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# Floquet Theory of the Quantum Dynamic Kingdon Trap

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**Abstract.** The dynamic Kingdon trap is an excellent device for the investigation of chaos and quantum chaos, both theoretically and experimentally. Since it may be interpreted as an electrodynamical version of a Penning-Malmberg trap, it is also suited for the study of strongly coupled periodically or aperiodically driven nonneutral plasmas. Floquet theory provides a natural framework for the quantum mechanics of the periodically driven dynamic Kingdon trap.

The dynamic Kingdon trap is an electrodynamical trap for the storage of charged particles [1]. It resembles an electrodynamical version of a Penning-Malmberg trap [2] and can be used to study nonlinear effects in forced nonneutral plasmas such as rf heating or phase transitions [1]. It can also be used to study classical and quantum chaos [3,4] in oscillating fields. Even a single charged particle stored in a dynamic Kingdon trap may experience a transition to chaos [1,5]. This makes the dynamic Kingdon trap an ideal system, both experimentally and theoretically, for studying quantum chaos effects.

In its simplest form the dynamic Kingdon trap consists of a straight wire surrounded by a conducting cylinder [1]. A superposition of an ac and a dc voltage is applied between the wire and the cylinder such that the dc voltage attracts a charged particle placed between wire and cylinder into the direction of the wire. The voltages induce surface charges on the wire of magnitudes $\sigma_{ac}$ and $\sigma_{dc}$, respectively. The main strength of the dynamic Kingdon trap is that it is capable of storing charged particles at zero angular momentum. We will exclusively focus on this case from now on. In addition we will assume that the particle has zero momentum parallel to the wire. Introducing the radial coordinate $r$ of the charged particle and the unit vector $\hat{r}$ in the radial direction, the force acting on a trapped charged particle of charge $Z$ is given by

$$\vec{F}(\vec{r},t) = \frac{Z}{2\pi\epsilon_0 r}\left[\sigma_{\rm dc} + \sigma_{\rm ac}\cos(\Omega t)\right]\hat{r}, \tag{1}$$

where $\Omega$ is the angular frequency of the applied ac voltage. Choosing

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$$l = \left| \frac{2Z\sigma_{\rm dc}}{\pi \epsilon_0 m \Omega^2} \right|^{1/2} \tag{2}$$

as the unit of length ($m$ is the mass of the trapped particle), and $2/\Omega$ as the unit of time, we define $\rho = r/l$ as the dimensionless radial coordinate and $\tau = \Omega t/2$ as the dimensionless time. Written in these dimensionless quantities, and defining the control parameter $\eta = \sigma_{\rm ac}/(2\sigma_{\rm dc})$, Newton's equation $m\ddot{\vec{r}} = \vec{F}(\vec{r}, t)$ yields the Kingdon equation [1,6]

$$\frac{d^2\rho}{d\tau^2} + [1 - 2\eta \cos(2\tau)] \frac{1}{\rho} = 0 \tag{3}$$

as the classical equation of motion of the charged particle. Phase-space portraits [3] of (3) reveal the presence of a dominant trapping island surrounded by a chaotic sea [1].

The force $\vec{F}$ acting on a charged particle in the dynamic Kingdon trap can be derived from the potential

$$V(r, t) = \left| \frac{Z\sigma_{dc}}{2\pi\epsilon_0} \right| \ln \left( \frac{r}{l} \right) [1 - 2\eta \cos(\Omega t)] \tag{4}$$

via $\vec{F} = -\vec{\nabla} V = -\hat{r} \partial V/\partial r$. We use this potential in the time-dependent Schrödinger equation for a single particle in the dynamic Kingdon trap:

$$i\hbar \frac{\partial \psi(\vec{r}, t)}{\partial t} = -\frac{\hbar^2}{2m} \Delta \psi(\vec{r}, t) + V(r, t)\, \psi(\vec{r}, t). \tag{5}$$

Because of the cylindrical symmetry of the trap and our assumption of zero angular momentum and zero momentum along the wire we have $\psi(\vec{r}, t) = \beta(r, t)$. Switching to dimensionless variables and defining $\varphi(\rho, \tau) = \rho^{\frac{1}{2}} \beta(\rho, \tau)$ we obtain the one-dimensional radial Schrödinger equation

$$i\alpha \frac{\partial \varphi(\rho, \tau)}{\partial \tau} = -\frac{\alpha^2}{2} \left[ \frac{\partial^2 \varphi(\rho, \tau)}{\partial \rho^2} + \frac{1}{4\rho^2} \varphi(\rho, \tau) \right] + [1 - 2\eta \cos(2\tau)] \ln(\rho) \varphi(\rho, \tau), \tag{6}$$

where we defined the effective dimensionless Planck constant $\alpha$, given by

$$\alpha = \left| \frac{\hbar \Omega \pi \epsilon_0}{Z\sigma_{\rm dc}} \right|. \tag{7}$$

Technically (6) is a linear partial differential equation with time-periodic coefficients. In this case Floquet's theory [7] applies. It states that solutions of (6) exist which are of the form

$$\phi_\mu(\rho, \tau) = \exp(-i\mu\tau)\, \Phi_\mu(\rho, \tau), \tag{8}$$

where $\mu$, in general a complex number, is the Floquet exponent and $\Phi_\mu(\rho,\tau)$ is a $\pi$-periodic function. These solutions of (6) are of prime importance to us, since we are interested in the long-time behavior of trapped quantum solutions. In this case it is convenient to study the wave function of the trapped particle at times that correspond to integer multiples of $\pi$. Given a starting state $|\varphi_0\rangle = |\varphi(\rho,\tau=0)\rangle$, the quantum states of the trapped particle at times $\tau_m = m\pi$ are given by

$$|\varphi(\tau_m)\rangle \equiv |\varphi_m\rangle = \hat{U}^m \, |\varphi_0\rangle, \quad m = 0,1,2,\dots\,, \tag{9}$$

where $\hat{U}$ is the one-cycle propagator, i.e. the time evolution operator of the system from $\tau = 0$ to $\tau = \pi$. Applying (9) to the Floquet solutions (8) for one cycle ($m = 1$) we have immediately

$$\hat{U} \, |\phi_\mu(\rho,\tau=0)\rangle = |\phi_\mu(\rho,\tau=\pi)\rangle = \exp(-i\mu\pi) \, |\phi_\mu(\rho,\tau=0)\rangle. \tag{10}$$

This equation shows that in addition to their mathematical significance, the Floquet solutions (8) at $\tau = \tau_m$ have a very direct physical meaning: they are the eigenfunctions of the one-cycle propagator $\hat{U}$. Floquet solutions of time-periodic quantum systems were studied in detail in the quantum literature. Among the first were Shirley [8] and Zeldovich [9]. The Floquet exponent $\mu$ is also known as the quasienergy, and the wave functions $\Phi_\mu(\rho,\tau)$ are known as the quasienergy wave functions [9].

Since according to (10) the Floquet states reproduce themselves up to a phase factor after propagation with $\hat{U}$, the Floquet states are non-spreading wave packets. Therefore, for periodically driven systems such as the dynamic Kingdon trap, the Floquet states are the closest analogues of the stationary states of a time-independent quantum system.

Computing the Floquet states of the dynamic Kingdon trap shows that they come basically in two varieties: (i) narrow states located inside of the trapping island, and (ii) broad states characterized by a large overlap with the chaotic sea. For sufficiently small $\alpha$ the states localized within the trapping island resemble harmonic oscillator states forming a systematic sequence of states characterized by an increasing number of zeros within the trapping island. This sequence is finite since at some point these Floquet states become so wide that they "spill out" of the trapping island and leak out into the chaotic sea. Even the "broad states" of the chaotic sea may harbor some surprises. It is possible that these states are not entirely delocalized over the chaotic sea, but show Anderson localization phenomena [10] akin to similar effects observed in the kicked rotor [11] and the hydrogen atom in a strong microwave field [12]. Computing the Wigner transforms

$$f_\mu(\rho,p) = \frac{1}{2\pi\alpha} \int \phi_\mu(\rho + s/2)\, \phi_\mu^*(\rho - s/2)\, e^{-ips/\alpha}\, ds \tag{11}$$

of the Floquet states localized inside of the trapping island shows that these states are not only localized inside of the island as far as their space coordinates are concerned, but are entirely localized inside of the island in phase space.

Of all of the quantum dynamical traps the dynamic Kingdon trap is perhaps the most interesting one from a nonlinear dynamics point of view. Loaded with only a single charged particle the trap shows a mixed phase space that possesses all of the classic phase-space morphology, including regular islands and a chaotic sea. Experimentally accessible quantum chaotic systems are rare. Due to its simplicity the dynamic Kingdon trap has a realistic chance of joining the currently small family of quantum chaos experiments. Among these the dynamic Kingdon trap is perhaps closest in spirit to the hydrogen atom in a strong microwave field [12,13]. Both systems are driven by external ac fields, and both systems show a mixed phase space. While it is known that the hydrogen atom in a strong microwave field possesses a true ionization continuum with unnormalizable quantum states, the nature of the quasienergy spectrum of the dynamic Kingdon trap is not currently known. Because of its simple, but nevertheless representative phase-space structure, the dynamic Kingdon trap has much to offer for theoretical and experimental quantum chaos research. Since the dynamic Kingdon trap resembles an electrodynamical version of a Penning-Malmberg trap [2] and since it is possible to store many charged particles simultaneously in a dynamic Kingdon trap [1], the dynamic Kingdon trap is also an excellent device for studying strongly forced nonneutral plasmas.

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