EFFECTS OF DAMPING CONSTANTS AND STRESS DISTRIBUTION ON THE RESONANCE RESPONSE OF MEMBERS

B. J. LAZAN
UNIVERSITY OF MINNESOTA

DECEMBER 1952

Statement A
Approved for Public Release

WRIGHT AIR DEVELOPMENT CENTER
NOTICES

When Government drawings, specifications, or other data are used for any purpose other than in connection with a definitely related Government procurement operation, the United States Government thereby incurs no responsibility nor any obligation whatsoever; and the fact that the Government may have formulated, furnished, or in any way supplied the said drawings, specifications, or other data, is not to be regarded by implication or otherwise as in any manner licensing the holder or any other person or corporation, or conveying any rights or permission to manufacture, use, or sell any patented invention that may in any way be related thereto.

The information furnished herewith is made available for study upon the understanding that the Government's proprietary interests in and relating thereto shall not be impaired. It is desired that the Judge Advocate (WCJ), Wright Air Development Center, Wright-Patterson Air Force Base, Ohio, be promptly notified of any apparent conflict between the Government's proprietary interests and those of others.
EFFECT OF DAMPING CONSTANTS AND STRESS DISTRIBUTION ON THE RESONANCE RESPONSE OF MEMBERS

B. J. Lazan
University of Minnesota

December 1952

Materials Laboratory
Contract No. AF 33(038)-18903
RDO No. 614-16

Wright Air Development Center
Air Research and Development Command
United States Air Force
Wright-Patterson Air Force Base, Ohio
FOREWORD

This report was prepared by the University of Minnesota under Contract No. AF 33(038)-18903 with the Wright Air Development Center, Wright-Patterson Air Force Base, Ohio. The work was initiated under Research and Development Order No. 614-16, "Fatigue Properties of Structural Materials", and was administered under the direction of the Materials Laboratory, Directorate of Research, Wright Air Development Center, with Mr. W. J. Trapp acting as project engineer.
ABSTRACT

The amplitude of vibration of a member at resonance, as defined by its resonance amplification factor, is analyzed in relationship to the damping properties of materials. Data are presented on damping energy to indicate the effect of stress magnitude, stress history and temperature. Based on the mathematical relationship found to exist between damping and stress magnitude the resonance amplification factors are determined for a variety of direct stress members and beams. It is shown that the amplification in vibration caused by resonance may be considered to be the product of three basic factors: (a) the mathematical factor, (b) the cross-sectional shape factor, and (c) the longitudinal stress-distribution factor. The first of these factors may be calculated from the damping and dynamic modulus properties of the material and the last two from the shape and loading characteristics of the member. Diagrams are presented to show these basic factors as functions of the damping exponent and other variables for members commonly encountered in engineering practice. Experimental data are presented to confirm the equations derived for resonance amplification factors of members having various shapes and stress distribution.

PUBLICATION REVIEW

This report has been reviewed and is approved.

FOR THE COMMANDING GENERAL:

M. E. SORTE
Colonel, USAF
Chief, Materials Laboratory
Directorate of Research

WADC TR 52-320
TABLE OF CONTENTS

INTRODUCTION ................................................. 1

VARIABLES WHICH AFFECT MAGNITUDE OF DAMPING ENERGY IN
MATERIALS AND PARTS ...................................... 1

MATHEMATICAL RELATIONSHIP BETWEEN DAMPING AND STRESS MAGNITUDE .... 2

RESONANCE AMPLIFICATION FACTOR UNDER UNIFORM DIRECT STRESS .......... 3

RESONANCE AMPLIFICATION FACTOR UNDER NONUNIFORM DIRECT STRESS .... 3

RESONANCE AMPLIFICATION FACTOR UNDER BENDING STRESS ................. 5

EXPERIMENTAL VERIFICATION FOR $A_r$-EQUATIONS .......................... 8

GENERAL OBSERVATIONS, SUMMARY, AND CONCLUSIONS ....................... 8

BIBLIOGRAPHY .................................................. 9

LIST OF TABLES

TABLE I SUMMARY OF VALUES FOR RESONANCE AMPLIFICATION-
FACTOR $A_r$ UNDER DIRECT STRESS ................................ 4

TABLE II SUMMARY OF VALUES FOR RESONANCE AMPLIFICATION-
FACTOR $A_r$ FOR BEAMS ........................................ 6

TABLE III COMPARISON OF THEORETICAL AND EXPERIMENTAL VALUES FOR FACTOR $A_r$ 9

LIST OF ILLUSTRATIONS

FIG. 1 D-S-N-A DIAGRAM FOR ALLOY S-516 INDICATING EFFECT OF STRESS
MAGNITUDE AND STRESS HISTORY ON DAMPING ENERGY .................... 2

FIG. 2 LONGITUDINAL STRESS-DISTRIBUTION FACTOR $K_s$ FOR VARIOUS COMMON
TYPES OF MEMBERS AND LOADINGS AS A FUNCTION OF DAMPING
EXponent n ................................................................ 7

FIG. 3 CROSS-SECTIONAL SHAPE FACTOR $K_s$ FOR VARIOUS COMMON SECTIONS
AS A FUNCTION OF DAMPING EXponent n ................................ 8

WADC TR 52-320 iv
Effect of Damping Constants and Stress Distribution on the Resonance Response of Members

By B. J. LAZAN, MINNEAPOLIS, MINN.

INTRODUCTION

Near-resonance vibration is generally considered to be a common cause for fatigue and other types of service failures in many and diverse fields of engineering. Even if actual failure can be avoided, the rough and noisy operation associated with the near-resonance condition frequently necessitates correction. Current trends toward higher speeds and decreased factor of safety have increased the importance of resonance vibration as a factor in design.

Considerable work has been done on the fatigue and other properties of materials which define their ability to withstand cyclic stress produced by resonance or other vibration conditions. However, relatively little has been done on the analysis of the properties of materials and other factors which govern the amplification in vibration that is caused by the near-resonance condition. Hence this paper is concerned with an analysis of factors which govern the relationship of the external force which excites a vibration (hereafter called the exciting force) to the internal force excited in the member at resonance (hereafter called the resonance excited force). The ratio of excited to exciting force shall be called \( A_n \), the resonance amplification factor, in this paper.

The general nature of the viscous damped resonance curve is well known (2) and will not be discussed here. Although the internal hysteresis damping of a material is fundamentally different in nature from viscous damping (3), and the resonance curves may be quite unsymmetrical (4), the resonance amplification factor nevertheless provides a convenient measure of the severity of the resonant condition. Consequently, this factor shall be used in this paper as a basis for analysis.

VARIABLES WHICH AFFECT MAGNITUDE OF DAMPING ENERGY IN MATERIALS AND PARTS

The energy absorption by an actual part, or its damping, may be due to several factors, among which are: (a) The inelastic behavior of the material as indicated by its damping capacity, (b) slippage and other structural or joint factors, and (c) aerodynamic effects. In general, there is very little data on the relative magnitude and importance of each of these three absorbers of damping energy. It is likely that in many applications the primary absorber of damping energy is the structure or joint factor and the contributions offered by the material damping are insignificant. However, there is little doubt that to generalize this statement, which is sometimes done, is highly misleading.

In many applications subjected to resonance vibrations it is difficult to include significant structural damping and one must rely on material damping. Even where significant structural and aerodynamic damping may be present it is shown by recent work (5) that material damping may still be sufficiently large to be highly significant as a limiter of resonance vibrations.

Since this paper is concerned primarily with the dynamic properties of materials, structural and aerodynamic damping shall not be discussed further and only the material damping factor will be analyzed in detail.

With this as background let us now discuss, as an indication of the variables which affect the magnitude of damping in materials, the dynamic properties of one particular material. For this discussion, data on the temperature-resistant material S-816 at three temperatures will be presented so as to illustrate a variety of trends and patterns of significance in resonance-vibration studies.

For brevity, the recently developed damping, elasticity, and fatigue testing machines (3, 6) used to procure the data on S-816 will not be described in this paper. Also, for brevity, diagrams showing the variation in damping energy absorbed by the material at various stress magnitudes as a function of a number of cycles of fatigue stress (5) are omitted. However, in order to establish mathematical relationships for later analysis, damping energy is plotted as a function of stress magnitude in Fig. 1, the different curves indicating the relationship for different stress histories and temperatures. The significance of the right-hand
FATIGUE STRENGTH

2 x 10^6 CYCLES

NUMBERS 2 - 6 INDICATE STRESS HISTORY
10^2 - 10^6 CYCLES

FIG. 1 D-S-N-A, Diagram for Alloy S-816 Indicating Effect of Stress Magnitude and Stress History on Damping Energy

ordinate of this figure will be discussed later. The series of curves shown at each temperature (solid lines represent room temperature, long dash 900 F, and short dash 1600 F) are identified with the numbers 2, 3, 4, 5, and 6 to indicate the damping values after 10^2, 10^3, 10^4, 10^6, and 10^8 cycles of stress. The stress level during the imposition of the stress history was the same as during the damping measurement. The flags enclosing the letters F.S. indicate the fatigue strengths of the material.

Due to space limitations, data for S-816 only are presented in this paper. It should be pointed out, however, that the damping values do not always decrease with an increase in number of cycles at room temperature as shown in Fig. 1. Many different patterns of behavior have been observed; for example, the damping of such powdered metals increases continuously with an increase in the number of cycles; 403 stainless displays an initial decrease followed by an increase to a peak after which the values decrease if the specimen does not previously fracture. Referring to Fig. 1, it may be seen that the log D versus log S is reasonably straight with slopes ranging from 2 to 8. Thus for S-816, exponent n varies between 2 and 8 depending on stress history, magnitude, and temperature.

In some of the earlier work (4, 8) exponent n was found to be approximately 3 for several materials at room temperature. However, most of the materials now under study at the University of Minnesota display exponents n which generally are greater than 3, particularly if various stress histories are included. An exponent n as large as 30 has been observed for one material at high stress.

In future work more careful analyses will be undertaken of stress history and temperature effects. For a given material it may be possible to handle these variables mathematically by considering J and n as functions of stress history and temperature, rather than as constants as assumed in the foregoing.
Resonance Amplification Factor Under Uniform Direct Stress

So far basic data only have been presented on the damping energy. Now what does such data mean in terms of vibration amplification of actual parts at resonance?

The effect of damping energy on the resonant behavior of a part may be specified in terms of the resonance amplification factor $A_0$, as briefly discussed previously. This factor is the ratio of the force which is excited during resonance vibration to the force exciting the vibration. Thus $A_0$ is a measure of the severity of a resonant condition. For most applications in which linearity may be assumed with reasonable accuracy this factor generally may be computed (1) from the equation:

$$ A_0 = \frac{2\pi W_0}{D_0} $$

where

- $W_0$ = elastic energy in member at maximum stress, in-lb
- $D_0$ = total damping energy in in-lb/cycle, $VD$
- $V$ = volume of material at stress

Therefore, for the special case of direct uniform stress for which the elastic energy $W_0 = VS^2/2E_o$

$$ A_0 = \frac{\pi S^2/E_o}{D} = \pi J S^{-1} $$

where $E_o$ = dynamic modulus of elasticity (5), psi.

If the modulus of elasticity $E_o$ is assumed to be reasonably constant, then $A_0$ is a function of the variables $S$ and $D$ only, and for this simplified case of direct uniform stress only, a grid of lines may be drawn in Fig. 1 as shown (see right-hand ordinate) to indicate the values of $A_0$. However, as shown in a previous publication (5) the modulus of elasticity does not remain constant but varies between 21 and 34 $\times$ 10^5 psi, depending on temperature, stress magnitude, and stress history. Therefore the right-hand scale in Fig. 3 includes provisions for determining $A_0$ for various values of $E_o$ as indicated at the lower end of this scale.

The wide range of $A_0$ values observed for S-816, as revealed by Fig. 1, is of considerable engineering significance. Depending upon stress magnitude and stress history, $A_0$ under direct uniform stress may be insignificantly small or as large as 150. At 900 F, for example, stress history alone may increase $A_0$ to over 100 times its virgin value. It is therefore apparent that the specification of $A_0$ for a given material and its comparison with other materials cannot be done in a simple manner, since two variables other than material and temperature are involved.

It is desirable to review for future reference the relationship between damping exponent $n$ and change in $A_0$ under increasing stress. As indicated previously, if $n = 2$ then $A_0$ is independent of stress magnitude since both elastic and damping energy increase as the square of stress. If, however, $n$ is larger than 2, which is the usual case, the damping energy increases more rapidly with stress than does the elastic energy resulting in a $A_0$ which decreases with increasing stress. These observations will be amplified in the analysis of cross-section shape and longitudinal stress-distribution effects in later sections.

Resonance Amplification Factor Under Nonuniform Direct Stress

All discussion to this point has been concerned primarily with the basic damping, elasticity, and fatigue properties of materials and the interpretation of these properties in terms of behavior of a resonant member under uniform direct stress only. In practice this rather desirable state of uniform stress is, of course, quite rare. Generally speaking, critical members are almost always stressed nonuniformly, in which case the equation $A_0 = \pi S^2/E_o D$ does not hold. Therefore, in the interest of providing wider engineering applicability for the type of data presented previously, this section is concerned with an analysis of the resonance amplification factor $A_0$ under nonuniform direct stress. Bending members are analyzed in the next section. Other types of members, such as those subjected to torsional stress, are not included in this paper for reasons of brevity, but may be analyzed in a similar manner.

Analyses of members under resonant vibrations have been made in the past (9, 8, 1) to indicate that fatigue strength is not the sole criterion for indicating resonance durability. However, an attempt is made in this section to undertake a more thorough treatment of the important variables and to generalize the equations expressing the resonance amplification factor of members. This type of analysis is also given in reference (10) published after the completion of this paper.

The general case of direct stress vibration covered in this section considers the inertia forces produced by and within the vibrating member. The force on a section of a member with internal inertia forces will be, of course, a function of the longitudinal location of the section. Although the member may have variable cross section as shown in the diagram in Table 1, it is assumed that the stress at a given cross section is uniformly distributed (although the total force on the section and the resultant stress may vary along the length of the member).

As indicated in Equation (2) the resonance amplification factor $A_0$ for a member generally may be computed from the equation $A_0 = 2\pi W_0/D_0$.

Referring to the figure and symbol definitions in Table 1 the total elastic energy $W_0 = \int dW_0 = \int W_0 dV = \frac{\pi S^2}{2E_o} A_0 dx$...

where $W_0$ is the unit elastic energy at any location and $dV$ is a small differential volume having thickness $dz$.

As indicated (3), some materials display sufficiently large variation in $E_o$ to necessitate considering this modulus a function of stress level and stress history. However, for most materials and stress ranges, $E_o$ may be assumed to be constant for most engineering calculations. Therefore $E_o$ may be moved from within the integral sign as indicated

$$ W_o = \frac{1}{2E_o} \int_0^\infty S^2 A_0 dx $$

Again, referring to the figure and symbols in Table 1, the total damping energy

$$ D_o = \int dD_o = \int D_o dV = J \int_0^\infty S_o A_0 dx $$

Combining Equations (2), (5), and (6)

$$ A_0 = \frac{\pi}{E_o J} \int_0^\infty S_o A_o dx $$

where $S_o$ is the stress at the section of maximum stress, as defined in Table 1.

For convenience in analysis and interpretation, it is desirable to define the following factors (justification for these definitions will appear later)

Material factor: $K_m = \frac{\pi}{E_o J S_o A_o}$

Longitudinal stress-distribution factor: $K_o = \int_0^\infty \left( \frac{S_o}{S} \right)^n A_o dx$
### TABLE I: SUMMARY OF VALUES FOR RESONANCE AMPLIFICATION FACTOR $A_r$ UNDER DIRECT STRESS

$A_r = K_{m}K_1K_a$, WHERE $K_{m} = \pi^{2}E_{m}J_{m}k_{m}^{2}$, $K_1 = a_{m}^{2}+1$, AND $K_a$ GIVEN BELOW IN SEVERAL CASES

<table>
<thead>
<tr>
<th>LONGITUDINAL SHAPE AND STRESS DISTRIBUTION IN MEMBER</th>
<th>VALUES OF $K_a$</th>
<th>DEFINITION OF SYMBOLS</th>
</tr>
</thead>
<tbody>
<tr>
<td>I GENERAL CASE OF DIRECT STRESS FOR EITHER (1) CONTINUOUS MEMBERS OR (2) MEMBERS WITH DISCONTINUITIES, CONSIDERING INTERNAL INERTIA FORCES.</td>
<td>$\int_{0}^{a} \frac{(S_x/S_m)^{n}A_x dx}{\int_{0}^{a} (S_x/S_m)^{n} dx}$</td>
<td>RESONANCE EXCITING FORCE $= F_0$.</td>
</tr>
<tr>
<td>IA CONTINUOUS MEMBERS HAVING CONSTANT CROSS-SECTIONAL AREA. $A_x = A_m$ IN GENERAL, $S_x \neq P_m/A_x$ IF INTERNAL INERTIA FORCES ARE SIGNIFICANT</td>
<td>$\int_{0}^{a} \frac{(S_x/S_m)^{n}A_x dx}{\int_{0}^{a} (S_x/S_m)^{n} dx}$</td>
<td>EXCITED FORCE AT SECTION $A_m$ OF MAX. STRESS $S_m = P_m = P_0A_r$.</td>
</tr>
<tr>
<td>IA1 INTERNAL INERTIA FORCES NEGLIGIBLE $S_x = P_m/A_m = S_m$</td>
<td>$K_a = 1$</td>
<td>$P_m A_m S_m$</td>
</tr>
<tr>
<td>IA2 INTERNAL INERTIA FORCES SUCH THAT: $S_x = \left(\frac{1}{2}\right)S_m$</td>
<td>$(n+1)/3$</td>
<td>SECTION OF MAX. STRESS $P_x = \frac{A_x}{S_x}$</td>
</tr>
<tr>
<td>IA3 INTERNAL INERTIA FORCES SUCH THAT: $S_x = \left(\frac{1}{2}\right)S_m$</td>
<td>$(mn+1)/(2m+1)$</td>
<td>$dx$</td>
</tr>
<tr>
<td>IA4 INTERNAL INERTIA FORCES SUCH THAT: $S_x = S_m \sin(\pi x/2a)$ (LONG. VIB. IN CANTILEVER ROD)</td>
<td>$\pi/4 \int_{0}^{\pi/2} \sin^{n} \theta d\theta$</td>
<td>REF. SECT.</td>
</tr>
<tr>
<td>IB CONTINUOUS MEMBERS HAVING VARIABLE AREA USE EQUATION I</td>
<td></td>
<td>FORCE ON REFERENCE SECTION</td>
</tr>
<tr>
<td>IB1 INTERNAL INERTIA FORCES NEGLIGIBLE $S_x = P_m/A_x$</td>
<td>$\int_{0}^{a} \frac{A_x^{n-1} dx}{\int_{0}^{a} A_x dx}$</td>
<td>IF INTERNAL INERTIA FORCES ARE NEGLIGIBLE; THEN $P = P_0$.</td>
</tr>
<tr>
<td>IB1a ROD HAVING TAPER &quot;s&quot; $A_x = A_m \left[1+\log\left(\frac{x}{a}\right)\right]$</td>
<td>$(n-2) \log 2$</td>
<td>IF EXTERNAL INERTIA FORCES ARE NEGLIGIBLE; THEN $P = P_0$.</td>
</tr>
<tr>
<td>IB2 INTERNAL INERTIA FORCES SUCH THAT: $S = S_m (x/a), S_m \sin(\pi x/2a)$, ETC. USE EQUATION I</td>
<td>$P = \frac{P_0}{\left(1.25^n\right)}$</td>
<td></td>
</tr>
<tr>
<td>IC DISCONTINUOUS MEMBERS HAVING CONSTANT CROSS-SECTION BETWEEN DISCONTINUITIES. $A_1 a_1 A_2 a_2$</td>
<td>$A_1 \int_{0}^{a_1} \frac{(S_x/S_m)^{n} dx + A_2 \int_{0}^{a_2} \frac{(S_x/S_m)^{n} dx + \ldots}{A_1 \int_{0}^{a_1} (S_x/S_m)^{n} dx + A_2 \int_{0}^{a_2} (S_x/S_m)^{n} dx + \ldots}}$</td>
<td>FOR TWO SECTION MEMBER (ONE DISCONTINUITY) $A_m = A_1 + A_2$.</td>
</tr>
<tr>
<td>IC1 INTERNAL FORCES NEGLIGIBLE $S_x = P_m/A_x$</td>
<td>$\left[a_1/a_2 + (A_1 + A_2)^{n-1}\right]$</td>
<td></td>
</tr>
<tr>
<td>IC2 INT. INERTIA FORCES NOT NEGLIGIBLE USE EQUATION IC.</td>
<td></td>
<td></td>
</tr>
<tr>
<td>ID DISCONTINUOUS MEMBERS WITH VARIABLE SECTION BETWEEN DISCONTINUITIES. USE EQUATION I.</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

WADC TR 52-320
In accordance with these definitions, the resonance amplification factor for the general case of direct stress may be written as

\[ A_\sigma = K_m K_z \]  \hspace{1cm} [10]

The material factor \( K_m \) is, of course, independent of cross-sectional shape and stress-distribution of the member; it depends only on the properties of the material \( E_d \), \( J \), and \( n \), and the operating stress \( S_m \). Referring to Equation [3], it may be seen that the material factor is identical to the resonance amplification factor \( A_\sigma \) under direct uniform stress. This relationship will be discussed again in connection with bending members.

The longitudinal stress distribution factor \( K_z \), separated from \( K_m \) as shown in Equations [7], [9], and [10], is a function of member shape (as specified by \( A_x \)), longitudinal stress distribution \( S_x \), member length \( a \), and damping exponent \( n \), and it is independent of damping constant \( J \).

In the analysis of bending members in a later section it will be observed that \( A_x \) also may be separated into material and longitudinal stress-distribution factors \( K_m \) and \( K_z \), with an additional \( K_\sigma \)-factor required to account for cross-sectional stress distribution. In the interest of generalizing, this \( K_\sigma \)-factor is included in the summary for direct-stress members given in Table 1, \( K_\sigma \) being equal to 1 for this case since a uniform stress distribution at a given cross section is assumed.

Referring to Table 1, the first row lists the general case of direct stress and subsequent rows cover various special direct-stress cases encountered in engineering practice. For each of the general, semigeneral, and special cases listed, the values of \( K_\sigma \) are given in the second column. For conciseness, the derivations for the \( K_\sigma \)-equations listed in this table are not shown in this paper. In general, these equations are rather simple mathematical reductions of the equation for general case I with appropriate substitutions for \( S_x \) and \( A_x \). The trends revealed by these \( K_\sigma \)-equations will be discussed later.

**Resonance Amplification Factor Under Bending Stress**

An analysis of the resonance amplification factor of beams, a structural component in which resonant vibrations are often of critical importance, is given in the following.

Referring to the figures in Table 2, the elastic and damping energy of a beam of general cross-section may be expressed as indicated in the following

Total elastic energy, \( W_o = \int 1/2 \cdot M_x \cdot d\theta = \frac{1}{2} \int_0^a M_x^2 \frac{dx}{E_d I_x} \)

and assuming \( E_d \) is constant as discussed previously

\[ W_o = \frac{1}{2} E_d \int_0^a S_x^2 \frac{dx}{I_x} \]  \hspace{1cm} [11]

Total damping energy \( D_o = J \int D \cdot dv = \int J \int DZ \cdot dy \cdot dx \)

but

\[ D = J \int S_y \cdot S_z \cdot Z \cdot dy \cdot dx \]

where \( S = \) stress at any point, \( S_x = \) maximum stress (at outer fibers) at any position \( x \), and \( t_x = \) maximum distance from neutral axis to outer fibers at any position \( x \).

Thus, since \( t_x \) and \( S_y \) are not functions of \( y \)

\[ D_o = \int \int J \int S_y \cdot S_z \cdot Z \cdot dy \cdot dx \]

\[ = J \int_0^a \left[ \int_{y=-t(x)}^{y=t(x)} y^2 Z \cdot dy \right] S_z^2 \frac{dx}{I_x} \]  \hspace{1cm} [12]

where \( t(x) \) and \( t'(x) \) are functions of \( x \) which define the location of the fibers most distant from neutral axis of the beam.

Combining Equations [2], [11], and [12]

\[ A_\sigma = \frac{2\pi}{E_d J} \int_0^a \int_{y=-t(x)}^{y=t(x)} y^2 Z \cdot dy \cdot dx \cdot \frac{S_z^2}{S_y} \cdot \frac{1}{I_x} \cdot dx \]

\[ = \frac{\pi}{E_d J S_m^{n-2}} \int_0^a \left[ \int_{y=-t(x)}^{y=t(x)} y^2 Z \cdot dy \cdot \frac{S_z^2}{S_y} \right] \cdot t_x \cdot dx \]  \hspace{1cm} [13]

Inspecting Equation [13], it is apparent that the first factor \( \pi/E_d J S_m^{n-2} \) is the material factor \( K_m \) discussed previously. It will be shown that, in general, the remaining part of this equation may be considered to be the product of two other factors: (a) The longitudinal stress-distribution factor \( K_z \), as discussed previously, and (b) the cross-sectional shape factor \( K_\sigma \), to be discussed later.

In the foregoing general case these two factors cannot be separated conveniently and only their product \((K_\sigma K_z)\) is specified. However, in most special cases discussed below, \( K_\sigma \) and \( K_z \) can be separated.

Consider now the special case of beams having constant cross-sectional moment of inertia \( I \) and neutral-axis distance \( t \) and \( t' \) within the length of the beam under consideration. Under these conditions the integral

\[ \int_{y=-t}^{t} y^2 Z \cdot dy \]

reduces to

\[ \int_{y=-t'}^{t'} y^2 Z \cdot dy \]

which is not a function of \( x \) and may therefore be removed from within the \( \int_0^a \) integral. Thus Equation [13] reduces to

\[ A_\sigma = \frac{\pi}{E_d J S_m^{n-2}} \int_0^a \left[ \int_{y=-t(x)}^{y=t(x)} y^2 Z \cdot dy \cdot \frac{S_z^2}{S_y} \right] \cdot t_x \cdot dx \]

Comparing Equation [14] with direct-stress Equation [7] (with \( A_x = \) const), it may be seen that the material factor

\[ K_m = \frac{\pi}{E_d J S_m^{n-2}} \]

and the longitudinal stress-distribution factor

\[ K_z = \frac{\int_0^a \left[ \int_{y=-t(x)}^{y=t(x)} y^2 Z \cdot dy \right] \cdot S_z^2}{\int_0^a \left[ \int_{y=-t(x)}^{y=t(x)} y^2 Z \cdot dy \right] \cdot t_x \cdot dx} \]

are identical to those defined in Equations [8] and [9]. All that remains to be accounted for in Equation [14] is therefore the last factor, which may be defined as follows

Cross-sectional shape factor \( K_\sigma = \frac{\int_{y=-t'}^{t'} y^2 Z \cdot dy}{\int_{y=-t'}^{t'} y^2 Z \cdot dy} \]  \hspace{1cm} [15]
TABLE II—SUMMARY OF VALUES FOR RESONANCE AMPLIFICATION FACTOR $A_r$ FOR BEAMS

$A_r = K_m \cdot K_c \cdot K_s$, WHERE $K_m = \pi^2 / \rho d^2 s m^{-1}$, AND $K_c$ AND $K_s$ ARE GIVEN BELOW FOR SEVERAL CASES

<table>
<thead>
<tr>
<th>LONGITUDINAL SHAPE AND STRESS DISTRIBUTION</th>
<th>VALUES FOR $K_c, K_s$ OR $K_c \cdot K_s$</th>
<th>$M$</th>
</tr>
</thead>
<tbody>
<tr>
<td>II. GENERAL CASE OF BENDING STRESS FOR EITHER (1) CONTINUOUS BEAMS OR (2) BEAMS WITH DISCONT. WITH SIGNIFICANT INTERNAL INERTIA FORCES CONSIDERED.</td>
<td>$K_c \cdot K_s \cdot I_{1}^{-n/2}$</td>
<td>$\int_0^a \left( \frac{S_x}{S_m} \right)^2 \left( \frac{I_x}{I_1} \right)^2 , dx$</td>
</tr>
<tr>
<td>II A. CONTINUOUS BEAM HAVING CONSTANT CROSS-SECTION. $I_x = I$ $K_c \cdot K_s \cdot I_{1}^{-n/2}$</td>
<td>$\int_0^a \left( \frac{S_x}{S_m} \right)^2 , dx$</td>
<td></td>
</tr>
<tr>
<td>SPECIAL CROSS-SECTIONAL SHAPE CONSIDERED</td>
<td>RECTANGLE</td>
<td>CIRCLE</td>
</tr>
<tr>
<td>VALUES OF $K_c \cdot K_s$</td>
<td>$K_c = 0.33(n+1)$</td>
<td>$K_c = 0.196(n+2)$</td>
</tr>
<tr>
<td>II A 1. CONSTANT MOMENT $S_x = S_m$; $K_s = 1$</td>
<td>$0.33(n+1)$</td>
<td>$0.196(n+2)$</td>
</tr>
<tr>
<td>II A 2. VARIABLE MOMENT SUCH THAT $S_x = (x/0)S_m$ $K_s = (n+1)/3$</td>
<td>$0.111(n+1)^2$</td>
<td>$0.065(n+1)(n+2)$</td>
</tr>
<tr>
<td>II A 3. VARIABLE MOMENT SUCH THAT $S_x = (x/0)^mS_m$ $K_s = (n+1)/3$</td>
<td>$0.33(n+1)(n+1)$</td>
<td>$0.196(m+1)(n+2)$</td>
</tr>
<tr>
<td>II A 4. VARIABLE MOMENT SUCH THAT $S_x = s_m \sin(\pi x/2a)$ $K_s = \pi^2 / 4 \int_0^{\pi/2} \sin^2 \theta , d\theta$</td>
<td>$0.26(n+1)$</td>
<td>$0.154(n+2)$</td>
</tr>
<tr>
<td>II B. DISCONTINUOUS BEAMS HAVING CONSTANT CROSS-SECTION BETWEEN DISCONTINUITIES</td>
<td>$K_c = 1/12$</td>
<td>$K_s = 1/12$</td>
</tr>
<tr>
<td>II B 1. CONSTANT MOMENT ON TWO SECTION MEMBER (1 SIMILAR TO 2)</td>
<td>$0.33(n+1)$</td>
<td>$0.196(n+2)$</td>
</tr>
<tr>
<td>$K_s = a_1 \frac{1}{a_2} + a_1 \frac{1}{a_2} + a_1 \frac{1}{a_2}$</td>
<td>$\frac{a_1}{a_2} + \frac{a_1}{a_2} + \frac{a_1}{a_2}$</td>
<td></td>
</tr>
</tbody>
</table>

WADG TR 52-320 6
Inspection of Equation [15] will show that the name assigned to this factor is descriptive of its nature.

Therefore it appears to be possible to define the resonance amplification factor of beam by the equation

\[ A_r = K_m K_r K_v \]  \hspace{1cm} \text{[16]}

In all cases \( K_v \) appears as a separate factor. Although for the case of beams of uniform cross section and other special cases it may be possible to separate the \( K_m \) and \( K_r \) factors as shown, in the most general case only the product \( K_r K_v \) can be specified.

Table 2 is a summary of values for the resonance amplification factor \( A_r \), for various general and special types of beams encountered in engineering practice. The arrangement of this table is similar to that of Table 1 and is self-explanatory. It should be noted that three cross-sectional shapes frequently encountered in engineering practice (the rectangle or square, circle, and diamond) are analyzed merely to indicate trends to be discussed later. Other shapes can, of course, also be analyzed by the same methods.

From Table 2 it may be noted that the equation for a vibrating beam (case IIA2) of rectangular cross section is

\[ A_r = \frac{\pi}{E_2 S_m} \left( \frac{n + 1}{3} \right)^3 \]

If this equation is used to compute the allowable force \( P_a \) at resonance and if a value of 3 is substituted for \( n \), then the resultant equation is in exact agreement with Marin's Equation [14] in reference (9).

Referring to Equation [14] of this paper, it may be seen that both the \( K_m \) and \( K_r \) factors are functions of damping exponent \( n \). Since this exponent is widely different for different materials, a range from 2 to 30 having been observed to date, it is desirable to diagram these factors as a function of \( n \) as shown in Figs. 2 and 3. These diagrams also reveal trends which provide a basis for discussing the interpretation of damping data in terms of resonance amplification factor as used in engineering design.

Fig. 2 diagrams the longitudinal stress-distribution factor \( K_r \) as a function of damping exponent \( n \) for various common shapes of members and types of loading. Members without discontinuities are shown in solid lines and members with one discontinuity (two section members) are shown in broken lines.

For all the cases referred to in Fig. 2, factor \( K_r \) equals 1 at \( n = 2 \), and, except for uniform direct stress, increases with increasing values of \( n \). The rate of increase in \( K_r \) with \( n \) is greatly dependent on the type of member and loading.

For example, for a freely vibrating cantilever member of constant cross section under either tension or bending stress (stress distribution assumed to be \( S_0 = S_m \sin \frac{\pi x}{2a} \) as indicated for cases IIA4 and IIA4) \( K_r \) increases from 1 to only 3.5 as damping exponent \( n \) increases from 1 to 30. As a second example, \( K_r \) for a tapered tension member (case IIBla) having a taper \( s \) of 4 displays the much larger increase from 1 to 40 in the same range of \( n \) values.

In the case of discontinuous members (broken lines), it is of course possible to have an infinite number of ratios of section area, and so on, at the discontinuities. For simplicity, members with one discontinuity only are considered, and for further simplification only a few ratios \( g \) of areas \( A_1 \) to \( A_2 \) and only a few ratios \( p \) of lengths \( a_1 \) to \( a_2 \) between discontinuities are included. Small values of ratio \( p \) were selected to indicate the characteristics of members with notches and other stress-concentration effects. For example, a member with a small notch (which allows relatively low working stress in the rest of the member) can attain \( K_v \) values as high as 80. This means that at the same nominal working stress a notched member may have 80 times the resonance amplification factor \( A_r \) of the unnotched member, if other factors remain constant.

To illustrate the effect of damping exponent \( n \) on the cross-sectional shape factor \( K_v \), three common sections were analyzed and diagrammed as shown in Fig. 3. The equations used to plot the characteristics of the diamond, circle, and rectangle are given in Table 2. The broken lines indicate the general trend for a modified diamond shape (with sharp upper and lower edges) and an I-beam. Since the exact shape of these sections governs the exact locations of the corresponding curves, these curves are qualitative indications only of the general trends.

It is significant to note from the curves that merely changing the cross-sectional shape of a beam from a diamond to an I-section (all other factors remaining constant) may decrease its resonance amplification factor very significantly, as much as 20 times under certain conditions. It is sometimes possible to make such shapes in cross-sectional shape in vacuum-tube components, structural beams, and other parts to reduce resonance amplification vibrations and increase service life without affecting functional behavior.

Referring again to the five shapes diagrammed in Fig. 3, it is apparent that the smaller the percentage volume at near peak stress, the larger is the \( K_v \) factor, particularly at large values of damping exponent \( n \). The physical reason for this mathematical observation may be explained on the basis of the ratio of the elastic-energy to damping energy. In general, it may be stated that if either the cross-sectional shape or longitudinal stress distribution are such as to cause a large percentage of the material to be at stress low enough to dissipate negligible damping energy, and yet at a stress high enough to store significant elastic energy...
energy, then relatively large A,-factor will result. This is so for diamond-shaped cross section (as compared to I-beams), for tapered members at constant force (as compared to constant section members), for cantilever beams (as compared to constant moment beams), and for stepped or notched (discontinuous) members (as compared to continuous members). In general, as much volume as possible should be exposed to near peak stress if A, is to be kept small.

An example of the method of using the three factors which govern the amplification in vibration caused by resonance is given in the following. The example considered is a cantilever beam of constant diamond cross-sectional shape made of S-816, vibrated at 900 F, and exposed to a maximum stress of 60,000 psi (virgin material assumed). From Equation [16]

\[ A_1 = K_m K_c K_a \]

From Fig. 1, \( K_m = 19 \) for virgin material at 900 F, at 60,000 psi and damping exponent \( n = 8 \).

From Fig. 2, \( K_c = 1.7 \) for a vibrating cantilever beam (case II A4) with a damping exponent of 8.

From Fig. 3, \( K_a = 7.8 \) for a diamond cross section of a material with a damping exponent of 8.

\[ A_1 = 19 + 1.7 \times 718 = 252 \]

If this resonance amplification factor is too large for the application under consideration, a change in cross section from diamond to I-section could be considered. The resultant reduction in \( K_c \)-factor from 7.8 to 1.7 would result in a reduction in resonance amplification from 252 to 55. Needless to say, the resultant reduction in fatigue stress at resonance may be sufficient to eliminate service-failure difficulties in critically stressed members.

**Experimental Verification for \( A_1 \)-Equations**

In order to procure at least partial experimental confirmation of the \( A_1 \) equations derived in the previous sections a series of tests was undertaken with a newly developed resonance-vibration testing machine described in reference (4). With this machine it is possible to excite resonant vibrations in various types of systems and at the same time maintain the desired excitation force by controlling the magnitude of the exciting force. Automatic controls maintain (a) a state of resonance and (b) the desired excited force, even under conditions of changing damping and dynamic stiffness in the system. The value of the resonance amplification factor \( A_1 \) may then be computed from the ratio of the excited force (indicated by an accelerometer output) to the exciting force (determined from a tachometer and a counter which indicate unbalance in a revolving eccentric).

Tests were undertaken in this machine to determine \( A_1 \) for a beam-type specimen made of alloy TP-2-B, this alloy being selected because its large damping exponent \( n = 13 \) provides a more critical test of the theory than a material with a small damping exponent. In these tests, made at a stress of 80,000 psi and a stress history of approximately 1000 cycles, three cross-sectional shapes were used, i.e., a square with the neutral axis parallel to one side, a circle, and a diamond shape (actually the same specimen as used for the square cross-section test, except that it was oriented so that the neutral axis was diagonally across two opposite corners). The experimental \( A_1 \)-values so determined and the theoretical \( A_1 \)-values calculated by means of the equations presented previously are indicated in Table 3.

It might appear from Table 3 that the check between the theoretical and experimental values of \( A_1 \) is rather poor. However, considering that in alloy TP-2-B (with a damping exponent \( n = 13 \)) a 3 per cent error in stress magnitude will result in a 29 per cent error in \( K_m \) and \( A_1 \), and that the number of other variables which affect damping (such as variability of stress history and rest, cyclic stress frequency, machining stress, and others discussed in reference 3), the check between theory and experiment is considered adequate.

**General Observations, Summary, and Conclusions**

In many machines and structures, it is highly desirable, if not essential, that the amplitude of near-resonance vibration encountered in service be reduced. In general, the most satisfactory approach is, of course, to reduce the magnitude of the exciting force or change its frequency so as to avoid the resonant condition. However, this cannot always be accomplished, in which case it becomes necessary either to make the machine more vibration-resistant or to increase the damping of the system in order to reduce the magnitude of the near-resonance vibration. Frequently, the methods for increasing the fatigue and dynamic strength of a part are exhausted without a satisfactory solution, in which case one must resort to increasing damping.

Sometimes it is possible to increase the damping of a vibrating system through more effective use of joint, aerodynamic, and other damping not directly related to the material, shape, and stress distribution in the members. However, this too may be quite difficult to accomplish, in which case it becomes necessary to increase the damping of the actual part. It is this last type of damping which is the subject of this paper.

It was shown in this paper that the amplification \( A_1 \) in vibration of a part at resonance is the product of three factors: (a) material factor \( K_m \), (b) longitudinal stress-distribution factor \( K_c \), and (c) damping exponent \( n \). See reference (5) for details on this material, including data on its damping, elasticity, and fatigue properties.
TABLE 3 COMPARISON OF THEORETICAL AND EXPERIMENTAL VALUES FOR FACTOR $A_r$ FOR TP-3-B AT A STRESS OF 80,000 PSI AND A STRESS HISTORY OF 1000 CYCLES

($D = 6.2 \times 10^{-9}$) therefore, $J = 6.2 \times 10^{-9}$ and $n = 13$

<table>
<thead>
<tr>
<th>Cross-sectional shape</th>
<th>$K_n$</th>
<th>$A_r$</th>
<th>$K_r$</th>
<th>$K_r$</th>
<th>$A_r$ from resonance test</th>
</tr>
</thead>
<tbody>
<tr>
<td>Square</td>
<td>14</td>
<td>2.2</td>
<td>4.8</td>
<td>150</td>
<td>280</td>
</tr>
<tr>
<td>Circle</td>
<td>14</td>
<td>2.2</td>
<td>9</td>
<td>280</td>
<td>340</td>
</tr>
<tr>
<td>Diamond</td>
<td>14</td>
<td>2.2</td>
<td>18</td>
<td>550</td>
<td>950</td>
</tr>
</tbody>
</table>

and (c) cross-sectional shape factor $K_n$. The critical relationships of these factors to damping exponent $n$ and other variables were derived and analyzed. Methods of decreasing the resonance amplification factor, based on the equations derived, were suggested and demonstrated. Experimental data were presented to verify the theory.

It is desirable at this point to generalize and interpret physically some of the mathematically derived $K_n$ and $K_r$-relationships discussed in the previous sections. This is done to provide a guide for those who find it necessary to reduce the resonance amplification factor of parts which encounter troublesome resonance vibration.

Probably the most important generalizations and conclusions apparent from Tables 1 and 2, and Figs. 2 and 3, are as follows:

(a) For materials with low damping exponent $n$, and many of the temperature-resistant materials may be so classified, the cross-sectional shape and longitudinal stress-distribution factors are extremely important. Relatively small changes in member shape and loading, which can sometimes be made through design without affecting functional behavior, cost, and so on, may greatly improve the ability of a part to withstand resonant conditions.

(b) For materials with high damping exponent $n$, and many of the temperature-resistant materials may be so classified, the cross-sectional shape and longitudinal stress-distribution factors are extremely important. Relatively small changes in member shape and loading, which can sometimes be made through design without affecting functional behavior, cost, and so on, may greatly improve the ability of a part to withstand resonant conditions.

(c) As a guide for decreasing the $K_n$ and $K_r$-factors for a part having large damping exponent $n$, an attempt should be made to design the part so that the percentage of material at peak or near-peak stress is as large as possible. Material at relatively low stress contributes a relatively large ratio of elastic energy to damping energy which tends to increase the resonance amplification factor.

It is felt that the analyses and observations discussed in this paper should provide the design engineer with improved perspective in dealing with problems involving resonant vibrations.

BIBLIOGRAPHY