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Geometric knot selection for radial scattered data approximation

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Abstract

Scattered exact and non-exact data are approximated by means of radial basis functions with compact support and the related knot selection is based on the information given by the discrete Gaussian curvature defined on a data triangulation. In case of non-exact data, a strategy to obtain a sign-reliable estimate of its distribution is given extending an approach already studied by the authors for non-exact 2D data.

1 Introduction

It is well known that, for any interpolation/approximation scheme, data shape preservation is often a desirable quality and, as a consequence, the determination of some criteria to establish the data shape is a very important topic. For this purpose, the use of the discrete curvature in case of exact 2D data is a standard approach. On the other hand, in case of non-exact data, the proposal in [6] allows the determination of a reasonable and sign-reliable discrete curvature estimate if the maximum data error is *a priori* given. In recent literature, interesting formulas have been introduced [3, 4] for defining the discrete Gaussian curvature when scattered 3D exact data are given and a related triangulation is assigned. Starting from these formulas, the approach considered in [6] is extended to the case of 3D scattered non-exact data in order to define a reasonable and sign-reliable estimate of the Gaussian curvature at the data points thereby obtaining important shape information. Thus we get some suggestions for determining the supports of the local radial basis functions [8] used in the approximation scheme together with the number, the position and the multiplicity of the related knots. The result is a good approximating surface (in particular with respect to its shape) with a high data reduction [2, 7].

The outline of the paper is as follows. In Section 2 the discrete Gaussian curvature is defined and an inequality is given to check its sign-reliability in case of non-exact data. In Section 3 the approximation scheme is presented and the knot selection strategy is given. Finally, in Section 4 some numerical results are presented to illustrate the features of the proposed approach.

2 Information about the shape

In this section, following the approach presented in [3, 4], we define the discrete Gaussian curvature (dGc) to obtain information about the shape suggested by the data. For this purpose, we need the following notation

- $\mathcal{P}_{xy} := \{\mathbf{X}_j = (x_j, y_j), j = 1, \ldots, N\} \subset \mathbb{R}^2$ is the set of the assigned distinct vertices on the xy-plane;
- $\mathcal{P} := \{\mathbf{P}_j = (\mathbf{X}_j, z_j), j = 1, \ldots, N\} \subset \mathbb{R}^3$ is the data set, with $z_j = f(\mathbf{X}_j)$;
- $\mathcal{T} := \{\mathbf{l}_j \in \mathbb{N}^3,\ 1 \leq l_{kj}, \leq N, k = 1, 2, 3, j = 1, \ldots, T\}$ is a given triangulation of $\mathcal{P}_{xy}$.

Thus, for any $\mathbf{X}_j \in \mathcal{P}_{xy}$ not belonging to the boundary of the convex hull of $\mathcal{P}_{xy}$ we can define the integral Gaussian curvature with respect to a related area S_j, [3]

$$\bar{K}_j := 2\pi - \sum_{k=1}^{n_j} \alpha_k^{(j)},$$

where the angles $\alpha_k^{(j)}, k = 1, \ldots, n_j$ are as follows

$$\alpha_k^{(j)} := \angle(\mathbf{e}_k^{(j)}, \mathbf{e}_{k+1}^{(j)}), \quad \mathbf{e}_k^{(j)} := \mathbf{V}_k^{(j)} - \mathbf{P}_j, \ k = 1, \ldots, n_j, \quad \mathbf{e}_{n_j+1}^{(j)} := \mathbf{e}_1^{(j)}$$

and $\{\mathbf{V}_1^{(j)}, \ldots, \mathbf{V}_{n_j}^{(j)}\} \subset \mathcal{P}$ is the set of ordered neighboring points of $\mathbf{P}_j$ given by the assigned triangulation. To derive the curvature at the vertex $\mathbf{P}_j$ from the above integral value, we normalize by the Voronoi area S_j [4]

$$K_j := \frac{\bar{K}_j}{S_j}. \tag{2.1}$$

If $\mathbf{X}_j$ is on the boundary of the convex hull of $\mathcal{P}_{xy}$, some auxiliary suitable "phantom" points should be defined in order to obtain a reliable estimate of the Gaussian curvature from (2.1).

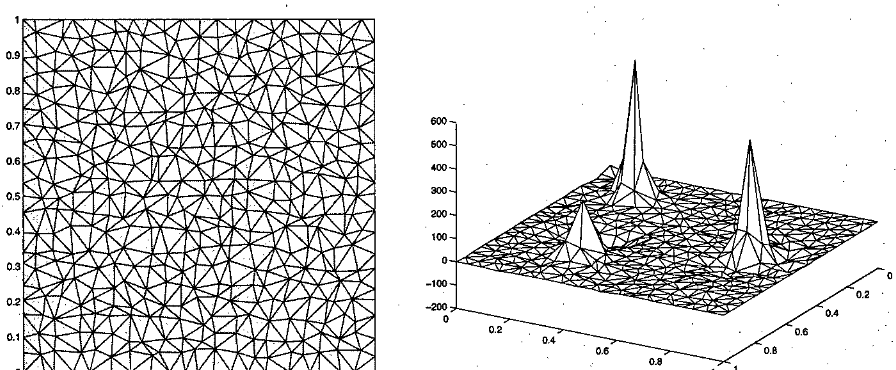

FIG. 1. The triangulation (left) and the discrete Gaussian curvature (right).

Shown on the left of Figure 1 is the Delaunay triangulation related to a set $\mathcal{P}_{xy}$ of 441 scattered vertices in the unit square and shown on the right is the discrete Gaussian curvature distribution related to the Franke function sampled on $\mathcal{P}_{xy}$.

In case of non-exact data, we need to check the sign-reliability of $\bar{K}_j$ for deriving some useful information about the shape suggested by the data. For this purpose, we use the theorem below, where

$$\begin{aligned} c_k^{(j)} &:= \frac{\mathbf{e}_k^{(j)} \cdot \mathbf{e}_{k+1}^{(j)}}{|\mathbf{e}_k^{(j)}|\,|\mathbf{e}_{k+1}^{(j)}|}, \ k = 1, \ldots, n_j, \\ \mathcal{K}_j &:= \tfrac{\pi}{2}\left(4 - n_j + \sum_{k=1}^{n_j} c_k^{(j)}\right), \end{aligned} \tag{2.2}$$

Remark 2.1 *$\mathcal{K}_j$ is an approximation of $\bar{K}_j$ obtained by replacing the angle $\alpha_k^{(j)}$ with $\frac{\pi}{2}(1 - c_k^{(j)}), k = 1, \ldots, n_j$.*

Theorem 2.2 *Let $\mathbf{P}_j \in \mathbb{R}^3, j = 1, \ldots, N$ be assigned distinct non-exact data points and let ϵ be a positive quantity such that $|\mathbf{P}_j - \mathbf{P}_j^e| \le \epsilon,\ j = 1, \ldots, N$, where $\mathbf{P}_j^e$ is the (unknown) exact data point corresponding to $\mathbf{P}_j$. If ϵ is sufficiently small and*

$$|\mathcal{K}_j| > 5\pi\epsilon \sum_{k=1}^{n_j} \frac{1}{|\mathbf{e}_k^{(j)}|}, \tag{2.3}$$

then

$$\mathcal{K}_j \mathcal{K}_j^e > 0,$$

where $\mathcal{K}_j^e$ is defined as $\mathcal{K}_j$ using the exact data points.

Proof: Let us consider a point $\mathbf{P}_j$ and its neighboring points $\{\mathbf{V}_1^{(j)}, \ldots, \mathbf{V}_{n_j}^{(j)}\} \subset \mathcal{P}$ and let us write the corresponding (unknown) exact points as follows

$$\begin{aligned} \mathbf{P}_j^e &:= \mathbf{P}_j - \epsilon_0 \mathbf{w}_0, \\ \mathbf{V}_k^{(j)e} &:= \mathbf{V}_k^{(j)} - \epsilon_k \mathbf{w}_k, k = 1, \ldots, n_j \end{aligned}$$

with $0 \le \epsilon_0, \epsilon_1, \ldots, \epsilon_{n_j} \le \epsilon$ and $|\mathbf{w}_0| = |\mathbf{w}_1| = \cdots = |\mathbf{w}_{n_j}| = 1$.

So, if ϵ is sufficiently small, we can define the non-zero vectors

$$\mathbf{e}_k^{(j)e} := \mathbf{V}_k^{(j)e} - \mathbf{P}_j^e$$

and we have

$$\mathbf{e}_k^{(j)e} = \mathbf{e}_k^{(j)} - \epsilon_k \mathbf{w}_k + \epsilon_0 \mathbf{w}_0.$$

Thus, if

$$\mathbf{c}_k^{(j)e} := \frac{\mathbf{e}_k^{(j)e} \cdot \mathbf{e}_{k+1}^{(j)e}}{|\mathbf{e}_k^{(j)e}|\,|\mathbf{e}_{k+1}^{(j)e}|},$$

using a first order Taylor approximation, we obtain

$$\mathbf{c}_k^{(j)e} = \mathbf{c}_k^{(j)}(1 + A_k) + \frac{1}{|\mathbf{e}_k^{(j)}|\,|\mathbf{e}_{k+1}^{(j)}|} B_k + \mathcal{O}(\epsilon^2)$$

where

$$\begin{aligned} A_k &= \epsilon_k \frac{\mathbf{e}_k^{(j)}\cdot\mathbf{w}_k}{|\mathbf{e}_k^{(j)}|^2} + \epsilon_{k+1}\frac{\mathbf{e}_{k+1}^{(j)}\cdot\mathbf{w}_{k+1}}{|\mathbf{e}_{k+1}^{(j)}|^2} - \epsilon_0\left(\frac{\mathbf{e}_k^{(j)}}{|\mathbf{e}_k^{(j)}|^2} + \frac{\mathbf{e}_{k+1}^{(j)}}{|\mathbf{e}_{k+1}^{(j)}|^2}\right)\cdot\mathbf{w}_0, \\ B_k &= -\epsilon_k\mathbf{e}_{k+1}^{(j)}\cdot\mathbf{w}_k - \epsilon_{k+1}\mathbf{e}_k^{(j)}\cdot\mathbf{w}_{k+1} + \epsilon_0(\mathbf{e}_k^{(j)} + \mathbf{e}_{k+1}^{(j)})\cdot\mathbf{w}_0. \end{aligned} \tag{2.4}$$

Thus, we can write

$$\mathcal{K}_j^e = \mathcal{K}_j\left(1 + \frac{\pi}{2\mathcal{K}_j}\sum_{k=1}^{n_j}\left(A_k c_k^{(j)} + \frac{B_k}{|\mathbf{e}_k^{(j)}||\mathbf{e}_{k+1}^{(j)}|}\right)\right) + \mathcal{O}(\epsilon^2).$$

So, if ϵ is sufficiently small, $\mathcal{K}_j\mathcal{K}_j^e > 0$ if

$$\frac{\pi}{2\mathcal{K}_j}\sum_{k=1}^{n_j}\left(A_k c_k^{(j)} + \frac{B_k}{|\mathbf{e}_k^{(j)}||\mathbf{e}_{k+1}^{(j)}|}\right) > -1$$

and this is true if

$$-\frac{\pi}{2|\mathcal{K}_j|}\sum_{k=1}^{n_j}\left(|A_k||c_k^{(j)}| + \frac{|B_k|}{|\mathbf{e}_k^{(j)}||\mathbf{e}_{k+1}^{(j)}|}\right) > -4/5. \tag{2.5}$$

Now, from (2.4) it is easy to verify that $|A_k| \le 2\epsilon(|\mathbf{e}_k^{(j)}|^{-1} + |\mathbf{e}_{k+1}^{(j)}|^{-1})$ and $|B_k| \le 2\epsilon(\mathbf{e}_k^{(j)}|^{-1} + |\mathbf{e}_{k+1}^{(j)}|^{-1})|\mathbf{e}_k^{(j)}||\mathbf{e}_{k+1}^{(j)}|$. Using these inequalities, after a little algebra, we obtain that, if ϵ is sufficiently small, (2.3) implies (2.5). □

If ϵ is an assigned small positive quantity such that $|\mathbf{P}_j - \mathbf{P}_j^e| \le \epsilon$, $j = 1, \ldots, N$, if (2.3) holds we use (2.1) to define K_j because we consider it sign-reliable. Otherwise, we try to get information about the sign of the Gaussian curvature at the point $\mathbf{P}_j$, repeating the check after substituting the neighboring points of $\mathbf{P}_j$ with other new suitable n_j points. In particular, these are chosen among the neighbors of all the $\mathbf{V}_k^{(j)}, k = 1, \ldots, n_j$ and they are uniformly spaced as much as possible with respect to the azimuth (defined relating to $\mathbf{P}_j$). If after this substitution (2.3) holds the new neighboring points are used to define K_j through (2.1), otherwise this strategy is repeated until we consider that the new neighbors are too far from $\mathbf{P}_j$. In the last case, we put the curvature value equal to 0.

3 Knot selection in radial approximation

Let $\phi : \mathbb{R}_{\ge 0} \to \mathbb{R}$, be a compactly supported radial basis function. We approximate the given data by the surface

$$z(\mathbf{X}) := a_0 + \sum_{l=1}^{M} a_l \phi\left(\frac{\| \mathbf{X} - \mathbf{X}_l^* \|_2}{\delta_l}\right),$$

where the set of knots $\{\mathbf{X}_l^*, l = 1, \ldots, M\} \subset \mathcal{P}_{xy}$ and the set of positive δ–parameters $\{\delta_l, l = 1, \ldots, M\}$ are previously chosen. The coefficients $a_0, \ldots, a_M$ are determined minimizing $\sum_{j=1}^{N}(z_j - z(\mathbf{X}_j))^2$. The knot number and their positions are selected considering the information given by the discrete Gaussian curvature distribution as defined in the previous section.

Inspired by the algorithm proposed in [6], the strategy for the $\mathbf{X}_l^*$ and $\delta_l, l = 1, \ldots, M$ choice can be summarized as follows:

- an input tolerance tol_G is given;
- a first set of distinct knots $\{\mathbf{X}_l^*, l = 1, \ldots, M_0\} \subset \mathcal{P}_{xy}$ with $M_0 \leq M$ is chosen. This is done selecting the areas where the absolute value of the discrete Gaussian curvature is greater than tol_G. A knot is located in the middle of an area if the sign of the related curvature is positive. In case of negative curvature, four knots are located near the boundary of the area also taking into consideration the suggestions given by the data distribution;
- initial values for the δ–parameters $\delta_l, l = 1, \ldots, M_0$ are determined considering the knot separation distance;
- the final set of knots is defined by possibly increasing the multiplicity of the previously selected knots. In this case, the δ–parameters associated to the same knot must be different.

Remark 3.1 *We observe that, to be sure that the least squares problem has a unique solution, it should be proved that the related collocation matrix is of full rank and this is clearly equivalent to the uniqueness of the corresponding interpolation problem (the only result we know about uniqueness of the radial interpolant defined with different scales is given in a submitted paper [1] where interesting sufficient conditions are given). However, we believe that the least squares problem is much more robust than the corresponding interpolation problem and in all the numerical experiments we have never had problems related to the rank of the collocation matrix (see also [5, 7]).*

4 Numerical results

In this section we use the compactly supported radial basis function [8]

$$\phi(r) := (1 - r)_+^3 (1 + 3r)$$

for checking the features of the proposed approach on two test functions. The first is the well known Franke function and the second is the function $z(\mathbf{X}) = 0.35(\sin(2\pi x) + \sin(2\pi y))$, $\mathbf{X} \in [0, 1]^2$. For both tests, $N = 441$ data points are considered. The exact data are obtained by evaluating the functions at the vertices represented on the left of Figure 1. The corresponding non-exact data are defined adding a random noise to the exact values. In particular, in the first test we have used $\epsilon = 0.07$ and in the second we have used $\epsilon = 0.08$, in $[0, 0.5]^2 \cup [0.5, 1]^2$ and $\epsilon = 0.008$, otherwise. The related discrete Gaussian curvature (dGc) distributions computed with the strategy sketched at the end of Section 2 are reported in Figure 2.

Figures 3 and 4 relate to the first test with exact and non-exact data, respectively. The distinct knots are $\mathbf{X}_1^* = (0.207, 0.205), \mathbf{X}_2^* = (0.449, 0.797), \mathbf{X}_3^* = (0.756, 0.349)$ and each of them is repeated three times with three different δ-parameter values, 0.6, 0.4, 0.3. The mean error $\sqrt{\sum_{j=1}^N (z_j - z(\mathbf{X}_j))^2 / N}$ is about 0.016 in Figure 3 and 0.025 in Figure

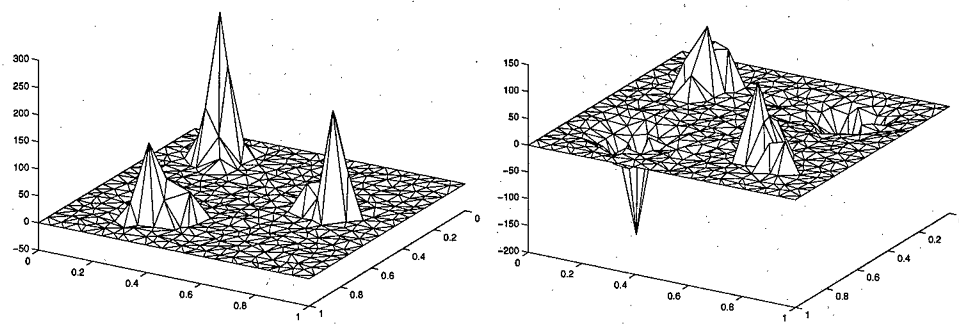

FIG. 2. dGc for the first (left) and second (right) set of non-exact data.

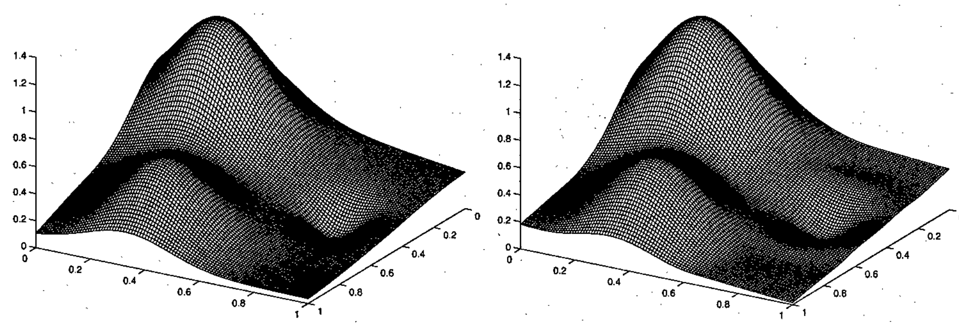

FIG. 3. The parent Franke surface (left) and its approximation (right).

4 (it was about 1/3 using only 3 distinct knots with all the δ–parameters equal to 0.6). Figures 5 and 6 relate to the second test. The distinct knots are $(0.258, 0.238)$, $(0.749, 0.737)$, $(0.950, 0.264)$, $(0.700, 0.264)$, $(0.756, 0.050)$, $(0.756, 0.300)$, $(0.050, 0.751)$, $(0.300, 0.751)$, $(0.264, 0.700)$, $(0.264, 0.950)$. The related δ–parameters are $0.8, 0.8, 0.6, 0.4, 0.6, 0.4, 0.6, 0.4, 0.4, 0.6$. The mean error is about 0.020 in Figure 5 and 0.026 in Figure 6.

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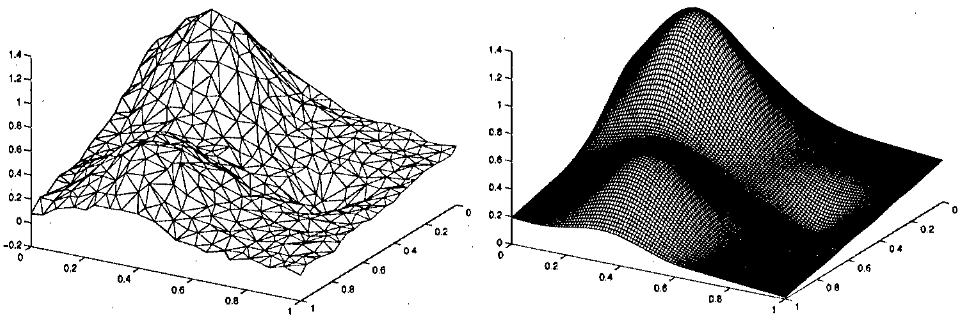

FIG. 4. The non-exact set of data (left) and its approximation (right).

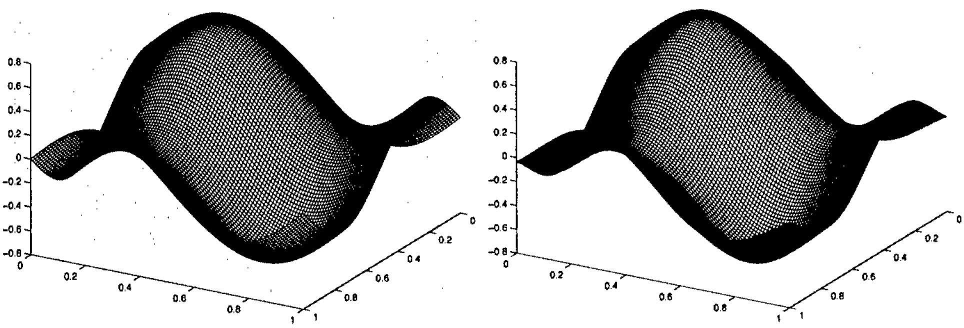

FIG. 5. The parent surface (left) and its approximation (right).

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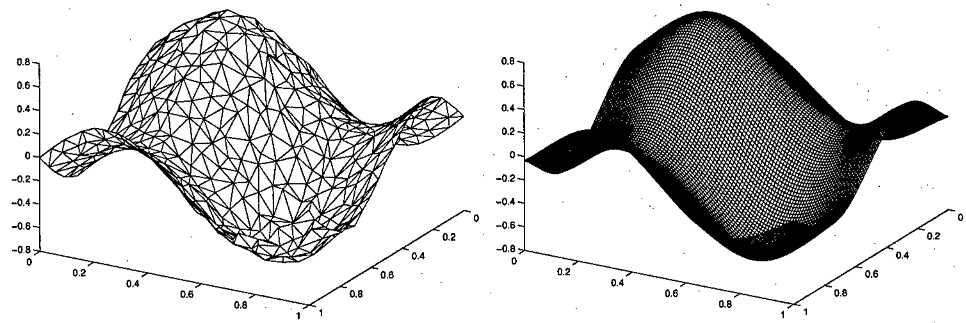

FIG. 6. The non-exact set of data (left) and its approximation (right).

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