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INVARIANT SUBSPACES OF COMPLETELY CONTINUOUS OPERATORS

by

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The spectral analysis of self adjoint and normal operators in a Hilbert space gives essentially all the information which is usually needed about the behavior of the operator. However, in case of general linear operators (even bounded) for which there is no spectral analysis available, one must look to some other methods to describe the structure of the operator.

The specific consideration of Hilbert spaces which is of such help in the investigation of self adjoint and normal operators does not seem to be of special importance when non-normal operators are considered. It is becoming more and more an accepted opinion that for such operators a general theory should be developed within the framework of Banach spaces.

One manner of attacking this problem is by the investigation of invariant subspaces. This method presents many difficulties which have yet to be overcome. The basic problem of the existence of such subspaces is not settled in general.

The present report is a contribution to this problem of existence for the special class of completely continuous operators in general Banach spaces.

INVARIANT SUBSPACES OF COMPLETELY CONTINUOUS OPERATORS[1]

by

N. Aronszajn and K. T. Smith

Let T be a linear bounded operator in a Banach space $\mathcal{B}$, $T(\mathcal{B}) \subset \mathcal{B}$. A closed linear subspace $\mathcal{L} \subset \mathcal{B}$ is said to be an invariant subspace of T if $T(\mathcal{L}) \subset \mathcal{L}$. $\mathcal{L}$ is a proper invariant subspace if $(0) \neq \mathcal{L} \neq \mathcal{B}$. If $\mathcal{B}$ is a Hilbert space and T is a self adjoint operator, an invariant subspace reduces T and hence the invariant subspaces coincide with the spectral subspaces. However, if T is only assumed to be a normal operator in a Hilbert space $\mathcal{B}$, then there may be invariant subspaces which do not reduce T.

At the present time the investigation of invariant subspaces is not very advanced and seems to present essential difficulties. In recent years interest in this study has increased since the subspaces appear in a natural way in connection with prediction theory (see A. N. Kolmogoroff [4] and N. Wiener [5]), and its interpretation in terms of unitary operators in a Hilbert space. Besides those cases which can be reduced essentially to the treatment of operators in a finite dimensional space, or self adjoint operators in a Hilbert space, there are very few for which the invariant subspaces have been completely described. Such a description was given by A. Beurling [2] in case of special isometric operators in a Hilbert space. For general bounded operators, even in a Hilbert space, it is not as yet known that there always exists a proper invariant subspace.

Some years ago, J. von Neumann informed the first author of this paper that in the early thirties he proved the existence of proper invariant subspaces for completely continuous operators in a Hilbert space; the proof was never published. In 1950, the first author found a proof of the theorem in this case which used orthogonal projections and hence could not be extended directly to Banach spaces. According to a conversation with J. von Neumann, this was essentially the same proof that he found earlier. Quite recently, the first author was able to give a proof for reflexive Banach spaces (which was

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not published). In the present paper the last proof is extended to general Banach spaces.[1]

Since the original proof for a Hilbert space differs in many aspects from the general proof, and since it was never published, it will be briefly indicated at the conclusion of this paper.

THEOREM. Let $\mathcal{B}$ be a Banach space and T a completely continuous operator in $\mathcal{B}$. There exist proper invariant subspaces of T.

Proof. We shall limit ourselves to an infinite dimensional space since in finite dimensional spaces our theorem is a simple consequence of the classical theory of elementary divisors.

Consider an aribtrary $f \neq 0$ in $\mathcal{B}$. The closed subspace $[T^n f]_0^\infty$ generated by f and its successive images, $Tf, T^2f, T^3f, \ldots,$ is clearly an invariant subspace of T. We can therefore limit ourselves to the case where

(1) $$[T^n f]_0^\infty = \mathcal{B}.$$

This formula implies the following properties:

(2) $\mathcal{B}$ is separable.

(3) All the elements $T^n f$ are $\neq 0$ and are linearly independent.

The proof of (2) is immediate. To prove (3), suppose that we have the relation $a_1T^{n_1}f + a_2T^{n_2}f + \ldots + a_kT^{n_k}f = 0$ with all the coefficients $\neq 0$, and $0 \leq n_1 < n_2 < \ldots < n_k$. It would follow that $T^{n_k}f = \frac{-1}{a_k}(a_1T^{n_1}f + \ldots + a_{k-1}T^{n_{k-1}}f)$ and hence all the $T^n f$'s would lie in the subspace generated by those with indices $n < n_k$ which is in contradiction to (1) and the infinite dimension of $\mathcal{B}$.

In the proof we shall not need the weak topology in $\mathcal{B}$, hence convergence will mean strong convergence in $\mathcal{B}$. We understand complete continuity (Banach's total continuity) in the sense that any bounded set is transformed

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1. In view of the theory of completely continuous operators as developed by S. Banach [1], this theorem obviously gives a new result only for completely continuous operators which are quasi-nilpotent (i.e. with spectrum reduced to the single point 0). A simple case of such an operator is given by the integral operator of Volterra type $Tf = \int_0^x f\,dx$.

by T into a relatively compact set (a set with compact closure).[1]

Since in every separable Banach space we can define an equivalent strictly convex norm (i.e. such that if $x \neq y$ and $\|x\| = \|y\| \neq 0$, then $\|x+y\| < \|x\| + \|y\|$ (see J. A. Clarkson [3]) we shall suppose, in view of (2) that the norm in $\mathcal{B}$ is strictly convex.

Consider an arbitrary finite dimensional subspace $\mathcal{L} \subset \mathcal{B}$. For every $x \in \mathcal{B}$ we can consider the minimal distance $\rho(x, \mathcal{L})$ from x to $\mathcal{L}$. Since $\mathcal{L}$ is of finite dimension, the shortest distance is certainly attained and in view of the strict convexity of the norm it is immediately proved that there exists a unique point $Px \in \mathcal{L}$ which realizes this minimal distance, i.e.

$$\|x-Px\| = \rho(x, \mathcal{L}) = \min_{y \in \mathcal{L}} \|x-y\| . \tag{4}$$

Px represents an operator in $\mathcal{B}$, in general non-linear. We shall call P the metric projection on $\mathcal{L}$, or briefly, (when this is not misleading) the projection on $\mathcal{L}$. We list here a few properties of this projection which are immediate consequences of its definition.

a-1) P is idempotent: $P^2 = P$.

a-2) P is homogeneous: $P(\alpha x) = \alpha P x$ for every α.

a-3) P is quasi-additive: $P(y+x) = y + Px$ for every $y \in \mathcal{L}$.

a-4) $\|Px - x\| \leq \|x\|$, $\|Px\| \leq 2\|x\|$.

a-5) $\big| \|x-Px\| - \|y-Py\| \big| \leq \|x-y\|$.

a-6) If $\mathcal{L}' \subset \mathcal{L}$ and P' is the projection on $\mathcal{L}'$, then $\|x-Px\| \leq \|x-P'x\|$.

Clearly a-5) is the general property of the shortest distance $\rho(x, \mathcal{A})$ from x to a fixed set $\mathcal{A}$

Consider now a sequence of closed subspaces $\mathcal{L}_k \subset \mathcal{B}$. We introduce the limit inferior of the sequence $\mathcal{L}_k$ as follows:

$$\lim \mathcal{L}_k = \text{set of all } x \in \mathcal{B} \text{ such that for some } x_k \in \mathcal{L}_k,\ x_k \to x. \tag{5}$$

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1. A more classical definition is: weak convergence of x_n to x implies strong convergence of Tx_n to Tx. In reflexive spaces, the two definitions are equivalent, but in non-reflexive Banach spaces the former implies the latter, the converse being in general false.

We now list two properties of this limit which can be immediately verified.

b-1) $\lim \mathcal{L}_k$ is a closed subspace.

b-2) If every $\mathcal{L}_k$ is finite dimensional, then $x \in \lim \mathcal{L}_k$ if and only if $P_k x \to x$, where P_k is the projection on $\mathcal{L}_k$.

We pass now to the actual proof of the theorem.

With f satisfying (1) we construct the k-dimensional subspace

$$\mathcal{L}^{(k)} = [T^n f]_0^{k-1} \quad . \tag{6}$$

We denote by $P^{(k)}$ the metric projection on $\mathcal{L}^{(k)}$. By (1) it is clear that $\lim \mathcal{L}^{(k)} = \mathcal{B}$ or that (see b-2))

$$P^{(k)} x \to x \quad \text{for all} \quad x \in \mathcal{B} \; . \tag{7}$$

We consider then the operator T_k in $\mathcal{L}^{(k)}$ defined by

$$T_k x = P^{(k)} T x \quad \text{for} \quad x \in \mathcal{L}^{(k)} . \tag{8}$$

We prove that T_k is linear. In fact, if $x = \sum_0^{k-1} \xi_i T^i f$, then

$$T_k x = P^{(k)} T x = P^{(k)} \left(\sum_0^{k-1} \xi_i T^{i+1} f \right) = \sum_0^{k-2} \xi_i T^{i+1} f + \xi_{k-1} P^{(k)} T^k f \; ;$$

we use here the properties a-3) and a-2). This shows that T_k is linear.

T_k being a linear operator in the k-dimensional space $\mathcal{L}^{(k)}$, we can use the classical result that it may be represented by a triangular matrix, which gives that there exists an increasing sequence of subspaces,

$$(0) = \mathcal{L}^{(k,0)} \subset \mathcal{L}^{(k,1)} \subset \ldots \subset \mathcal{L}^{(k,k)} = \mathcal{L}^{(k)} \tag{9}$$

where $\mathcal{L}^{(k,i)}$ is an invarian subspace of T_k of dimension i.

We denote by $P^{(k,i)}$ the projection on $\mathcal{L}^{(k,i)}$.

LEMMA.[1] Let $\{k_m\}$ and $\{i_m\}$ be sequences of integers such that $k_m \nearrow \infty$ and $0 \leq i_m \leq k_m$. Further, let $x_m \in \mathcal{L}^{(k_m, i_m)}$. If $Tx_m \to y$ then $y \in \lim \mathcal{L}^{(k_m, i_m)}$

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1. The construction of subpaces (9) and the lemma are valid for any linear operator T (even not continuous).

In fact, we have $P^{(k_m)}Tx_m = Tx_m \in \mathcal{L}^{(k_m,i_m)}$. On the other hand, by a-5) and (7)

$$\|Tx_m - P^{(k_m)}Tx_m\| \leq \|y - P^{(k_m)}y\| + \|Tx_m - y\| \longrightarrow 0 .$$

$$\|y - P^{(k_m)}Tx_m\| \leq \|y - Tx_m\| + \|Tx_m - P^{(k_m)}Tx_m\| \longrightarrow 0,$$

which proves the lemma.

<u>Corollary 1.</u>[1] <u>For any sequences $\{k_m\}$ and $\{i_m\}$ satisfying the conditions of the lemma, $\lim \mathcal{L}^{(k_m,i_m)}$ is an invariant subspace of</u> T.

In fact, if $x \in \lim \mathcal{L}^{(k_m,i_m)}$, i.e. if for some $x_m \in \mathcal{L}^{(k_m,i_m)}$, $x_m \longrightarrow x$, we have by continuity of T, $Tx_m \longrightarrow Tx$, and by the lemma, $Tx \in \underline{\lim}\, \mathcal{L}^{(k_m,i_m)}$.

<u>Corollary 2.</u>[2] <u>If the lim of every subsequence of $\{\mathcal{L}^{(k_m,i_m)}\}$ is $\neq (0)$, then for any bounded sequence $\{x_m\}$, $x_m \in \mathcal{L}^{(k_m,i_m)}$, we have</u> $Tx_m \longrightarrow 0$.

By complete continuity of T, the sequence $\{x_m\}$ is transformed into a relatively compact sequence $\{Tx_m\}$. Therefore it is enough to prove that if any subsequence $\{Tx_{m_j}\}$ converges to some y, then $y = 0$. But this follows from our hypothesis, since by the lemma, $y \in \underline{\lim}\, \mathcal{L}^{(k_{m_j},i_{m_j})}$.

We choose now an arbitrary real a with

(10) $$0 < a < 1, \qquad \|Tf\| > a\|T\|\,\|f\| .$$

Since $f \in \mathcal{L}^{(k)}$, we have by (9) and a-6)

$$\|f\| = \|f - P^{(k,0)}f\| \geq \|f - P^{(k,1)}f\| \geq \ldots \geq \|f - P^{(k,k)}f\| = 0 .$$

There exists therefore for each $k = 1, 2, \ldots,$ a unique index $i(k)$, $0 \leq i(k) < k$, such that

(11) $$\|f - P^{(k,i(k))}f\| \geq a\|f\| > \|f - P^{(k,i(k)+1)}f\| .$$

Let u_k, $k = 1, 2, \ldots,$ be an element of $\mathcal{L}^{(k,i(k)+1)}$ such that

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1. This corollary is valid for any <u>bounded</u> linear T.

2. It is only in this corollary that complete continuity is essential for our proof.

(12) $$\|u_k\| = 1, \quad P^{(k,i(k))}u_k = 0$$

Such an element can be obtained from an arbitrary element $v \in \mathcal{L}^{(k,i(k)+1)} - \mathcal{L}^{(k,i(k))}$ by putting $u_k = \|v - P^{(k,i(k))}v\|^{-1}(v - P^{(k,i(k))}v)$. property (12) is then proved by using a-2) and a-3).

Since the dimensions of $\mathcal{L}^{(k,i(k))}$ and $\mathcal{L}^{(k,i(k)+1)}$ differ by 1, every element $y \in \mathcal{L}^{(k,i(k)+1)}$ is representable in a unique way in the form $y = x + \beta u_k$ with $x = P^{(k,i(k))}y$. Correspondingly, we shall put

(13) $$P^{(k,i(k)+1)}f = x_k + \beta_k u_k, \quad P^{(k,i(k)+1)}Tf = x'_k + \beta'_k u_k, \quad x_k \text{ and } x'_k \in \mathcal{L}^{(k,i(k))}.$$

We have, by a-4)

(14) $$\|x_k\| = \|P^{(k,i(k))}P^{(k,i(k)+1)}f\| \leq 4\|f\|, \quad \|x'_k\| \leq 4\|Tf\|.$$

We prove now the following statements:

I. For every sequence $k_m \nearrow \infty$, $\lim \mathcal{L}^{(k_m,i(k_m))} \neq \mathcal{B}$.

II. For some sequence $k'_m \nearrow \infty$, $\lim \mathcal{L}^{(k'_m,i(k'_m)+1)} \neq (0)$.

III. If for every sequence $k_m \nearrow \infty$, $\lim \mathcal{L}^{(k_m,i(k_m))} = (0)$, then for every sequence $k'_m \nearrow \infty$, $\lim \mathcal{L}^{(k'_m,i(k'_m)+1)} \neq \mathcal{B}$.

Proof of I. If $\lim \mathcal{L}^{(k_m,i(k_m))} = \mathcal{B}$, then by b-2) $P^{(k_m,i(k_m))} \to I$ which contradicts (11).

Proof of II. If our statement were not true, we would have by Corollary 2 that the bounded sequence $\{P^{(k,i(k)+1)}f\}$ (see a-4)) is transformed into a sequence $\{TP^{(k,i(k)+1)}f\}$ converging to 0. Since $Tf = T(f - P^{(k,i(k)+1)}f) + TP^{(k,i(k)+1)}f$, we get $\|Tf\| = \lim \|T(f - P^{(k,i(k)+1)}f)\| \leq \liminf \|T\| \, \|f - P^{(k,i(k)+1)}f\|$ which, by (11), gives $\|Tf\| \leq a\|T\| \, \|f\|$ in contradiction to (10).

Proof of III. Suppose that for some $\{k'_m\} \nearrow \infty$, $\lim \mathcal{L}^{(k'_m,i(k'_m)+1)} = \mathcal{B}$. By b-2) we have $P^{(k'_m,i(k'_m)+1)}f \to f$ and $P^{(k'_m,i(k'_m)+1)}Tf \to Tf$. By (13) we have then $f = \lim(x_{k'_m} + \beta_{k'_m}u_{k'_m})$, $Tf = \lim(x'_{k'_m} + \beta'_{k'_m}u_{k'_m})$; hence,

$Tf = \lim(Tx_{k'_m} + \beta_{k'_m}Tu_{k'_m})$ and $T^2f = \lim(Tx'_{k'_m} + \beta'_{k'_m}Tu_{k'_m})$. By (14) and

Corollary 2, it follows $Tf = \lim \beta_{k'_m} Tu_{k'_m}$, $T^2 f = \lim \beta'_{k'_m} Tu_{k'_m}$. Hence $\beta'_{k'_m} / \beta_{k'_m}$ converge to some γ and $T^2 f = \gamma Tf$ in contradiction to (3).

We achieve the proof of our theorem as follows. If there is any sequence $k_m \nearrow \infty$ such that $\mathcal{S} = \underline{\lim}\, \mathcal{L}^{(k_m, i(k_m))} \neq (0)$, then in view of Statement I and Corollary 1, $\mathcal{S}$ is a proper invariant subspace. If there is no such sequence $\{k_m\}$ then by Statement II we choose a sequence $k'_m \nearrow \infty$ so that $\mathcal{S}' = \underline{\lim}\, \mathcal{L}^{(k'_m, i(k'_m)+1)} \neq (0)$. By Statement III and Corollary 1, $\mathcal{S}'$ is then a proper invariant subspace.

<u>Proof in case of a Hilbert space $\mathcal{B}$</u>. In this proof we use weak and strong convergence of elements and operators in $\mathcal{B}$, denoted by the symbols $\rightharpoonup$ and $\rightarrow$. The simplifying feature in the present case is that the metric projections coincide with usual orthogonal projections and hence are linear. The $\mathcal{L}^{(k)}$ can now be any increasing sequence of subspaces with union dense in $\mathcal{B}$; f may be any element $\neq 0$ and belonging to $\mathcal{L}^{(1)}$. The operator T_k is now the restriction of $P^{(k)} T P^{(k)}$ to $\mathcal{L}^{(k)}$. The subspaces $\mathcal{L}^{(k,i)}$ are defined as before.

The lemma is replaced by the following: <u>If</u> $P^{(k_m, i_m)} \rightharpoonup Q$, <u>then</u> $QTQ = TQ$ (the operator Q is necessarily positive with bound ≤ 1).

In the proof the fact is used that $P^{(k,i)} T P^{(k,i)} = P^{(k)} T P^{(k,i)}$ and that $P^{(k)} \rightarrow I$.

As Corollary, we obtain that: $T(Q(\mathcal{B})) \subset \mathcal{S}$ where $\mathcal{S}$ is the closed subspace of all x's with $Qx = x$. Hence $\mathcal{S}$ is a proper invariant subspace except when $Q = 0$ or $Q = I$, or else when $0 \neq Q \neq I$ and T vanishes on the range of Q. In the last case every one-dimensional subspace of $Q(\mathcal{B})$ is clearly an invariant subspace.

The proof is continued by defining $i(k)$ as in (11); the number a need not be restricted by the second part of (10). We then choose a subsequence $\{k_m\}$ so that for some Q and Q', $P^{(k_m, i(k_m))} \rightharpoonup Q$, and $P^{(k_m, i(k_m)+1)} \rightharpoonup Q'$. In view of the corollary, it remains only to investigate the case when both Q and Q' are 0 or I. Here we use the general lemma that: <u>if projections converge weakly to a projection, they converge strongly</u>.

We then prove the following statements: I) $Q \neq I$; II) $Q' \neq 0$; III) *If* $Q = 0$, *then* $Q' \neq I$.

The proofs of I) and II) are immediate. For the proof of III) we notice that otherwise $I = Q' - Q$, and hence I would be the strong limit of the one-dimensional projections $P^{(k_m, M(k_m)+1)} - P^{(k_m, M(k_m))}$ which is impossible. This completes the proof.

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