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THE TENSOR VIRIAL EQUATIONS*

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Abstract

The tensor virial equations of motion are developed. They are found to be tensor equations of second rank which upon contraction give the usual scalar virial equation. The tensor virial equations may be applied to anisotropic systems in the same manner that the scalar virial equation is applied to isotropic systems. Several applications of the tensor virial equations are considered. The diffusion of ions through a magnetic field and the diffusion of molecules through a gas are calculated. The derivation of the Navier - Stokes equations and the Reynolds stress tensor for a turbulent flow is developed, leading in a natural way to Prandtl's mixing length ideas. The dynamics of a self gravitating oscillating gas cloud is investigated. The expressions for gravitating homogeneous ellipsoidal regions are worked out for use in problems involving inverse square interactions, e.g. galactic dynamics, clouds of charged particles, etc. Because of the applicability to hydromagnetics, the stress tensor formulation of the tensor virial equations is developed in the last section.

Introduction

0.1 The use of the virial theorem in kinetic theory is well known. It does not seem to have been remarked with sufficient generality that the virial equation can be extended to a set of equations between tensor components. As the formulas of the first section show, these relations are essentially equations of motion for the ordinary moment of inertia tensor of an assembly of particles. The usual scalar virial theorem results from the tensor equations by the process of contraction. The tensor virial equations are useful in problems of the dynamics of complex systems, especially in hydrodynamics and transport theory where the actual integration of the equations of motion is not practical. One may then obtain approximate information by investigating the behavior of the moment of inertia of the system or of a subsystem, e.g. an eddy. The tensor form of the equations permits one to obtain results for anisotropic motions wherever the scalar virial can be applied to isotropic motions.

The method has proved useful in astrophysical problems, particularly those dealing with the dynamics of a finite region, e.g. the dynamics of an individual star, clusters of stars, and galaxies[2]. We shall treat some illustrative examples in transport theory. We shall find that the main advantage will be a means of sidestepping uninteresting detail, such as the precise density distribution, allowing one to compute directly the more pertinent quantities, e.g. mean diffusion or expansion rates, mean pressures and viscous stresses etc.

We shall derive the Navier - Stokes equations from the tensor

virial equations and show that the parallelism of viscous and turbulent stresses is inherent in the theory without any artificiality. Thus, it forms a basis for the turbulence theories to be found elsewhere[7-9]. The generality of our results makes them applicable to the construction of a statistical magnetohydrodynamic theory of the same nature as existing turbulence theories.

It is well known that the scalar virial theorem holds in quantum mechanics; an extension of the tensor virial theorem to the quantum mechanical case should be possible, but will not be attempted here.

Lagrangian Formulation

1.1 Consider a system of particles in a space with the cartesian coordinates x^i. We shall for the moment fix our attention on the νth particle. We represent its mass by ${}_\nu m^i$ and its position by ${}_\nu x^i$. We assume that all velocities are small compared to the speed of light. The Lagrangian is written

$$ {}_\nu L = {}_\nu T - {}_\nu V \tag{1} $$

where ${}_\nu T$ and ${}_\nu V$ represent the kinetic and potential energies.

Let us introduce the symbol ${}_\nu Q_i$ to denote those forces for which a scalar potential does not exist. In this category we include all forces of constraint. The ${}_\nu Q_i$ are defined as the differential coefficients in the expression for the work

$$ \delta\, {}_\nu W = {}_\nu Q_i \delta\, {}_\nu x^i \tag{2} $$

Repeated italic indices imply summation convention; Greek do not. The $\delta_\nu x^i$ represent an arbitrary displacement of the νth particle under the effect of all those forces not included in ${}_\nu V$.

Lagrange's equations for the νth particle become

$$ \frac{d}{dt}\left(\frac{\partial_\nu L}{\partial_\nu \dot{x}^i}\right) - \frac{\partial_\nu L}{\partial_\nu x^i} = {}_\nu Q_i \tag{3} $$

Multiplying by ${}_\nu x^j$ gives us the tensor equation of second rank which may be written as

$$ \frac{d}{dt}\left({}_\nu x^j \frac{\partial_\nu L}{\partial_\nu \dot{x}^i}\right) = {}_\nu \dot{x}^j \frac{\partial_\nu L}{\partial_\nu \dot{x}^i} + {}_\nu x^j \frac{\partial_\nu L}{\partial_\nu x^i} + {}_\nu x^j\, {}_\nu Q_i \tag{4} $$

Consider now the sum over a number of particles in the space. We shall indicate the summation by $\sum_\nu$. In practice this sum is usually effected by summing over all particles in a simply connected region of space, though in some diffusion problems

(cf. 2.12) this is not the case. We define the tensors

$$J_{ij} = \sum_{\nu} {}_{\nu}x^{i} \frac{\partial L}{\partial_{\nu}\dot{x}^{j}} \tag{5}$$

$$2T_{ij} = \sum_{\nu} {}_{\nu}\dot{x}^{i} \frac{\partial L}{\partial_{\nu}\dot{x}^{j}} \tag{6}$$

$$\Phi_{ij} = \sum_{\nu} {}_{\nu}x^{i} \frac{\partial L}{\partial_{\nu}x^{j}} \tag{7}$$

$$\Omega_{ij} = \sum_{\nu} {}_{\nu}x^{i} \frac{\partial L}{\partial_{\nu}x^{j}} \tag{8}$$

and the scalars

$$L = \sum_{\nu} {}_{\nu}L \tag{9}$$

$$T = \sum_{\nu} {}_{\nu}T \tag{10}$$

$$V = \sum_{\nu} {}_{\nu}V \tag{11}$$

Thus, L, T, and V are the Lagrangian, kinetic energy, and potential energy respectively of the particles over which we are summing. (4) may be rewritten as

$$\frac{d}{dt} J_{ij} = 2T_{ij} + \Phi_{ij} + \Omega_{ij} \tag{12}$$

These tensors admit of a straight forward interpretation. The diagonal components of J_{ij} are just

$$2J_{\alpha\alpha} = \frac{d}{dt} \sum_{\nu} {}_{\nu}m \, ({}_{\nu}x^{\alpha})^2 \tag{13}$$

We do not use summation convention on Greek indices. The diagonal terms of J_{ij} are, then, essentially the rate of change of the inertia tensor. The off diagonal terms represent the angular momentum of the system. Now $\partial L/\partial_{\nu}\dot{x}^i$ is just ${}_{\nu}m_{\nu}\dot{x}^j$. Thus

$$2T_{ij} = \sum_{\nu} {}_{\nu}m_{\nu}\dot{x}^i_{\nu}\dot{x}^i \tag{14}$$

The spur of T_{ij} is simply T, the total kinetic energy of the system. Hence, we call T_{ij} the <u>kinetic tensor</u>. Φ_{ij}, on the other hand, is given by

$$\Phi_{ij} = - \sum_{\nu} {}_{\nu}x^i \frac{\partial V}{\partial_{\nu}x^j}$$

so that, if V is a homogeneous function of degree n of the ${}_{\nu}x^i$, Euler's theorem gives

$$\Phi_{ii} = - nV \tag{15}$$

i.e. the spur of Φ_{ij} is some number --n times the potential energy of the system. Hence we call Φ_{ij} the <u>potential tensor</u> of the system.

(5) may be resolved into its symmetric and antisymmetric parts. We define

$$I_{ij} = \frac{1}{2} \sum_{\nu} {}_{\nu}m_{\nu}x^i_{\nu}x^j \tag{16}$$

$$K_{ij} = \frac{1}{2} \sum_{\nu} {}_{\nu}m \left[{}_{\nu}x^i_{\nu}\dot{x}^j - {}_{\nu}x^j_{\nu}\dot{x}^i \right] \tag{17}$$

so that

$$\frac{d}{dt} I_{ij} = \frac{1}{2} (J_{ij} + J_{ji}) \tag{18}$$

$$K_{ij} = \frac{1}{2} (J_{ij} - J_{ji}) \tag{19}$$

Further, we let

$$\Upsilon_{ij} = \frac{1}{2}\left[\Phi_{ij} + \Phi_{ji}\right] \tag{20}$$

$$\Xi_{ij} = \frac{1}{2}\left[\Phi_{ij} - \Phi_{ji}\right] \tag{21}$$

$$M_{ij} = \frac{1}{2}\left[\Omega_{ij} + \Omega_{ji}\right] \tag{22}$$

$$N_{ij} = \frac{1}{2}\left[\Omega_{ij} - \Omega_{ji}\right] \tag{23}$$

Thus we have the symmetric and anti-symmetric equations

$$\frac{d^2}{dt^2} I_{ij} = 2T_{ij} + \Upsilon_{ij} + M_{ij} \tag{24}$$

$$\frac{d}{dt} K_{ij} = \Xi_{ij} + N_{ij} \tag{25}$$

The symmetric equation gives the manner in which the moment of inertia tensor, I_{ij}, varies. The antisymmetric equation gives the rate of change of the angular momentum K_{ij} of the region.

It is well known that if the interaction of the particles may be expressed in terms of central force fields, they do not transfer angular momentum. Hence, in the absence of external fields we have

$$\Xi_{ij} = N_{ij} = 0 \tag{26}$$

so that (14) and (15) may be rewritten

$$\frac{d^2}{dt^2} I_{ij} = 2T_{ij} + \Phi_{ij} + \Omega_{ij} \tag{27}$$

$$\frac{d}{dt} K_{ij} = 0 \tag{28}$$

(See Appendix A and B for a more general development and further discussion.)

1.2 To obtain the conventional scalar virial equation we contract (1.2.5) and obtain

$$\frac{d}{dt} J = 2T + \Phi + \Omega \tag{1}$$

where in general we define

$$X \equiv X_{ii}$$

We note from (1.2.8) that

$$\begin{aligned} J &= J_{ii} \\ &= \frac{d}{dt} I_{ii} \\ &= \frac{1}{2} \frac{d}{dt} \sum_{\nu} {}_{\nu}m_{\nu}r^2 \end{aligned} \tag{2}$$

where

$${}_{\nu}r^2 = \sum_{i=1}^{3} ({}_{\nu}x^i)^2$$

Using (2) we may write in place of (1)

$$\frac{d^2}{dt^2} I = 2T + \Phi + \Omega \tag{3}$$

Applications

2.1 It will be our purpose in this section to illustrate some of the methods of application of the tensor virial equations to transport problems and associated kinetic phenomena.

2.11 Consider a simply connected cloud of electrons and protons in a uniform magnetic field. We assume that the net charge of the cloud is zero and recombination is negligible. The Lagrangian for the νth particle is

$$ {}_\nu L = \frac{1}{2}\, {}_\nu m\, {}_\nu v^i\, {}_\nu v^i + \frac{{}_\nu q}{c}\, {}_\nu A^i\, {}_\nu v^i \tag{1} $$

where ${}_\nu A^i$ is the vector potential representing the magnetic field, ${}_\nu q$ the charge. It follows that

$$ \frac{\partial_\nu L}{\partial_\nu v^i} = {}_\nu m\, {}_\nu v^i + \frac{{}_\nu q}{c}\, {}_\nu A^i $$

$$ \frac{\partial_\nu L}{\partial_\nu x^i} = \frac{{}_\nu q}{c}\, {}_\nu v^j\, \frac{\partial_\nu A^j}{\partial_\nu x^i} $$

We obtain from (1.1.12), after some rearranging

$$ \frac{d}{dt} \sum_\nu {}_\nu m\, {}_\nu v^i\, {}_\nu x^j = \sum_\nu \left\{ {}_\nu m\, {}_\nu v^i\, {}_\nu v^j - {}_\nu x^j \frac{{}_\nu q}{c} \left[\frac{d_\nu A^i}{dt} - {}_\nu v^k \frac{\partial A^k}{\partial x^i} \right] \right\} \tag{2} $$

If we assume the magnetic field to be of magnitude B and in the positive z direction, then we may take

$$ A_x = -\frac{1}{2} By,\ A_y = \frac{1}{2} Bx,\ A_z = 0,\ \frac{\partial A_i}{\partial t} = 0 \tag{3} $$

Thus, we obtain the three equations

$$\frac{d}{dt}\sum_{\nu} {}_{\nu}m_{\nu}v_{x\nu}x^{j} = \sum_{\nu}\left\{{}_{\nu}m_{\nu}v_{x\nu}v^{j} + {}_{\nu}x^{j}\,\frac{{}_{\nu}q}{c}\,{}_{\nu}v_{y}\,B\right\} \tag{4}$$

$$\frac{d}{dt}\sum_{\nu} {}_{\nu}m_{\nu}v_{y\nu}x^{j} = \sum_{\nu}\left\{{}_{\nu}m_{\nu}v_{y\nu}v^{j} - {}_{\nu}x^{j}\,\frac{{}_{\nu}q}{c}\,{}_{\nu}v_{x}\,B\right\} \tag{5}$$

$$\frac{d}{dt}\sum_{\nu} {}_{\nu}m_{\nu}v_{z\nu}x^{j} = \sum_{\nu} {}_{\nu}m_{\nu}v_{z\nu}v^{j} \tag{6}$$

Summing over the whole cloud the off diagonal terms average out because there is no correlation between ${}_{\nu}v^{i}$, and ${}_{\nu}v^{j}$, or ${}_{\nu}x^{i}$ and ${}_{\nu}x^{j}$, $(i \neq j)$ as we sum over ν. We are left with the diagonal terms

$$\frac{d}{dt}\sum_{\nu} {}_{\nu}m_{\nu}v_{x\nu}x = \sum_{\nu}\left\{{}_{\nu}m({}_{\nu}v_{x})^{2} + \frac{{}_{\nu}q}{c}\,{}_{\nu}x_{\nu}v_{y}\,B\right\} \tag{7}$$

$$\frac{d}{dt}\sum_{\nu} {}_{\nu}m_{\nu}v_{y\nu}y = \sum_{\nu}\left\{{}_{\nu}m({}_{\nu}v_{y})^{2} - \frac{{}_{\nu}q}{c}\,{}_{\nu}y_{\nu}v_{x}\,B\right\} \tag{8}$$

$$\frac{d}{dt}\sum_{\nu} {}_{\nu}m_{\nu}v_{z\nu}z = \sum_{\nu} {}_{\nu}m({}_{\nu}v_{z})^{2} \tag{9}$$

We see from (9) that we have free expansion in the z direction.

Consider (7). We must compute $\sum_{\nu} {}_{\nu}q_{\nu}x_{\nu}v_{y}\,B/c$. We shall describe the average position of a particle between successive collisions by means of the center of mass of its trajectory. The projection on the xy plane of the velocity is

$${}_{\nu}v_{xy} \equiv \sqrt{({}_{\nu}v_{x})^{2} + ({}_{\nu}v_{y})^{2}} \tag{10}$$

The radius of curvature of the projection on the xy plane is

$${}_{\nu}\Lambda = \frac{c\,{}_{\nu}v_{xy}\,{}_{\nu}m}{{}_{\nu}qB} \tag{11}$$

We shall describe the projection of the trajectory onto the xy plane as shown in Figure 1. The $\xi\eta$ axes are oriented so that the trajectory is symmetric about the ξ axis. The η component of the moment of the trajectory about $\xi = a$ is

$$I_\eta (a) = \int (\xi - a)\, ds$$

where ds is an element of length along the trajectory. We have

$$I_\eta (a) = \int_{-\psi/2}^{\psi/2} (\Lambda \cos\omega - a)\, d\omega$$

since

$$\xi = \Lambda \cos\omega \text{ and } ds = \Lambda\, d\omega$$

where ω is measured from the positive ξ axis. We obtain

$$I_\eta (a) = \Lambda \left[2\Lambda \sin\frac{\psi}{2} - a\psi \right]$$

if $\xi = a$, $\eta = 0$ is the center of mass of the trajectory, $I_\eta (a) = 0$ giving

$$a = \frac{\sin\frac{\psi}{2}}{\frac{\psi}{2}} \tag{12}$$

Returning to the xy coordinate system, if we take the center of mass of the trajectory to have an abscissa x_o, then the center of the trajectory is seen from Fig. 2 to be at $x_o + a\cos\varphi$. The trajectory is traversed in the direction indicated for $_\nu q > 0$, $B > 0$. If θ is the angle between the positive x axis and the line drawn through the center of the trajectory and a given point on the trajectory, then for the point on the trajectory

$$x = x_o + a\cos\varphi + \Lambda\cos\theta \tag{13}$$

$$v_y = -v_{xy}\cos\theta \tag{14}$$

Hence, using (12), (13), and (14)

$$xv_y = -v_{xy}\left(x_o\cos\theta + \Lambda \frac{\sin\frac{\psi}{2}}{\frac{\psi}{2}} \cos\varphi\cos\theta + \Lambda\cos^2\theta\right) \tag{15}$$

We shall now average over the trajectory by operating with

$$\frac{1}{\psi}\int_{\pi+\varphi-\psi/2}^{\pi+\varphi+\psi/2} d\theta$$

We obtain

$$(xv_y)_\theta = v_{xy}\left[x_o\cos\varphi\frac{\sin\frac{\psi}{2}}{\frac{\psi}{2}} + \Lambda\frac{\sin^2\frac{\psi}{2}}{\left(\frac{\psi}{2}\right)^2}\cos^2\varphi - \frac{\Lambda}{2}\left(1 + \frac{\sin\psi}{\psi}\cos 2\varphi\right)\right] \tag{16}$$

where the subscript θ indicates that the average over θ has been carried out.

We now average over all possible orientations of the projection of the trajectory by operating with

$$\frac{1}{2\pi}\int_0^{2\pi} d\varphi$$

We obtain

$$(xv_y)_{\theta\varphi} = \frac{\Lambda v_{xy}}{2}\left[\frac{\sin 2\frac{\psi}{2}}{\left(\frac{\psi}{2}\right)^2} - 1\right] \tag{17}$$

Now, if the mean free path of a particle is λ, then the probability of its traveling a distance s between collisions is $\exp\left[-s/\lambda\right](1/\lambda)$. But $\psi = s/\Lambda$. Thus, the average over s gives

$$(xv_y)_{\theta\varphi s} = \frac{\Lambda v_{xy}}{2} \int_0^\infty \exp(-s/\lambda) \left[\frac{\sin^2\left(\frac{s}{2\Lambda}\right)}{\left(\frac{s}{2\Lambda}\right)^2} - 1 \right] d\left(\frac{s}{\lambda}\right)$$

$$= \frac{\Lambda v_{xy}}{2} \left[\frac{2\Lambda}{\lambda} \int_0^\infty du \exp\left(-\frac{2\Lambda}{\lambda} u\right) \frac{\sin^2 u}{u^2} - 1 \right] \tag{18}$$

which may be integrated[1] to give

$$(xv_y)_{\theta\varphi s} = \frac{\Lambda v_{xy}}{2} \left\{ \frac{2\Lambda}{\lambda} \left[\text{Arctan} \frac{\lambda}{\Lambda} - \frac{\Lambda}{2\lambda} \ln\left(1 + \frac{\lambda^2}{\Lambda^2}\right) \right] - 1 \right\} \tag{19}$$

Using (11) we obtain

$$\frac{{}_\nu q}{c}\, {}_\nu x\, {}_\nu v_y\, B = {}_\nu m \frac{({}_\nu v_{xy})^2}{2} \left\{ \frac{2\Lambda}{\lambda} \left[\text{Arctan} \frac{\lambda}{\Lambda} - \frac{\Lambda}{2\lambda} \ln\left(1 + \frac{\lambda^2}{\Lambda^2}\right) \right] - 1 \right\}$$

Thus

$$\sum_\nu \left[{}_\nu m ({}_\nu v_x)^2 + \frac{{}_\nu q}{c}\, {}_\nu x\, {}_\nu v_y\, B \right] = \sum_\nu {}_\nu m \left\{ ({}_\nu v_x)^2 \right.$$

$$\left. + \frac{1}{2} ({}_\nu v_{xy})^2 \left[\frac{2\Lambda}{\lambda} \left(\text{Arctan} \frac{\lambda}{\Lambda} - \frac{\Lambda}{2\lambda} \ln \left(1 + \frac{\lambda^2}{\Lambda^2}\right)\right) - 1 \right] \right\} \tag{20}$$

However, we expect the velocity to be statistically isotropic over the xy plane. Thus

$$({}_\nu v_x)^2 = \frac{1}{2} ({}_\nu v_{xy})^2 \tag{21}$$

and we obtain

$$\sum_\nu \left[{}_\nu m ({}_\nu v_x)^2 + \frac{{}_\nu q}{c}\, {}_\nu x\, {}_\nu v_y\, B \right] = f(\lambda/\Lambda) \sum_\nu {}_\nu m ({}_\nu v_x)^2 \tag{22}$$

where

$$f(\lambda/\Lambda) \equiv \frac{2\Delta}{\lambda}\left[\text{Arctan}\,\frac{\lambda}{\Lambda} - \frac{\Delta}{2\lambda}\ln\left(1 + \frac{\lambda^2}{\Lambda^2}\right)\right] \tag{23}$$

We note that

$$f(x) \sim 1 - \frac{1}{6}x^2 + \frac{1}{15}x^4 + 0^6\ (x)$$

$$f(x) \sim \frac{\pi}{x} - \frac{1}{x^2}\left[2 + \ln(1 + x^2)\right] + \frac{2}{3x^4} + 0^{-6}\ (x)$$

We rewrite (7), (8), and (9) as

$$\frac{d}{dt}\sum_{\nu} {}_{\nu}m_{\nu}v_{x\nu}x = f(\lambda/\Lambda)\sum_{\nu} {}_{\nu}m({}_{\nu}v_x)^2 \tag{24}$$

$$\frac{d}{dt}\sum_{\nu} {}_{\nu}m_{\nu}v_{y\nu}y = f(\lambda/\Lambda)\sum_{\nu} {}_{\nu}m({}_{\nu}v_y)^2 \tag{25}$$

$$\frac{d}{dt}\sum_{\nu} {}_{\nu}m_{\nu}v_{z\nu}z = \sum_{\nu} {}_{\nu}m({}_{\nu}v_z)^2 \tag{26}$$

(24), (25), and (26) are not sufficient to determine the form of $\rho(x, y, z)$. They represent a restriction on the system of particles under consideration, though they by no means determine a unique state of the system. They give no information concerning the form of the spatial distribution of the particles. Hence in order to use (24), (25), and (26) we must supply a form of distribution over space. (24), (25), and (26) will then tell us how the scale of the distribution varies with time. With three equations it is obvious that the form of the distribution may be characterized by three parameters, one parameter describing the scale of the distribution in each of the three directions.

At first thought it seems a serious drawback that our equations will not supply us with the form of the spatial distribution as well as its variation with time. However, it must

be remembered that the main difficulty in problems in transport theory when the solution is attempted from the Boltzmann equation is the tremendous mathematical complexity of the problem. The complexity is brought about by our having to determine simultaneously the form of the spatial distribution as well as the variation with time. Hence we must cut down on the amount of information that we seek to obtain from our calculations, supplying the deficit by some informed estimates. The weakness of our method is, then, the source of its main advantage. In most cases the form of the spatial distribution is not as important to us as the variation in time and is usually known fairly well anyway. The theory of stochastic processes, for instance, tells us that in most cases the spatial distribution will be gaussian for sufficiently large values of time. The end result is that in practice we may assume a form for the spatial distribution characterized by three units of scale, one in each direction, and determine the variation with time of the three units of scale and ultimately of the spatial distribution .

For the problem at hand, any initial spatial distribution will eventually diffuse into a gaussian. However, the purpose of this paper is to exhibit the general mehtods rather than to solve accurately specific problems. Therefore, to save computation we shall assume that the particles are constrained to a homogeneous distribution within a rectangular parallelepiped with sides $a_{xy}(t)$ and $a_z(t)$ and center at the origin. To compare this artificial problem with a real problem in nature we identify the $a_{xy}(t)$ and $a_z(t)$ as characteristic scales of the distributions, say as twice the mean deviation of the final gaussian distribution.

The expansion is restricted by the equation of continuity to

$$v_x(x,t) = \kappa_{xy}(t)\, x \tag{27}$$

$$v_y(y,t) = \kappa_{xy}(t)\, y \tag{28}$$

$$v_z(z,t) = \kappa_z(t)\, z \tag{29}$$

It follows that

$$\frac{d\, a_{xy}}{dt} = \kappa_{xy}(t)\, a_{xy} \tag{30}$$

etc.

Now, in $\sum_\nu {}_\nu m_\nu v_{x\nu} x$ all motions of smaller scale than the dilatation average out, leaving us with

$$\sum_\nu {}_\nu m_\nu v_{x\nu} x = \int dV \rho\, \kappa_{xy}(t)\, x^2$$

$$= \frac{1}{12} M\, \kappa_{xy}(t)\, a_{xy}^2(t) \tag{31}$$

$$\sum_\nu {}_\nu m_\nu v_{z\nu} z = \frac{1}{12} M\, \kappa_z(t)\, a_z^2(t) \tag{32}$$

where M is the total mass of the cloud.

$$M = \sum_\nu {}_\nu m \tag{33}$$

We let $\overline{(v_x)}^2$ represent the mean square velocity in the x and also in the y direction (cf. (21)), and $\overline{(v_z)}^2$ in the z direction. Then

$$\sum_\nu {}_\nu m({}_\nu v_x)^2 = M\, \overline{(v_x)}^2 \tag{34}$$

etc. $\overline{(v_x)}^2$ and $\overline{(v_z)}^2$, it must be remembered, include the small scale motions as well as the dilatation velocity. Hence the total energy density $\overline{(v)}^2/2$ is given by

$$\frac{1}{2}\overline{v^2} = 2\left(\frac{1}{2}\overline{(v_x)^2}\right) + \frac{1}{2}\overline{(v_z)^2} \tag{35}$$

Conservation of energy implies that $\overline{v^2}$ is independent of t.

Using (31) and (34), we may now write (24) and (25) as

$$\frac{d}{dt}\left[\kappa_{xy}(t)\ a_{xy}^2\ (t)\right] = 12\ f(\lambda/\Lambda)\ \overline{(v_x)^2}$$

(32) gives (26) as

$$\frac{d}{dt}\left[\kappa_z(t)\ a_x^2\ (t)\right] = 12\ \overline{(v_z)^2}$$

Using (30) we obtain

$$\frac{d}{dt}\left[a_{xy}(t)\ \frac{d}{dt}\ a_{xy}(t)\right] = 12\ f(\lambda/\Lambda)\ \overline{(v_x)^2} \tag{36}$$

$$\frac{d}{dt}\left[a_z(t)\ \frac{d}{dt}\ a_z(t)\right] = 12\ \overline{(v_z)^2} \tag{37}$$

Let us consider several special solutions of (36) and (37). If B is sufficiently small or ρ sufficiently large, then $\lambda \ll \Lambda$ and $f(\lambda/\Lambda) \sim 1$. Hence the diffusion is unhindered by B and we have an isotropic adiabatic expansion. We take

$$a_{xy}(t) = a_z(t) = a(t)$$

$$\overline{(v_x)^2} = \overline{(v_z)^2} = \frac{1}{3}\ \overline{v^2}$$

In place of (36) and (37) we obtain the single equation

$$\frac{d}{dt}\left[a(t)\frac{d}{dt}\ a(t)\right] = 4\ \overline{v}^2$$

which may be written

$$a\frac{d^2a}{dt^2} + \left(\frac{da}{dt}\right)^2 = 4\ \overline{v}^2 \tag{38}$$

It should be noted that sufficient expansion will eventually increase λ to where it is no longer less than Λ.

If we consider the other extreme of a strong field and/or a low density, $\lambda \gg \Lambda$, and $f(\lambda/\Lambda) \sim 0$. The cloud expands only in the z direction. The energy of the expansion is

$$\frac{1}{2} \int dV \rho \left[K_z(t) \; z \right]^2 = \frac{M}{24} \left(\frac{da_z}{dt} \right)^2 \tag{39}$$

The energy of all motions of smaller scale than the parallelepiped is, then,

$$\frac{1}{2} M \overline{v}^2 - \frac{M}{24} \left(\frac{da_z}{dt} \right)^2 \tag{40}$$

(We note the division of the motion into two components depending upon the scale relative to the region considered. More will be said on this in 2.21.) Assuming this energy to be distributed isotropically over the three dimensions, we have T_{zz} as just the energy in the dilatation velocity plus one third of the energy in the small scale motion,

$$\begin{aligned} T_{zz} &= \frac{M}{24} \left(\frac{da_z}{dt} \right)^2 + \frac{1}{3} \left[\frac{1}{2} M \overline{v}^2 - \frac{M}{24} \left(\frac{da_z}{dt} \right)^2 \right] \\ &= \frac{M}{6} \left[\overline{v}^2 + \frac{1}{6} \left(\frac{da_z}{dt} \right)^2 \right] \end{aligned} \tag{41}$$

Thus, (37) becomes

$$a_z \frac{d^2 a_z}{dt^2} + \frac{1}{3} \left(\frac{da_z}{dt} \right)^2 = -4 \overline{v}^2 \tag{42}$$

Now let us put no restrictions on λ but constrain the particles so that there is no expansion in the z direction. This can be done by considering the region confined between two material planes perpendicular to the z axis.

Analogous to (39), it can be shown that the energy of the expansion in the x and y directions is each given by $(M/24)(da_{xy}/dt)^2$. Thus, the motions of smaller scale than the region have an energy

$$\frac{1}{2} M \overline{v}^2 - \frac{M}{12} \left(\frac{da_{xy}}{dt}\right)^2 \tag{43}$$

And, assuming isotropy, T_{xx} and T_{yy} are just equal to the energy of the dilatation plus one third of the small scale energy given in (43).

$$\begin{aligned} T_{xx} = T_{yy} &= \frac{M}{24} \left(\frac{da_{xy}}{dt}\right)^2 + \frac{M}{3} \left[\frac{1}{2} \overline{v}^2 - \frac{1}{12} \left(\frac{da_{xy}}{dt}\right)^2 \right] \\ &= \frac{M}{6} \left[\overline{v}^2 + \frac{1}{12} \left(\frac{da_{xy}}{dt}\right)^2 \right] \end{aligned} \tag{44}$$

(36) becomes

$$a_{xy} \frac{d^2 a_{xy}}{dt^2} + \left[1 - \frac{1}{3} f(\lambda/\Lambda) \right] \left(\frac{da_{xy}}{dt}\right)^2 = 4f(\lambda/\Lambda)\, \overline{v}^2 \tag{45}$$

(38), (42), and (45) are all of the same general form

$$y \frac{d^2 y}{dx^2} + \alpha \left(\frac{dy}{dx}\right)^2 = \beta \tag{46}$$

where α and β are constants if we neglect the variation of λ with the expansion. To integrate (46) we let $p \equiv dy/dx$ and write

$$\frac{d^2 y}{dx^2} = \frac{dp}{dx} = p \frac{dp}{dy} \tag{47}$$

We obtain

$$\left(\frac{dy}{dx}\right)^2 = \frac{\beta}{\alpha} \left[1 - \frac{C}{y^{2\alpha}} \right] \tag{48}$$

where C is the constant of integration.

Let us assume that when $t = 0$ there is no expansion and $y = y_o$. (48) becomes

$$\frac{dy}{dx} = \sqrt{\frac{\beta}{\alpha}\left[1 - \left(\frac{y_o}{y}\right)^{2\alpha}\right]} \tag{49}$$

Thus, we obtain the solution by quadrature,

$$x = \sqrt{\frac{\alpha}{\beta}} \int_{y_o}^{y} \frac{y^{\alpha} dy}{\sqrt{y^{2\alpha} - y_o^{2\alpha}}} \tag{50}$$

For (38) we have $\alpha = 1$ and $\beta = 4\,\overline{v}^2$. (49) and (50) give the rate of expansion of each face of the parallelepiped as

$$\frac{1}{2}\frac{d}{dt}\,a(t) = \sqrt{\overline{v}^2}\left\{1 - \left[\frac{a(o)}{a(t)}\right]^2\right\}^{1/2} \tag{51}$$

and the length of each side as

$$a^2(t) - a^2(o) = 4\,\overline{v}^2\,t^2 \tag{52}$$

For large values of t we note that the velocity of expansion

$$\frac{1}{2}\,\frac{d}{dt}\,a(t) \sim \sqrt{\overline{v}^2} \tag{53}$$

in agreement with the usual result for expansion into a vacuum.

For (42), $\alpha = 1/3$, $\beta = 4\,\overline{v}^2$. (49) and (50) give

$$\frac{1}{2}\frac{da_z}{dt} = \sqrt{3\,\overline{v}^2}\left\{1 - \left[\frac{a_z(o)}{a_z(t)}\right]^{2/3}\right\}^{1/2} \tag{54}$$

$$t = \frac{1}{2\sqrt{3\,\overline{v}^2}}\left[a^{2/3}(t) - a^{2/3}(o)\right]^{1/2}\left[a^{2/3}(t) + 2a^{2/3}(o)\right] \tag{55}$$

For large values of t, the rate of expansion is

$$\frac{1}{2}\frac{da_z}{dt} \sim \sqrt{3\,\overline{v}^2} \tag{56}$$

Finally, for (46)

$$\alpha = 1 - \frac{1}{3}\,f(\lambda,\Lambda)$$

$$\beta = 4f(\lambda,\Lambda)\,\overline{v}^2$$

We obtain the diffusion rate as

$$\frac{1}{2}\frac{da_{xy}}{dt} = \left[\frac{f(\lambda/\Lambda)\ \overline{v}^2}{1 - \frac{1}{3} f(\lambda/\Lambda)}\right]^{1/2} \left\{1 - \left[\frac{a_{xy}(o)}{a_{xy}(t)}\right]^{2(1-f(\lambda/\Lambda)/3}\right\}^{1/2} \tag{57}$$

For large values of t,

$$\frac{1}{2}\frac{da_{xy}}{dt} \sim \left[\frac{f(\lambda/\Lambda)\ \overline{v}^2}{1 - \frac{1}{3} f(\lambda/\Lambda)}\right]^{1/2} \tag{58}$$

We note that if $\lambda \ll \Lambda$, $f(\lambda/\Lambda) \sim 1$ and

$$\frac{1}{2}\frac{da_{xy}}{dt} \sim \sqrt{\frac{3\ \overline{v}^2}{2}} \tag{59}$$

(50) results in an integral which must be evaluated numerically except for special values of $f(\lambda/\Lambda)$.

If one wishes to consider further complications such as a uniform distribution of neutral particles which inhibit the diffusion of the charged particles, one introduces an additional force into $\sum_{\nu} {}_{\nu}F^{i}{}_{\nu}x^{j}$ due to collisions of the charged with the neutral particles.

2.12 As an example of a diffusion problem consider a space filled with a homogeneous distribution of neutral particles. At time t_o we mark every particle within the rectangle with sides a_x, a_y, and a_z and center at the origin. We ask how these marked particles will be spread out at some subsequent time t. As has been pointed out, the tensor virial equations do not determine the form of the spatial distribution, but, upon assuming some spatial distribution, indicate how it will vary with time. It is convenient to express this idea by stating that we use the tensor virial equations to determine the dynamical conditions for

a given scale without inquiring into fluctuations of smaller scale. Now, it can be shown from the theory of stochastic processes that the asymptotic distribution as $t \longrightarrow \infty$ is a gaussian centering about the origin. Obviously our initial step function distribution rounds its corners and spreads out with increasing time. It is not our purpose here to go into so much detail. Thus, we describe the distribution by the three lengths $\alpha_x(t)$, $\alpha_y(t)$, and $\alpha_z(t)$. For $t = t_o$ we identify them with a_x, a_y, and a_z. They correspond to approximately twice the standard deviation of the distribution.

The tensor virial equations reduce to

$$\frac{d}{dt}\left[\alpha_i(t)\frac{d}{dt}\alpha_j(t)\right] = 0 \tag{1}$$

since the pressure on the boundaries just balances the kinetic tensor. The solution of (1) is readily shown to be

$$\begin{aligned}\alpha_i(t) &= \alpha_i(t_o)\left(\frac{t}{t_o}\right)^{1/2} \\ &= a_i\left(\frac{t}{t_o}\right)^{1/2}\end{aligned} \tag{2}$$

$$\frac{d}{dt}\alpha_i(t) = \frac{a_i}{2t_o}\left(\frac{t_o}{t}\right)^{1/2} \tag{3}$$

We must now evaluate t_o. The rate of increase of $\alpha_\beta(t)$ at each face of the parallelepiped is $(1/2)(d\alpha_\beta/dt)$. At time t_o the expansion occurs only within a distance L of each face, where L is the mean free path; the particles farther into the parallelepiped are as yet undisturbed. Of all the particles in the parallelepiped, a fraction $L/\left[\alpha_\beta(t_o)/2\right]$ are contained in the slab of thickness L and normal to the β direction. Half of

these particles are moving outward across the face with a velocity $\sqrt{\overline{(v_\rho)^2}}$. Therefore, the fraction $L/a_\beta(t_o)$ of the particles have a velocity $\sqrt{\overline{(v_\rho)^2}}$ outward across a face normal to the β direction. We write that the rate of change of the characteristic length $a_\beta(t)/2$ is

$$\frac{d}{dt}\left[\frac{a_\beta(t)}{2}\right] = \frac{L}{a_\beta(t_o)} \sqrt{\overline{(v_\beta)^2}} \tag{4}$$

of time t_o. Putting $t = t_o$ in (3) and comparing with (4), we obtain

$$t_o = \frac{a_\beta^2}{4L\sqrt{\overline{(v_\beta)^2}}} \tag{5}$$

Thus, from (2) and (3) we obtain

$$a_i(t) = 2\left[L\sqrt{\overline{(v_\beta)^2}}\ t\right]^{1/2} \tag{6}$$

$$\frac{d}{dt}\, a_i(t) = \left[\frac{L\sqrt{\overline{v_\beta^2}}}{t}\right]^{1/2} \tag{7}$$

One advantage of this method is obviously the doing away with the infinite diffusion rate obtained at $t = t_o$ if the problem were formulated in the conventional manner in terms of $\nabla\rho$. The thermal diffusion equation is altered to give a finite rate of propagation of thermal disturbances.

2.2 Let us use (1.1.12) to derive the equations of motion of a finite region of fluid. We consider a finite region moving with the fluid. We denote the center of mass of the region by the cartesian coordinates $\bar{x}^i$. We shall locate points within the region relative to the center of mass of the region by the coordinates ξ^i, so that the coordinates of the νth particle become

$$_{\nu}x^{i} = {}_{\nu}\bar{x}^{i} + {}_{\nu}\xi^{i} \tag{1}$$

The velocity of the νth particle is

$$_{\nu}v^{i} = \frac{d_{\nu}x^{i}}{dt} \tag{2}$$

which we shall find convenient to write as

$$_{\nu}v^{i} = \bar{v}^{i} + {}_{\nu}u^{i} \tag{3}$$

Clearly

$$\sum_{\nu} {}_{\nu}m_{\nu}\xi^{i} = \sum_{\nu} {}_{\nu}m_{\nu}u^{i} = 0 \tag{4}$$

We refer to the $_{\nu}u^{i}$ as the <u>local</u> and $\bar{v}^{i}$ as the <u>translocal velocity field</u> of the region. Thus, the local field is the portion of the velocity composed of fluctuations of smaller scale than the region. Or, in other words, the local field is the portion which does average out; the translocal field is the portion of the velocity field which does not average out over the region.

From (1.1.5) - (1.1.7) and (4) it follows that

$$J^{ij} = M\bar{x}^{i}\bar{v}^{j} + \sum_{\nu} {}_{\nu}m_{\nu}\xi^{i}{}_{\nu}u^{j}$$

$$\sum_{\nu} {}_{\nu}x^{i}{}_{\nu}F^{j} = \bar{x}^{i} \sum_{\nu} {}_{\nu}F^{j} + \sum_{\nu} {}_{\nu}\xi^{i}{}_{\nu}F^{j}$$

$$2T^{ij} = M\bar{v}^{i}\bar{v}^{j} + \sum_{\nu} {}_{\nu}m_{\nu}u^{i}{}_{\nu}u^{j}$$

(1.1.12) becomes after some rearranging of terms

$$M\frac{d\bar{v}^{i}}{dt} = \sum_{\nu} {}_{\nu}F^{i} + \frac{1}{\bar{x}^{j}}\left\{\sum_{\nu} {}_{\nu}m_{\nu}u^{i}{}_{\nu}u^{j} + \sum_{\nu} {}_{\nu}\xi^{j}{}_{\nu}F^{i} - \frac{d}{dt}\sum_{\nu} {}_{\nu}m_{\nu}\xi^{j}{}_{\nu}u^{i}\right\} \tag{5}$$

But, from Newton's equations we have

$$M \frac{d\bar{v}^i}{dt} = \sum_\nu {}_\nu F^i \tag{6}$$

Thus

$$\frac{d}{dt} \sum_\nu {}_\nu m_\nu \xi^i {}_\nu u^j = \sum_\nu {}_\nu m_\nu u^i {}_\nu u^j + \sum_\nu {}_\nu \xi^i {}_\nu F^j \tag{7}$$

for the region, independently of the motion of its center of mass.

We shall now use (7) to evaluate $\sum_\nu {}_\nu F^j$. We decompose ${}_\nu F^i$ into two portions,

$${}_\nu F^i = {}_\nu F^i_o + {}_\nu f^i \tag{8}$$

${}_\nu F^i_o$ represents the mean external force e.g. gravitational or electric fields. We shall call it the <u>translocal force field</u>. Finally ${}_\nu f^i$ denotes the short range forces of collision of the molecules.

Since ${}_\nu F_o{}^j$ is independent of position in the region, it being the average over the region and at most a function of the type of particle, we have

$$\sum_\nu {}_\nu F_o{}^j {}_\nu \xi^i = 0 \tag{9}$$

from (4). (7) becomes

$$\frac{d}{dt}\left(\sum_\nu {}_\nu m_\nu u^j {}_\nu \xi^i\right) = \left(\sum_\nu {}_\nu m_\nu u^i {}_\nu u^j\right) + \sum_\nu {}_\nu \xi^i {}_\nu f^j \tag{10}$$

Now, the ${}_\nu f^i$, being collisional forces, cancel out over the interior of the region. The only contribution can come from the surface of the region where collisions take place with particles not in the region, so that the equal and opposite force on the other colliding particle is not included in $\sum_\nu$. Thus, we find it convenient to define the usual stress tensor σ^{ij} such that

$\sigma^{i\alpha} dS_\alpha$ is the force in the i direction across an element of area dS_α normal to the α direction. $\sigma^{i\alpha}$ represents the force exerted by the matter on the positive side of dS_α on the matter on the negative side. This is the customary definition in elasticity[5] and electromagnetic theory. We shall replace $\sum_\nu$ by integration over the elements of volume dV.

The introduction of a stress tensor and the associated processes and parameters such as integration, differentiation, pressure, viscosity, etc. requires a limited form of continuity in our hitherto unrestricted system. Our notion of infinitesimal becomes that of the physical infinitesimal, viz. that the smallest elements of volume dV that we consider must be sufficiently large as to contain a large number of particles. If n represents the number of particles in dV, then the fluctuation of n is of the order of $\sqrt{n}$. We must require that

$$\sqrt{n} << n$$

in order that our averaging process over dV have a smoothing effect. To be treated as an infinitesimal, dV, of course, must be of smaller scale than the phenomena in which we are interested. The alternative to these restrictions on dV is to consider an ensemble of systems so that the average may be carried out over the given dV in each member system rather than just a single system.

We rewrite (10) as

$$\frac{d}{dt}\left(\int dV \rho \xi^i u^j\right) = \int dV(\rho u^i u^j) + \int dS_k \xi^i \sigma^{jk} \qquad (11)$$

Gauss' theorem gives

$$\frac{d}{dt}\left(\int dV\rho\xi^{i}u^{j}\right) = \int dV(\rho u^{i}u^{j}) + \int dV\frac{\partial}{\partial\xi^{k}}(\xi^{i}\sigma^{jk}) \tag{12}$$

$$= \int dV(\rho u^{i}u^{j}) + \int dV\sigma^{ji} + \int dV\xi^{i}\frac{\partial\sigma^{jk}}{\partial\xi^{k}} \tag{13}$$

$-\partial\sigma^{jk}/\partial\xi^{k}$ represents the force in the i direction per unit volume. The translocal portion of $-\partial\sigma^{jk}/\partial\xi^{k}$, given to accelerating the entire region, cannot contribute to $\int dV\xi^{i}\partial\sigma^{jk}/\partial\xi^{k}$. This is readily shown if we note that the translocal portion of $-\partial\sigma^{jk}/\partial\xi^{k}$ is defined as

$$\left(\frac{\partial\sigma^{jk}}{\partial\xi^{k}}\right)_{1} = \rho\frac{d\bar{v}^{j}}{dt} \tag{14}$$

Then

$$\int dV\xi^{i}\left(\frac{\partial\sigma^{jk}}{\partial\xi^{k}}\right)_{1} = \frac{d\bar{v}^{j}}{dt}\int dV\rho\xi^{i}$$

$$= 0$$

by (4).

The remaining portion of $\partial\sigma^{jk}/\partial\xi^{k}$ is local to the region and is responsible for changing the shape of the region. Since σ^{ij} is the only stress field present, it follows that if the local component of $\partial\sigma^{jk}/\partial\xi^{k}$ vanishes, then the region is in equilibrium. We have $dJ^{ij}/dt = 0$ or,

$$\frac{d}{dt}\int dV\rho\xi^{j}u^{i} = 0$$

and (13) becomes

$$0 = \int dV\rho u^{i}u^{j} + \int dV\sigma^{ij} \tag{15}$$

which has the solution*

$$\sigma^{ij} = -\rho u^i u^j \tag{16}$$

$$= -2T^{ij} \tag{17}$$

From (8) and (17) we obtain

$$\sum_\nu \nu F^i = \sum_\nu \nu F_o{}^i + \int dS_k \sigma^{ik}$$

$$= \sum_\nu \nu F_o{}^i - \int dV \frac{\partial}{\partial \xi^k}(\rho u^i u^k)$$

(6) becomes

$$M\frac{d\bar{v}^i}{dt} = \sum_\nu \nu F_o{}^i - \int dV \frac{\partial}{\partial \xi^k}(\rho u^i u^k) \tag{18}$$

Consider now $(\rho u^i u^j)$. If there is no shearing of the region, we expect no correlation over the region between u^i and u^j. Hence $(\rho u^i u^j)$ is zero if $i \neq j$. The diagonal terms are nonzero and give the pressure. If, however, there is an overall shearing of the region, we see that there will be a correlation between u^i and u^j for $i \neq j$. Let the shearing be represented by

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*We see, then, that $2T^{ij}$ represents the integral over the region of the Reynolds or kinetic stresses, which may be represented by pressure, $2T^{\alpha\alpha}$, and shearing stress $2T^{\alpha\beta}$. Φ^{ij}, as in (1.1.12), represent the integral over the region of all other stress fields, e.g. gravity, electric fields, etc. If we denote by Y^{ij} the sum of the Reynolds and all other stress fields, then (1.1.12) may be written

$$\frac{d}{dt} J_{ij} = Y_{ij}$$

If Y_{ij} vanishes, so also does dJ_{ij}/dt. This will be discussed at greater length in 3.1.

$\partial\bar{v}^{\alpha}/\partial\bar{x}^{\beta}$. If the average scale in the β direction of the local velocity field u^i is denoted by L^{β}, then with a velocity u^{β} we expect to find associated a velocity $-L^{\beta}\partial\bar{v}^{\alpha}/\partial\bar{x}^{\beta}$ in the α direction. Hence, we write

$$(\rho u^{\alpha}u^{\beta}) = -(\rho u^{\beta}L^{\beta})\frac{\partial\bar{v}^{\alpha}}{\partial\bar{x}^{\beta}} - (\rho u^{\alpha}L^{\alpha})\frac{\partial\bar{v}^{\beta}}{\partial\bar{x}^{\alpha}} \tag{19}$$

for $\alpha \neq \beta$. We define

$$^{\alpha\alpha}{}_{\mu} = (\rho u^{\alpha}L^{\alpha}) \tag{20}$$

and write more generally

$$(\rho u^{\alpha}u^{\beta}) = -\sigma^{\alpha\alpha}\delta^{\beta}_{\alpha} - {}^{\beta\beta}{}_{\mu}\frac{\partial\bar{v}^{\alpha}}{\partial\bar{x}^{\beta}} - {}^{\alpha\alpha}{}_{\mu}\frac{\partial\bar{v}^{\beta}}{\partial\bar{x}^{\alpha}} \tag{21}$$

Thus, we rewrite (18) as

$$M\frac{d\bar{v}^{\alpha}}{dt} = \sum_{\nu}\nu F_o{}^{\alpha} + \int dV\frac{\partial}{\partial\bar{x}^{\alpha}}\sigma^{\alpha\alpha} + \int dV\sum_{\beta}\frac{\partial}{\partial\bar{x}^{\beta}}{}^{\beta\beta}{}_{\mu}\frac{\partial\bar{v}^{\alpha}}{\partial\bar{x}^{\beta}} + \int dV\frac{\partial}{\partial\bar{x}^{i}}\left({}^{\alpha\alpha}{}_{\mu}\frac{\partial\bar{v}^{i}}{\partial\bar{x}^{\alpha}}\right) \tag{22}$$

which is the equation of motion for a finite region of fluid.

2.21 Let us consider some special cases of the equations of motion of a finite volume of fluid expressed in (2.2.22). If we consider an incompressible flow, then

$$\frac{\partial\bar{v}^{i}}{\partial\bar{x}^{i}} = 0$$

and the last term in (2.2.22) becomes

$$\int dV\frac{\partial}{\partial\bar{x}^{i}}\left({}^{\alpha\alpha}{}_{\mu}\frac{\partial\bar{v}^{i}}{\partial x^{\alpha}}\right) = \int dV\frac{\partial\bar{v}^{i}}{\partial x^{\alpha}}\frac{\partial}{\partial\bar{x}^{i}}{}^{\alpha\alpha}{}_{\mu} \tag{1}$$

If there are no compressibility effects, then to a fair degree of

approximation the variation of ${}^{\alpha\alpha}\mu$ in the direction of the flow may be neglected. Hence, we are able to rewrite (2.2.22) as

$$\mathbf{M}\frac{d\bar{v}^{\alpha}}{dt} = \sum_{\nu} {}^{\nu}F_{c}{}^{\alpha} + \int dV \frac{\partial}{\partial \bar{x}^{\alpha}} \sigma^{\alpha\alpha} + \int dV \sum_{\beta} \frac{\partial}{\partial \bar{x}^{\beta}} \left({}^{\beta\beta}\mu \frac{\partial \bar{v}^{\alpha}}{\partial \bar{x}^{\beta}} \right) \qquad (2)$$

We note that

$$\sigma^{\alpha\alpha} = - p^{\alpha\alpha}$$

where $p^{\alpha\alpha}$ is the pressure in the α direction. Thus, if there is an approach to isotropy in the local motions, we write

$$p = -\frac{1}{3}\sigma^{i}_{\cdot i} \qquad (3)$$

$$= \frac{1}{3}(\rho u^{i}u_{i}) \qquad (4)$$

and the second term in (2) becomes

$$\int dV \frac{\partial}{\partial \bar{x}^{\alpha}} \sigma^{\alpha\alpha} \cong - dV \frac{\partial p}{\partial \bar{x}^{\alpha}} \qquad (5)$$

It is of interest to compare (4) with Batchelor's value[6] computed by Fourier transform methods from the theory of isotropic turbulence.

Again, if there is an approach to isotropy in the local motions, we define

$$\bar{u} = (\sqrt{u^{i}u_{i}}) \qquad (6)$$

$$L = (\sqrt{L^{i}L_{i}}) \qquad (7)$$

so that we may write

$$u^{\alpha} = \frac{1}{\sqrt{3}}\bar{u} \qquad (8)$$

$$L^{\alpha} = \frac{1}{\sqrt{3}}L \qquad (9)$$

Assuming no correlation between u^α and L^α, we obtain

$$^{\alpha\alpha}\mu = \frac{1}{3}\rho\bar{u}L \tag{10}$$

$$\equiv \mu \tag{11}$$

and rewrite (2) for the case of isotropic local motions as

$$M\frac{d\bar{v}^\alpha}{dt} = \sum_\nu \nu F_o{}^\alpha - \int dV \frac{\partial p}{\partial \bar{x}^\alpha} + \int dV \frac{\partial}{\partial \bar{x}^i}\left(\mu \frac{\partial \bar{v}^\alpha}{\partial \bar{x}^i}\right) \tag{12}$$

If we let the size of the region approach zero, we obtain the usual Navier - Stokes equations for an incompressible fluid,

$$\frac{d\bar{v}^\alpha}{dt} = F^\alpha - \frac{1}{\rho}\frac{\partial p}{\partial x^\alpha} + \frac{\partial}{\partial x^i}\left(\nu\frac{\partial}{\partial x^i}\bar{v}^\alpha\right) \tag{13}$$

where F^α is the force per unit mass and ν is μ/ρ. p and μ are due only to the thermal motions, i.e. motions local to a scale of zero.

2.22 It should be noted that $^{\alpha\alpha}\mu$ and $p^{\alpha\alpha}$ or $|\sigma^{\alpha\alpha}|$, and therefore p and μ are monotonically increasing functions of the size of the region considered. $\bar{v}^\alpha$ and its space and time derivatives, on the other hand, are monotonically decreasing functions of the size of the region.

It is of interest to note how with the tensor virial equation the velocity field v^i naturally resolves itself into the translocal and local velocity fields $\bar{v}^i$ and u^i. We note that only the statistical characteristics of the local field appear in the field equations of the translocal field, (2.2.22) Thus, we are not surprised to see the emergence of Prandtl's mixing length concept in computing $(\rho u^i u^j)$ for $i \neq j$. Altogether, then, it

would seem that the statistical representation of Prandtl's mixing length theory and the more extensive treatment carried out in Heisenberg's field equation[7,8] and elsewhere[9] follow quite naturally from the tensor virial equations of motion.

2.3 The tensor virial equations are particularly suited to investigation of the dynamics of finite regions of matter. The following examples are typical of those encountered in astrophysics in the treatment of gas clouds, star clusters[2], etc. and in diffusion problems. In the treatment of the dynamics of gas clouds the tensor virial equations are convenient because even with very artificial constraints the question of whether a term is to be included as a potential energy or as a kinetic energy is easily decided.

2.31 Consider the radial oscillation of a gas cloud in its own gravitational field. The tensor virial equations do not give us the radial distribution matter. To prevent the mathematics from becoming too complex we introduce the constraint that the density be homogeneous within the sphere of radius $r_o(t)$ about the origin and zero outside. Thus, the radial velocity at any point within the cloud is

$$v(r) = \frac{r}{r_o}\frac{dr_o}{dt} \tag{1}$$

The motion of the system is thus describable by the single parameter r_o. Hence, we use the scalar virial equation (1.2.3). It is readily shown that

$$V = -\frac{3GM^2}{5r_o} \tag{2}$$

$$I = \frac{3}{10} M r_o^2 \tag{3}$$

where M is the mass of the sphere. The contribution of the radial oscillations to T is

$$T_1 = \frac{3}{10} M \left(\frac{dr_o}{dt}\right)^2 \tag{4}$$

Let us assume that the motions of smaller scale than r_o follow a polytrope law so that

$$\overline{(v)}^2 = \overline{(v)}_o^2 \left(\frac{r_{oo}}{r_o}\right)^{A - 1}$$

where

$$r_{oo} = r_o(t_o).$$

A corresponds to the usual γ written when only thermal motions are considered. For instance, if there is no dissipation, A = 5/3 for turbulence and for the thermal motions in monatomic gases. If more than one type of local motion is present, we write

$$2T_2 = M \sum_a \overline{(v_a)}_o^2 \left(\frac{r_{oo}}{r_o}\right)^{3(A_a - 1)} \tag{5}$$

(1.2.3) becomes

$$\frac{d^2 r_o}{dt^2} = \frac{5}{3} \sum_a \frac{\overline{(v_a)}_o^2 \; r_{oo}^{3(A_a-1)}}{r_o^{3A_a-2}} - \frac{GM}{r_o^2} \tag{6}$$

after cancelling out a factor of 3M/10. Upon integration we obtain the energy equation

$$\epsilon = \frac{3}{10}\left(\frac{dr_o}{dt}\right)^2 + \sum_a \frac{\overline{(v_a)}_o^2}{3(A_a - 1)}\left(\frac{r_{oo}}{r_o}\right)^{3(A_a-1)} - \frac{3GM}{5r_o} \tag{7}$$

where ϵ is the total energy per unit mass. We obtain $r_o(t)$ by quadrature as

$$t - t_o = \int_{r_{oo}}^{r_o} \frac{dr_o}{\left[\frac{10}{3}\epsilon - \frac{10}{3}\sum_a \frac{\overline{(v_a)_o^2}}{3(A_a - 1)}\left(\frac{r_{oo}}{r_o}\right)^{3(A_a - 1)} + \frac{2GM}{r_o}\right]^{1/2}} \tag{8}$$

2.32 One finds in his applications of the tensor virial equations that he usually uses rectangular parallelepipedal or ellipsoidal regions. In the case that the region is an isolated one, the parallelepiped becomes inconvenient because of the extreme complexity of the force field expressions from which the potential tensor is computed.[12] The expressions for the contribution to the potential tensor of the mutual inverse square interactions of the particles composing a homogeneous ellipsoid are recorded here. If external forces are present there will, of course, be an additional contribution to the potential tensor. The expressions derived in this section have been applied elsewhere to problems in galactic dynamics.[2]

Consider particles of matter distributed uniformly throughout the interior of an ellipsoid with semi-axes a^α. Without loss of generality we orient our axes so that

$$a^1 \geq a^2 \geq a^3 \tag{1}$$

It can be shown[12] for gravitational interaction that

$$\Phi^{\alpha\alpha} = -\frac{3}{10} GM^2 (a^\alpha)^2 N^\alpha \tag{2}$$

where M is the mass of the ellipsoid and

$$N^1 = \frac{2}{\left[(a^1)^2 - (a^3)^2\right]^{3/2}} \left\{\frac{1}{k^2}\left[v - E(\omega,k)\right]\right\} \tag{3}$$

$$N^2 = \frac{2}{\left[(a^1)^2 - (a^3)^2\right]^{3/2}} \left\{\frac{E(\omega,k)}{k^2(1-k^2)} - \frac{1}{(1-k^2)}\frac{\operatorname{sn} v \operatorname{cn} v}{\operatorname{dn} v} - \frac{v}{k^2}\right\} \tag{4}$$

$$N^3 = \frac{2}{\left[(a^1)^2 - (a^3)^2\right]^{3/2}} \left\{ \frac{\mathrm{snv\ dnv}}{\mathrm{cnv}} - E(\omega,k) \right\} \frac{1}{1 - k^2} \tag{5}$$

where

$$\sin\omega = \mathrm{snv} = \sqrt{1 - \left(\frac{a^3}{a^1}\right)^2}\ , \quad \mathrm{cnv} = \frac{a^3}{a^1}\ , \quad \mathrm{dnv} = \frac{a^2}{a^3} \tag{6}$$

$$k^2 = \frac{(a^1)^2 - (a^2)^2}{(a^1)^2 - (a^3)^2}\ , \quad v = F(\omega,k) \tag{7}$$

$F(\omega,k)$ and $E(\omega,k)$ are Legendre's elliptic integrals of the first and second kinds respectively.

If $\overline{(v^\alpha)^2}$ represents the mean velocity in the α direction, then for equilibrium of the ellipsoid, (1.1.12) gives

$$\overline{(v^\alpha)^2} = \frac{3}{10}\, GM\, (a^\alpha)^2\, N^\alpha \tag{8}$$

If the gas cloud has rotational symmetry e.g. a spheroidal galaxy, and

$$a^1 = a^2 > a^3, \tag{9}$$

then the $\Phi^{\alpha\alpha}$ reduce to

$$\Phi^{11} = \Phi^{22} = -\frac{3GM^2}{5a}\, \Psi^{11}\left(\frac{a^3}{a^1}\right) \tag{10}$$

$$\Phi^{33} = -\frac{3GM^2}{5a}\, \Psi^{33}\left(\frac{a^3}{a^1}\right) \tag{11}$$

where

$$\Psi^{11}(y) = \frac{1}{2(1 - y^2)^{3/2}}\left[\sin^{-1}(1-y^2)^{1/2} - y(1 - y^2)^{1/2}\right] \tag{12}$$

$$\Psi^{33}(y) = \frac{y^2}{(1 - y^2)^{3/2}}\left[\frac{1}{y}(1 - y^2)^{1/2} - \sin^{-1}(1 - y^2)^{1/2}\right] \tag{13}$$

(8) reduces to

$$\overline{(v^1)^2} = \overline{(v^2)^2} = \frac{3\,GM}{5a^1}\Psi^{11}\left(\frac{a^3}{a^1}\right) \tag{14}$$

$$\overline{(v^3)^2} = \frac{3GM}{5a^1} \Psi^{33}\left(\frac{a^3}{a^1}\right) \tag{15}$$

It would be well to reemphasize the fact that the $\overline{(v^\alpha)^2}$ are the mean square values of the local velocity field and need not be random in the usual sense. We only require that the average of v^α over the region vanish. Thus the rotational velocity of the region as well as smaller scale turbulence and thermal velocities are part of the local velocity field and make up $\overline{(v^\alpha)^2}$. For instance, if we consider a spheriodal galaxy in which the velocities local to the rotation are isotropic, it follows that the rotation velocity v_ω is

$$\overline{(v_\omega)^2} = 2\left[\overline{(v^1)^2} - \overline{(v^3)^2}\right]$$

$$- \frac{6GM}{5a^1}\left[\Psi^{11}\left(\frac{a^3}{a^1}\right) - \Psi^{33}\left(\frac{a^3}{a^1}\right)\right]$$

from (14) and (15).

Stress Tensor Formulation

3.1 Having discussed the tensor virial equations in terms of the kinetic and potential tensors, let us now state the relations in terms of stress tensors. This formulation is particularly useful when dealing with problems in magnetohydrodynamics.[13] We shall assume ponderamotive forces representable by the general stress tensor Z^{ij}. Then (1.1.12) may be written*

$$\frac{d}{dt}\int dV \rho v^i x^j = \int dV \rho v^i v^j + \int dS_k x^j \sigma^{ik} + \int dV x^j \frac{\partial}{\partial x^k} Z^{ik} \tag{1}$$

in cartesian coordinates. It should be noted that we cannot

*See formulation from Newton's equations in Appendix A.

express the contribution of Z^{ij} to the potential tensor as a surface integral. The net force on the region may be expressed as $\int dS_k Z^{jk}$, but in computing the potential tensor the position at which the force is exerted on the matter field is important because of ξ^i under the integral. Hence we have action at a distance appearing explicitly. Using (2.2.1), (2.2.3), and (2.2.4) we may rewrite (1) as

$$M \frac{d\bar{v}^i}{dt} = \int dS_k \sigma^{ik} + \int dV \frac{\partial}{\partial \bar{x}^k} Z^{ik}$$

$$+ \frac{1}{\bar{x}^j} \left\{ - \frac{d}{dt} \int dV \rho u^i \xi^j + \int dV \rho u^i u^j + \int dS_k \xi^j \sigma^{ik} \right.$$

$$\left. + \int dV \xi^j Z^{ik}_{,k} \right\}$$

where the comma indicates differentiation. From Newton's equations

$$M \frac{d\bar{v}^i}{dt} = \int dS_k (\sigma^{ik} + Z^{ik})$$

Thus, we have for the local motion the tensor equation valid in any coordinate system.

$$\frac{d}{dt} \int dV \rho u^i \xi^j (q^n) = \int dV \rho u^i u^j + \int dS_k \xi^j (q^n) \sigma^{ik}$$

$$+ \int dV \xi^j (q^n) Z^{ik}_{,k} \tag{2}$$

from which we see that <u>the tensor virial equations for the exterior velocity of a given region are independent of the interior velocity field and its time derivatives.</u> This was noted in passing in a more restricted case in 2.2.

We note that

$$\int dS_k \xi^j \sigma^{ik} = \int dV \sigma^{ij} + \int dV \xi^j \sigma^{ik}_{,k} \tag{3}$$

Using (3) and (2.2.16), (2) may be rewritten as

$$\frac{d}{dt} \int dV \rho\, u^i \xi^j(q^n) = \int dV \xi^j(q^n) \left[\sigma^{ik}_{,k} + Z^{ik}_{,k}\right] \tag{4}$$

or

$$\frac{d}{dt} \int dV \rho\, u^i \xi^j(q^n) = \int dS_k \xi^j(q^n) \; (\sigma^{ik} + Z^{ik}) \tag{5}$$

$$- \int dV (\sigma^{ij} + Z^{ij})$$

If we are considering an isolated system, the surface integral vanishes and

$$\frac{d}{dt} \int dV \rho\, u^i \xi^j(q^n) = - \int dV (\sigma^{ij} + Z^{ij}) \tag{6}$$

$-Z^i_i$ is just the energy per unit volume of the stress field Z^{ij}. Thus, if we contract (6), we have from 1.2 and (2.2.17) that

$$\frac{dJ}{dt} = 2T + S \tag{7}$$

where T is the total kinetic energy of the local motions and

$$S = -Z^i_i \tag{8}$$

The symmetric and anti-symmetric parts of (5) may be written as in 1.1 to give

$$\frac{d^2}{dt^2} I^{ij} = \frac{1}{2} \int dS_k \left[\xi^j(q^n)(\sigma^{ik} + Z^{ik}) + \;^{i}(q^n)(\sigma^{jk} + Z^{jk})\right] \tag{9}$$

$$- \int dV (\sigma^{ij} + Z^{ij})$$

$$\frac{d}{dt} K^{ij} = \frac{1}{2} \int dS_k \left[\xi^j(q^n)(\sigma^{ik} + Z^{ik}) - \xi^i(q^n)(\sigma^{jk} + Z^{jk})\right] \tag{10}$$

where

$$I^{ij} = \frac{1}{2} \int dV \rho \xi^{i}(q^{n}) \xi^{j}(q^{n}) \tag{11}$$

$$K^{ij} = \frac{1}{2} \int dV \rho \left[\xi^{i}(q^{n}) u^{j} - \xi^{j}(q^{n}) u^{i} \right] \tag{12}$$

For an isolated system the surface integrals vanish giving

$$\frac{d^2}{dt^2} I^{ij} = - \int dV \ (\delta^{ij} + Z^{ij}) \tag{13}$$

$$\frac{d}{dt} K^{ij} = 0 \tag{14}$$

Contracting these two equations, we obtain

$$\frac{d^2 I}{dt} = 2T + S \tag{15}$$

$$\frac{d}{dt} K \equiv 0 \tag{16}$$

3.11 In the previous section we have set up the tensor virial equations in terms of a general stress tensor Z^{ij}. It would be well if we note briefly the form of Z^{ij} for various fields. It is readily shown[9] that for electromagnetic fields the components of the three dimensional Maxwell stress tensor are given by

$$Z^{ij} = H^{i}B^{j} + D^{i}E^{j} - g^{ij}S \tag{1}$$

$$S = \frac{1}{2} g_{ij} (H^{i}B^{j} + D^{i}E^{j}) \tag{2}$$

in MKSQ units. g^{ij} is the metric tensor. S is the energy density of the field.

In most problems in magnetohydrodynamics it is possible to entirely neglect the displacement current[10]. The electromagnetic

field becomes, then, a magnetic field and is a pure stress field with no associated inertia to be included in our equations. We drop the terms involving the electric field so that

$$Z^{ij} = H^i B^j - g^{ij} S \tag{3}$$

$$S = \frac{1}{2} g_{ij} H^i B^j \tag{4}$$

Further simplification may be carried out since the permeability of the conducting fluid usually approximates very closely to empty space and is in any case isotropic and homogeneous.

In the case of a gravitational field we have the field intensity Ψ^i given in terms of a scalar ψ according to

$$\Psi_i = -\psi_{,i} \tag{5}$$

so that

$$g^{ij} \psi_{,i,j} = 4\pi G \rho \tag{6}$$

$$\Psi^i_{,i} = -4\pi G \rho \tag{7}$$

The force per unit volume is

$$F^i = \rho \Psi^i$$

$$= -\frac{1}{4\pi G} \Psi^i \Psi^j_{,j} \tag{8}$$

Thus, in order that

$$F^i = Z^{ik}_{,k} \tag{9}$$

the stress tensor is defined as

$$Z^{ij} = -\frac{1}{4\pi G} \left[\Psi^i \Psi^j - g^{ij} W \right] \tag{10}$$

where

$$W = \frac{1}{2}\Psi^{i}\Psi_{j} \tag{11}$$

In order to derive (8) from (9) and (10) we note from (4) that the curl of Ψ^{i} is zero.

The total energy of the field is

$$S = -Z^{i}{}_{i}$$

$$= -\frac{W}{4\pi G}$$

$$= -\frac{1}{8\pi G}\Psi^{i}\Psi_{i} \tag{12}$$

Conclusion

4.1 It was pointed out in 2.22 that the ideas basic to Prandtl's and Heisenberg's formulations of turbulence theory arise as a natural consequence of the tensor virial formulation of the equations of motion. Let us summarize the more important of these principles.

The effect of multiplying by ξ^{i} before summing over the particles in the region is to cancel out the effects of force and velocity fluctuations of larger scale than the region considered. The result is a separation of all effects into the local and translocal components. It was then shown in 3.1 that the local tensor virial equations are independent of the translocal field. (2.2.18) shows that the translocal field depends only on the statistical properties of the local field. We found in 2.2 that the statistical

properties of the local field relevant to a calculation of the translocal field may be constructed in a natural way from a characteristic length and velocity of the local field, both of which depend upon the scale of the region used.

Appendix A

In 1.1 the tensor virial equations were developed in cartesian coordinates. Let us now very briefly outline their development in a general coordinate system. Let the coordinates of the νth particle be ${}_\nu q^i$. Let the position of the νth particle relative to the origin be represented by the vector $\xi^i({}_\nu q^k)$. We note that

$$d\xi^i({}_\nu q^k) = \xi^i_{,j}\, d_\nu q^j \tag{1}$$

The Lagrangian equations of motion are

$$\frac{d}{dt}\left(\frac{\partial L}{\partial_\nu \dot{q}^i}\right) - \frac{\partial L}{\partial_\nu q^i} = {}_\nu Q^i \tag{2}$$

and are readily seen to be covariant and of rank one because L is a scalar, dt is an invariant, and the ${}_\nu Q_i$ satisfy (1.1.2) in which $\delta_\nu W$ is a scalar. Multiplying by $\xi^j({}_\nu q^k)$, (2) may be written as

$$\frac{d}{dt}\left[\xi^j({}_\nu q^k)\frac{\partial L}{\partial_\nu \dot{q}^i}\right] = \xi^i_{,j}\, {}_\nu \dot{q}^j \frac{\partial L}{\partial_\nu \dot{q}^i} + \xi^j({}_\nu q^k)\frac{\partial L}{\partial_\nu q^i} + \xi^j({}_\nu q^k)\, {}_\nu Q_i \tag{3}$$

(1.1.5) – (1.1.8) are redefined as

$$J^j_{\cdot i} = \sum_\nu \xi^j({}_\nu q^k)\frac{\partial L}{\partial_\nu \dot{q}^i} \tag{4}$$

$$2T^j_i = \sum_\nu \xi^j_{,k}\, {}_\nu \dot{q}^k \frac{\partial L}{\partial_\nu \dot{q}^i} \tag{5}$$

$$\Phi^j_{\cdot i} = \sum_\nu \xi^j({}_\nu q^k)\frac{\partial L}{\partial_\nu q^i} \tag{6}$$

$$\Omega^{j}_{.i} = \sum_{\nu} \xi^{j}\,({}_{\nu}q^{k})\,{}_{\nu}Q_{i} \tag{7}$$

Thus (3) is the mixed tensor equation of rank two

$$\frac{d}{dt}\,J^{j}_{.i} = 2T^{j}_{i} + \Phi^{j}_{.i} + \Omega^{j}_{.i} \tag{8}$$

Had we begun with Newton's equations

$${}_{\nu}F^{i} = {}_{\nu}m\frac{d}{dt}\,{}_{\nu}v^{i} \tag{9}$$

which are contravariant, we should have obtained the contravariant form of (8), in which

$$J^{ij} = \sum_{\nu} {}_{\nu}m\xi^{j}({}_{\nu}q^{k})\,\dot{\xi}^{i}({}_{\nu}q^{n}) \tag{10}$$

$$2T^{ij} = \sum_{\nu} {}_{\nu}m\dot{\xi}^{i}({}_{\nu}q^{k})\,\dot{\xi}^{j}({}_{\nu}q^{n}) \tag{11}$$

$$\Phi^{ij} = \sum_{\nu} {}_{\nu}F^{i}\xi^{j}({}_{\nu}\xi^{k}) \tag{12}$$

Appendix B

In 1.1 we tacitly assumed that the dynamics of our system of particles could be described in terms of the space coordinates ${}_{\nu}q^{i}$ of each particle and the time derivatives ${}_{\nu}\dot{q}^{i}$. These coordinates do not include spin coordinates for the practical reason that we would then be faced with working in a space with a much more complicated geometry. It follows, then, that if dipoles, quadrupoles, etc. are to be considered, they must be decomposed into two, four, or more "elementary" particles with suitable constraints.

With this decomposition it follows that the field of any particle must be spherically symmetric in at least the proper coordinate system of the particle. Hence the angular momentum of the system is conserved and the portion of Φ^{ij} due to the interaction with each other of the particles in the region is symmetric in its indices. We have denoted the symmetric portion of Φ^{ij} by Υ^{ij} in (1.1.20). Thus, if ${}_{\eta\nu}V\ ({}_{\eta\nu}r)$ denotes the potential energy of the interaction of the νth and ηth particle, we have

$$ {}_{\eta\nu}V = {}_{\nu\eta}V $$

We define V_1 to be the potential energy of the interaction of the particles with themselves. Thus

$$ V_1 = \frac{1}{2}\sum_{\eta,\nu} {}_{\eta\nu}V\ ({}_{\eta\nu}r) \tag{1} $$

where

$$ ({}_{\eta\nu}r)^2 = g^{ij}\left[\xi^{i}({}_{\nu}q^{k}) - \xi^{i}({}_{\eta}q^{k})\right]\left[\xi^{j}({}_{\nu}q^{l}) - \xi^{j}({}_{\eta}q^{l})\right] $$

V_1 is the local portion of V.

If we define L_1 as the local portion of L, i.e. due to the interaction of the particles in the region with each other, then from (1.1.7) we have in cartesian coordinates that

$$\Upsilon^{ij} - \sum_{\nu} {}_{\nu}x^{i} \frac{\partial L_1}{\partial_{\nu}x^{j}}$$

$$\Upsilon^{ij} = -\sum_{\nu} {}_{\nu}x^{i} \frac{\partial V_1}{\partial_{\nu}x^{j}}$$

$$= -\sum_{\eta,\nu} {}_{\nu}x^{i} \frac{\partial}{\partial_{\eta\nu}r} \left[{}_{\eta\nu}V({}_{\eta\nu}r) \right] \frac{\partial_{\eta\nu}r}{\partial_{\nu}x^{j}}$$

$$= -\sum_{\eta,\nu} {}_{\eta\nu}r \frac{\partial_{\eta\nu}V({}_{\eta\nu}r)}{\partial_{\eta\nu}r} \frac{({}_{\nu}x^{i} - {}_{\eta}x^{i})\,{}_{\nu}x^{j}}{({}_{\eta\nu}r)^{2}}$$

$$= -\frac{1}{2}\sum_{\eta,\nu} {}_{\eta\nu}r \frac{\partial_{\eta\nu}V({}_{\eta\nu}r)}{\partial_{\eta\nu}r} \left\{ \frac{({}_{\nu}x^{i} - {}_{\eta}x^{i})({}_{\nu}x^{j} - {}_{\eta}x^{j})}{({}_{\eta\nu}r)^{2}} + \frac{({}_{\nu}x^{i} - {}_{\eta}x^{i})({}_{\nu}x^{j} + {}_{\eta}x^{j})}{({}_{\eta\nu}r)^{2}} \right\}$$

$$= -\frac{1}{2}\sum_{\eta,\nu} {}_{\eta\nu}r \frac{\partial_{\eta\nu}V({}_{\eta\nu}r)}{\partial_{\eta\nu}r} \frac{({}_{\nu}x^{i} - {}_{\eta}x^{i})({}_{\nu}x^{j} - {}_{\eta}x^{j})}{({}_{\eta\nu}r)^{2}} \qquad (2)$$

We see that $\Upsilon^{ij} = \Upsilon^{ji}$ as required. Let us diagonalize Υ^{ij} by referring it to its principal axes. We have, then, only the three terms $\Upsilon^{\alpha\alpha}$. Repeated Greek indices do not imply summation convention. $\Upsilon^{\alpha\alpha}$ is the contribution to the potential tensor of the forces and displacements in the α direction.

If the ${}_{\eta\nu}V({}_{\eta\nu}r)$ are homogeneous of degree n, then

$$\Upsilon^{\alpha\alpha} := -\frac{n}{2}\sum_{\eta,\nu} {}_{\eta\nu}V({}_{\eta\nu}r) \left[\frac{{}_{\nu}x^{\alpha} - {}_{\eta}x^{\alpha}}{{}_{\eta\nu}r} \right]^{2} \qquad (3)$$

We define the diagonal tensor of the second rank $(\gamma^{\alpha})^2$ by

$$(\gamma^{\alpha})^2 = \frac{\sum_{\eta,\nu} \eta\nu V({}_{\eta\nu}r)\left(\frac{{}_{\nu}x^{\alpha} - {}_{\eta}x^{\alpha}}{{}_{\eta\nu}r}\right)^2}{\sum_{\eta,\nu} \eta\nu V({}_{\eta\nu}r)} \tag{4}$$

Using (1) and (4), (3) may be written

$$\gamma^{\alpha\alpha} = -\, n(\gamma^{\alpha})^2\, V_1 \tag{5}$$

The γ^{α} may be interpreted in a sense as the direction cosines of the eccentricity of the region. We note from (5) that

$$\sum_{\alpha=1}^{3} (\gamma^{\alpha})^2 = 1$$

Dynamical symmetry of the region gives $\gamma^1 = \gamma^2 = \gamma^3$. An elongation of the region in the β direction makes γ^{β} larger than the other two γ^i, etc.

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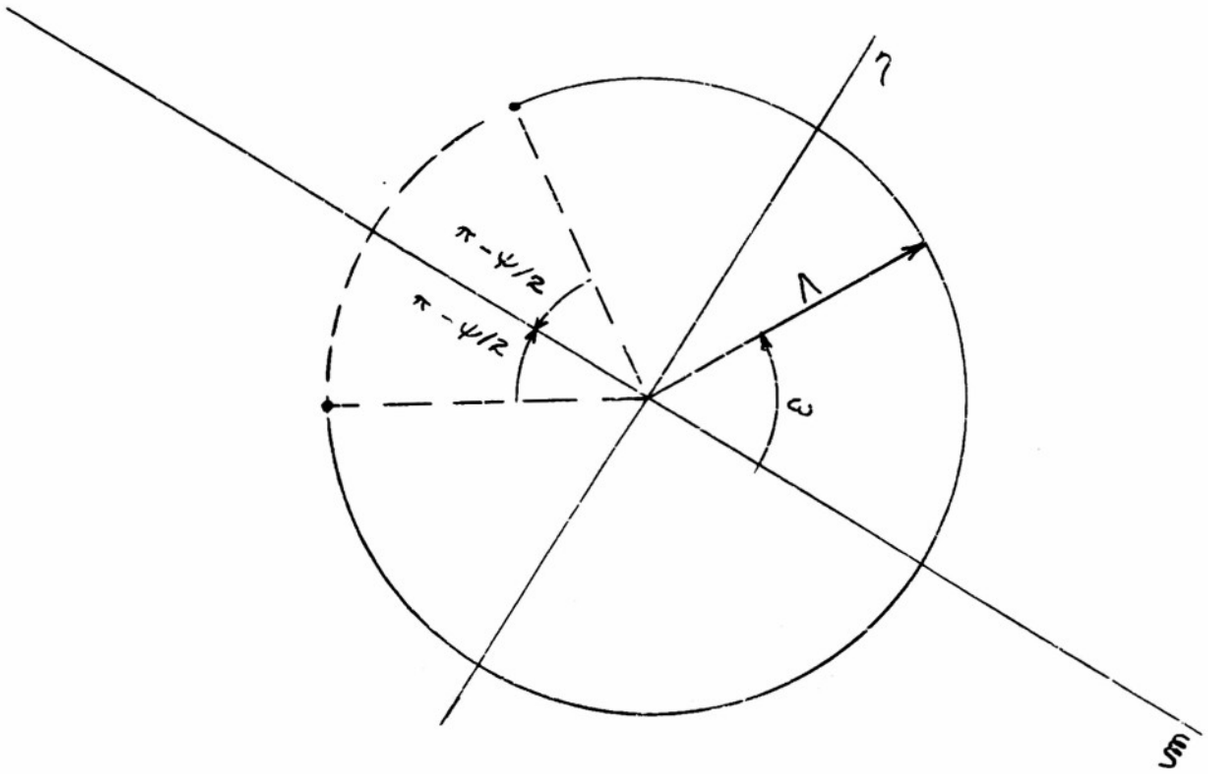

Figure 1

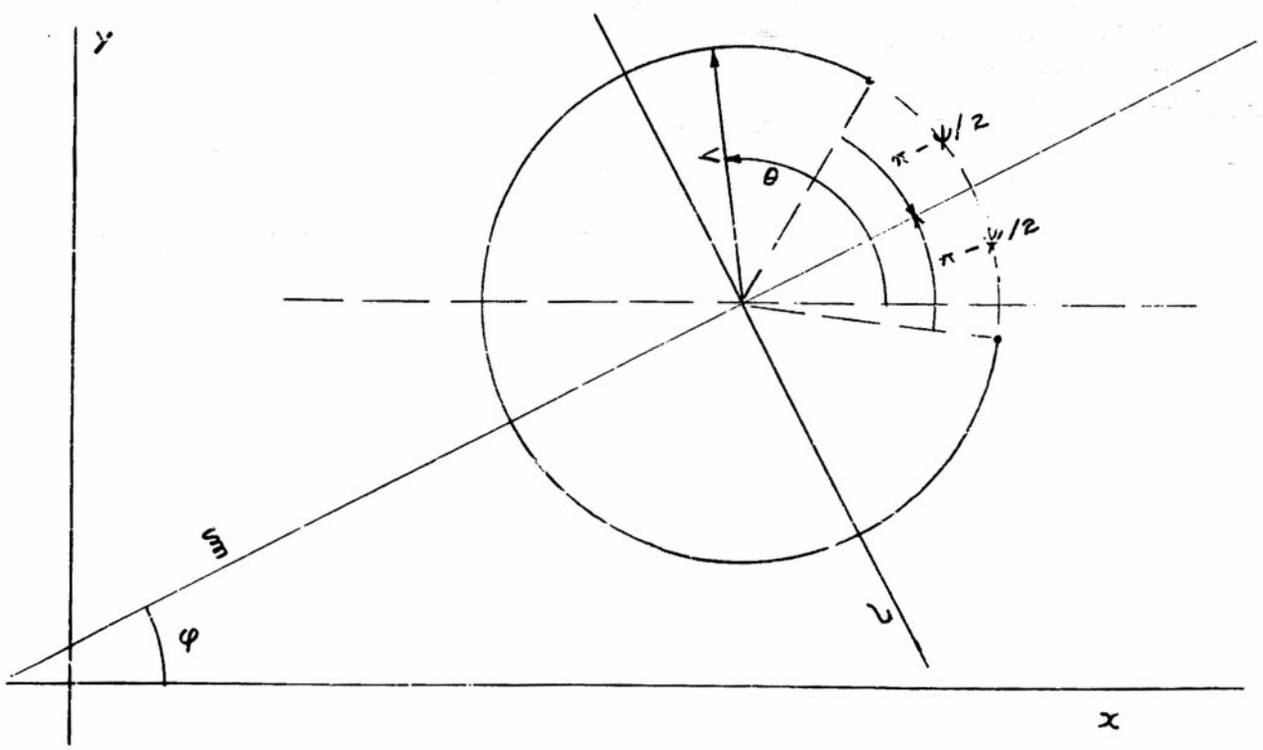

Figure 2