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Orthogonal distance fitting of parametric curves and surfaces

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Abstract

Fitting of parametric curves and surfaces to a set of given data points is a relevant subject in various fields of science and engineering. In this paper, we review the current orthogonal distance fitting algorithms for parametric models in a well organized and easily understandable manner, and present a new algorithm. Each of these algorithms estimates the model parameters minimizing the square sum of the error distances between the model feature and the given data points. The model parameters are grouped and simultaneously estimated in terms of form, position, and rotation parameters. The form parameters determine the shape of the model feature, and the position/rotation parameters describe the rigid body motion of the model feature. The new algorithm is applicable to any kind of parametric curve and surface. We give fitting examples for circle, cylinder, and helix in space.

1 Introduction

The use of parametric curves and surfaces is very common and model fitting to a set of given data points is a relevant subject in various fields of science and engineering. For fitting of curves and surfaces, orthogonal distance fitting is of primary concern because of the applied error definition, namely the shortest distance from the given point to the model feature [5, 9]. While there are orthogonal distance fitting algorithms for explicit [3], and implicit models [2, 7] in the literature, we are considering in this paper fitting algorithms for parametric models [4, 6, 8, 10, 11] (Fig. 1).

The goal of the orthogonal distance fitting is the estimation of the model parameters minimizing the performance index

$$\sigma_0^2 = (\mathbf{X} - \mathbf{X}')^{\mathrm{T}}\mathbf{P}^{\mathrm{T}}\mathbf{P}(\mathbf{X} - \mathbf{X}') \tag{1.1}$$

or

$$\sigma_0^2 = \mathbf{d}^{\mathrm{T}}\mathbf{P}^{\mathrm{T}}\mathbf{P}\mathbf{d}\,, \tag{1.2}$$

where $\mathbf{X}^{\mathrm{T}} = (\mathbf{X}_1^{\mathrm{T}}, \ldots, \mathbf{X}_m^{\mathrm{T}})$ and $\mathbf{X}'^{\mathrm{T}} = (\mathbf{X}_1'^{\mathrm{T}}, \ldots, \mathbf{X}_m'^{\mathrm{T}})$ are the coordinates vectors of the m given points and of the m corresponding points on the model feature, respectively. Moreover, $\mathbf{d}^{\mathrm{T}} = (d_1, \ldots, d_m)$ is the distances vector with $d_i = \|\mathbf{X}_i - \mathbf{X}_i'\|$, $\mathbf{P}^{\mathrm{T}}\mathbf{P}$ is the weighting matrix. We are calling the fitting algorithms based on the performance indexes (1.1) and (1.2) *coordinate-based algorithm* and *distance-based algorithm*, respectively.

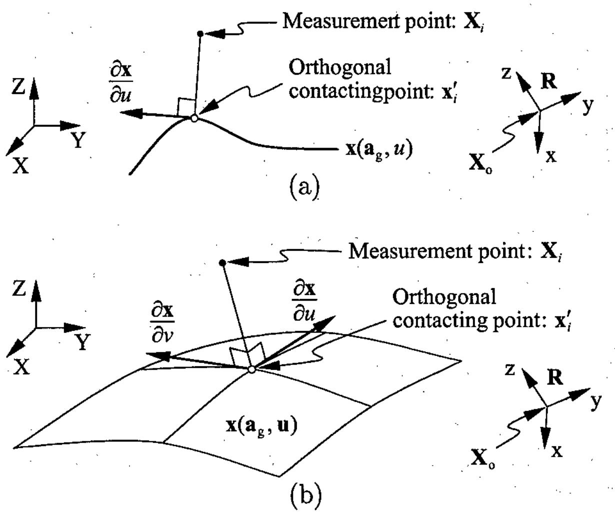


FIG. 1. Parametric features, and the orthogonal contacting point $\mathbf{x}'_i$ in frame xyz from the given point $\mathbf{X}_i$ in frame XYZ: (a) Curve; (b) Surface.

In this paper, the model parameters $\mathbf{a}$ are grouped and simultaneously estimated in three categories. First, the *form parameters* $\mathbf{a}_\mathrm{g}$ (e.g. three axis lengths a, b, c of an ellipsoid) describe the shape of the standard model feature defined in model coordinate system xyz (Fig. 1)

$$\mathbf{x} = \mathbf{x}(\mathbf{a}_\mathrm{g}, \mathbf{u}) \qquad \text{with} \qquad \mathbf{a}_\mathrm{g} = (a_1, \ldots, a_l)^\mathrm{T} . \tag{1.3}$$

The form parameters are invariant to the rigid body motion of the model feature. The second and the third parameters groups, respectively the *position parameters* $\mathbf{a}_\mathrm{p}$ and the *rotation parameters* $\mathbf{a}_\mathrm{r}$, describe the rigid body motion of the model feature in machine coordinate system XYZ:

$$\begin{aligned} &\mathbf{X} = \mathbf{R}^{-1}\mathbf{x} + \mathbf{X}_\mathrm{o} \qquad \text{or} \qquad \mathbf{x} = \mathbf{R}(\mathbf{X} - \mathbf{X}_\mathrm{o}), \\ \text{where} \quad &\mathbf{R} = \mathbf{R}_\kappa \mathbf{R}_\varphi \mathbf{R}_\omega = (\mathbf{r}_1 \;\; \mathbf{r}_2 \;\; \mathbf{r}_3)^\mathrm{T}, \qquad \mathbf{R}^{-1} = \mathbf{R}^\mathrm{T}, \\ &\mathbf{a}_\mathrm{p} = \mathbf{X}_\mathrm{o} = (X_\mathrm{o}, Y_\mathrm{o}, Z_\mathrm{o})^\mathrm{T}, \qquad \text{and} \qquad \mathbf{a}_\mathrm{r} = (\omega, \varphi, \kappa)^\mathrm{T}. \end{aligned} \tag{1.4}$$

A subproblem of the orthogonal distance fitting of a parametric model is the finding of the location parameters $\{\mathbf{u}_i\}_{i=1}^m$, which represent the nearest points $\{\mathbf{X}'_i\}_{i=1}^m$ on the model feature from each given point $\{\mathbf{X}_i\}_{i=1}^m$. The model parameters $\mathbf{a}$ and the location parameters $\{\mathbf{u}_i\}_{i=1}^m$ will generally be estimated through iteration. By the *total method* [6, 10], $\mathbf{a}$ and $\{\mathbf{u}_i\}_{i=1}^m$ will be simultaneously determined, while they are to be separately estimated by the *variable-separation method* [4, 8, 11] in a nested iteration scheme. There could be four combinations for algorithmic approaches as shown in Table 1. One of the algorithmic approaches in Table 1 results in an obviously underdetermined linear system for iteration, thus, it has no practical application. We describe and compare the realistic three algorithmic approaches in the following sections.

Algorithmic approaches	Distance-based algor.	Coordinate-based algor.
Total method	Underdetermined system	I (ETH [6, 10])
Variable-separation method	II (NPL [4, 11])	III (FhG, this paper)

TAB. 1. Orthogonal distance fitting algorithms for parametric models.

2 Orthogonal distance fitting algorithm I (ETH)

The ETH algorithm [6, 10] is based on the performance index (1.1), and simultaneously estimates the model parameters $\mathbf{a}$ and the location parameters $\{\mathbf{u}_i\}_{i=1}^m$ for the nearest points on the model feature. We introduce the new estimation parameters vector $\mathbf{b}$ containing $\mathbf{a}$ and $\{\mathbf{u}_i\}_{i=1}^m$ as follows,

$$\mathbf{b}^{\mathrm{T}} = (\mathbf{a}^{\mathrm{T}}, \mathbf{u}_1^{\mathrm{T}}, \ldots, \mathbf{u}_m^{\mathrm{T}}) = (\mathbf{a}_{\mathrm{g}}^{\mathrm{T}}, \mathbf{a}_{\mathrm{p}}^{\mathrm{T}}, \mathbf{a}_{\mathrm{r}}^{\mathrm{T}}, \mathbf{u}_1^{\mathrm{T}}, \ldots, \mathbf{u}_m^{\mathrm{T}}).$$

The parameters vector $\mathbf{b}$ minimizing the performance index (1.1) can be determined by the Gauss-Newton method

$$\mathbf{P}\frac{\partial \mathbf{X}'}{\partial \mathbf{b}}\bigg|_k \Delta\mathbf{b} = \mathbf{P}(\mathbf{X}-\mathbf{X}')|_k\,, \qquad \mathbf{b}_{k+1} = \mathbf{b}_k + \alpha\Delta\mathbf{b}\,, \tag{2.1}$$

with the Jacobian matrices of each point $\mathbf{X}'_i$ on the model feature, from (1.3) and (1.4)

$$\begin{aligned}\mathbf{J}_{\mathbf{X}'_i,\mathbf{b}} &= \frac{\partial \mathbf{X}}{\partial \mathbf{b}}\bigg|_{\mathbf{X}=\mathbf{X}'_i} = \left(\mathbf{R}^{-1}\frac{\partial \mathbf{x}}{\partial \mathbf{b}} + \frac{\partial \mathbf{R}^{-1}}{\partial \mathbf{b}}\mathbf{x} + \frac{\partial \mathbf{X}_{\mathrm{o}}}{\partial \mathbf{b}}\right)\bigg|_{\mathbf{u}=\mathbf{u}_i} \\ &= \left(\mathbf{R}^{-1}\frac{\partial \mathbf{x}}{\partial \mathbf{a}_{\mathrm{g}}} \;\bigg|\; \mathbf{I} \;\bigg|\; \frac{\partial \mathbf{R}^{-1}}{\partial \mathbf{a}_{\mathrm{r}}}\mathbf{x} \;\bigg|\; \mathbf{0}_1, \cdots, \mathbf{0}_{i-1},\, \mathbf{R}^{-1}\frac{\partial \mathbf{x}}{\partial \mathbf{u}},\, \mathbf{0}_{i+1}, \cdots, \mathbf{0}_m\right)\bigg|_{\mathbf{u}=\mathbf{u}_i}.\end{aligned}$$

A disadvantage of the ETH algorithm is that the storage space and the computing time cost increase very rapidly with the number of the data points, unless the sparse linear system (2.1) is handled beforehand by a sparse matrix algorithm.

3 Orthogonal distance fitting algorithm II (NPL)

The NPL algorithm [4, 11] is based on the performance index (1.2), and separately estimates the model parameters $\mathbf{a}$ and the location parameters $\{\mathbf{u}_i\}_{i=1}^m$ in a *nested iteration* scheme

$$\min_{\mathbf{a}} \min_{\{\mathbf{u}_i\}_{i=1}^m} \sigma_0^2\left(\{\mathbf{X}'_i(\mathbf{a},\mathbf{u})\}_{i=1}^m\right).$$

The inner iteration determines the location parameters $\{\mathbf{u}'_i\}_{i=1}^m$ for the minimum distance points $\{\mathbf{X}'_i\}_{i=1}^m$ on the current model feature from each given point $\{\mathbf{X}_i\}_{i=1}^m$, and, the outer iteration updates the model parameters. In this paper, in order to implement the parameters grouping of $\mathbf{a}^{\mathrm{T}} = (\mathbf{a}_{\mathrm{g}}^{\mathrm{T}}, \mathbf{a}_{\mathrm{p}}^{\mathrm{T}}, \mathbf{a}_{\mathrm{r}}^{\mathrm{T}})$, we have modified the initial NPL algorithm.

3.1 Orthogonal contacting point

For each given point $\mathbf{x}_i = \mathbf{R}(\mathbf{X}_i - \mathbf{X}_{\mathrm{o}})$ in frame xyz, we determine the orthogonal contacting point $\mathbf{x}'_i$ on the standard model feature (1.3). Then, the orthogonal contacting point $\mathbf{X}'_i$ in frame XYZ to the given point $\mathbf{X}_i$ will be obtained through a backward transformation of $\mathbf{x}'_i$ into XYZ. We are searching the location parameters $\mathbf{u}$ which minimizes the error distance between the given point $\mathbf{x}_i$ and the corresponding point $\mathbf{x}$ on the model feature (1.3)

$$D = (\mathbf{x}_i - \mathbf{x}(\mathbf{a}_g, \mathbf{u}))^{\mathrm{T}}(\mathbf{x}_i - \mathbf{x}(\mathbf{a}_g, \mathbf{u}))\,. \tag{3.1}$$

The first order necessary condition for a minimum of (3.1) as a function of $\mathbf{u}$ is

$$\mathbf{f}(\mathbf{x}_i, \mathbf{x}(\mathbf{a}_g, \mathbf{u})) = \frac{1}{2}\begin{pmatrix} D_u \\ D_v \end{pmatrix} = -\begin{pmatrix} (\mathbf{x}_i - \mathbf{x}(\mathbf{a}_g, \mathbf{u}))^{\mathrm{T}}\mathbf{x}_u \\ (\mathbf{x}_i - \mathbf{x}(\mathbf{a}_g, \mathbf{u}))^{\mathrm{T}}\mathbf{x}_v \end{pmatrix} = \mathbf{0}\,. \tag{3.2}$$

The condition (3.2) means that the error vector $(\mathbf{x}_i - \mathbf{x})$ and the surface tangent vectors $\partial\mathbf{x}/\partial\mathbf{u}$ at $\mathbf{x}$ should be orthogonal. We solve (3.2) for $\mathbf{u}$ by using the Newton method (how to derive the Jacobian matrix $\partial\mathbf{f}/\partial\mathbf{u}$ is shown in Section 4).

$$\left.\frac{\partial\mathbf{f}}{\partial\mathbf{u}}\right|_k \Delta\mathbf{u} = -\mathbf{f}(\mathbf{u})|_k\,, \qquad \mathbf{u}_{k+1} = \mathbf{u}_k + \alpha\Delta\mathbf{u}\,. \tag{3.3}$$

3.2 Orthogonal distance fitting

We update the model parameters $\mathbf{a}$ minimizing the performance index (1.2) by using the Gauss-Newton method (outer iteration)

$$\mathbf{P}\left.\frac{\partial\mathbf{d}}{\partial\mathbf{a}}\right|_k \Delta\mathbf{a} = -\mathbf{P}\mathbf{d}|_k\,, \qquad \mathbf{a}_{k+1} = \mathbf{a}_k + \alpha\Delta\mathbf{a}\,.$$

From $d_i = \|\mathbf{X}_i - \mathbf{X}_i'\|$, and equations (1.3) and (1.4), we derive the Jacobian matrices of each *orthogonal distance* d_i

$$\begin{aligned}\mathbf{J}_{d_i,\mathbf{a}} = \frac{\partial d_i}{\partial\mathbf{a}} &= -\left.\frac{(\mathbf{X}_i - \mathbf{X}_i')^{\mathrm{T}}}{\|\mathbf{X}_i - \mathbf{X}_i'\|}\frac{\partial\mathbf{X}}{\partial\mathbf{a}}\right|_{\mathbf{u}=\mathbf{u}_i'} \\ &= -\frac{(\mathbf{X}_i - \mathbf{X}_i')^{\mathrm{T}}}{\|\mathbf{X}_i - \mathbf{X}_i'\|}\left.\left(\mathbf{R}^{-1}\left(\frac{\partial\mathbf{x}}{\partial\mathbf{a}} + \frac{\partial\mathbf{x}}{\partial\mathbf{u}}\frac{\partial\mathbf{u}}{\partial\mathbf{a}}\right) + \frac{\partial\mathbf{R}^{-1}}{\partial\mathbf{a}}\mathbf{x} + \frac{\partial\mathbf{X}_o}{\partial\mathbf{a}}\right)\right|_{\mathbf{u}=\mathbf{u}_i'}.\end{aligned}$$

With (1.4) and (3.2) at $\mathbf{u}=\mathbf{u}_i'$,

$$(\mathbf{X}_i - \mathbf{X}_i')^{\mathrm{T}}\mathbf{R}^{-1}\left.\frac{\partial\mathbf{x}}{\partial\mathbf{u}}\right|_{\mathbf{u}=\mathbf{u}_i'} = (\mathbf{x}_i - \mathbf{x}_i')^{\mathrm{T}}\left.\frac{\partial\mathbf{x}}{\partial\mathbf{u}}\right|_{\mathbf{u}=\mathbf{u}_i'} = \mathbf{0}^{\mathrm{T}}\,,$$

$$\text{and}\quad \mathbf{J}_{d_i,\mathbf{a}} = -\frac{(\mathbf{X}_i - \mathbf{X}_i')^{\mathrm{T}}}{\|\mathbf{X}_i - \mathbf{X}_i'\|}\left(\mathbf{R}^{-1}\left.\frac{\partial\mathbf{x}}{\partial\mathbf{a}_g}\right|_{\mathbf{u}=\mathbf{u}_i'} \;\middle|\; \mathbf{I} \;\middle|\; \frac{\partial\mathbf{R}^{-1}}{\partial\mathbf{a}_r}\mathbf{x}_i'\right)$$

is the resultant Jacobian matrix for d_i. A drawback of the NPL algorithm is that the convergence and the accuracy of 3D-curve fitting (e.g. fitting of a circle in space) are relatively poor. 2D-curve fitting or surface fitting with the NPL algorithm do not suffer from such problems.

4 Orthogonal distance fitting algorithm III (FhG)

At the Fraunhofer Institute IPA (FhG-IPA), a new orthogonal distance fitting algorithm for parametric models is developed, which minimizes the performance index (1.1) in a nested iteration scheme (variable-separation method). The new algorithm is a generalized extension of an orthogonal distance fitting algorithm for implicit plane curves [1]. Interested readers are referred to [2] for the orthogonal distance fitting of implicit surfaces and plane curves. The location parameter values $\{\mathbf{u}_i'\}_{i=1}^m$ for the minimum distance points $\{\mathbf{X}'_i\}_{i=1}^m$ on the current model feature from each given point $\{\mathbf{X}_i\}_{i=1}^m$ are to be found by the algorithm described in Section 3.1 (inner iteration). In this section, we intend to describe the outer iteration which updates the model parameters $\mathbf{a}$ minimizing the performance index (1.1) by using the Gauss-Newton method

$$\mathbf{P}\frac{\partial \mathbf{X}'}{\partial \mathbf{a}}\bigg|_k \Delta\mathbf{a} = \mathbf{P}(\mathbf{X}-\mathbf{X}')|_k\,, \qquad \mathbf{a}_{k+1} = \mathbf{a}_k + \alpha\Delta\mathbf{a}\,, \tag{4.1}$$

with the Jacobian matrices of each *orthogonal distance point* $\mathbf{X}'_i$, from (1.3) and (1.4)

$$\begin{aligned}\mathbf{J}_{\mathbf{X}'_i,\mathbf{a}} = \frac{\partial \mathbf{X}}{\partial \mathbf{a}}\bigg|_{\mathbf{X}=\mathbf{X}'_i} &= \left(\mathbf{R}^{-1}\left(\frac{\partial \mathbf{x}}{\partial \mathbf{a}} + \frac{\partial \mathbf{x}}{\partial \mathbf{u}}\frac{\partial \mathbf{u}}{\partial \mathbf{a}}\right) + \frac{\partial \mathbf{R}^{-1}}{\partial \mathbf{a}}\mathbf{x} + \frac{\partial \mathbf{X}_o}{\partial \mathbf{a}}\right)\bigg|_{\mathbf{u}=\mathbf{u}'_i} \\ &= \mathbf{R}^{-1}\frac{\partial \mathbf{x}}{\partial \mathbf{u}}\frac{\partial \mathbf{u}}{\partial \mathbf{a}}\bigg|_{\mathbf{u}=\mathbf{u}'_i} + \left(\mathbf{R}^{-1}\frac{\partial \mathbf{x}}{\partial \mathbf{a}_g}\bigg|_{\mathbf{u}=\mathbf{u}'_i} \;\bigg|\; \mathbf{I} \;\bigg|\; \frac{\partial \mathbf{R}^{-1}}{\partial \mathbf{a}_r}\mathbf{x}'_i\right).\end{aligned} \tag{4.2}$$

The derivative matrix $\partial\mathbf{u}/\partial\mathbf{a}$ at $\mathbf{u}=\mathbf{u}'_i$ in (4.2) describes the variational behavior of the location parameters $\mathbf{u}'_i$ for the orthogonal contacting point $\mathbf{x}'_i$ in frame xyz relative to the differential changes of the parameters vector $\mathbf{a}$. Purposefully, we derive $\partial\mathbf{u}/\partial\mathbf{a}$ from the condition (3.2). Because (3.2) has an implicit form, its derivatives lead to

$$\frac{\partial \mathbf{f}}{\partial \mathbf{u}}\frac{\partial \mathbf{u}}{\partial \mathbf{a}} + \frac{\partial \mathbf{f}}{\partial \mathbf{x}_i}\frac{\partial \mathbf{x}_i}{\partial \mathbf{a}} + \frac{\partial \mathbf{f}}{\partial \mathbf{a}} = \mathbf{0} \quad \text{or} \quad \frac{\partial \mathbf{f}}{\partial \mathbf{u}}\frac{\partial \mathbf{u}}{\partial \mathbf{a}} = -\left(\frac{\partial \mathbf{f}}{\partial \mathbf{x}_i}\frac{\partial \mathbf{x}_i}{\partial \mathbf{a}} + \frac{\partial \mathbf{f}}{\partial \mathbf{a}}\right), \tag{4.3}$$

where $\partial\mathbf{x}_i/\partial\mathbf{a}$ is, from $\mathbf{x}_i = \mathbf{R}(\mathbf{X}_i - \mathbf{X}_o)$,

$$\frac{\partial \mathbf{x}_i}{\partial \mathbf{a}} = \frac{\partial \mathbf{R}}{\partial \mathbf{a}}(\mathbf{X}_i - \mathbf{X}_o) - \mathbf{R}\frac{\partial \mathbf{X}_o}{\partial \mathbf{a}} = \left(\mathbf{0} \;\Big|\; -\mathbf{R} \;\Big|\; \frac{\partial \mathbf{R}}{\partial \mathbf{a}_r}(\mathbf{X}_i - \mathbf{X}_o)\right).$$

The other three matrices $\partial\mathbf{f}/\partial\mathbf{u}$, $\partial\mathbf{f}/\partial\mathbf{x}_i$, and $\partial\mathbf{f}/\partial\mathbf{a}$ in (3.3) and (4.3) are to be directly derived from (3.2). The elements of these three matrices are composed of simple linear combinations of components of the error vector $(\mathbf{x}_i - \mathbf{x})$ with elements of the following three vector/matrices $\partial\mathbf{x}/\partial\mathbf{u}$, $\mathbf{H}$, and $\mathbf{G}$ (XHG matrix):

$$\frac{\partial \mathbf{x}}{\partial \mathbf{u}} = (\mathbf{x}_u \;\; \mathbf{x}_v)\,, \qquad \mathbf{H} = \begin{pmatrix} \mathbf{x}_{uu} & \mathbf{x}_{uv} \\ \mathbf{x}_{vu} & \mathbf{x}_{vv} \end{pmatrix}, \qquad \mathbf{G} = \begin{pmatrix} \mathbf{G}_0 \\ \mathbf{G}_1 \\ \mathbf{G}_2 \end{pmatrix} = \frac{\partial}{\partial \mathbf{a}_g}\begin{pmatrix} \mathbf{x} \\ \mathbf{x}_u \\ \mathbf{x}_v \end{pmatrix}, \tag{4.4}$$

$$\frac{\partial \mathbf{f}}{\partial \mathbf{u}} = (\mathbf{x}_u \;\; \mathbf{x}_v)^{\mathrm{T}}(\mathbf{x}_u \;\; \mathbf{x}_v) - \begin{pmatrix} (\mathbf{x}_i - \mathbf{x})^{\mathrm{T}}\mathbf{x}_{uu} & (\mathbf{x}_i - \mathbf{x})^{\mathrm{T}}\mathbf{x}_{uv} \\ (\mathbf{x}_i - \mathbf{x})^{\mathrm{T}}\mathbf{x}_{vu} & (\mathbf{x}_i - \mathbf{x})^{\mathrm{T}}\mathbf{x}_{vv} \end{pmatrix},$$

$$\frac{\partial \mathbf{f}}{\partial \mathbf{x}_i} = -(\mathbf{x}_u \;\; \mathbf{x}_v)^{\mathrm{T}}\,, \qquad \frac{\partial \mathbf{f}}{\partial \mathbf{a}} = \left(\begin{matrix} \mathbf{x}_u^{\mathrm{T}}\mathbf{G}_0 - (\mathbf{x}_i - \mathbf{x})^{\mathrm{T}}\mathbf{G}_1 \\ \mathbf{x}_v^{\mathrm{T}}\mathbf{G}_0 - (\mathbf{x}_i - \mathbf{x})^{\mathrm{T}}\mathbf{G}_2 \end{matrix} \;\Bigg|\; \mathbf{0} \;\Bigg|\; \mathbf{0}\right).$$

Now (4.3) can be solved for $\partial\mathbf{u}/\partial\mathbf{a}$ at $\mathbf{u}=\mathbf{u}'_i$, and the Jacobian matrix (4.2) and the linear system (4.1) can be completed and solved for the parameter update $\Delta\mathbf{a}$.

We would like to stress that only the standard model equation (1.3), without involvement of the position/rotation parameters, is required in (4.4). The overall structure of the FhG algorithm remains unchanged for all dimensional fitting problems of parametric models. All that is necessary for a new parametric model is to derive the XHG matrix of (4.4) from (1.3) of the new model feature, and to supply a proper set of initial para-

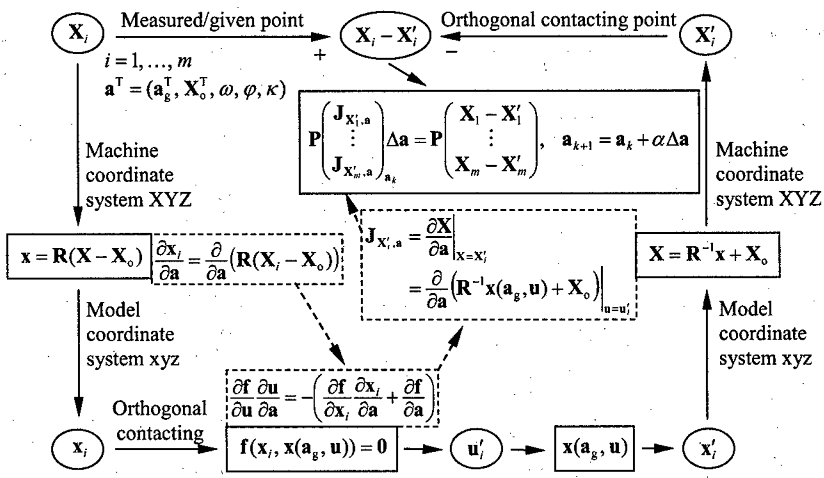


FIG. 2. Information flow with the FhG algorithm.

X	5	6	5	5	3	2	0	−1	−1	0	3	4	7	9
Y	1	3	4	6	5	4	2	0	−2	−5	−7	−8	−10	−9
Z	−3	−1	1	3	5	7	9	11	11	11	11	11	11	10

TAB. 2. Fourteen coordinate triples representing a helix.

meter values $\mathbf{a}_0$ for iteration (4.1). An overall schematic information flow with the FhG algorithm is shown in Fig. 2. The FhG algorithm shows robust and fast convergence with 2D/3D-curve and surface fitting. The storage space and computing time cost are proportional to the number of data points. A disadvantage of the FhG algorithm is that it additionally requires the second derivatives $\partial^2\mathbf{x}/\partial\mathbf{a}_g\partial\mathbf{u}$ as shown in (4.4).

As a fitting example, we show the orthogonal distance fitting of a helix. The standard model feature (1.3) of a helix in frame xyz can be described as follows. $\mathbf{x}(\mathbf{a}_g, u) = \mathbf{x}(r, h, u) = (r\cos u, r\sin u, hu/2\pi)^{\mathrm{T}}$, with a constraint on the position and rotation parameters

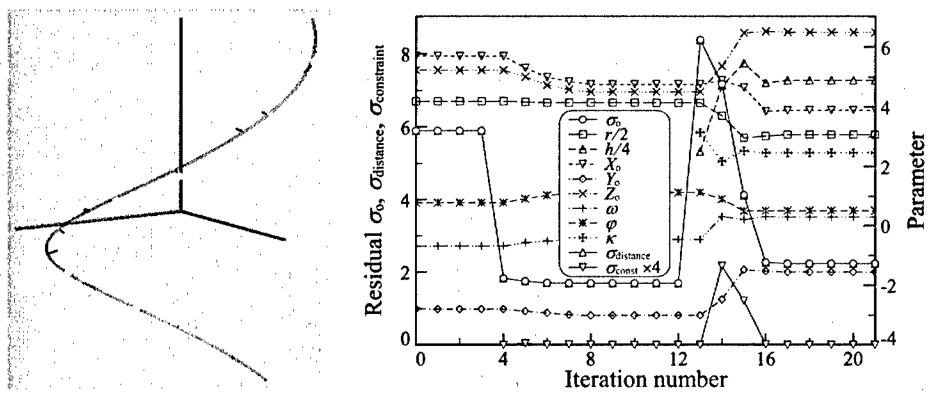
$$f_c(\mathbf{a}_p, \mathbf{a}_r) = (\mathbf{X}_o - \overline{\mathbf{X}})^{\mathrm{T}}\mathbf{r}_3(\omega, \varphi) = 0\,,$$

where r and h are respectively the radius and elevation of a helix. $\overline{\mathbf{X}}$ is the gravitational center of the given points set and $\mathbf{r}_3$ (see (1.4)) is the vector of direction cosines of the z-axis. We have obtained the initial parameter values from a 3D-circle fitting, and a cylinder fitting, successively. The helix fitting to the points set in Table 2 with the initial values of $h = 10$ and $\kappa = \pi$ terminated after 0.22s, 8 iteration cycles for $\|\Delta\mathbf{a}\| = 3.2\times10^{-7}$ with a Pentium 133 MHz PC (Table 3, Fig. 3). They were 0.33s, 10 iteration cycles for $\|\Delta\mathbf{a}\| = 3.6\times10^{-7}$ with the ETH algorithm, and, 1.05s, 61 iteration cycles for $\|\Delta\mathbf{a}\| = 8.8\times10^{-7}$ with the NPL algorithm. The computing cost with the ETH algorithm increases rapidly with the number of the data points. The NPL algorithm showed slow convergences with the 3D-circle and the helix fitting (3D-curve fitting).

Parameters $\hat{\mathbf{a}}$	σ_0	r	h	X_o
3D-Circle	5.8913	8.3850	−−	5.6999
$\sigma(\hat{\mathbf{a}})$	−−	0.7355	−−	0.9939
Cylinder	1.6925	8.2835	−−	4.7596
$\sigma(\hat{\mathbf{a}})$	−−	0.2738	−−	0.7465
Helix	2.2301	6.1368	19.5811	3.8909
$\sigma(\hat{\mathbf{a}})$	−−	0.4238	1.3214	0.5488

Y_o	Z_o	ω	φ	κ
−2.7923	5.2333	−0.6833	0.7882	−−
0.8421	0.8821	0.1177	0.1375	−−
−3.0042	4.5081	−0.4576	1.1327	−−
0.4525	0.6513	0.3049	0.2116	−−
−1.5560	6.4871	0.3003	0.5114	2.4602
0.3934	0.7500	0.0880	0.0663	0.2881

TAB. 3. Results of the orthogonal distance fitting to the points set in Table 2.



(a) (b)

FIG. 3. Orthogonal distance fitting to the points set in Table 2: (a) Helix fit; (b) Convergence of the fit. Iteration number 0–3: 3D-circle, 4–12: circular cylinder, and 13–: helix fit with the initial value of $h=10$ and $\kappa=\pi$.

5 Summary

In this paper, we have reviewed the current orthogonal distance fitting algorithms for parametric curves and surfaces in an easily understandable manner, and presented a new algorithm. By each of the algorithms the model parameters are grouped and simultaneously estimated in terms of form/position/rotation parameters. The ETH algorithm demands a large amount of storage space and high computing cost, and the NPL algorithm shows relatively poor performance with 3D-curve fitting. The new algorithm, the FhG algorithm, has no such drawbacks of the ETH algorithm or of the NPL algorithm. A disadvantage of the FhG algorithm is that it requires the second derivatives $\partial^2\mathbf{x}/\partial\mathbf{a}_g\partial\mathbf{u}$. The FhG algorithm does not require a necessarily good set of initial parameter values, which could also be internally supplied as demonstrated with the fitting examples. From the viewpoint of implementation and application to a new model feature, the FhG algorithm is universal and very efficient. Merely the standard model equation (1.3) of the new model feature is eventually required, which has only few form parameters. The functional interpretation and treatment of the position/rotation parameters are basically identical for all parametric models. The storage space and the computing time cost are proportional to the number of given data points. Together with other orthogonal distance fitting algorithms for implicit models [2], the FhG algorithm is certified by the German federal authority PTB [5, 9], with a certification grade that the parameter estimation accuracy is higher than 0.1μm for length unit, and 0.1μrad for angle unit for all parameters of all tested model features.

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