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A NOTE ON THE COMPATIBILITY OF DISTRIBUTION FUNCTIONS

by

George Marsaglia

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A Note on the Compatibility of Distribution Functions

By George Marsaglia
Montana State University and University of North Carolina

A class of random variables is generally defined in either of two ways. One, as a class of measurable functions over a space with a probability measure, the other, by defining each element in terms of the properties it shares with the other members of the class - these properties being characterized by distribution functions. The latter development usually takes this form: associated with each finite subset $t_1, \ldots, t_n$ of some set $T$ we have a distribution function $F_{t_1, \ldots, t_n}$. This system of distribution must satisfy the well known consistency relations.

Let the integer $k$ be fixed. Suppose that associated with each finite subset $t_1, \ldots, t_n$ of $T$, with $n \leq k$, we have a distribution function $F_{t_1, \ldots, t_n}$ and that this system satisfies the consistency relations. Call such a system a $k$-fold system of distribution functions.

Can a $k$-fold system of distribution functions always be extended to define a class of random variables?

This question merits some consideration. Most of the convergence criteria for sequences of random variables are determined by the 2-fold distribution functions, as is the covariance function of a stochastic process. One may seek an example of a stochastic process with a particular property determined by the $k$-fold system of distribution functions, and having specified a satisfactory $k$-fold system, may ask whether a stochastic process exists with this $k$-fold system.

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The question also appears to have some significance in the axiomatic development of probability in terms of random experiments. We refer particularly to Cramer, *Mathematical Methods of Statistics*, paragraph 14.2. An affirmative answer to the above questions would seem necessary for the appropriateness of Axiom 3 of that paragraph.

The following example shows that, at least for 2-fold systems, an extension is not always possible.

Let $x_1, x_2, x_3$ be any three elements from $\mathcal{X}_T$, $t \in T$. Let $a \neq b$ be two real numbers. Then the relations

$$P \left[ x_1 = a, x_j = b \right] = P \left[ x_1 = b, x_j = a \right] = \frac{1}{2}, \ i = 1, 2, 3; \ j = 1, 2, 3; \ i \neq j,$$

uniquely determines a consistent 2-fold system of distribution functions. But no distribution function for $x_1, x_2, x_3$ exists which is consistent with this system. For any such distribution function will be completely specified by the eight values

$$P \left[ x_1 = c, x_2 = d, x_3 = e \right]$$

where $c, d,$ and $e$ are each equal to $a$ or $b$. Since at least two of them, say $c$ and $d$, are equal, we have, by the consistency requirement,

$$P \left[ x_1 = c, x_2 = c, x_3 = e \right] \leq P \left[ x_1 = c, x_2 = c \right] = 0$$

The question of the compatibility of distribution functions must thus be confined to specific $k$-fold systems and specific values of $k$. 