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A Class of Totally Positive Blending B-Bases

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Abstract. Totally positive blending bases present good shape preserving properties when they are used in CAGD. Among these bases there exist special bases, called B-bases, which have optimal shape preserving properties. In particular, the corresponding control polygon is nearest to the curve among all the control polygons; thus many geometrical properties are similar to the ones of the curve. Examples of totally positive blending B-bases are the Bernstein polynomials and the B-spline basis. Our purpose is to construct new classes of such bases starting from compactly supported totally positive scaling functions.

§1. Introduction

One of the main goals in Computer Aided Geometric Design (CAGD) is to predict or control the *shape* of a curve by studying or specifying the *shape* of the control polygonal arc formed by certain points which define the curve, typically the coefficients when the curve is expressed in terms of a particular basis. This is possible when we choose as a basis a system of functions $\boldsymbol{v} = (v_0, \dots, v_n)$ with suitable shape preserving properties. This means that the geometrical properties of the curve in $\mathbb{R}^2$

$$\gamma(x) = \sum_{i=0}^{n} P_i v_i(x), \quad x \in I \subset \mathbb{R}, \tag{1}$$

constructed on the control points $P_i \in \mathbb{R}^2$, $i = 0, \dots, n$, are implied by the geometrical properties of the control polygon $P_0 \dots P_n$. The shape preserving properties of each representation (1) depend on the characteristic of the system $\boldsymbol{v}$.

The bases commonly used in CAGD, such as Bernstein bases, B-splines, β-splines, nonuniform rational splines (NURBS), are blending totally positive systems. This means that the collocation matrix

$$M\begin{pmatrix} v_0, \dots, v_n \\ x_0, \dots, x_s \end{pmatrix} := (v_i(x_j))_{i=0}^{n}\,{}_{j=0}^{s} \tag{2}$$

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for any sequence $x_0 < \cdots < x_s$, $x_i \in I$, $i = 0, \ldots, s$, is totally positive (*i.e.*, all its minors are non-negative), and the basis functions add to one, that is

$$\sum_{i=0}^{n} v_i(x) = 1, \quad x \in I. \tag{3}$$

The importance of blending totally positive systems is due to the fact that they enjoy two properties which are usually demanded for curve control: the convex hull (CH) and the variation diminishing (VD) properties (see, for instance, [5,6]). As a consequence, in many ways the shape of the curve γ mimics the shape of the control polygon $P_0 \ldots P_n$. However, blending totally systems usually do not enjoy a property which is also important: the end-point interpolation (EPI) property.

Bases which simultaneously satisfy the VD, CH and EPI properties can be obtained by considering blending B-bases [5].

Following [5], a totally positive (TP) system $\boldsymbol{u}$ of linearly independent functions is said to be a B-basis if any totally positive basis $\boldsymbol{v}$ of the space U generated by $\boldsymbol{u}$ satisfies the condition

$$\boldsymbol{v} = \boldsymbol{u}A, \quad A \text{ nonsingular totally positive matrix.} \tag{4}$$

In [4] it was proved that if there exists a blending TP basis in U, then there exists a *unique* blending B-basis for that space. B-bases have *optimal properties* in the geometric context [5], that is, in particular, the control polygon with respect to the B-basis is nearest to the curve among all the control polygons with respect to any other TP basis.

Some examples of B-bases are given in [4,5]; in particular, the B-spline basis is the blending B-basis in the space of the polynomial splines of degree m, on a given interval with a prescribed sequence of knots.

At this point, it is worthwhile to remark that in the case of cardinal splines (knots at the integers), this basis is related to the cardinal B-spline N^m, defined by $N^m = N^{m-1} * N^0$, where N^0 is the characteristic function of $[0, 1)$ and $*$ denotes the convolution product (see, for instance [8]).

On the other hand, N^m is a scaling function, that is the solution of the functional equation

$$N^m(x) = \frac{1}{2^m} \sum_{i=0}^{m+1} \binom{m+1}{i} N^m(2x - i), \quad x \in \mathbb{R}. \tag{5}$$

In this paper, we analyse the more general problem of the construction of blending B-bases considering, instead of N^m, a scaling function satisfying a functional equation more general than (5):

$$\varphi(x) = \sum_{i \in \mathbb{Z}} a_i \varphi(2x - i), \quad x \in \mathbb{R}, \tag{6}$$

where the mask $\boldsymbol{a} = \{a_i\}_{i\in\mathbb{Z}}$ satisfies the following conditions:

$$\sum_{i\in\mathbb{Z}} a_{2i+1} = \sum_{i\in\mathbb{Z}} a_{2i} = 1. \tag{7}$$

It is known that a solution φ of (6) exists if the mask $\boldsymbol{a}$ satisfies further conditions, in addition to (7). In particular, if:

i) $\boldsymbol{a}$ is compactly supported on $[0, m+1]$ (with $a_0 a_{m+1} \neq 0$),

ii) the symbol

$$p(z) = \sum_{i=0}^{m+1} a_i z^i \tag{8}$$

has roots with negative real part (Hurwitz polynomial),

then there exists [8] a *unique* scaling function solution of (6), whose support is $[0, m+1]$, such that

$$\sum_{i\in\mathbb{Z}} \varphi(x-i) = 1, \quad x \in \mathbb{R}. \tag{9}$$

Moreover, the functions $\{\varphi(\cdot - i), i \in \mathbb{Z}\}$ are linearly independent and totally positive on $\mathbb{R}$.

The aim of this paper is to construct new classes of blending B-bases, from a given system $\{\varphi(\cdot - i), i \in Z\}$, where Z is a finite subset of $\mathbb{Z}$ and φ is a scaling function. In Section 2 some preliminaries are outlined, whereas in Section 3 this construction is specialized to the new classes of scaling functions introduced in [10]. Finally, Section 4 is devoted to some examples.

§2. Preliminaries

Let $I = [\alpha, \beta]$, with α, β integers, be a finite interval of $\mathbb{R}$ and let φ be a compactly supported scaling function, whose support is $[0, L]$, associated with a mask $\boldsymbol{a}$ enjoying the properties *i)* and *ii)* of the previous section. Then, the system of $n = \beta - \alpha + L - 2$ functions

$$\boldsymbol{\Phi} := \{\varphi(x-i), \alpha - L + 1 \le i \le \beta - 1\}, \quad x \in [\alpha, \beta], \tag{10}$$

constitutes a blending (cf. (9)) TP basis in the space U_Φ generated by itself, and fulfils some interesting shape preserving properties.

Indeed, because of the properties of φ mentioned above, the basis $\boldsymbol{\Phi}$ satisfies the CH and the VD properties. Thus, $\boldsymbol{\Phi}$ preserves *monotonicity* and *convexity*, that is, any straight line cuts the curve γ_Φ no more often than it cuts the control polygon [7]. Further shape preserving properties can be deduced by the generalized VD property for TP bases (see [2]).

It is rather natural to wonder whether $\boldsymbol{\Phi}$ is a B-basis, too. To this end we can use the following proposition from [4].

Proposition A. *A TP basis $B = (\zeta_0, \dots, \zeta_n)$ is a B-basis if and only if the following conditions hold:*

$$\inf \left\{ \frac{\zeta_i(x)}{\zeta_j(x)} \mid x \in I, \zeta_j(x) \neq 0 \right\} = 0,$$

for all $i \neq j$.

Clearly, Proposition A provides a useful test to check if a TP basis is a B-basis. If the check fails, one can construct the unique blending B-basis of the space U_Φ by means of the procedure given in [4, Th 3.6 and Th. 4.2].

§3. Construction of B-bases of Scaling Functions

One of the main advantages of the cardinal B-spline as scaling function is that its mask has an explicit expression (cf. (5)). A wide generalization of the cardinal B-splines was developed in [10], where a new family of scaling functions has been introduced by means of a *new family of masks*, which have an explicit expression. These scaling functions depend on certain free parameters, have prescribed smoothness and, as for the cardinal B-splines, are compactly supported, totally positive and centrally symmetric. They were introduced as follows.

Let $\boldsymbol{H}$ denote the set of all compactly supported and centrally symmetric masks whose symbol is a Hurwitz polynomial. In [10] it was proved that a mask $\boldsymbol{a}$ belongs to $\boldsymbol{H}$ if and only if its coefficients are of the type

$$a_i^{(m,k)} = \sum_{r=0}^{k/2} b_r^{(r)} \binom{m+1-2r}{i-r}, \qquad i = 0, 1, \dots, m+1, \tag{11}$$

where $m = 2, 3, \dots$, k is an even integer such that $1 \leq k \leq m$, and

$$b_i^{(r)} = b_i^{(r-1)} - \binom{k-2r+2}{i-r+1} b_{r-1}^{(r-1)}, \quad r = 0, 1, \dots, K, \quad K := \frac{k}{2} - 1, \quad i = r+1, \dots, K+1, \tag{12}$$

and $b_i^{(0)}$, $i = 0, \dots, k$, are such that

$$\begin{cases} b_{k-r}^{(0)} = b_r^{(0)}, \quad r = 0, 1, \dots, k, \\ b_{\frac{k}{2}}^{(0)} = 2^{k-m} - 2\sum_{i=0}^{K} b_i^{(0)}, \\ \det\,(b_{2i-j}^{(0)}; i, j = 1, \dots, p) > 0, \quad p = 1, \dots, k \end{cases} \tag{13}$$

(assume $\binom{l}{i} = 0$ for $i < 0$ or $i > l$).

Due to the properties of $\boldsymbol{a} \in \boldsymbol{H}$, the scaling function $\varphi_{m,k}$, which is the solution of the scaling equation

$$\varphi_{m,k}(x) = \sum_{i=0}^{m+1} a_i^{(m,k)} \varphi_{m,k}(2x - i), \quad x \in \mathbb{R}, \tag{14}$$

is compactly supported on $[0, m+1]$ and centrally symmetric, and the functions $\{\varphi_{m,k}(\cdot - i), i \in \mathbb{Z}\}$ are linearly independent, normalized and TP. Moreover, recalling that a scaling function belongs to $C^r(\mathbb{R})$ if and only if the symbol can be factored as

$$p(z) = (z+1)^{r+1} q_{m-r}(z), \quad q_{m-r}(1) = 2^{-r}, \tag{15}$$

(see [8]), one can prove that $\varphi_{m,k} \in C^{m-k}(\mathbb{R})$.

Remark. *Choosing suitably the coefficients $b_i^{(0)}$, the $\varphi_{m,k}$ reduces to the cardinal B-spline of degree m, and the $\varphi_{m,k}$ can be viewed as a generalization of the cardinal B-splines. In particular, for $k = 1$, the unique family of scaling functions that we obtain are the cardinal B-splines. Moreover, in the case when $m = 3$, the coefficients of the mask (11) are a subset of those of the filters exploited by Burt and Adelson in vision analysis* [1].

Following the procedure outlined in the previous section, any of the scaling functions $\varphi_{m,k}$ can be used to construct a blending TP basis $\boldsymbol{\Phi}_{m,k}$ defined on a finite interval. Observe that a space is suitable for design purposes if it has a blending TP basis.

By means of Proposition A, it is easy to show that the basis $\boldsymbol{\Phi}_{m,k}$ is not a B-basis. Then to obtain a blending B-basis starting from the functions $\varphi_{m,k}(x - i)$, we have to apply the procedure given in [4]. The corresponding algorithm can be illustrated as follows. Let

$$u_i^0 = \varphi_{m,k}(x - i), \quad i = \alpha - m, \ldots, \beta - 1,$$

where the values of $\varphi_{m,k}$ can be evaluated by means of the cascade algorithm [12]. For $j = 0, \ldots, m-2$, define iteratively

$$u_i^{j+1} := \begin{cases} u_i^j - \inf\left(\dfrac{u_i^j}{u_{i-1}^j}\right) u_{i-1}^j, & i = m, m-1, \ldots, j+1, \\ u_i^j, & i = j, j-1, \ldots, 0. \end{cases}$$

Then, let

$$v_i^0 = u_i^{m-1}, \qquad i = \alpha - m, \ldots, \beta - 1,$$

and for $j = 0, \ldots, m-2$ define iteratively

$$v_i^{j+1} := \begin{cases} v_i^j - \inf\left(\dfrac{v_i^j}{v_{i+1}^j}\right) v_{i+1}^j, & i = 0, 1, \ldots, \beta - 2 - j, \\ v_i^j, & i = \beta - 1 - j, \ldots, \beta - 1. \end{cases}$$

The system $\Phi^B_{m,k} = \{b_i := v_i^{m-1}, i = \alpha - m, \dots, \beta - 1\}$, forms a B-basis. The system $\{d_i b_i, i = \alpha - m, \dots, \beta - 1\}$, where d_i, $i = \alpha - m, \dots, \beta - 1$ are positive constants such that $d_{\alpha-m} b_{\alpha-m} + \dots + d_{\beta-1} b_{\beta-1} = 1$, is the required blending B-basis.

We remark that one of the difficulties in applying this method lies in the evaluation of $\inf(u_i^j / u_{i-1}^j)$ and $\inf(v_i^j / v_{i+1}^j)$. For instance, in the examples of Section 4, the infimums has been evaluated by extrapolating the values that the involved functions u_i and v_i assume in a suitable right neighbourhood of α and in a suitable left neighbourhood of β, respectively.

§4. Examples

For $k = 2$, the mask (11) depends on a free parameter $b_0^{(0)}$, which for computational convenience we chose as a dyadic fraction: $b_0^{(0)} = 2^{-h}$. Thus, the explicit expression of the mask coefficients becomes

$$a_{j,m}^{(h)} = 2^{-h} \left[\binom{m+1}{j} + 4(2^{h-m} - 1) \binom{m-1}{j-1} \right], \tag{16}$$

($j = 0, 1, \dots, m+1$, $m \geq 2$, $h > m - 1$), which corresponds to the symbol

$$p_{m,h}(z) = 2^{-h}(1+z)^{m-1}(z^2 + (2^{h-m+2} - 2)z + 1). \tag{17}$$

Observe that the second term in the mask (16) can be seen as a perturbation of the mask of the cardinal B-spline to which (16) reduces when $h = m$.

Given the interval $I = [\alpha, \beta]$, we can construct the family of blending TP bases

$$\mathbf{\Phi}_{m,h} = \{\varphi_{m,h}(x - i), \alpha - m \leq i \leq \beta - 1\}, \tag{18}$$

where $m \geq 2$ and $h > m - 1$. In Fig. 1 the basis $\mathbf{\Phi}_{3,4}$ defined on the interval $[0, 4]$ is displayed (dashed line) together with the corresponding blending B-basis (solid line) obtained by means of the procedure outlined in the previous section.

For $k = 4$, the symbol $p(z)$ depends on two free parameters, that is $b_0^{(0)}$ and $b_1^{(0)}$, which again, for computational convenience, we choose as dyadic fractions: $b_0^{(0)} = 2^{-h}$, $b_1^{(0)} = 2^{l-h}$; $h, l \in \mathbb{R}$ are arbitrary numbers such that $h > m - 2 + \log_2(1 + 2^{l-1})$, in order to fulfil the third of (13). Thus, the symbol has the form

$$p_{m,h,l}(z) = 2^{-h}(1+z)^{m-3} \left(z^4 + 2^l z^3 + (2^{-m+4+h} - 2 - 2^{l+1}) z^2 + 2^l z + 1 \right), \tag{19}$$

where $m > 3$, and the coefficients $a_{i,m}^{(h,l)}$, $0 \leq i \leq m+1$, of the corresponding mask are

$$a_{i,m}^{(h,l)} = \frac{1}{2^h} \left[\binom{m+1}{i} + (2^l - 4) \binom{m-1}{i-1} + (2^{-m+4+h} - 2^{l+2}) \binom{m-3}{i-2} \right]. \tag{20}$$

Also in this case, the mask of the cardinal B-spline N^m can be obtained setting suitably the parameters h and l, that is, $h = m$ and $l = 2$. In Fig. 2 the blending TP basis $\mathbf{\Phi}_{5,6,2}$ defined on the interval $[0, 6]$ is displayed (dashed line) together with the corresponding blending B-basis (solid line) obtained.

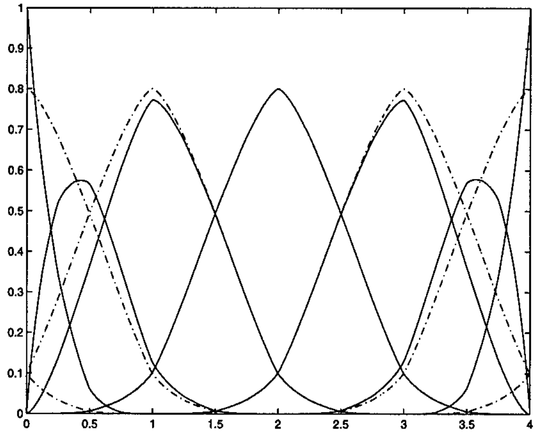

Fig. 1. The blending B-basis $\Phi^B_{3,4}$ (solid line) and the blending TP basis $\Phi_{3,4}$ (dashed line) in the interval $[0,4]$.

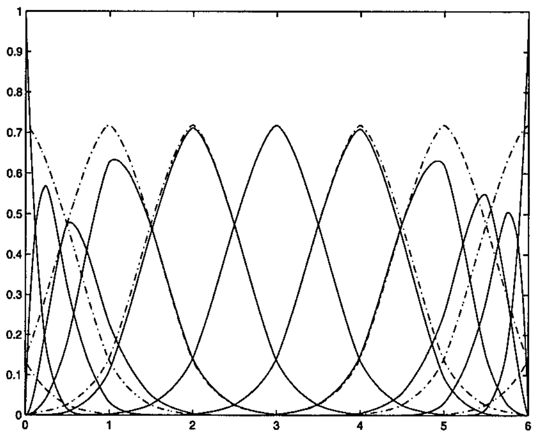

Fig. 2. The blending B-basis $\Phi^B_{5,6,2}$ (solid line) and blending TP basis $\Phi_{5,6,2}$ (dashed line) in the interval $[0,6]$.

Remark. *When the scaling function is just N^m, the procedure outlined here gives the basis of the cardinal B-splines as defined in* [11].

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