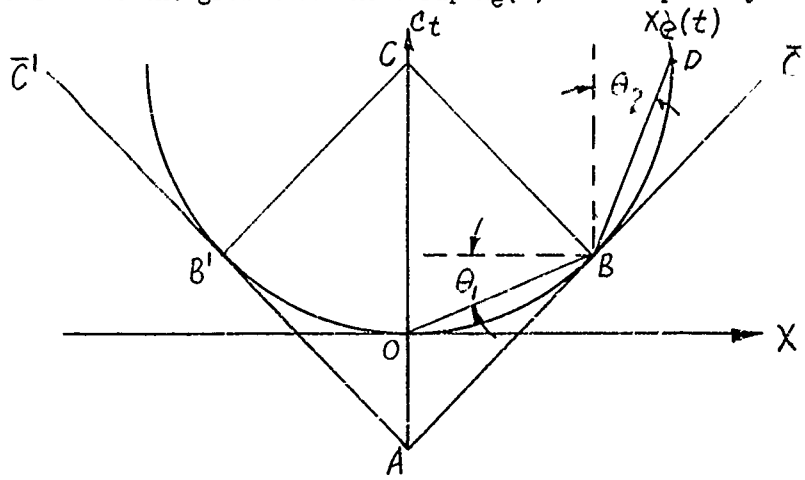

FIG. 1.

The lines AB, AB' are the traces of the characteristic Mach cone tangent to x(t) in the plane y = 0. Since the equation of motion is hyperbolic with constant leading coefficients, any disturbance travels in the fluid field at the constant velocity c. In the region B' OB, the edge of S moves outward at a velocity $\dot{x}_e > c$, and therefore, in the region BOB'C, no point on the body surface can be influenced by the fact that the body has finite width. The pressure in that region can be determined from one-dimensional wave theory (Ref. 3, p. 41) and is

$$p \approx \rho c V\left(1-\frac{1}{2}\frac{V^2}{\dot{x}_e^2}\right) \quad , \qquad V \ll \dot{x}_e$$

and since $V \ll c$ and $\dot{x}_e \geq c$ for this domain, the pressure can be approximated very closely by

$$p \approx \rho c V$$

To investigate the flow in regions $\bar{O}B\bar{C}$, $OB'\bar{C}'$ it is convenient to approximate the curve OB by a straight line whose angle θ with the horizontal x-axis is small, since $\tan\theta_1 \leq \frac{V}{c}$. The curve BD is

approximated by a straight line whose angle with the vertical (ct) axis is θ_0. The point D is any convenient point on $x_0(t)$. The origin of the coordinate system is now shifted to B. The flow in the region CBC is conical in the sense introduced by Busemann (Ref. 4). It is characterized by the absence of any length parameter in the equation of motion or the geometry of the field. The velocity and pressure depend only on the parameters $\xi = x/ct$ and [illegible] In the $\xi\,\eta$ plane, the Mach cone from B appears as the [illegible] circle. The boundary BD is a segment on the ξ axis. The [illegible] of contact of the Mach cone from B with the plane waves from [illegible] the points FF' on the unit circle (Fig. [illegible]), situated at

$$\xi_1(F) = e^{i\psi} \qquad \xi_2(F') = e^{-i\psi}$$

$$\cos\psi = \frac{V}{c}$$

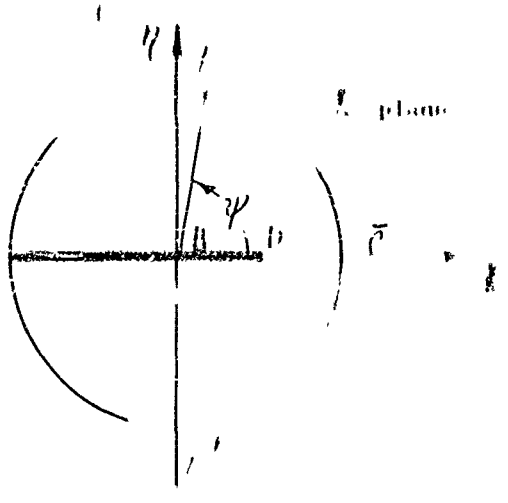

FIG.

The differential equation for the pressure, obtained by introducing $p(\xi,\eta)$ into the wave equation, Eq. 5, and transformed to polar coordinates σ, γ in the $\xi\,\eta$ plane is

$$(1-\sigma^2)p_{\sigma\sigma} + \frac{1-2\sigma^2}{\sigma}p_\sigma + \frac{1}{\sigma^2}p_{\gamma\gamma}$$

The angular coordinate γ is now left unchanged, while the radial coordinate is transformed as suggested by Tchaplygine (Ref. 3, p. 28)

$$S = \frac{1-\sqrt{1-\sigma^2}}{\sigma} \tag{6}$$

The unit circle and the origin remain invariant and rays from the origin are transformed into themselves, although not point by point. In particular, the edge D which was situated at the point $\sigma_D = \tan\theta_2$ is now defined by

$$S_D = \cot\theta_2\left(1-\sqrt{1-\tan^2\theta_2}\right)$$

In the S, γ plane the pressure and the velocity components satisfy the equation

$$p_{ss} + \frac{p_s}{s} + \frac{p_{\gamma\gamma}}{s^2} = 0$$

The functions p, u, v are therefore real parts of analytic functions P, U, V of the complex variable $\zeta = Se^{i\gamma}$. Since v is constant along CD, $\frac{\partial\gamma}{\partial\xi}$ vanishes there. It follows from the momentum equation that $\frac{\partial P}{\partial\xi} = 0$, but along the real axis

$$\frac{\partial P}{\partial y} = \frac{1}{\sigma}\frac{\partial P}{\partial\gamma}$$

and because of the invariance of angles under the Tchaplygine transformation $\frac{\partial P}{\partial\eta}$ vanishes. From the Cauchy-Riemann conditions, it follows that the harmonic conjugate p^* of p is constant along CD and may be set equal to zero. On F' $\overline{CF}$ where $|\gamma|<\psi$, no disturbance can affect the pressure field, so that the pressure vanishes there. Since the pressure is continuous in the field, the pressure along FCF' is the same as behind the waves from OB, that is $p = \rho cV$ along CF and $-\rho cV$ along CF'. To simplify conditions along the real axis, it is convenient to introduce the homographic transformation

$$w = \frac{\zeta - S_D}{1-\zeta S_D} \tag{7}$$

which leaves the unit circle and the real axis invariant, and puts the point D at the origin. The points FF' are now located at

$$w_{FF'} = \tan\theta_2 \pm i\sqrt{1-\tan^2\theta_2} = e^{\pm i\alpha}$$

Along the positive real w-axis p is zero, and since it is harmonic it must be antisymmetric with respect to the real axis. With respect to the negative real w-axis, it is symmetric by virtue of Schwartz's reflection principle, since p^* vanishes there. The w-plane (Fig. 3) is therefore a double-sheeted Riemann surface with

a cut along CD. A reflection of P(w) in CD defines P(w) in the lower sheet. The solution in the physical plane has a discontinuity along CD, where the body can support a pressure discontinuity, but it is continuous in the fluid, along the positive real axis.

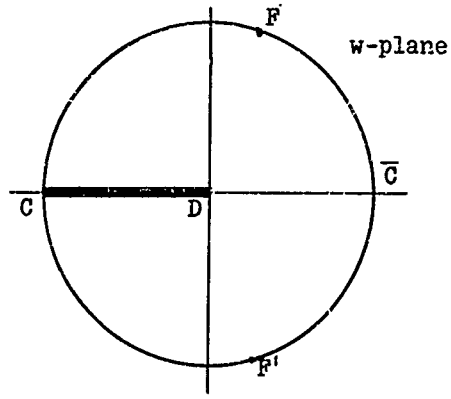

FIG. 3.

If the double-sheeted surface is unwound by the transformation $\lambda = \sqrt{w}$ the points F, F' are mapped into four points $\lambda_\ell = \pm e^{\pm i\frac{\alpha}{2}}$, and the boundary conditions for P (λ) are given on the unit - λ circle as follows

$$Re\ P(\lambda) = 0, \quad -\frac{\alpha}{2} < \arg\lambda < \frac{\alpha}{2}, \quad \frac{2\pi-\alpha}{2} < \arg\lambda < \frac{2\pi+\alpha}{2}$$

$$Re\ P(\lambda) = \rho c V, \quad \frac{\alpha}{2} < \arg\lambda < \frac{2\pi-\alpha}{2}$$

$$Re\ P(\lambda) = -\rho c V, \quad \frac{2\pi+\alpha}{2} < \arg\lambda < \frac{4\pi-\alpha}{2}$$

The function which satisfies these conditions is (Ref. 3, p. 79)

$$P(\lambda) = \frac{i\rho c V}{\pi} \log_e \frac{(\lambda-\lambda_1)(\lambda-\lambda_4)}{(\lambda-\lambda_2)(\lambda-\lambda_3)}$$

The pressure distribution on the body surface is found by retracing back through the transformations in Eq. 8, 7, and 6. The result for small θ_1 ($\tan\theta_1 \approx \theta_1$) is

$$p \approx \frac{2\rho c V}{\pi} \tan^{-1}\left[A(1+B\theta_1)\right]$$

where

$$A = \frac{1+sS_D}{1-s}\sqrt{\frac{s+S_D}{2(1+sS_D)(1+S_D^2)}}\ , \quad B = \frac{1}{2}\left(\frac{1+3S_D^2}{1+S_D^2}\right),$$

$$S_D = \cot\theta_2\left(1-\sqrt{1-\tan^2\theta_2}\right), \quad s = \frac{1-\sqrt{1-\sigma^2}}{\sigma},$$

$$\sigma = \frac{x}{ct}$$

Several factors restrict the usefulness of this solution. First, the boundary of the body has been approximated by straight line segments. For this approximation to be reasonable it is necessary to make these segments very short. In this connection Evvard's method (Ref. 5) based on the appropriate fundamental solution of the wave equation might be used in order to eliminate approximation of the boundary by straight line segments. Coupled to this restriction is the fact that to extend the solution any further requires the use of complicated methods of superimposing other solutions and then matching boundary conditions. Because of the complexities of such a solution and the questionable accuracy of the solution already at hand, such an attempt does not seem to be justified. Since the two-space dimensional case is essentially of academic interest, it is worth noting that the problem can be treated by using the fundamental solution of the wave equation. This has been done by R. H. Owens using Riesz's method (Ref. 6). It should be noted here that the retarded potential cannot be formulated in two dimensions because of Huygen's principle.

Riesz's method is presented in Appendix B. The advantage of the method is that it clearly shows what boundary and initial conditions are necessary to solve a problem in hyperbolic differential equations. Recalling that in m dimensions the hyperbolic distance R is given by

$$R^2 = \sum_{j=1}^{m}\left(\xi^j - x^j\right)^2$$

and since

$$\Box_m R^{2-m} = 0 \qquad (\Box = \text{D'Alembertian})$$

R^{2-m} satisfies this m-dimensional wave equation. However, an attempt to build up a solution from this so-called fundamental solution fails when $m > 3$ because any integral involving R^{2-m} diverges. However, an integral of $R^{\alpha+2-m}$ where $\alpha > m - 4$ will converge (the use of analytic continuation is the basis for Riesz's method) and

its limit as $\alpha \to 0$ may be considered as the analytic continuation of the integral of $R^{\alpha+2-m}$ into the domain $0 \leq \alpha \leq m - 4$, i.e., it defines the integral of R^{2-m}. In order to perform this analytic continuation it is necessary to integrate by parts. However, the integral is multiple, and Green's theorem is used, which corresponds to integration by parts. For $m = 4$ the familiar retarded potential solution is obtained. However this can be done more simply, and constitutes the basis for the formulation of the problem by R. H. Korkegi.

KORKEGI'S SOLUTION FOR THREE-SPACE DIMENSIONS

The formulation of the problem is again essentially that of L. Trilling. The first few instants of water entry during which only a very small part of the sphere is submerged will be considered. The depth of penetration is small compared to the radius of the wetted area; hence, to the order of approximation of this analysis, boundary conditions will be satisfied in the plane of the undisturbed water surface rather than on the curved surface of the sphere. Since the surface of contact of sphere and water is the only disturbance present, the analytic problem is that of determining the flow field that is due to an expanding disk of disturbances. In addition, the problem will be restricted to considering the time during which the rate of expansion of the wetted surface exceeds the rate of wave propagation in water and the free surface of the water beyond the area of contact is undisturbed. The following assumptions are made:

1. That the flow is irrotational and isentropic.

2. That the velocity of the body V and the rate of wave propagation in water c are constant.

3. That V/c is much smaller than unity.

4. That boundary conditions are satisfied in the plane of the undisturbed water surface.

5. That the sphere is considered rigid.

The condition of irrotationality, $\nabla \times \bar{U} = 0$, is identically satisfied by a potential such that $\bar{U} = \nabla \varphi$. ($\bar{U}$ is the velocity vector in the flow field). With a system of coordinates x, y, z (or r, α, z) fixed in space, the analytic problem consists of the following partial differential equation, boundary, and initial conditions. Figure 4 shows the area of contact of sphere and water represented by a disk.

(8) $\nabla^2 \varphi - 1/c^2 \, \varphi_{tt} = 0$

(9a) $\varphi(r, 0, t) = 0 \quad , \quad r > R(t)$

(9b) $\varphi_z(r, 0, t) = 0 \quad , \quad r > R(t)$

(9c) $\varphi_z(r, 0, t) = V \quad , \quad r < R(t)$

$\varphi(r, z, 0) = \varphi_t(r, z, 0) = 0 \quad , \quad r^2 + z^2 \neq 0$

where

$R(t)$ = the instantaneous radius of the disk

x, y, z = coordinates of the field point

ξ, η, ζ = coordinates of a variable point in space

t = fixed time

τ = variable time

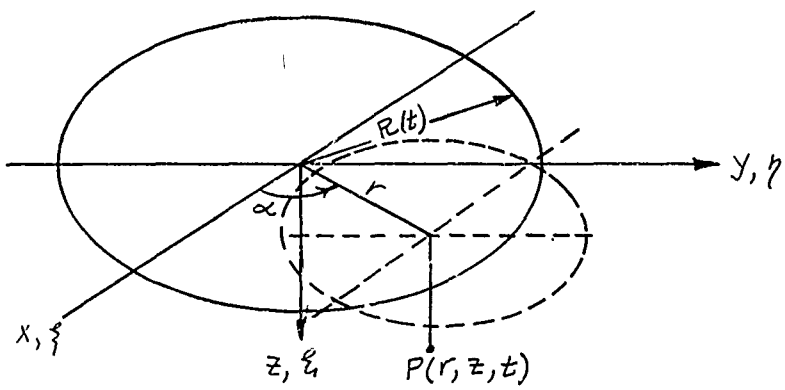

FIG. 4.

A fundamental solution of the wave equation (Eq. 8) is the simple source

$$\varphi(r_1, t) = \frac{f(t - r_1/c)}{r_1}$$

It represents the instantaneous potential at time t at a field point a distance r_1 away from a source of strength f. Since the wave equation is linear, solutions can be constructed by superimposing simple sources. In particular, the solution for a distribution of point sources in the plane $\zeta = 0$ is

$$\varphi(x,y,z,t) = \iint_A \frac{f(\xi,\eta,0,t-\frac{r_1}{c})}{r_1}\, dA$$

where

$$r_1 = \sqrt{(x-\xi)^2 + (y-\eta)^2 + z^2}$$

It is found that the instantaneous pulse strength f is directly proportional to the instantaneous and local value of φ_z. This is shown by Lagerstrom (Appendix C) and yields

$$f(\xi,\eta,0,\tau) = -\frac{\varphi_z}{2\pi}(\xi,\eta,0,\tau)$$

Thus, the solution can be written in terms of a retarded potential as follows

$$\text{(10a)} \quad \varphi(x,y,z,t) = -\frac{1}{2\pi}\iint_A \frac{\varphi_z(\xi,\eta,0,t-\frac{r_1}{c})}{\sqrt{(x-\xi)^2+(y-\eta)^2+z^2}}\, d\xi\, d\eta$$

or

$$\text{(10b)} \quad \varphi(x,y,z,t) = -\frac{1}{2\pi}\int_z^{ct}\int_0^{2\pi} \varphi_z(\xi,\eta,0,t-\frac{r_1}{c})\, d\theta\, dr_1$$

where

$$\theta = \tan^{-1}\left(\frac{y-\eta}{x-\xi}\right)$$

The limits of integration of Eq. 10b are obtained as follows: if there are no disturbances prior to time $\tau = 0$, then, at time $\tau = t$, the field point P (Fig. 5), is influenced by disturbances occurring within a sphere of radius ct about P; since disturbances are limited to the plane $\zeta = 0$, the region of integration is the circular area defined by the intersection of the sphere ct with the plane $\zeta = 0$. The integrand of Eq. 10a and b is determined by the boundary conditions of the problem.

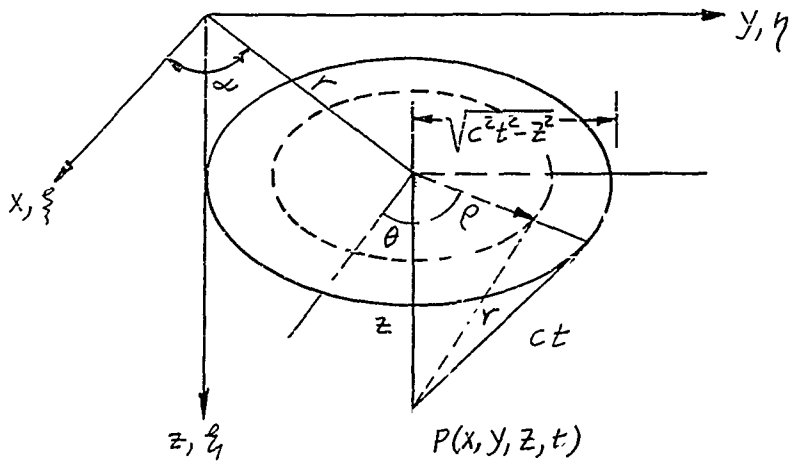

FIG. 5.

For a sphere of radius a entering water with a vertical velocity V the instantaneous radius of the wetted surface for small depths of penetration is approximately (see Fig. 6)

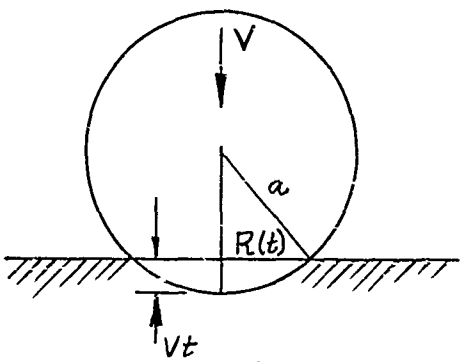

FIG. 6.

(11) $$R(t) = \sqrt{2aVt}$$

The rate of growth of the radius of the wetted area is hence

(12) $$\dot{R}(t) = \sqrt{\frac{aV}{2t}}$$

When the rate of growth becomes sonic $\dot{R}(t_c) = c$, the time at which the rate of growth is sonic is

(13) $$t_c = \frac{aV}{2c^2}$$

Since this analysis will hold for times during which the rate of growth is supersonic, it must be required that

$$t \leq t_c \quad , \quad R(t) \leq R(t_c) = \frac{aV}{c}$$

It is convenient to use dimensionless variables (denoted by primes) as follows

(14a) $$R'(t) = R(t) \Big/ \frac{aV}{c} \leq 1$$

(14b) $$t' = t \Big/ \frac{aV}{2c^2} \leq 1$$

The dimensionless rate of growth becomes the Mach number

$$M = \frac{\dot{R}(t)}{c} = \sqrt{\frac{aV}{2c^2}}\, t^{-\frac{1}{2}} = t'^{-\frac{1}{2}}$$

Now the boundary conditions (Eq. 9) are applied to the retarded potential solution (Eq. 10a) in order to determine the potential at an arbitrary field point. Since the problem is axially symmetric, no generality is lost by choosing a point in the vertical plane y = 0. Because of the time dependency involved, not all pulses emitted in the circular region of integration illustrated in Fig. 5 are felt at the field point P at time t. Hence it is necessary to determine the region of integration within the circular area of radius $\sqrt{c^2t^2 - z^2}$, with pulses of strength $\varphi_z = V$ (Eq. 9c) which will contribute to the potential at P (x, 0, z, t). This region is bounded by the locus of the points of intersection of the instantaneous Mach cone (see Fig. 7a) with the expanding disk (representing the wetted surface of the sphere) for all times between $\tau = 0$ and $\tau = t - z/c$. From Fig. 7a it can be seen that pulses emitted for $0 < \tau < \tau_1$, and $\tau > \tau_2$ have not yet reached P at time t; only those emitted for $\tau_1 < \tau < \tau_2$ have influence. Hence, the area in the plane z = 0 in which disturbances emitted will reach the field point P(x, 0, z) at time t is bounded by the curve $\eta(\xi)$ (Fig. 7b) given by the equation

$$R^2(\tau) = \xi^2 + \eta^2$$

where

$$R^2(\tau) = 2aV\tau = 2aV\left(t - \frac{r}{c}\right) = 2aV\left(t - \frac{1}{c}\sqrt{(x-\xi)^2 + \eta^2 + z^2}\right)$$

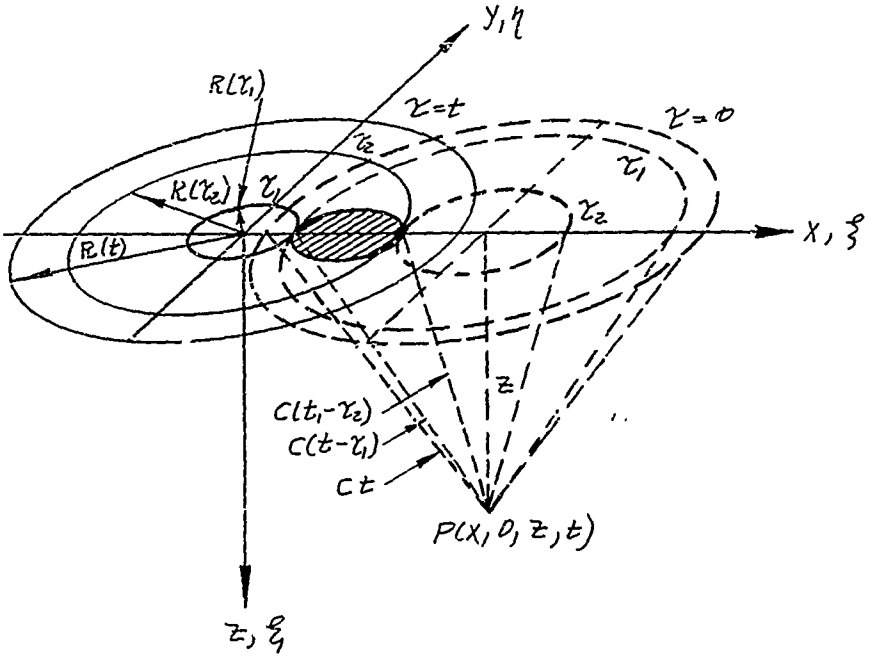
y, η
R(τ₁)
τ=t
τ=0
τ₂
τ₁
R(τ₂)
τ₁
τ₂
R(t)
x, ξ
z
C(t₁-τ₂)
C(t-τ₁)
Ct
P(x, 0, z, t)
z, ζ

FIG. 7a.

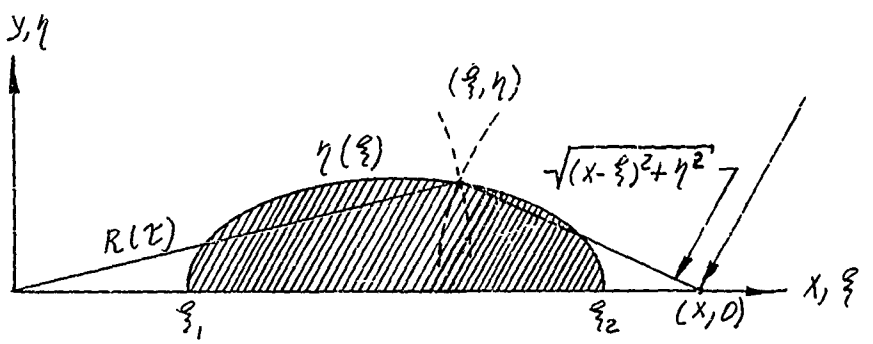
y, η
(ξ, η)
η(ξ)
√((x-ξ)²+η²)
R(τ)
x, ξ
ξ₁
ξ₂
(x, 0)

FIG. 7b.

This equation can be expressed in dimensionless form as follows

$$t' - 2\sqrt{(x'-\xi')^2 + \eta'^2 + z'^2} = \xi'^2 + \eta'^2$$

where $t' = t\Big/\frac{aV}{2c^2}$ and all space quantities are reduced to dimensionless form through division by $\frac{aV}{c}$, that is, $\xi' = \xi\Big/\frac{aV}{c}$, $x' = x\Big/\frac{aV}{c}$. Solving for $\eta'(\xi')$ the equation is obtained for the curve bounding the area of integration as follows

$$(15) \quad \eta'(\xi') = \left\{ f_1(\xi') - \left[f_1^2(\xi') - f_2(\xi') \right]^{\frac{1}{2}} \right\}^{\frac{1}{2}}$$

where

$$f_1(\xi') = 2 + t' - \xi'^2$$

$$f_2(\xi') = (t' - \xi'^2)^2 - 4\left[z'^2 + (x' - \xi')^2 \right]$$

Hence, from the retarded potential solution of the wave equation (Eq. 8) and the boundary conditions (Eq. 9b and c), the potential at an arbitrary field point (x,0,z) at time t, given in terms of dimensionless coordinates, is

$$(16) \quad \varphi(x,0,z,t) = -\frac{aV^2}{\pi c} \int_{\xi_1'}^{\xi_2'} d\xi' \int_0^{\eta'(\xi')} \frac{d\eta'}{\sqrt{(x'-\xi')^2 + \eta'^2 + z'^2}}$$

where $\eta'(\xi')$ is given in Eq. 15 and the limits ξ_1' and ξ_2' are the real roots of the quartic equation obtained by setting $f_2(\xi') = 0$. In dimensional coordinates the potential is

$$(17) \quad \varphi(x,0,z,t) = -\frac{V}{\pi} \int_{\xi_1}^{\xi_2} d\xi \int_0^{\eta(\xi)} \frac{d\eta}{\sqrt{(x-\xi)^2 + \eta^2 + z^2}}$$

where

$$\eta(\xi) = \left\{ 2\left(\frac{aV}{c}\right)^2 + 2aVt - \xi^2 - 2\frac{aV}{c}\left[\left(\frac{aV}{c}\right)^2 + 2aVt + x^2 + z^2 - 2x\xi\right]^{\frac{1}{2}} \right\}^{\frac{1}{2}}$$

The region of influence of disturbances due to contact of the sphere with the water surface is sketched in Fig. 8. The dashed line represents the front of the compression wave moving out into the fluid from the disturbed region.

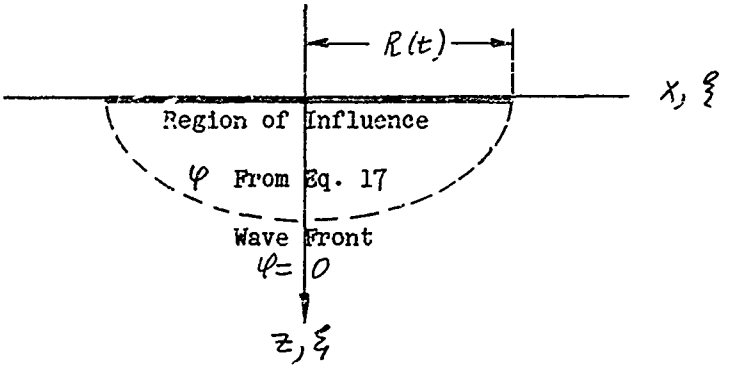

FIG. 8.

Due to the complexity of its limits, the double integral for the potential (Eq. 17) does not lend itself to an exact solution for an arbitrary field point. For x = 0, however, the potential and the pressure can be readily evaluated. In this case the area of integration degenerates to a circle about the origin. The result is given in terms of dimensionless space and time coordinates

$$(18) \quad p(0,0,z',t') = \frac{\rho c V}{\sqrt{1+z'^2+t'}}$$

Fortunately, the integral for the potential can be simplified for the evaluation of the pressure distribution on the wetted surface of the sphere. Since the integrand is a continuous function of ξ and η for every value of z, the potential in the plane z = 0 can be directly evaluated.

Now polar coordinates are introduced in the xy-plane (see Fig. 5) with the field point P(r,0,t) as origin

$$\rho = \sqrt{(x-\xi)^2+(y-\eta)^2} \quad , \quad \theta = \tan^{-1}\left(\frac{\eta-y}{\xi-x}\right)$$

Since $r_1 \longrightarrow \rho$ as $z \longrightarrow 0$, the retarded potential solution (Eq. 10b) becomes

$$(19) \quad \varphi(r,0,t) = -\frac{1}{2\pi}\int_0^{ct}\int_0^{2\pi} \varphi_z(\rho,\theta,t-\rho/c)\,d\theta\,d\rho$$

where $r^2 = x^2 + z^2$

Because of axial symmetry, the potential is independent of α (see Fig. 4). The domain of dependence of P(r,0,t) in the xy-plane is the area bounded by the circle of radius ct about P. However, as for the case of an arbitrary field point, only those pulses inside the area defined by $\rho_1(\theta)$ will contribute to the potential at P(r,0,t). (Figure 9 shows the area of integration for the point P(r,0) at time t.)

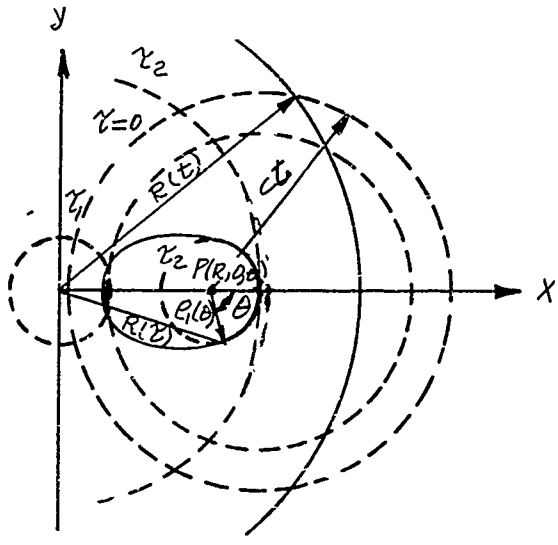

FIG. 9.

This can be expressed mathematically as follows

(20) $$\varphi_z(\rho, \theta, t - \rho/c) = V \quad , 0 \leq \rho \leq \rho_1(\theta)$$

$$\varphi_z(\rho, \theta, t - \rho/c) = 0 \quad , \quad \rho_1(\theta) < \rho < ct$$

Hence, Eq. 19 becomes

(21) $$\varphi(r, 0, t) = -\frac{V}{2\pi} \int_0^{2\pi} \rho_1(\theta)\, d\theta$$

The equation giving the boundary of the area of integration is (see Fig. 9)

$$R^2(\tau) = \rho_1^2(\theta)\sin^2\theta + (r + \rho_1(\theta)\cos\theta)^2$$
$$= \rho_1^2(\theta) + 2r\rho_1(\theta)\cos\theta + r^2$$

Since the square of the instantaneous radius of the disk (wetted surface) is

$$R^2(\tau) = 2aV\tau = 2aV(t - \rho_1/c)$$

In dimensionless coordinates is obtained

$$(22)\quad \rho_1'(\theta) = -(1 + r'\cos\theta) + \left[(1 + r'\cos\theta)^2 + (t' - r'^2)\right]^{\frac{1}{2}}$$

where

$$t' = t\Big/\frac{aV}{2c^2}\ , \quad r' = r\Big/\frac{aV}{c}$$

Since $r' \leq t' \leq 1$ for supersonic expansion of the disk, $\rho_1'(\theta) > 0$ for $0 \leq \theta \leq 2\pi$, i.e., the area of integration always encloses the point P(r,0,t) on the disk.

For the two limiting cases of a point first at the center of the disk ($r' = 0$), and second on the edge ($r' = t'$), the area of integration degenerates to a circle of radius $\rho_1' = -1 + \sqrt{1 + t'}$ and then vanishes, respectively.

The equation for the potential at a point r on the disk, given in terms of dimensionless quantities is hence

$$(23)\quad \varphi(r,0,t) = \frac{aV^2}{c}\left\{1 - \frac{1}{2\pi}\int_0^{2\pi}\left[(1 + r'\cos\theta)^2 + (t' - r'^2)\right]^{\frac{1}{2}} d\theta\right\}$$

To the order of approximation of this analysis, the pressure is given by

$$(24)\quad p = \rho\varphi_t = -\rho\frac{2c^2}{aV}\varphi_{t'}$$

Hence, the pressure corresponding to the potential of Eq. 23, and given in terms of the shock pressure, is

$$(25) \qquad \frac{p(r,0,t)}{\rho c V} = \frac{1}{\pi}\int_0^{\pi} \frac{d\theta}{\sqrt{(1+r'\cos\theta)^2 + (t'-r'^2)}}$$

This is a complete elliptic integral of the first kind, which can be reduced to the Legendre standard form (see Appendix D) yielding

$$(26) \qquad \frac{p(r,0,t)}{\rho c V} = A(r',t')\int_0^{\frac{\pi}{2}} \frac{d\varphi}{\sqrt{1-k^2(r',t')\sin^2\varphi}}$$

$$= A(r',t')\, K[\alpha(r',t')]$$

where

$$A(r',t') = \frac{2}{\pi[(1+t')^2-(2r')^2]^{1/4}}$$

$$k(r',t') = \left\{\frac{1}{2}\left[1 - \frac{1+t'-2r'^2}{\sqrt{(1+t')^2-(2r')^2}}\right]\right\}^{1/2} = \sin\alpha(r',t')$$

Values of the elliptic integral $K(\alpha)$ are tabulated by Jahnke and Emde (Ref. 7).

The time history of the pressure distribution on the sphere during impact is plotted in terms of the dimensionless radius of the wetted area $r' = \frac{r}{\frac{aV}{c}}$ (actually the projection of the wetted area in the plane of the undisturbed water surface), with the dimensionless time $t' = \frac{t}{\frac{aV}{2c^2}}$ as a parameter in Fig. 1.

It is to be recalled that this analysis is valid only for times during which the rate of expansion of the wetted area of the sphere exceeds the speed of wave propagation in water, i.e., for $t < \frac{aV}{2c^2}$ or $t' < 1$. For this case the free surface of the water beyond the area of contact is undisturbed. If the expansion rate were less than the speed of wave propagation, compression waves would propagate outward from the edge of the area of contact and disturb the free surface inside the wave front (mathematically this means that $\varphi_z \neq 0$ for $R(t) < r < \frac{aV}{2c} + ct$).

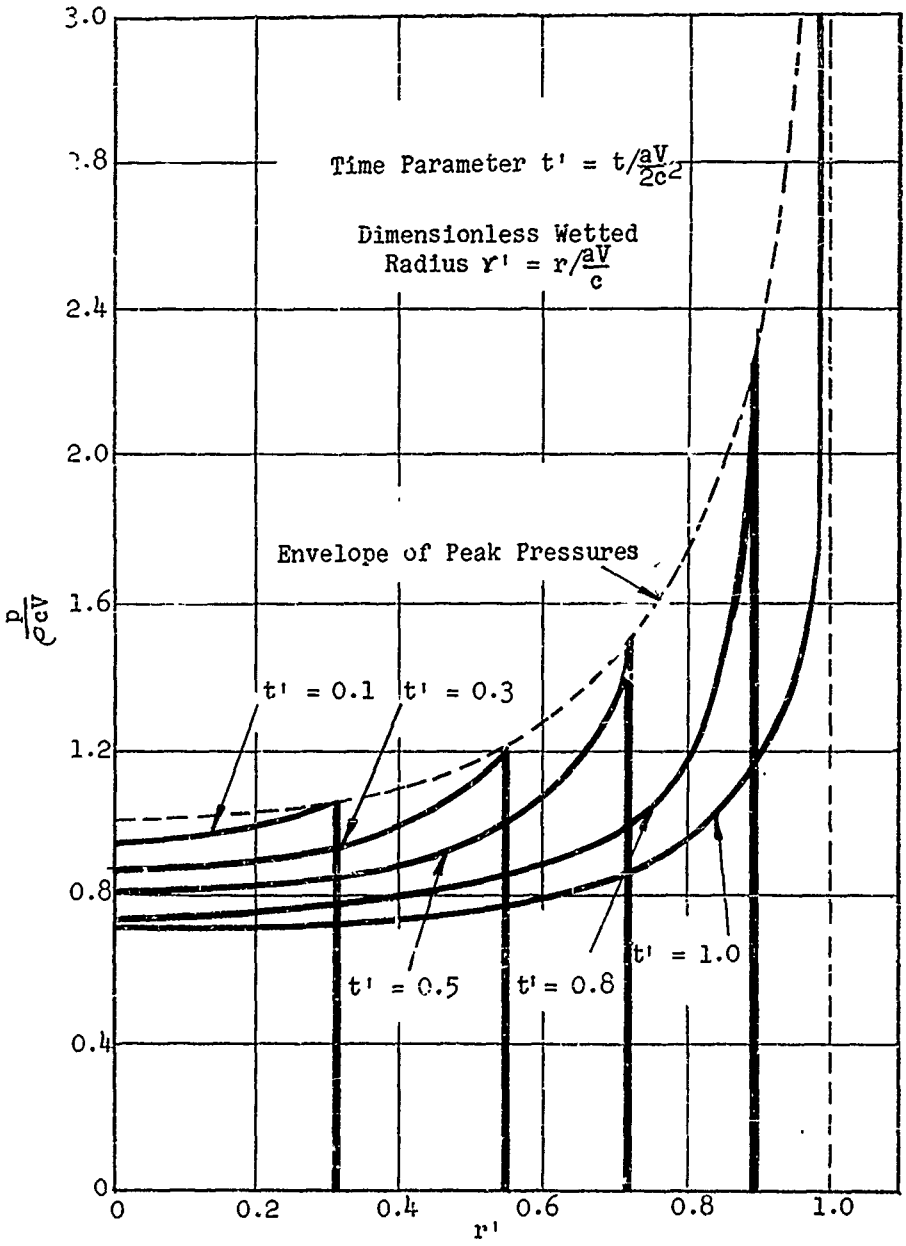

FIG. 10. Time History of the Pressure Distribution Along the Radius of the Wetted Surface of a Sphere During the First Few Instants of Impact With a Plane Water Surface.

From incompressible theory the pressure at the stagnation point of a sphere is given in terms of the dynamic pressure by (Ref. 1)

$$\frac{p_i}{\frac{1}{2}\rho V^2} = \sqrt{\frac{a}{Vt}}$$

As a function of the dimensionless time parameter $t' = t/\frac{aV}{2c^2}$, the incompressible pressure can be expressed in terms of the shock pressure as follows

$$(27) \qquad \frac{p_i}{\rho c V} = \frac{1}{\sqrt{2t'}}$$

From Eq. 25 the compressible stagnation pressure ($r' = 0$) is

$$(28) \qquad \frac{p_c}{\rho c V} = \frac{1}{\sqrt{1+t'}}$$

Equation 27 clearly shows the nature of the singularity of the incompressible pressure formula at the instant of impact ($t' = 0$) while Eq. 28 indicates that the compressible pressure formula has the finite value $\rho c V$.

Equations 28 and 29 are plotted in Fig. 11, with a predicted curve for the compressible pressure when the rate of expansion of the wetted surface of the sphere is subsonic. With increasing time or a decreasing rate of expansion, it is expected that the compressible and incompressible pressures approach each other. When the rate of expansion of the wetted surface becomes much smaller than the speed of wave propagation in water, the effects of compressibility are negligible, hence, incompressibility theory is quite adequate for pressure predictions.

EXPERIMENTAL WORK

The theoretical work at this Station has been concerned with the entry of a rigid body into water and the subsequent motion of the fluid, taking into account the compressibility of the water. No account of the fact that an actual object is not strictly rigid but deformable has been taken in any of the theoretical work. Yet the structural effects, such as deformation of the nose at impact or even failure at high entry velocities, is of extreme practical importance.

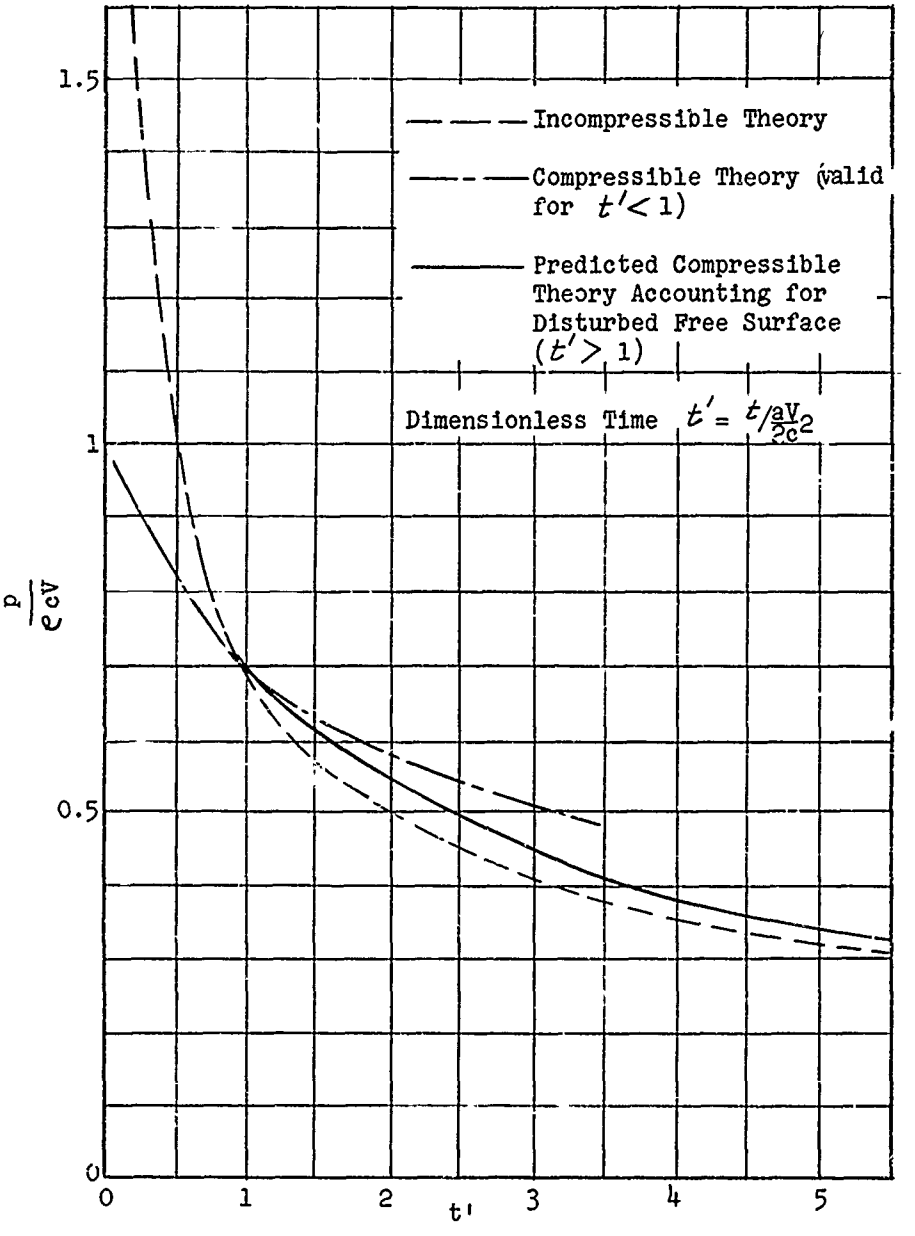

FIG. 11. Comparison of the Compressible and Incompressible Pressure; Time History at the Stagnation Point of a Sphere.

The difficulties which arise in any experimental efforts to isolate the compressibility effects are fairly well recognized. The fact that the missile acts as a compressible medium has been noted above. It is also true that the water surface is not a strictly smooth plane. In addition, there is evidence that air is entrapped between the body and the water. All these factors make it difficult to separate the true compressibility effect of the water and modify all experimental results. Most of the experimental work yields order-of-magnitude results which may act as guides to the actual design of missiles and future theoretical and experimental work.

The normal component of force at the point of impact for a blunt-nosed missile entering the water at an angle θ arises from the pressure $\rho cV \sin\theta$ where $V \sin\theta$ is the velocity of the missile normal to the water surface. In order to study the peak impact pressures when the nose of a projectile first comes in contact with the water, a set of pressure plugs were distributed over the nose of a hemispherically shaped missile (Ref. 8). The plugs are thin phosphor-bronze diaphragms supported peripherally on an accurately reamed shoulder. For entry velocities of 500 fps or more the plugs used were 0.02 inch thick and 0.250 inch in diameter. Application of a sufficiently large pressure gives the diaphragm a permanent set. The pressure plugs were calibrated statically. Figure 11 of Ref. 8 shows a plot of the pressure at the impact point as a function of velocity. The agreement with that calculated by the formula $\rho cV \sin\theta$ is very good, considering the fact that the static calibration of the plugs was used. It is true that the natural frequency of the membrane is theoretically high. For these membranes the frequency should be 25,000 cps. The effective frequency as an inelastic, deformable membrane was not measured, so the true dynamic response is not known but is estimated to be about 5,000 cps. For a hemispherical head with a radius a of 1 foot, entry velocity V of 500 fps, and c of 4,800 fps, the duration of the ρcV pressure is roughly twice the time at which the growth of the radius of the wetted area of the sphere is just sonic or $t = \frac{aV}{c^2}$ = 22 times 10^{-6} second. This is because of the reflection of the pressure wave from the point where the velocity of the contact point is just equal to c. The analogy for the pressure plug is a spring-mass system which can move only in one direction just as though the mass were restricted by a rachet. Hence the time for one-fourth cycle of the motion of the membrane needs to be considered. For 5,000 cps this time is t = 50 microseconds, as compared to a duration of ρcV of t = 22 microseconds. Therefore the plug should not be deformed as much as the static calibration would indicate. However, as an order of magnitude, the check is very good.

A laboratory setup for obtaining pressure-time measurements has been established at NOTS by C. R. Nisewanger (Ref. 9). Nisewanger's method consists of making direct pressure measurements on bodies by means of electromechanical transducers (gages) set flush into the surface of the bodies. The experimental body is a 12-inch-diameter hollow dural hemisphere mounted on a suitable carriage and guided during a fall of 11 feet into a tank of water by two vertical rails. The velocity at impact is about 24 fps.

For these values of diameter and velocity the ρcV pressure to be expected would be about 1,600 psi, and have a duration of about 1/2 microsecond. The resonant frequency of the gages used here is 10^5 cps. Thus even though the resonant frequency of the gages is much higher than that of the pressure plugs, the lower entry velocities give a ρcV pressure duration of only 1/2 microsecond. For a 1/2-inch-diameter gage the maximum pressure obtained was 120 psi, as compared to the theoretical value of 1,600 psi. However, this experimental value of pressure was registered nearly 13 microseconds after the initial impact, which means that the peak pressure might have been modified considerably by trapped air. It should be emphasized that the experimental work was done with great care and precision. The presence of trapped air seems to be indicated by this experimental procedure.

Appendix A

VON KÁRMÁN'S DERIVATION

A simple derivation for the impulsive piston pressure $p = \rho cV$ is given by Von Kármán (Ref. 10). When a flat plate strikes a water surface, the pressure will have an infinite value if the water is considered as incompressible, since a finite mass of water is given a certain amount of kinetic energy in zero time.

It is possible to obtain an approximate value for the maximum pressure taking compressibility into account in the following manner: The propagation of momentary increase of pressure in a fluid takes place at the speed of sound in the fluid, designated by c. Therefore the mass of fluid accelerated in the time dt is $\rho Scdt$, where S is the surface of the fluid struck by the body. Since the velocity of this mass is increased from zero to V in the time dt, the force acting is ρScV and the pressure is $p = \rho cV$,

Appendix B

OWENS' TREATMENT OF THE TWO-SPACE DIMENSION PROBLEM BY RIESZ'S METHOD

DEFINITIONS, CONVENTIONS, AND THEOREMS

The Riesz method is presented in detail in Ref. 6. The advantage of this method is that it shows clearly which boundary and initial conditions are needed to solve a problem in hyperbolic equations. As a preliminary, the definitions, conventions, and theorems necessary for the application of Riesz's method are presented.

Definition: Riemann-Liouville integral (R-L integral)

$$I^{\alpha} f \equiv \overset{x}{\underset{0}{I}}{}^{\alpha} f = \frac{1}{\Gamma(\alpha)} \int_0^x f(\xi)(x-\xi)^{\alpha-1} d\xi \equiv \frac{1}{\Gamma(\alpha)} \int_0^x f(x-\eta)\eta^{\alpha-1} d\eta$$

where α is a complex number. This operator associates to each function $f(x)$, defined for $x > o$, a new function $I^{\alpha}f(x)$ defined in the same domain.

Lemma: For fixed, bounded f with n continuous derivatives, where n is arbitrary, and for fixed x, $I^{\alpha}f$ is an analytic function of α.

Proof: $I^{\alpha}f$ as defined converges for $Re\,\alpha > 0$ and is analytic in α. So when $Re\,\alpha > 0$ integration by parts yields

$$\text{(B1)}\quad I^{\alpha} f = \frac{f(0)\,x^{\alpha}}{\Gamma(\alpha+1)} + \frac{1}{\Gamma(\alpha+1)} \int_0^x f'(\xi)(x-\xi)^{\alpha}\, d\xi$$

However, this expression equals the original expression in $Re\,\alpha > 0$ but it converges for $Re\,\alpha > -1$. The principle of analytic continuation allows the use of Eq. B1 as the <u>definition</u> of $I^{\alpha}f$ in $Re\,\alpha > -1$, for it is necessary to show only that the old <u>and</u> new expressions are analytic in α and that they agree in a common domain. In the same way $I^{\alpha}f$ can be defined for $Re\,\alpha > -n$ since repeated integration by parts will allow extension of the domain of analyticity (1 unit to the left each time) throughout the complex plane.

The use of analytic continuation is the basis of Riesz's method and hence the reason for including the previous proof.

<u>Properties of $I^{\alpha}f$:</u>

1. $I^{1} f = \int_0^x f(\xi)\, d\xi$

2. $I^{n} f = (n-1)! \int_{\xi_1=0}^{x} \int_{\xi_2=0}^{\xi_1} \int_{\xi_3=0}^{\xi_2} \cdots \int_{\xi_n=0}^{\xi_{n-1}} f(\xi_n)\, d\xi_n\, d\xi_{n-1} \cdots d\xi_1$

3. $I^{0} f = f(x)$

4. $I^{\alpha} f = f(0) \sum_{j=1}^{n} \frac{x^{\alpha+j-1}}{\Gamma(\alpha+j)} + I^{\alpha+n} f^{(n)}(x)$

$$5.\ I^{\frac{1}{2}} f = \frac{1}{\sqrt{\pi}} \int_0^x \frac{f(\xi)}{\sqrt{x-\xi}}\, d\xi$$

$$6.\ I^{-\frac{1}{2}} f = \frac{x^{-\frac{1}{2}} f(0)}{\sqrt{\pi}} + \frac{1}{\sqrt{\pi}} \int_0^x \frac{f'(\xi)\, d\xi}{\sqrt{x-\xi}} = \frac{d}{dx} \frac{1}{\sqrt{\pi}} \int_0^x \frac{f(\xi)}{\sqrt{x-\xi}}\, d\xi$$

$$7.\ I^{-n} f = f^{(n)}(x) = \frac{d^n f}{dx^n}$$

8. For special α, i.e., $\alpha = 0$ or $-n$, $I^{\alpha}f$ depends only on the values of $f(\xi)$ in an arbitrarily small neighborhood of $\xi = x$, but for general α, $I^{\alpha}f$ depends on the whole range of values of $f(\xi)$ for $0 < \xi < x$.

<u>Theorem</u>: If $Re\,\beta > 0$, $Re\,\alpha$ arbitrary, then $I^{\alpha} I^{\beta} = I^{\alpha+\beta}$

<u>Application</u>: Abel's integral equation is

$$f(x) = \int_0^x g(\xi)(x-\xi)^{-\beta}\, d\xi \quad , \quad 0 < \beta < 1$$

The problem is to find g(x) if f(x) is known. The equation can be written

$$f(x) = \Gamma(1-\beta) \frac{1}{\Gamma(1-\beta)} \int_0^x g(\xi)(x-\xi)^{(1-\beta)-1}\, d\xi = \Gamma(1-\beta) I^{1-\beta} g$$

Since $1-\beta>0$ one gets

$$g(x) = \frac{1}{\Gamma(1-\beta)} I^{-(1-\beta)} f(x) = \frac{1}{\Gamma(1-\beta)} I^{-1} I^{\beta} f$$

on using the last theorem again since $\beta>0$. From property 7

$$g(x) = \frac{1}{\Gamma(1-\beta)} \frac{d}{dx} I^{\beta} f = \frac{1}{\Gamma(1-\beta)} \frac{d}{dx} \frac{1}{\Gamma(\beta)} \int_0^x f(\xi)(x-\xi)^{\beta-1} d\xi$$

Since

$$\Gamma(1-\beta)\Gamma(\beta) = \frac{\pi}{\sin \pi\beta}$$

one obtains

$$g(x) = \frac{\sin \pi\beta}{\pi} \frac{d}{dx} \int_0^x f(\xi)(x-\xi)^{\beta-1} d\xi$$

which is the required solution.

<u>Definition</u>: The Riesz integral in m independent variables.

(B2) $$I_m^{\alpha}(f, D, P) = \frac{1}{H_m(\alpha)} \int_D f(Q) R^{\alpha-m} dV_m \quad , \quad \alpha \text{ complex}$$

where

$f = f(x^1, x^2, \ldots, x^m)$, i.e., m independent variables, the superscripts being distinguishing marks, not exponents.

D = the m-dimensional volume bounded by the m-1-dimensional hypercone with apex at $P(x^1, x^2, \ldots, x^m)$ whose equation is $R = 0$ and $x^1 \geq \xi^1 > 0$.

R = the hyperbolic distance of the arbitrary point $Q(\xi^1, \xi^2, \ldots, \xi^m)$ from P, given by $R^2 = (\xi^1 - x^1)^2 - (\xi^2 - x^2)^2 - \ldots - (\xi^m - x^m)^2$

$dV_m = d\xi^1 d\xi^2 \ldots d\xi^m$, an m-dimensional volume element

$H_m(\alpha)$ = a constant defined by

$$\text{(B3)} \quad H_m(\alpha) = 2^{\alpha-1} \pi^{\frac{m-2}{2}} \Gamma\left(\frac{\alpha}{2}\right) \Gamma\left(\frac{\alpha+2-m}{2}\right)$$

Remarks: $I_m f$ depends on α, f, P (corresponding to the upper limit of the R-L integral) and on a domain D (corresponding to interval 0, x of the R-L integral). The volume D is bounded by the hypercone and the hyperplane $\xi^1 = 0$. $I_m^\alpha f$ reduces to the R-L integral for m = 1. Lastly, the Riesz integral is m-fold, but by introducing mean values for f it can be reduced to a double integral.

Theorem:

$$\text{(B4)} \quad I_m^0 f(f, D, P) \equiv f(P) = f(x^1, x^2, \ldots, x^m)$$

In particular this is true even if D is a small part of a cone including the apex P. This is analogous to the R-L integral in which $I_m^0 f$ depends on values of $f(\xi)$ only, in a small neighborhood of $\xi = x$.

Theorem:

$$\text{(B5)} \quad I_m^2 f = \frac{\sqrt{\pi}}{\Gamma\left(\frac{m-1}{2}\right)} I_0^1 \left[\rho^{m-2} (1+\rho)^{\frac{2-m}{2}} \int_0^{x^1} m(\rho, \lambda) \lambda \, d\lambda \right]$$

where $m(\rho,\lambda)$ is a mean value of f. The essential point is that this formula represents $I_m^2 f$ in terms of a simple R-L integral over the interval (0,1).

Theorem:

$$\text{(B6)} \int_D (\varphi \Box_m \psi - \psi \Box_m \varphi)\, dV_m = \int_B \left(\varphi \frac{\partial \psi}{\partial \nu} - \psi \frac{\partial \varphi}{\partial \nu}\right) dV_{m-1}$$

where D is an m-dimensional volume and B is an (m - 1)-dimensional volume, i.e., the "surface" enclosing D. Differentiation in the co-normal direction is represented by $\frac{\partial}{\partial \nu}$, the normal being the outward-drawn normal to the "surface." The m-dimensional wave operator defined below is represented by $\Box_m$. This is Green's theorem in hyperbolic geometry.

If the components of the normal are given by $\bar{n} = (n_1, n_2, \ldots, n_m)$ the co-normal components are defined by $\text{con}\bar{n} = (n_1, -n_2, \ldots, -n_m)$. This follows from the hyperbolic metric associated with the hyperbolic differential equation. On the plane $\xi^1 = 0$, the normal has components $\bar{n} = (-\xi^1, 0, \ldots, 0)$ hence $\text{con}\bar{n} = (-\xi^1, 0, \ldots, 0)$ so that $\frac{\partial}{\partial \nu} = -\frac{\partial}{\partial \xi^1}$. On the plane $\xi^2 = 0$, $\bar{n} = (0, -\xi^2, 0, \ldots, 0)$ provided $x^2 > 0$ (i.e., provided the apex of the cone is on the positive side of the plane $\xi^2 = 0$) so that the normal extends in the negative direction (see sketch in next section). Hence $\text{con}\bar{n} = (0, \xi^2, 0, \ldots, 0)$ and $\frac{\partial}{\partial \nu} = \frac{\partial}{\partial \xi^2}$. The expression for $\frac{\partial}{\partial \nu}$ on the "curved surface" of the cone will not be needed.

Recalling that in m-dimensions $R^2 = (\xi^1 - x^1)^2 - \sum_{j=2}^{m} (\xi^j - x^j)^2$, one can compute $\Box_m R^{\alpha+2-m} \equiv \frac{\partial^2}{\partial x'^2} R^{\alpha+2-m} - \sum_{k=2}^{m} \frac{\partial^2}{\partial x^{k2}} R^{\alpha+2-m}$ $= \alpha(\alpha-m+2) R^{\alpha-m+2}$ from which it follows that for $\alpha = 0$, $\Box_m R^{2-m} = 0$ and hence R^{2-m} satisfies this m-dimensional wave equation (note that $\Box_m$ is defined above). An attempt to build up a solution from this so-called fundamental solution fails when $m > 3$ because any integral involving R^{2-m} diverges. However an integral of $R^{\alpha+2-m}$ where $\alpha > m-4$ will converge, and its limit as $\alpha \to 0$ may be considered as the analytic continuation of the integral of $R^{\alpha+2-m}$ into the domain $0 \leq \alpha \leq m-4$, that is, it defines the integral of R^{2-m}. This analytic continuation is obtained by integration by parts as done in the continuation of the simple R-L integral. However, this involves a multiple integral, and the tool available is Green's theorem, the use of which corresponds to integration by parts.

TWO-SPACE DIMENSIONAL CASE

Riesz's method for m = 3 will be applied as illustrated in Fig. 12 to give physical meaning to the process. The case m = 4 will be treated in the language of m = 3.

Characteristic cone, vertex at P(ct, y, x):

$$c^2(\lambda - t)^2 - (\eta - y)^2 - (\xi - x)^2 = 0$$

"Reflected" cone, vertex at $\bar{P}$(ct, - y, x):

$$c^2(\lambda - t)^2 - (\eta + y)^2 - (\xi - x)^2 = 0$$

"Hyperbolic" distances from P, $\bar{P}$ to $Q(c\lambda, \eta, \xi)$ (Q not on cone) are respectively:

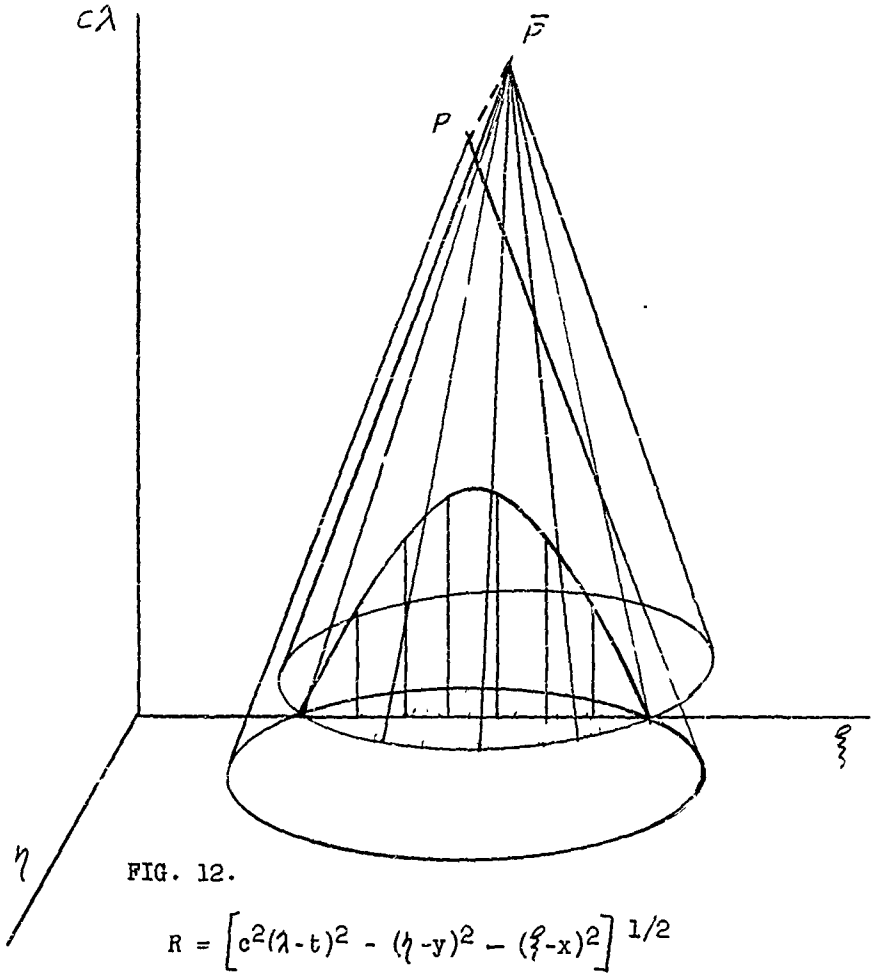

FIG. 12.

$$R = \left[c^2(\lambda - t)^2 - (\eta - y)^2 - (\xi - x)^2\right]^{1/2}$$

$$\bar{R} = \left[c^2(\lambda - t)^2 - (\eta + y)^2 - (\xi - x)^2\right]^{1/2}$$

Let D_1, $\bar{D}_1$, be the respective interiors of the two cones for which $\lambda, \eta > 0$. Let s, $\bar{s}$ be the surfaces $\lambda = 0$ and interior of the cones; $s_2 = \bar{s}_2$ the surface $\eta = 0$ and interior of the cones; s_1, $\bar{s}_1$ the remaining "curved" surfaces of the cones. Let $B = s + s_1 + s_2$, $\bar{B} = \bar{s} + \bar{s}_1 + \bar{s}_2$ be the total surfaces of the two figures.

The two-space dimensional problem is

(B7) $\Box_3 \varphi = \varphi_{tt} - c^2(\varphi_{xx} + \varphi_{yy}) = 0$

(B8) $\varphi(0,y,x) = \varphi_t(0,y,x) = 0, \quad x^2+y^2 \neq 0$ (initial conditions)

(B9) $\varphi_y(t,0,x) = V, \quad |x| < x(t) \; ; \; \varphi(t,0,x), \quad |x| > x(t)$

(boundary conditions)

Now Green's theorem is applied where φ satisfies $\Box_3 \varphi = 0$ and $\psi = R^{\alpha-1}$, $\Box_3 \psi = \alpha(\alpha-1) R^{\alpha-3}$

giving

(B10) $$\alpha(\alpha-1)\int_{D_1} \varphi R^{\alpha-3} dV = \int_B \left(\varphi \frac{\partial}{\partial \nu} R^{\alpha-1} - R^{\alpha-1} \frac{\partial}{\partial \nu} \varphi\right) ds$$

However

$$I_3^{\alpha} \varphi = \frac{1}{H_3(\alpha)} \int_{D_1} \varphi R^{\alpha-3} dV$$

from Eq. B2 and

$$H_3(\alpha+2) = \alpha(\alpha-1) H_3(\alpha) = 2\pi \Gamma(\alpha+1)$$

from Eq. B3 and from the properties of the Γ-function (duplication formula). Thus Eq. B10 becomes

(B11) $$2\pi\Gamma(\alpha+1) I_3^{\alpha} \varphi = \int_B \left(\varphi \frac{\partial}{\partial \nu} R^{\alpha-1} - R^{\alpha-1} \frac{\partial \varphi}{\partial \nu}\right) ds$$

Now $I_3^0 \varphi = \varphi(t,y,x)$ from Eq. B4 so that Eq. B11 becomes

$$\varphi(t,y,x) = \lim_{\alpha \to 0} \frac{1}{2\pi\Gamma(\alpha+1)} \int_B \left(\varphi \frac{\partial R^{\alpha-1}}{\partial \nu} - R^{\alpha-1} \frac{\partial \varphi}{\partial \nu}\right) ds$$

Thus, the solution $\varphi(t, y, x)$ depends only on the values of φ on the boundaries of the cone D_i. Since $R = 0$ on S_1, and with $\alpha > 2$, the part of $\int_S$ on S_1 vanishes, which is the reason for not expressing $\frac{\partial}{\partial \nu}$ on S_1. On S, $\frac{\partial}{\partial \nu} = -\frac{\partial}{\partial c\lambda}$ and on S_2, $\frac{\partial}{\partial \nu} = \frac{\partial}{\partial \eta}$. Then Eq. B12 becomes

$$\text{(B13)} \quad \varphi(t, y, x) = \lim_{\alpha \to 0} \left\{ \frac{1}{2\pi\Gamma(\alpha+1)} \iint_S \left[-\varphi(0, \eta, \xi) \frac{\partial R^{\alpha-1}}{\partial c\lambda} + R^{\alpha-1} \frac{\partial}{\partial c\lambda} \varphi(0, \eta, \xi) \right] d\xi\, d\eta \right.$$
$$\left. + \frac{1}{2\pi\Gamma(\alpha+1)} \iint_{S_2} \left[\varphi(\lambda, 0, \xi) \frac{\partial R^{\alpha-1}}{\partial \eta} - R^{\alpha-1} \frac{\partial \varphi(\lambda, 0, \xi)}{\partial \eta} \right] dc\lambda\, d\xi \right\}$$

The reflected cone will now be used but the solution has been written in the form of Eq. B13 to call attention to the fact that if the problem had no boundary conditions, and hence the cone were not cut up by the surface $\eta = 0$, the integral over S_2 would not occur and the solution φ would be given completely by the initial conditions which appear in $\iint_S$. In this case the reflected cone would not be needed. However, the occurrence of mixed boundary

conditions renders the reflected cone a useful device enabling one to represent the solution in terms of one boundary value.

Since the initial conditions of Eq. B8 are zero, we have from Eq. B11

$$\text{(B14)}\quad 2\pi\Gamma(\alpha+1)I_3^{\alpha}\varphi=\iint_{S_2}\left[\varphi(\lambda,0,\xi)\frac{\partial R^{\alpha-1}}{\partial\eta}-R^{\alpha-1}\frac{\partial\varphi(\lambda,0,\xi)}{\partial\eta}\right]d\lambda\, d\xi$$

Using the reflected cone, one gets the analogous relationship

$$\text{(B15)}\quad 2\pi\Gamma(\alpha+1)I_3^{\alpha}\bar{\varphi}=\iint_{\bar{S}_2}\left[\varphi(\lambda,0,\xi)\frac{\partial \bar{R}^{\alpha-1}}{\partial\eta}-\bar{R}^{\alpha-1}\frac{\partial\varphi}{\partial\eta}(\lambda,0,\xi)\right]d\lambda\, d\xi$$

where

$$I_3^{\alpha}\bar{\varphi}\equiv\frac{1}{H_3(\alpha)}\int_{\bar{D}_1}\varphi\,\bar{R}^{\alpha-3}\,dV$$

From the definitions of S_2, $\bar{S}_2$, R, and $\bar{R}$ it can be observed that on $S_2 = \bar{S}_2$ where $\eta = 0$, $R = \bar{R}$ and $R_\eta = -\bar{R}_\eta$. Making these substitutions in Eq. B15, and adding Eq. B14 and Eq. B15 gives

$$2\pi\Gamma(\alpha+1)\left(I_3^{\alpha}\varphi+I_3^{\alpha}\bar{\varphi}\right)=-\iint_{S_2}2R^{\alpha-1}\frac{\partial}{\partial\eta}\varphi(\lambda,0,\xi)\,d\lambda\, d\xi$$

<u>Lemma</u>: $I_m^{0}\bar{\varphi}=0$ that is

$$\lim_{\alpha\to 0} I_m^{0}\bar{\varphi}=\lim_{\alpha\to 0}\frac{1}{H_m(\alpha)}\int_{\bar{D}}\varphi\,\bar{R}^{\alpha-m}\,dV_m=0$$

Using this lemma, Eq. B17 becomes

$$\varphi(t,y,x)=\lim_{\alpha\to 0}\frac{-1}{\pi\Gamma(\alpha+1)}\iint_{S_2}R^{\alpha-1}\varphi_y(\lambda,0,\xi)\,d\lambda\, d\xi$$

which is the solution to the problem. For m = 3 the integral converges so that by carrying out this limiting process directly yields

$$\varphi(t,y,x) = -\frac{1}{\pi}\iint_{S_2} \frac{\varphi_y(\lambda,0,\xi)\,d\lambda\,d\xi}{\sqrt{c^2(\lambda-t)^2-y^2-(\xi-x)^2}}$$

Evaluation of this integral is usually very difficult and is not considered here. It should be mentioned that in this evaluation, with these mixed boundary conditions, the method developed by J. C. Evvard (Ref. 5) may be used when the point P(ct, 0, x) is in the "subsonic" region and only when y = 0.

THREE-SPACE DIMENSIONAL CASE

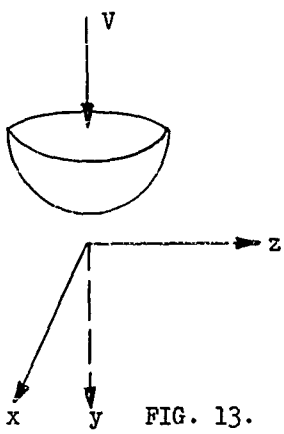

FIG. 13.

Notation:

V = entry velocity of body

c = velocity of sound in water

ρ = density of water

φ = velocity potential

p = hydrodynamic pressure

Assumptions:

1. That entry velocity remains constant and $\frac{V}{c} \ll 1$.
2. That c and ρ remain constant.

3. That flow is isentropic, hence inviscid.

4. That effects of splash and of gravitation waves are ignored, and the water surface remains plane.

5. That the body is very blunt at the point of impact: hence the boundary conditions are applied on the water surface and not on the surface of the body.

6. That perturbation velocities of the fluid are small so that squares of these velocities are neglected.

The problem to be investigated is that of the flow pattern and the hydrodynamic pressure induced by the body entering the water. The investigation will treat only the first few moments of contact and therefore only a small penetration of the body into the water. Assumptions 4 and 5 are made for this purpose. In particular, the pressure history, for the short interval considered, is desired.

Since the water is compressible, this treatment differs from the classical one which assumes an incompressible fluid.

The flow is initially irrotational and remains irrotational, so that we may assume a velocity potential φ from which perturbation velocities are given by $u = \varphi_x$, $v = \varphi_y$, $W = \varphi_z$.

From the equations of hydrodynamics which are linearized, by neglecting squares of velocities and using the assumptions, the following problem is obtained whose solution answers to the assumptions and description above.

$$\varphi_{tt} - c^2 (\varphi_{xx} + \varphi_{yy} + \varphi_{zz}) \equiv \Box_4 \varphi = 0$$

$$p = -\varphi_t \rho$$

$$\left.\begin{aligned} \varphi(x,y,z,0) &= 0 \\ \varphi_t(x,y,z,0) &= 0 \end{aligned}\right\} x^2 + y^2 + z^2 \neq 0 \text{ (initial conditions)}$$

$$\left.\begin{aligned} \varphi_y(x,0,z,t) &= V, \quad x^2 + z^2 < X(t)^2 \\ \varphi(x,0,z,t) &= 0, \quad x^2 + z^2 > X(t)^2 \end{aligned}\right\} \text{(boundary conditions)}$$

The expression $X(t)$ describes the radius of the ring of contact between the body and the plane water surface and may be determined from the geometry of the body. Hence $x^2 + z^2 < X(t)^2$ corresponds to points on the body in the disturbed fluid surface and $x^2 + z^2 > X(t)^2$ to points in the undisturbed surface. (Since boundary conditions are applied on the fluid surface rather than on the body surface, the problem is that of an expanding disk on the surface, sending downward pulses corresponding to velocity V.)

The following definitions are needed:

Characteristic cone, vertex at $P(x,y,z,ct)$:

$$c^2(\lambda - t)^2 - (\xi - x)^2 - (\eta - y)^2 - (\zeta - z)^2 = 0$$

"Reflected" cone, vertex at $P(x,-y,z,ct)$:

$$c^2(\lambda - t)^2 - (\xi - x)^2 - (\eta + y)^2 - (\zeta - z)^2 = 0$$

"Hyperbolic" distances from P, $\bar{P}$ to $Q(\xi, \eta, \zeta, c\lambda)$ are respectively

$$R = \left\{ c^2(\lambda - t)^2 - (\xi - x)^2 - (\eta - y)^2 - (\zeta - z)^2 \right\}^{\frac{1}{2}}$$

$$\bar{R} = \left\{ c^2(\lambda - t)^2 - (\xi - x)^2 - (\eta + y)^2 - (\zeta - z)^2 \right\}^{\frac{1}{2}}$$

The regions D_1, $\bar{D}_1$, S, $\bar{S}$, $S_2 = \bar{S}_2$, S_1, $\bar{S}_1$, $B = S + S_1 + S_2$ and $\bar{B} = \bar{S} + \bar{S}_1 + \bar{S}_2$ are defined as in the two-dimensional case and are described later (Fig. 14.).

Applying Green's theorem again where

$$\Box_4 \varphi = 0$$

and

$$\psi = R^{\alpha-2}, \quad \Box_4 \psi = \alpha(\alpha-2) R^{\alpha-4}$$

one obtains

$$\text{(B20)} \quad \alpha(\alpha-2)\int_{D_1} \varphi R^{\alpha-4} dV_4 = \int_B \left(\varphi \frac{\partial R^{\alpha-2}}{\partial \nu} - R^{\alpha-2}\frac{\partial \varphi}{\partial \nu}\right) dV_3$$

but

$$I_4^{\alpha}\varphi = \frac{1}{H_4(\alpha)}\int_D \varphi R^{\alpha-4} dV_4$$

from Eq. B2 and

$$H_4(\alpha+2) = \alpha(\alpha-2) H_4(\alpha) = 2^{\alpha+1}\pi\,\Gamma\left(\tfrac{\alpha}{2}+1\right)\Gamma\left(\tfrac{\alpha}{2}\right)$$

from Eq. B3 and from the properties of the Γ-function. Thus Eq. B20 becomes

$$\text{(B21)} \quad 2^{\alpha+1}\pi\,\Gamma\left(\tfrac{\alpha}{2}+1\right)\Gamma\left(\tfrac{\alpha}{2}\right) I_4^{\alpha}\varphi = \int_B \left(\varphi \frac{\partial R^{\alpha-2}}{\partial \nu} - R^{\alpha-2}\frac{\partial \varphi}{\partial \nu}\right) dV_3$$

Now take $\alpha > 3$. Since $R = 0$ on S_1 so is $\frac{\partial R^{\alpha-2}}{\partial \nu}$ and hence $\int_{S_1} = 0$, On S, $\frac{\partial}{\partial \nu} = -\frac{\partial}{\partial \zeta}$ and on S_2, $\frac{\partial}{\partial \nu} = \frac{\partial}{\partial \eta}$. Hence Eq. B2 becomes

$$\text{(B22)}\quad I_4^{\alpha}\varphi = \frac{1}{2^{\alpha+1}\pi\Gamma(\frac{\alpha}{2}+1)\Gamma(\frac{\alpha}{2})}\left\{\int_S\left[-\varphi(\xi,\eta,\zeta,0)\frac{\partial R}{\partial C\lambda}^{\alpha-2} + R^{\alpha-2}\frac{\partial\varphi(\xi,\eta,\zeta,0)}{\partial C\lambda}\right]d\xi\, d\eta\, d\zeta\right.$$
$$\left.+\int_{S_2}\left[\varphi(\xi,0,\zeta,\lambda)\frac{\partial R}{\partial\eta}^{\alpha-2} - R^{\alpha-2}\frac{\partial\varphi}{\partial\eta}(\xi,0,\zeta,\lambda)\right]d\xi\, d\zeta\, dC\lambda\right\}$$

Since the initial conditions are zero, the first integral vanishes and Eq. B22 becomes

$$\text{(B23)}\quad I_4^{\alpha}\varphi = \frac{1}{2^{\alpha+1}\pi\Gamma(\frac{\alpha}{2}+1)\Gamma(\frac{\alpha}{2})}\int_{S_2}\left[\varphi(\xi,0,\zeta,\lambda)\frac{\partial R}{\partial\eta}^{\alpha-2} - R^{\alpha-2}\frac{\partial\varphi(\xi,0\zeta,\lambda)}{\partial\eta}\right]d\xi\, d\zeta\, dC\lambda$$

This process applied to the "reflected" cone gives

$$\text{(B24)}\quad I_4^{\alpha}\bar{\varphi} = \frac{1}{2^{\alpha+1}\pi\Gamma(\frac{\alpha}{2}+1)\Gamma(\frac{\alpha}{2})}\int_{\bar{S}_2}\left[\varphi(\xi,0,\zeta,\lambda\frac{\partial\bar{R}}{\partial\eta}^{\alpha-2} - \bar{R}^{\alpha-2}\frac{\partial\varphi(\xi,0,\zeta,\lambda)}{\partial\eta}\right]d\xi\, d\zeta\, dC\lambda$$

From the definitions of S_2, $\bar{S}_2$, R, $\bar{R}$ on $S_2 = \bar{S}_2$ where $\eta = 0$, $R = \bar{R}$ and $\frac{\partial R}{\partial\eta} = -\frac{\partial\bar{R}}{\partial\eta}$. Making these substitutions in Eq. B24, and adding Eq. B23 and Eq. B24 gives

$$\text{(B25)}\quad I_4^{\alpha}\varphi + I_4^{\alpha}\bar{\varphi} = \frac{-2}{2^{\alpha+1}\pi\Gamma(\frac{\alpha}{2}+1)\Gamma(\frac{\alpha}{2})}\cdot\int_{S_2}R^{\alpha-2}\frac{\partial\varphi(\xi,0,\zeta,\lambda)}{\partial\eta}d\xi\, d\zeta\, dC\lambda$$

Using the lemma that $I_m^0\bar{\varphi} = 0$ and Eq. B4, Eq. B25 becomes

(B26)
$$\varphi(x,y,z,t) = \lim_{\alpha \to 0} \frac{-1}{2^{\alpha}\pi\Gamma(\frac{\alpha}{2}+1)\Gamma(\frac{\alpha}{2})} \int_{S_2} R^{\alpha-2} \frac{\partial\varphi(\xi,0,\zeta,\lambda)}{\partial\eta} d\xi\, d\zeta\, d\lambda$$

where the "surface" of integration, S_2, is the volume bounded by the surface $c^2(\lambda-t)^2 - (\xi-x)^2 - (\zeta-z)^2 = y^2$.

Again it can be seen that the solution is completely expressed in terms of the boundary conditions (and initial conditions if different from zero). In this case the integral in Eq. B26 diverges for $\alpha = 0$. However, the factor preceding the integral was constructed in such a manner that the existence of the limit is insured.

Evaluation of this expression (Eq. B26) for φ is quite involved. The special point P (0,0,0,ct) in the supersonic region can be treated quite simply by putting $R = \left[c^2(t-\lambda)^2 - \xi^2 - \zeta^2\right]^{\frac{1}{2}}$, introducing polar coordinates in the (ξ, ζ) plane, using the symmetry of $\frac{\partial\varphi}{\partial\eta}$, namely $\frac{\partial\varphi}{\partial\eta}(\rho\cos\theta, 0, \rho\sin\theta, \lambda) \equiv \psi(\rho,\lambda)$, integrating with respect to θ, then with respect to ρ and finally letting $\alpha \to 0$. However, in an attempt to be more general, mean values will be used to express $\varphi(x,y,z,t)$ for the general point P(x,y,z,ct). For this purpose put $H_4(\alpha+2) \equiv 2^{\alpha+1}\pi\Gamma(\frac{\alpha}{2}+1)\Gamma(\frac{\alpha}{2})$ for the factor before the integral in Eq. B26 and consider

(B27) $$\varphi(x,y,z,t) = \lim_{\alpha \to 0} \frac{-2}{H_4(\alpha+2)} \int_{S_2} R^{\alpha-2} \frac{\partial\varphi(\xi,0,\zeta,\lambda)}{\partial\eta} dV_3$$

Since S_2 is the "surface" $\eta = 0$, introduction of the δ -function makes Eq. B27 take the form

$$\text{(B28)} \quad \varphi(x,y,z,t) = \lim_{\alpha \to 0} \frac{1}{H_4(\alpha+2)} \int_{D_1} \left[-2 \frac{\partial \varphi(\xi, 0, \zeta, \lambda)}{\partial \eta} \delta(\eta) \right] R^{\alpha-2} dV_4$$

and by definition from Eq. 2, this can be written

$$\text{(B29)} \quad \varphi(x,y,z,t) = \lim_{\alpha \to 0} I_4^{\alpha+2} \left[-2 \frac{\partial \varphi(\xi, 0, \zeta, \lambda)}{\partial y} \delta(\eta) \right]$$

$$= I_4^2 \left[-2 \frac{\partial \varphi(\xi, 0, \zeta, \lambda)}{\partial y} \delta(\eta) \right]$$

Now Eq. B5 provides a formula for $I_4^2 f$ with which one can discuss the mean value, $m(\rho, \bar{\lambda})$, of the function f mentioned below.

Referring to Fig. 14, one can make the following definitions:

$$r^2 = \sum_{j=2}^{m} (\xi^j - x^j)^2 = \text{Euclidean distance}$$

from $Q(\xi^1, \xi^2, \ldots, \xi^m)$ to the "center line" of the cone D in the plane $\xi^1 =$ constant.

(B30)

$$\bar{\lambda} = x^1 - \xi^1$$

$$\rho = \frac{r}{\bar{\lambda}}$$

$$R^2 = (x^1 - \xi^1)^2 - r^2 = \bar{\lambda}^2 - \rho^2 \bar{\lambda}^2 = \bar{\lambda}^2 (1 - \rho^2)$$

For this problem M = 4 and

$$r^2 = (\xi - x)^2 + (\eta - y)^2 + (\zeta - z)^2$$

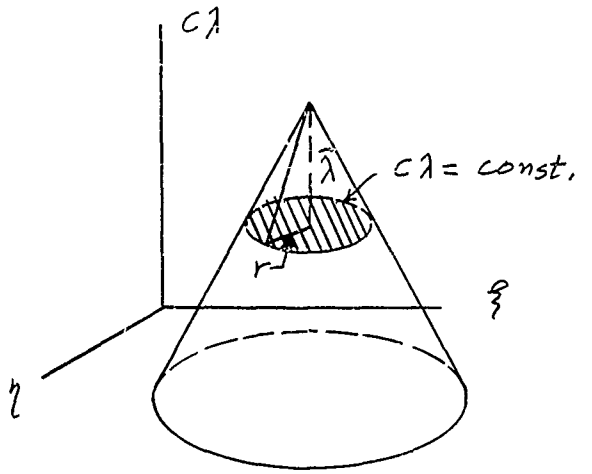

FIG. 14.

(B31) $\bar{\lambda} = ct - c\lambda$

and Eq. B2 becomes

(B32) $$I_4^2 f = \frac{\sqrt{\pi}}{\Gamma(\frac{3}{2})} \underset{0}{\overset{1}{I}}{}_1^0 \left\{ \rho^2 (1+\rho)^{-1} \int_0^{ct} m(\rho, \bar{\lambda})\, \bar{\lambda}\, d\bar{\lambda} \right\}$$

However, $\underset{0}{\overset{1}{I}}{}_1^0$ gives the function inside the braces of Eq. B32 with the argument replaced by one, by virtue of property 3 of R-L integrals. Putting $\rho = 1$ one gets

$$I_4^2 f = \frac{\sqrt{\pi}}{\Gamma(\frac{3}{2})} \left[(1)^2 (2)^{-1} \int_0^{ct} m(1, \bar{\lambda})\, \bar{\lambda}\, d\bar{\lambda} \right.$$

which gives

(B33) $$I_4^2 f = \int_0^{ct} m(1, \bar{\lambda})\, \bar{\lambda}\, d\bar{\lambda}$$

If the cone is generalized to m dimensions, it can be observed that the locus of points r = constant, $\bar{\lambda}$ = constant represents the (m-2)-dimensional surface of an (m-1)-dimensional hypersphere contained in the (m-1)-dimensional hyperplane ξ^1 = constant whose center is at the point $(\xi^1, x^2, \ldots, x^m)$, with $\xi^1 = x^1 - \bar{\lambda}$.

Definition: The mean value of f is formed by integrating f over the surface of this sphere and dividing by the area of the surface.

By induction it can be shown that

$$V_{m-1} = \frac{r^{m-1}}{m-1} \frac{2\pi^{\frac{m-1}{2}}}{\Gamma\left(\frac{m-1}{2}\right)}$$

represents the volume of an (m-1)-dimensional sphere. This formula holds for m = 3 (area of a circle) and m = 4 (volume of an ordinary sphere.)

Now assume that the formula is valid for an (m-1)-dimensional sphere and prove that it holds for an m-dimensional sphere. By induction it will then hold for all dimensions. By the volume of an m-dimensional sphere is meant the "area" of any cross section multiplied by the elementary thickness and integrated throughout this sphere (see Fig. 15). The area of this cross section is the volume of the (m-1)-dimensional sphere. Thus

$$V_m = 2\int_0^{\frac{\pi}{2}} \frac{(r\cos\theta)^{m-1}}{m-1} \frac{2\pi^{\frac{m-1}{2}}}{\Gamma\left(\frac{m-1}{2}\right)} (r\cos\theta)\,d\theta$$

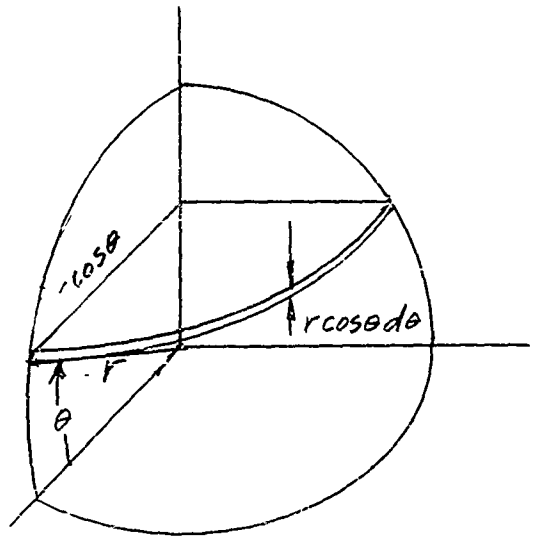

FIG. 15.

Using the relation $\int_0^{\frac{\pi}{2}} \cos^m \theta \, d\theta = \frac{\sqrt{\pi}}{2} \frac{\Gamma\left(\frac{m+1}{2}\right)}{\Gamma\left(\frac{m+2}{2}\right)}$ this becomes

(B34) $$V_m = \frac{r^m}{m} \frac{2\pi^{\frac{m}{2}}}{\Gamma\left(\frac{m}{2}\right)}$$

which has the same form as the formula for an (m-1)-dimensional sphere and the assertion is proven.

By considering the area as the radial derivative of the volume (and putting m-1 for m in Eq. B34 one gets for the area of the (m-2)-dimensional surface of an (m-1)-dimensional sphere

(B35) $$S_{m-2} = r^{m-2} \frac{2\pi^{\frac{m-1}{2}}}{\Gamma\left(\frac{m-1}{2}\right)}$$

and by the above definition

$$\text{(B36)} \quad m(\rho, \bar{\lambda}) \equiv \frac{1}{S_{m-2}} \int_{r, \bar{\lambda} \text{ fixed}} f(x^1 - \bar{\lambda}, \xi^2, \ldots, \xi^m)\, dw$$

where dw is the surface element of the (m-2)-dimensional surface of the sphere whose center is the point $(x^1 - \bar{\lambda}, x^2, \ldots, x^m)$. For the present case $m = 4$, $\rho = 1$ (implying $r = \bar{\lambda}$ from Eq. B30) and the center is the point $(x, y, z, ct - \bar{\lambda})$. Then Eq. B36 becomes

$$m(1 - \bar{\lambda}) = \frac{1}{S_2} \int_{r = \bar{\lambda} \text{ fixed}} f(Q)\, dw = \frac{1}{4\pi \bar{\lambda}^2} \int f(\xi, \eta, \zeta, ct - \bar{\lambda})\, dw$$

where use has been made of Eq. B35. Substituting this in Eq. B34 gives

$$\text{(B37)} \quad I_4^2 f = \frac{1}{4\pi} \int_0^{ct} \frac{d\bar{\lambda}}{\bar{\lambda}} \int_{r = \bar{\lambda}} f(\xi, \eta, \zeta, ct - \bar{\lambda})\, dw$$

Applying this formula to Eq. B29 where $f = -2 \frac{\partial \varphi(\xi, 0, \zeta, \lambda)}{\partial y} \delta(\eta)$ the solution given by Eq. B24 then becomes

$$\text{(B38)} \quad \varphi(x, y, z, t) = -\frac{1}{2\pi} \int_0^{ct} \frac{d\bar{\lambda}}{\bar{\lambda}} \int_{r = \bar{\lambda}} \frac{\partial \varphi}{\partial y}\left(\xi, 0, \zeta, \frac{ct - \bar{\lambda}}{c}\right) \delta(\eta)\, dw$$

Appendix C

INSTANTANEOUS PULSE STRENGTH

The potential equation is

$$\varphi(x,y,z,t) = \iint_A \frac{f(\xi,\eta,0,t-r_1/c)}{r_1}\,dA$$

Following Lagerstrom, pc coordinates are introduced (Fig. 16)

$$\rho = \sqrt{(x-\xi)^2+(y-\eta)^2}$$

$$\theta = \tan^{-1}\frac{y-\eta}{x-\xi}$$

and

$$r_1^2 = \rho^2 + z^2$$

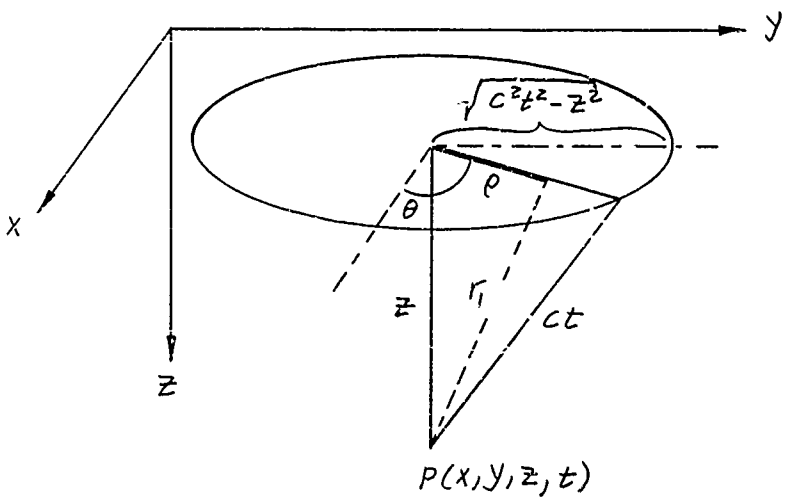

FIG. 16.

For fixed z

$$r_1 dr_1 = \rho d\rho$$

hence the element of area is

$$dA = \rho d\rho d\theta = r_1 dr_1 d\theta$$

The potential equation becomes

$$\varphi(x,y,z,t) = \int_z^{ct} \int_0^{2\pi} f(\rho,\theta,t-\frac{r_1}{c}) d\theta dr$$

Differentiating with respect to z

$$\frac{\partial\varphi(x,y,z,t)}{\partial z} = \int_0^{2\pi} d\theta \frac{\partial}{\partial z} \int_z^{ct} f(\rho,\theta,t-\frac{r_1}{c}) dr_1$$

Now, from

$$r_1^2 = \rho^2 + z^2 \quad , \quad \frac{\partial\rho}{\partial z} = -\frac{z}{\rho} \quad , \quad \frac{\partial r_1}{\partial z} = \frac{z}{r_1}$$

Therefore

$$\varphi_z(x,y,z,t) = \int_0^{2\pi} d\theta \left(\int_z^{ct} \frac{\partial f}{\partial z} dr_1 - f\Big]_{r_1=z} \right)$$

but

$$\frac{\partial f}{\partial z} = \frac{\partial f}{\partial \rho} \frac{\partial \rho}{\partial z} = -\frac{z}{\rho} \frac{\partial f}{\partial \rho}$$

Hence

$$\varphi_z(x,y,z,t) = -z \int_0^{2\pi} \int_z^{ct} \frac{1}{\rho} \frac{\partial f}{\partial \rho} dr_1 d\theta - \int_0^{2\pi} f(x,y,z,t-\frac{z}{c}) d\theta$$

Setting $z = 0$ one obtains

$$\varphi_z(x, y, 0, t) = -2\pi f(x, y, 0, t)$$

The instantaneous pulse strength is thus directly proportional to φ_z.

Appendix D

LEGENDRE STANDARD FORM

The Legendre standard form of the elliptic integral of the first kind

$$\text{(D1)} \quad I = \int_0^{\pi} \frac{d\theta}{\sqrt{(1+r'\cos\theta)^2 + (t'-r'^2)}}$$

This equation can be reduced to the Legendre standard form by the following four transformations:

<u>Transformation 1</u>

Let $\xi = \cos\theta$

Upon substitution into Eq. D1 and factorization of the expression under the radical one obtains

$$\text{(D2)} \quad I = \frac{1}{r'}\int \frac{d\xi}{\sqrt{-(\xi-\alpha)(\xi-\beta)(\xi-\gamma)(\xi-\delta)}} = \frac{1}{r'}\int \frac{d\xi}{\sqrt{\Gamma(\xi)}}$$

where

$$\alpha = -\frac{1}{r'}\left(1 + i\sqrt{t'-r'^2}\right)$$

$$\beta = -\frac{1}{r'}\left(1 - i\sqrt{t'-r'^2}\right)$$

$$\gamma = 1 \quad , \quad \delta = -1$$

Transformation 2

To rid the radical of odd powers of the variable of integration, the procedure outlined by Kármán and Biot (Ref. 11), for a fourth-degree polynomial is followed, and the homographic transformation made

$$\xi = \frac{p + qz}{1 + z}$$

This yields the integral

$$I = \frac{1}{r'} \int \frac{(q - p)\,dz}{\sqrt{-G(z)}}$$

where $G(z) = [(p-\alpha)(p-\beta) + (q-\alpha)(q-\beta)z^2][(p-\gamma)(p-\delta) + (q-\gamma)(q-\delta)z^2]$

and the requirement that odd powers of z vanish determines p and q as follows

$$p = -a - \sqrt{a^2 - 1}$$

$$q = -a + \sqrt{a^2 - 1}$$

where

$$a = \frac{1 + t'}{2r'}$$

Upon factoring one can rewrite the integral as follows

$$\text{(D3)} \quad I = k \int \frac{dz}{\sqrt{-(1 + g^2 z^2)(1 + h^2 z^2)}}$$

where

$$K = \frac{q - p}{r'\sqrt{(p-\alpha)(p-\beta)(p-\gamma)(p-\delta)}}$$

$$g^2 = \frac{(q-\alpha)(q-\beta)}{(p-\alpha)(p-\beta)} = \frac{\sqrt{a^2-1} + \bar{a}}{\sqrt{a^2-1} - \bar{a}}$$

$$h^2 = \frac{(q-\gamma)(q-\delta)}{(p-\gamma)(p-\delta)} = \frac{\sqrt{a^2-1} - \bar{a}}{\sqrt{a^2-1} + \bar{a}}$$

$$\bar{a} = \frac{1-t'}{2r'}$$

Transformation 3

Since $a > 1$ and $\bar{a} > 0$ $(r' \leq t' \leq 1)$, therefore $|g^2| > |h^2|$. Following Kármán and Biot, putting

$$z = \frac{\xi}{g} \quad ; \quad c^2 = -\frac{h^2}{g^2}$$

where $c < 1$; hence, Eq. D3 becomes

$$\text{(D4)} \qquad I = \frac{K}{g} \int \frac{d\xi}{\sqrt{(c^2\xi^2+1)(1+\xi^2)}}$$

Transformation 4

Lastly, applying the transformation

$$\xi = \frac{1}{c \cos\varphi}$$

Equation D4 becomes

$$I = \frac{K}{g\sqrt{1+c^2}} \int \frac{d\varphi}{\sqrt{1 - \frac{c^2}{1+c^2} \sin^2\varphi}}$$

or, with the limits of integration

(D5) $$I = B(r', t') \int_0^{\frac{\pi}{2}} \frac{d\varphi}{\sqrt{1 - k^2(r', t') \sin^2 \varphi}}$$

where

$$B(r', t') = \frac{2}{\left[(1+t')^2 - (2r')^2\right]^{\frac{1}{4}}}$$

$$k^2(r', t') = \frac{1}{2} \left\{ 1 - \frac{1 + t' - 2r'}{\left[(1+t')^2 - (2r')^2\right]^{1/2}} \right\}$$

Equation D5 is a complete elliptic integral of the first kind.

NOMENCLATURE

A	Area
a	Radius of sphere
C, c	Velocity of sound in water
F	Surface of fluid struck by the body
F, F'	Points on the unit circle of Fig. 2
f	Instantaneous pulse strength
M	Mach number
m	Number of dimensions
P	Field point
P, U, V p, u, v	Functions of the complex variable
p	Pressure
R	Hyperbolic distance
$R(t)$	Instantaneous radius of the disk
$\dot{R}(t)$	Rate of growth of radius of wetted area
r	A point on the disk
S	Wetted surface of the body
S'	Horizontal free surface of the liquid
t	Fixed time
t_c	Time at which rate of expansion of wetted surfaces is sonic

$\bar{U}$ Velocity vector in flow field

V Normal velocity of disk

$w = \dfrac{\xi - S_D}{1 - \xi S_D}$

x Coordinate

x, ξ

θ Entry angle

λ

ξ, η Coordinate axis of the $\xi\eta$ plane

ξ, η, ζ Coordinates of a variable point in space

ρ Density of water

σ, τ Polar coordinates

τ Variable time

φ Velocity potential

ψ Angle between BD and BF in Fig. 2

REFERENCES

1. New York University, Applied Mathematics Panel. The Force of Impact on a Sphere Striking a Water Surface, by M. Shiffman and D. C. Spencer. NYU, New York, February 1945. (AMP Report 421 R/AMG).

2. Lamb, H. Hydrodynamics. New York, Dover Publications, 1945. P. 476.

3. National Advisory Committee for Aeronautics. Linearized Supersonic Theory of Conical Wings, by P. A. Lagerstrom. Washington, NACA, January 1950. (Technical Note 1685.)

4. -------. Infinitesimale Kegelige Überschallströmung, by A. Rusemann. Tr. from Jahrbuch der Luftfahrtforschung, Vol. 7B, No. 3 (1943) pp. 105-121. Washington, NACA, 1947. (Technical Memorandum 1100.)

5. -------. Distribution of Wave Drag and Lift in the Vicinity of Wing Tips at Supersonic Speeds, by John C. Evvard. Washington, NACA, July 1947. (Technical Note 1382.)

6. Riesz, M. "L'Integrale de Riemann-Liouville et le Problems de Cauchy," ACTA MATHEMATICA, 1949.

7. Jahnke, E. and F. Emde. Tables of Functions, 4th ed. New York, Dover Publications, 1945.

8. U. S. Naval Ordnance Test Station, Inyokern. Scale Factors in Water Entry, by J. H. Wayland. Inyokern, Calif., NOTS, 18 April 1947. (NAVORD Report 978.)

9. -------. Laboratory Measurements of Pressure Versus Time at Water Impact, by C. R. Nisewanger. China Lake, Calif., NOTS, January 1952. (Technical Memorandum 614.)

10. National Advisory Committee for Aeronautics. The Impact on Seaplane Floats During Landing, by Th. von Kármán. Washington, NACA, October 1929. (Technical Note 321.)

11. von Kármán, Th., and M. A. Biot. Mathematical Methods in Engineering, 1st ed. New York, McGraw-Hill, 1940. P. 121.

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