

## Langevin method for shot noise in single-electron tunneling

A. N. Korotkov

Department of Physics, State University of New York, Stony Brook,  
 NY 11794-3800, USA and

Nuclear Physics Institute, Moscow State University, Moscow 119899, Russia

**Abstract.** The shot noise in correlated single-electron tunneling can be calculated using the Langevin approach. It is equivalent to the Fokker–Plank-type approach in the “orthodox” framework, however, the advantage is a natural possibility to describe also the fluctuations in the quantum frequency range.

Correlated single-electron tunneling [1] remains an attractive topic during last decade. The theoretical study of the shot noise in this regime is interesting because the noise reflects rather directly the correlations in the charge transport, and also because the noise determines the ultimate sensitivity of single-electron devices.

The shot noise in single-electron tunneling is due to the randomness of tunneling events. The earlier developed formalism [2] of noise calculation is of the Fokker–Plank type and is based on the deterministic master equation of “orthodox” theory [3] (we assume sufficiently large tunnel resistances,  $R_j \gg h/e^2$ ):

$$\dot{\sigma} = \Gamma \sigma, \quad (1)$$

where the element  $\sigma_n$  of the vector  $\sigma$  is the probability to find the system in the charge state  $n \equiv \{n_1, \dots, n_L\}$  (which is characterized by the numbers  $n_i$  of excess electrons in each of  $L$  internal nodes of the system) and

$$\Gamma_{mn} = \Gamma_{m \leftarrow n} - \delta_{mn} \sum_k \Gamma_{k \leftarrow n}, \quad \Gamma_{m \leftarrow n} = \sum_j \Gamma_{m \leftarrow n}^j, \quad (2)$$

where  $\Gamma_{m \leftarrow n}^j$  are the corresponding tunneling rates and the summation over the junction number  $j$  is necessary when an electron can tunnel to (from) an internal node from (to) different external electrodes.

To find the mutual spectral density for two processes  $X(t)$  and  $Y(t)$  we can calculate first the correlation function  $K_{XY}(\tau) = \langle X(t+\tau)Y(t) \rangle - \langle X \rangle \langle Y \rangle$  (brackets denote the averaging over time) and then take the Fourier transform  $S_{XY}(\omega) = 2 \int_{-\infty}^{+\infty} K_{XY}(\tau) \exp(i\omega\tau) d\tau$ . If both  $X$  and  $Y$  are functions of the charge state  $n$  (for example, potential of a node) then the correlation function is given by the simple expression

$$K_{XY}(\tau) = \theta(\tau) \sum_{m,n} X(m) \sigma(\tau, m|n) Y(n) \sigma_n^{st} + \theta(-\tau) \sum_{m,n} Y(m) \sigma(m, -\tau|n) \times X(n) \sigma_n^{st} - \langle X \rangle \langle Y \rangle, \quad (3)$$

where  $\sigma(\tau, m|n)$  is the retarded Green’s function of Eq. (1) being the probability to find the system in the state  $m$  at  $t = \tau > 0$  if at  $t = 0$  it was in the state  $n$ ,  $\langle X \rangle = \sum_n X(n) \sigma_n^{st}$ , and  $\sigma_n^{st}$  is the stationary distribution,  $\Gamma \sigma^{st} = 0$ ,  $\sum_n \sigma_n^{st} = 1$ .

However, if  $X$  and/or  $Y$  represent the current through a tunnel junction or in an external lead, Eq. (3) should be modified. For example, if  $X(t)$  is the current contribution corresponding to tunneling events  $\Gamma_{m \leftarrow n}^j$  while  $Y(t)$  corresponds to  $\Gamma_{m' \leftarrow n'}^{j'}$ , then (similar to Ref. [2])

$$K_{XY}(\tau)/\tilde{e}_{\pm}^j \tilde{e}_{\pm}^{j'} = \theta(\tau) \Gamma_{m \leftarrow n}^j \sigma(\tau, n|m') \Gamma_{m' \leftarrow n'}^{j'} \sigma_n^{st} + \theta(-\tau) \Gamma_{m' \leftarrow n'}^{j'} \\ \times \sigma(n', -\tau|m) \Gamma_{m \leftarrow n}^j \sigma_n^{st} - \Gamma_{m \leftarrow n}^j \sigma_n^{st} \Gamma_{m' \leftarrow n'}^{j'} \sigma_n^{st} + \delta_{mm'} \delta_{nn'} \delta_{jj'} \delta(\tau) \Gamma_{m \leftarrow n}^j \sigma_n^{st}. \quad (4)$$

Here the last term is responsible for the high-frequency limit. The effective charges  $\tilde{e}_{\pm}^j$  and  $\tilde{e}_{\pm}^{j'}$  are determined by the direction of electron tunneling,  $\tilde{e}_{+}^j = -\tilde{e}_{-}^j$ , and by the circuit capacitances [2] (so that  $\tilde{e}^j = e$  only if the current through junction  $j$  is measured). Any current-current correlation function can be written as a sum of  $K_{XY}(\tau)$  given by Eq. (4) over all possible transitions between charge states (such a sum is a counterpart of Eq. (3) in which the sum is written explicitly).

The expressions for spectral densities directly follow from Eqs. (3) and (4) because the Fourier transformation affects only the evolution operator  $\sigma(\tau, m|n)$ , and the corresponding Green's function in the frequency representation is simply obtained from Eq. (1):  $\sigma(\omega, m|n) = [(-i\omega \mathbf{1} - \Gamma)^{-1}]_{mn}$ , where  $\mathbf{1}$  is the unity matrix. For example, Eq. (4) leads to the following spectral density:

$$S_{XY}(\omega)/\tilde{e}_{\pm}^j \tilde{e}_{\pm}^{j'} = 2 \Gamma_{m \leftarrow n}^j [(-i\omega \mathbf{1} - \Gamma)^{-1}]_{nm'} \Gamma_{m' \leftarrow n'}^{j'} \sigma_n^{st} \\ + 2 \Gamma_{m' \leftarrow n'}^{j'} [(i\omega \mathbf{1} - \Gamma)^{-1}]_{n'm} \Gamma_{m \leftarrow n}^j \sigma_n^{st} + 2 \delta_{nn'} \delta_{mm'} \delta_{jj'} \Gamma_{m \leftarrow n}^j \sigma_n^{st}. \quad (5)$$

This method allows to calculate all spectral densities within the framework of “orthodox” theory, and at least for the single-electron transistor the numerical procedure is rather trivial [2] because the matrix  $\Gamma$  is three-diagonal and the matrix inversion is straightforward.

The alternative method of calculations can be based on the Langevin-type approach [3]. Because of the discrete nature of the charge states, the random term cannot be simply added (in analogy with the standard Langevin method) to some evolution equation for the “coordinate”  $n(t)$ , but should be introduced into the master equation (similar to Ref. [4]). The derivation of the formalism can be understood in the following way. Let us consider the ensemble of  $M$  ( $M \gg 1$ ) independent similar circuits, and let us average all magnitudes over this ensemble. Then the average (over time) currents and voltages will not change (due to ergodicity), but the spectral densities of fluctuations (second order magnitudes) will decrease  $M$  times. Hence, to calculate spectral densities of the initial system, we can take the leading ( $\sim M^{-1}$ ,  $M \rightarrow \infty$ ) order of the spectral density of magnitudes averaged over the ensemble.

In contrast to the single system, the dynamics of the large ensemble is easily described using the Langevin approach. At any moment of time  $t$  the ensemble can be characterized by “coordinates”  $M\sigma_n(t)$  which represent the numbers of participants being in different charge states  $n$  (notice that now  $\sigma_n(t)$  is not a probability but the fluctuating coordinate). While in the stationary state the average number of transitions from state  $m$  to state  $n$  during small time  $\Delta t$  is given by  $M\Gamma_{m \leftarrow n} \sigma_n^{st} \Delta t$ , the r.m.s. of this number is obviously  $(M\Gamma_{m \leftarrow n} \sigma_n^{st} \Delta t)^{1/2}$ . Hence, the recipe is the following [4]: for each average flux  $M\Gamma_{m \leftarrow n} \sigma_n^{st}$  in the space of charge states, we should add in the master equation the

random  $\delta$ -correlated (white) flux  $\xi_{m \leftarrow n}^j(t)$  with the corresponding “seed” spectral density given by Schottky-like formula,

$$\dot{\sigma}_m(t) = \sum_n \Gamma_{mn} \sigma_n(t) + \xi_m(t), \quad \xi_m(t) = \sum_{n,j} \xi_{m \leftarrow n}^j(t) - \xi_{n \leftarrow m}^j(t), \quad (6)$$

$$S_{\xi_{m \leftarrow n}^j \xi_{m' \leftarrow n'}^{j'}}(\omega) = 2M^{-1} \delta_{mm'} \delta_{nn'} \delta_{jj'} \Gamma_{m \leftarrow n}^j \sigma_n^{st}. \quad (7)$$

For the fluxes in opposite directions ( $m \leftarrow n$  and  $n \leftarrow m$ ) we should apply  $\xi(t)$  for each direction, so that the random flux does not vanish even if the net average flux is zero.

Because of the linearity of Eqs. (6)–(7) the final spectral densities of the averaged (over  $M$ ) magnitudes are obviously proportional to  $1/M$ . Hence, rescaling to the single system can be done formally assuming  $M = 1$  in Eqs. (6)–(7). So, instead of keeping  $M$  and rescaling at the final stage, we will use  $M = 1$  in all equation below.

Using the standard procedure we find the Fourier transform

$$\sigma_m(\omega) = \left[ (-i\omega \mathbf{1} - \Gamma)^{-1} \right]_{mn} \xi_n(\omega). \quad (8)$$

Then for the occupation–occupation spectral density we obtain the expression

$$\begin{aligned} S_{\sigma_m \sigma_n} &= \sum_{m'n'} \left[ (-i\omega \mathbf{1} - \Gamma)^{-1} \right]_{mm'} \left[ (i\omega \mathbf{1} - \Gamma)^{-1} \right]_{nn'} S_{\xi_{m'} \xi_{n'}} \\ &= 2 \left[ (-i\omega \mathbf{1} - \Gamma)^{-1} \right]_{mn} \sigma_n^{st} + 2 \left[ (i\omega \mathbf{1} - \Gamma)^{-1} \right]_{nm} \sigma_m^{st}, \end{aligned} \quad (9)$$

which coincides with the result of Fokker-Plank approach (Fourier transform of Eq. (3) without X and Y factors).

The technique is similar for the current–current fluctuations. The case of Eqs. (4) and (5) corresponds to currents

$$X(t) = \tilde{e}_\pm^j \left[ \Gamma_{m \leftarrow n}^j \sigma_n(t) + \xi_{m \leftarrow n}^j(t) \right], \quad Y(t) = \tilde{e}_\pm^{j'} \left[ \Gamma_{m' \leftarrow n'}^{j'} \sigma_{n'}(t) + \xi_{m' \leftarrow n'}^{j'}(t) \right], \quad (10)$$

and the straightforward calculations using Eqs. (7) and (8) lead to Eq. (5). Thus, the Fokker-Plank method is equivalent to the Langevin method within the “orthodox” framework. However, in contrast to the former approach, the Langevin method easily allows phenomenological generalization for the fluctuations in the range of high (“quantum”) frequencies,  $\hbar\omega \sim eV$ .

Let us remind that in “orthodox” theory [1] the tunneling rate  $\Gamma = I_0(W/e)/e[1 - \exp(-W/T)]$  is determined by the energy gain due to tunneling  $W = eV_b - e^2/2C_{eff}$ , where  $I_0(v)$  is the “seed”  $I$ - $V$  curve of the junction (in the linear case  $I_0(v) = v/R_j$ ),  $V_b$  is the voltage across the junction before the tunneling,  $C_{eff}$  is the effective junction capacitance (which also accounts for the environment), and  $T$  is the temperature. The generalization of the Langevin method is the substitution of Eq. (7) by the equation

$$\begin{aligned} S_{\xi_{m \leftarrow n}^j \xi_{m' \leftarrow n'}^{j'}}(\omega) &= \delta_{mm'} \delta_{nn'} \delta_{jj'} [\tilde{\Gamma}^+ + \tilde{\Gamma}^-] \sigma_n^{st}, \\ \tilde{\Gamma}^\pm &= \frac{I_{0,j}(W_{m \leftarrow n}^j/e \pm \hbar\omega/e)}{e \left[ 1 - \exp\left(- (W_{m \leftarrow n}^j \pm \hbar\omega)/T\right) \right]}, \end{aligned} \quad (11)$$

which is derived for the individual tunneling event within the standard tunneling hamiltonian technique averaging the quantum current–current correlator and then taking the Fourier transform. Eq. (11) can be considered as a generalization of the fluctuation-dissipation theorem and equations of Ref. [3] for the case of single-electron tunneling. (Actually, the only difference is that we separate fluctuations corresponding to two directions of tunneling. In absence of the Coulomb blockade they could be summed together leading to the standard factor [3]  $\coth((W \pm \hbar\omega)/2T)$  instead of the denominator of Eq. (11).)

Equations (6) and (11) represent a phenomenological generalization in which the low frequency behavior is treated by the master equation while high frequency properties are taken into account for individual tunneling events. At high frequencies,  $\omega \gg \Gamma$ , the occupation–occupation and occupation–current spectral densities vanish, while for the current–current spectral density instead of Eq. (5) we get  $S_{XY}(\omega) = \tilde{e}_{\pm}^j \tilde{e}_{\pm}^{j'} S_{\xi_{m \leftarrow n}^j \xi_{m' \leftarrow n'}^{j'}}(\omega)$ , because the first terms of Eq. (10) are too slow to give a contribution. This result coincides with the result of Ref. [4]. The advantage of the Langevin approach is the possibility to obtain spectral densities in the “orthodox” and “quantum” frequency ranges using the same formalism while before they were necessarily being treated on different footing.

Notice that in the “quantum” frequency range the current spectral density does not correspond directly to the available power because of the contribution from zero-point oscillations. The contribution to be subtracted from  $S_{\xi_{m \leftarrow n}^j}$  is equal to  $2\hbar\omega \text{Re}G(\omega)$ , where the active conductance

$$\text{Re}G(\omega) = \left( \tilde{e}_{\pm}^j \right)^2 / 2\hbar\omega \left[ \tilde{\Gamma}^+ - \tilde{\Gamma}^- \right] \tag{12}$$

corresponds to the lowest order of photon-assisted tunneling.

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