TITLE: Galerkin Method for Solving of Singular Integral Equation of Diffraction Problem

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ADP013889 thru ADP013989
Galerkin Method for Solving of Singular Integral Equation of Diffraction Problem*

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1 The statement of the diffraction problem

Let $P = \{ x : 0 \leq x_1 \leq a, 0 \leq x_2 \leq b, 0 \leq x_3 \leq c \}$ be a resonator with perfectly conducting boundary. Let $Q$ be a three-dimensional body, located in $P$. $Q$ is characterized by tensor permittivity $\varepsilon$ and constant permeability $\mu_0$. We suppose that components of $\varepsilon$ are smooth functions in $Q$ and $\left( \frac{1}{\varepsilon} - \mathbf{I} \right)$ is invertible in $Q$, $Q \cap \partial P = \emptyset$. Let $P/Q$ be homogeneous and isotropic medium. Incident and diffraction fields depend on time variable as $e^{-j\omega t}$.

We will find electromagnetic diffraction fields $E$ and $H$, satisfying Maxwell's equations in $P \setminus \partial Q$:

$$\begin{align*}
\text{rot} \mathbf{H} &= -j\omega \mathbf{E} + \mathbf{j}_E^0 \\
\text{rot} \mathbf{E} &= j\omega \mu_0 \mathbf{H} - \mathbf{j}_H^0 .
\end{align*}$$

(1)

The complete field should have continuous tangent components at $\partial Q$:

$$\left[ \mathbf{n} \times \mathbf{E} \right]_{\partial Q} \bigm\vert_{\partial Q} = 0 .$$

(2)

2 Integro-differential equations for the diffraction problem

We will express the solution of the stated problem in terms of vector potentials $\mathbf{A}_E$ and $\mathbf{A}_H$ [4]:

$$\begin{align*}
\mathbf{A}_E &= \int_Q \mathbf{G}_E(x, y) \mathbf{j}_E^0(y) dy, \\
\mathbf{A}_H &= \int_Q \mathbf{G}_H(x, y) \mathbf{j}_H^0(y) dy , \\
\mathbf{E} &= j\omega \mathbf{E}_0 \mathbf{A}_E - \frac{1}{j\omega \varepsilon_0} \text{grad div} \mathbf{A}_E - \text{rot} \mathbf{A}_H , \\
\mathbf{H} &= j\omega \mu_0 \mathbf{A}_H - \frac{1}{j\omega \mu_0} \text{grad div} \mathbf{A}_H + \text{rot} \mathbf{A}_E .
\end{align*}$$

(3)

Here $\mathbf{j}_E^0 = \mathbf{j}_E^0 + \mathbf{j}_E^0$, $\mathbf{j}_H^0 = \mathbf{j}_E^0 + \mathbf{j}_H^0$, ($\mathbf{j}_E^0$, $\mathbf{j}_H^0$ are polarization currents), $\mathbf{G}_E, \mathbf{G}_H$ are Green functions for Helmholtz equation, conforming to the arbitrary currents $\mathbf{j}_E^0, \mathbf{j}_H^0$.

$\mathbf{G}_E, \mathbf{G}_H$ are known [3] to have the form of diagonal tensors (the components of $\mathbf{G}_E$ are written out below):

$$\begin{align*}
G^1_E &= \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \frac{2\pi e_a}{\partial r \gamma \gamma Y} \cos\left( \frac{\pi n}{a} x_1 \right) \sin\left( \frac{\pi n}{b} x_2 \right) \cos\left( \frac{\pi n}{b} y_1 \right) \sin\left( \frac{\pi m}{b} y_2 \right) \left( \text{sh} \gamma x_3 \text{sh} \gamma (c - y_3), x_3 < y_3 \right) \\
&\quad \text{sh} \gamma y_3 \text{sh} \gamma (c - x_3), x_3 > y_3 \\
G^2_E &= \sum_{n=1}^{\infty} \sum_{m=0}^{\infty} \frac{2\pi e_a}{\partial r \gamma \gamma Y} \sin\left( \frac{\pi n}{a} x_1 \right) \cos\left( \frac{\pi n}{b} x_2 \right) \sin\left( \frac{\pi n}{b} y_1 \right) \cos\left( \frac{\pi m}{b} y_2 \right) \left( \text{sh} \gamma x_3 \text{sh} \gamma (c - y_3), x_3 < y_3 \right) \\
&\quad \text{sh} \gamma y_3 \text{sh} \gamma (c - x_3), x_3 > y_3 \\
G^3_E &= \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \frac{\pi e_a}{\partial r \gamma \gamma Y} \sin\left( \frac{\pi n}{a} x_1 \right) \sin\left( \frac{\pi n}{b} x_2 \right) \sin\left( \frac{\pi m}{b} y_1 \right) \cos\left( \frac{\pi m}{b} y_2 \right) \left( \text{ch} \gamma x_3 \text{ch} \gamma (c - y_3), x_3 < y_3 \right) \\
&\quad \text{ch} \gamma y_3 \text{ch} \gamma (c - x_3), x_3 > y_3
\end{align*}$$

*supported by Russian Foundation for Basic Research, grant 01–01–00053
Here \( \gamma = \sqrt{\left(\frac{\pi n}{a}\right)^2 + \left(\frac{\pi m}{b}\right)^2 - k_0^2} \) (the proper branch for square root is chosen as in \([2, \S 2.3]\), \( \epsilon_0 = 1 \) and \( \epsilon_n = 2 \) for \( n = 1, 2, 3, \ldots \)).

We can obtain the following integro-differential equations (under the condition \( \mu = \mu_0 \hat{I} \) in \( P \)):

\[
\mathbf{E}(x) = \mathbf{E}_0(x) + k_0^2 \int_\mathcal{Q} \nabla E \left[ \frac{\xi(y)}{\epsilon_0} - \hat{I} \right] \mathbf{E}(y) dy + \nabla \cdot \left( \int_\mathcal{Q} \epsilon(y) \nabla \mathbf{E} \left[ \frac{\xi(y)}{\epsilon_0} - \hat{I} \right] \mathbf{E}(y) dy \right),
\]

and we have

\[
\mathbf{H}(x) = \mathbf{H}_0(x) - i\omega \epsilon_0 \hat{I} \left( \int_\mathcal{Q} \epsilon(y) \nabla \cdot \nabla \mathbf{E} \left[ \frac{\xi(y)}{\epsilon_0} - \hat{I} \right] \mathbf{E}(y) dy, \ x \in \mathcal{Q} \right).
\]

We can extract singularity of Green function \( \mathcal{G} \). Using Fourier transformation and interpolation polynomials we can obtain:

\[
\mathcal{G}(x, y) = \frac{1}{4\pi|x - y|} \cdot \hat{I} + \text{diag}\{g_1(x, y), g_2(x, y), g_3(x, y)\},
\]

where \( g_k \) are smooth functions.

### 3 Galerkin method

Let us introduce the following auxiliary function

\[
\mathcal{G}(x, y) = -\sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \frac{4}{a b n m} \sin\left(\frac{\pi n}{a} x_1\right) \sin\left(\frac{\pi m}{b} y_1\right) \sin\left(\frac{\pi n}{a} x_2\right) \sin\left(\frac{\pi m}{b} y_2\right) \times \begin{cases} \sinh x_3 \sinh (c - y_3), x_3 < y_3 \\ \sinh y_3 \sinh (c - x_3), x_3 > y_3 \end{cases}
\]

The derivatives of \( \mathcal{G} \) are connected to the derivatives of \( G_k \) through the equalities:

\[
\frac{\partial G_k}{\partial x_i} = \frac{\partial \mathcal{G}}{\partial y_i}, \quad i = 1, 2, 3.
\]

Before describing the method itself we should make some transformations of equation (5). Denoting \( \left(\frac{\xi(y)}{\epsilon(y)} - \hat{I}\right) \) as \( \xi \) and \( \left(\frac{\xi(y)}{\epsilon(y)} - \hat{I}\right) \mathbf{E} \) as \( \mathbf{J} \) we obtain the following equation

\[
A \mathbf{J} := \xi \mathbf{J}(x) - k_0^2 \int_\mathcal{Q} \nabla \mathbf{E} \left[ \frac{\xi(y)}{\epsilon(y)} - \hat{I} \right] \mathbf{J}(y) dy - \nabla \cdot \left( \int_\mathcal{Q} \epsilon(y) \nabla \mathbf{E} \left[ \frac{\xi(y)}{\epsilon(y)} - \hat{I} \right] \mathbf{J}(y) dy \right) = \mathbf{E}_0(x)
\]

We can write vector equation (8) as a system of three scalar equations:

\[
\sum_{i=1}^{3} \xi_i \mathbf{J}^i(x) - k_0^2 \int_\mathcal{Q} \nabla \mathbf{E}_i \mathbf{J}(y) dy - \nabla \cdot \left( \int_\mathcal{Q} \epsilon(y) \nabla \mathbf{E} \left[ \frac{\xi(y)}{\epsilon(y)} - \hat{I} \right] \mathbf{J}(y) dy \right) = \mathbf{E}_0^i(x), \quad l = 1, 2, 3.
\]

We will determine the components of approximate solution \( \mathbf{J} \) in the following way:

\[
\mathbf{J}^1 = \sum_{k=1}^{N} a_k \mathbf{f}_k^1(x), \quad \mathbf{J}^2 = \sum_{k=1}^{N} b_k \mathbf{f}_k^2(x), \quad \mathbf{J}^3 = \sum_{k=1}^{N} c_k \mathbf{f}_k^3(x),
\]

where \( f_k^l \) are basis "hat"-functions dependent essentially on \( x^l \). The explicit form of \( f_k^l \) is given below.

Let \( \Pi \) be a parallelepiped: \( \Pi = \{x : a_1 \leq x^1 \leq b_1, a_2 \leq x^2 \leq b_2, a_3 \leq x^3 \leq c_3 \} \). We will cover \( \Pi \) with smaller parallelepipeds

\[
\Pi_{k_{1m}} = \{x : x_1^{k_{1m}} \leq x^1 \leq x_1^{k_{1m}+1}, x_2^2 \leq x^2 \leq x_2^{k_{1m}+1}, x_3^{k_{1m}} \leq x^3 \leq x_3^{k_{1m}+1}\}
\]

\[
x_1^{k} = a_1 + \frac{a_2 - a_1}{n} k, \quad x_2^{k} = b_1 + \frac{b_2 - b_1}{n} l, \quad x_3^{k} = c_1 + \frac{c_2 - c_1}{m} m;
\]

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where \( k = 1, \ldots, n - 1; \quad l, m = 0, 1, \ldots, \frac{n}{2} - 1. \)

Denoting \( (x_k - x_{k-1}) \) as \( h^2 \) we get the formulas for \( f_{klm}^1 \):\[
\begin{align*}
f_{klm}^1 &= \begin{cases} 
\frac{x_k - x_{k+1}}{x_k - x_{k-1}}, & \text{if } x \in [x_{k-1}; x_k] \text{ and } x \in \Pi_{klm}^1 \\
\frac{x_k - x_{k+1}}{x_k - x_{k-1}}, & \text{if } x \in [x_k; x_{k+1}] \text{ and } x \in \Pi_{klm}^1 \\
0, & \text{if } x \notin \Pi_{klm}^1
\end{cases}
\end{align*}
\] (12)

or
\[
f_{klm}^2 = \begin{cases} 1 - \frac{1}{h^2} |x - x_k|, & \text{if } x \in \Pi_{klm}^2 \\
0, & \text{if } x \notin \Pi_{klm}^2
\end{cases}
\] (13)

Functions \( f_{klm}^1 \) and \( f_{klm}^2 \) should be determined by similar formulas. Since
\[
f_{klm}^1|_{z \in (x_k, x_{k+1})} = 0, \quad f_{klm}^2|_{z \in (x_k, x_{k+1})} = 0, \quad f_{klm}^3|_{z \in (x_{k-1}, x_k)} = 0,
\] (14)
every component of approximate vector solution vanishes at some side of \( Q \). However the constructed set of basis functions does satisfy the necessary approximation condition.

Introducing total enumeration for basis functions we get
\[
f_{k1}^1, f_{k2}^2, f_{k3}^3; \quad k = 1, \ldots, N,
\]
where \( N = \frac{1}{4} (n^3 - n^2) \).

It is convenient to represent the augmented matrix for determining unknown coefficients \( a_k, b_k, c_k \) in block form:
\[
\begin{pmatrix}
A_{11} & A_{12} & A_{13} & B_1 \\
A_{21} & A_{22} & A_{23} & B_1 \\
A_{31} & A_{32} & A_{33} & B_1 
\end{pmatrix}
\] (15)

where columns \( B_k \) and matrices \( A_{kl} \) are determined by formulas:
\[
B_k = (E_k^0, f_k^0);
\] (16)
\[
A_{kl} = (\xi_k f_l^1, f_l^1) - \delta_{kl} k_0^2 \left( \int_Q G_E^k (x, y) f_l^1 (y) dy, f_l^1 (x) \right) - \\
\left( \frac{\partial}{\partial x_k} \int_Q \frac{\partial}{\partial x_l} G_E^k (x, y) f_l^1 (y) dy, f_l^1 (x) \right),
\] (17)

\( k = 1, 2, 3; \quad i = 1, \ldots, N. \) \((f,g)\) determines the scalar product in \( L_2, \) \( (f,g) = \int_Q f(x) g(x) dx. \)

Applying the formulas of integration by parts to both internal and external integrals and taking into account (7) and (14) we obtain:
\[
A_{kl} = \int_{n_j^i \cap n_i^j} \xi_k f_l^1 (x) f_l^1 (x) dx - \delta_{kl} k_0^2 \int_{n_j^i \cap n_i^j} G_E^k (x, y) f_l^1 (y) f_l^1 (x) dy dx - \\
\int_{n_i^j \cap n_j^i} G(x, y) \frac{\partial}{\partial x_l} f_l^1 (y) \frac{\partial}{\partial x_k} f_l^1 (x) dy dx.
\] (18)

References

