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TITLE: International Conference on Curves and Surfaces [4th], Saint-Malo, France, 1-7 July 1999. Proceedings, Volume 2. Curve and Surface Fitting

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Interpolating Functions on Lines in 3-Space

Martin Peternell and Helmut Pottmann

Abstract. Given straight lines L_i, $i = 1, \dots, N$, in Euclidean 3–space with associated function values f_i, we study the interpolation problem of constructing a smooth real valued function F which interpolates values f_i at given data lines L_i. The function F shall be defined on the entire set of lines or at least on lines contained in a domain of interest in 3–space.

§1. Introduction

The problem of constructing an interpolating function F for data lines L_i and corresponding function values f_i is a scattered data interpolation problem in the set of lines $\mathcal{L}$ in Euclidean 3-space E^3.

A variety of solutions of scattered data interpolation problems for data points $X_i \in U$ with $U = \mathbb{R}^n$ or $U \subset \mathbb{R}^n$ are known, see [3]. Extensions to spheres and other surfaces in $\mathbb{R}^3$ are described in [2] and references therein.

Scattered data interpolation on lines is quite different, since the set of lines $\mathcal{L}$ is not a Euclidean space. It is a result of classical geometry that the set of lines $\bar{\mathcal{L}}$ of projective extension P^3 of Euclidean 3-space E^3 is a 4–dimensional quadratic variety M_2^4 in projective P^5. Thus, the general formulation of the problem is as follows: Construct a function $F : M_2^4 \to \mathbb{R}$ interpolating values f_i to corresponding data lines L_i. For practical purposes it is sufficient to construct (or represent) functions on subsets of M_2^4 which correspond to domains of interest in E^3, containing all data lines.

The solution presented here will be the following. We restrict to specific four-dimensional subsets $\mathcal{L}_0$ of M_2^4. These subsets possess parametrizations $\mathbb{R}^4 \to \mathcal{L}_0$ with the property that distances between lines in $\mathcal{L}_0$ are induced by special positive quadratic forms in $\mathbb{R}^4$. This fact allows us to apply well–known methods in $\mathbb{R}^4$ to solve interpolation (or also approximation) problems.

Applications include light field rendering in computer graphics [4]. Considering motion planning in robotics, the method applies to represent a distance function of robot arms (lines) to obstacles. The first motivation for studying functions on lines came from five axis milling. There, the question occurs of how to represent axis positions (lines) of the cutting tool.

Curve and Surface Fitting: Saint-Malo 1999
Albert Cohen, Christophe Rabut, and Larry L. Schumaker (eds.), pp. 351–358.

ISBN 0-8265-1357-3.

§2. Lines in Space

An oriented line L in Euclidean 3-space E^3 is determined by a point $\mathbf{p}$ and a unit direction vector $\mathbf{l}$ ($\|\mathbf{l}\| = 1$). Together with the moment vector

$$\bar{\mathbf{l}} = \mathbf{p} \times \mathbf{l}, \tag{1}$$

we obtain a representation of L by a sixtuple

$$\mathbf{L} = (\mathbf{l}; \bar{\mathbf{l}}) = (l_1, l_2, l_3; l_4, l_5, l_6). \tag{2}$$

These l_i's are called normalized Plücker coordinates of L. By (1), these coordinates are not independent, but satisfy the Plücker relation

$$\mathbf{l} \cdot \bar{\mathbf{l}} = l_1 l_4 + l_2 l_5 + l_3 l_6 = 0. \tag{3}$$

Substituting $\mathbf{l}$ by $-\mathbf{l}$ leads to coordinate vector $-\mathbf{L}$ which defines the same line but with opposite orientation. To get more information about the structure of lines in space, it is necessary to study the set of lines $\bar{\mathcal{L}}$ in the projective extension P^3 of E^3.

E^3 is extended to P^3 by adding points and lines at infinity. Using the analytical model $\mathbb{R}^4$, points in P^3 are one dimensional subspaces of $\mathbb{R}^4$. Thus, we will use the following notation for points in P^3,

$$(x_0, x_1, x_2, x_3)\mathbb{R} := (\lambda x_0, \ldots, \lambda x_3), \lambda \in \mathbb{R}.$$

Let $\omega : x_0 = 0$ be the plane at infinity. We write briefly $(x_0, \mathbf{x})\mathbb{R}$, with $\mathbf{x} \in \mathbb{R}^3$ for points in P^3. The transition from homogeneous to Cartesian coordinates is given by

$$(x_0, x_1, x_2, x_3) \mapsto (\frac{x_1}{x_0}, \frac{x_2}{x_0}, \frac{x_3}{x_0}),$$

which is obviously only possible for points not at infinity.

A line L in P^3 usually is spanned by two points $(p_0, \mathbf{p})\mathbb{R}$ and $(q_0, \mathbf{q})\mathbb{R}$. Homogeneous Plücker coordinates are obtained by

$$\mathbf{L} = (l_1, \ldots, l_6) = (p_0 \mathbf{q} - q_0 \mathbf{p}, \mathbf{p} \times \mathbf{q}). \tag{4}$$

If we substitute $(p_0, \mathbf{p})$ by $\lambda(p_0, \mathbf{p})$, we get $\lambda \mathbf{L}$ such that the l_i's are only determined up to a scalar multiple. This proves homogeneity of $\mathbf{L}$.

If L is not in ω, the relation to definition (2) is obtained as follows. Let $(p_0, \mathbf{p})\mathbb{R}$ be a proper point on L such that we can switch to Cartesian coordinates $\mathbf{p}$ by letting $p_0 = 1$. Further, let $(q_0, \mathbf{q})\mathbb{R}$ be the intersection point $\omega \cap L$ which implies $q_0 = 0$. Inserting this into (4) gives (2) up to a normalization of the direction vector $\mathbf{q} = \mathbf{l}$ of the line L.

If L is in ω, its Plücker coordinates are $(\mathbf{o}, \mathbf{a})\mathbb{R}$ with $\mathbf{o} = (0, 0, 0)$ and some not vanishing vector $\mathbf{a}$. We can interprete L as the line at infinity of a pencil of parallel planes $\mathbf{a} \cdot \mathbf{x} = c$, with $c \in \mathbb{R}$. All these planes possess $\mathbf{a}$ as normal vector.

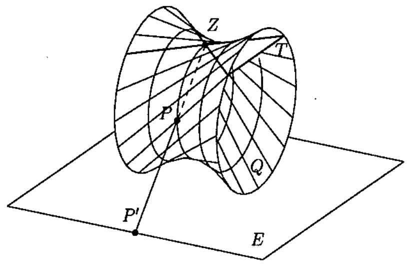

Fig. 1. Stereographic projection of a hyperboloid Q.

Since $\mathbf{L}$ and $\lambda\mathbf{L}$ define the same line in P^3, homogeneous Plücker coordinates (4) define points $\mathbf{L}\mathbb{R}$ in P^5. But only those 6-tuples $(x_1, \ldots, x_6)\mathbb{R}$ are Plücker coordinates of a line X in P^3, which satisfy

$$x_1x_4 + x_2x_5 + x_3x_6 = 0.$$

This quadratic variety is called the Klein quadric M_2^4, where upper and lower indices denote dimension and degree of this variety. The maximal dimension of its subspaces is 2. It is a point model of the set of lines $\bar{\mathcal{L}}$ of P^3. The bijection

$$\gamma : \bar{\mathcal{L}} \to M_2^4$$

from lines $L \subset P^3$ to points $\mathbf{L}\mathbb{R}$ of M_2^4 is called Klein mapping.

The image points $(\mathbf{o}, \mathbf{a})\mathbb{R}$ of lines at infinity lie in the plane $E_\omega : x_1 = x_2 = x_3 = 0$ which is entirely contained in M_2^4. All lines passing through the origin $O = (1, 0, 0, 0)\mathbb{R}$ have Plücker coordinates $\mathbf{L} = (\mathbf{1}, \mathbf{o})$. This can be checked by letting $\mathbf{p} = (0, 0, 0)$ in formula (1). The corresponding image points in P^5 lie in the plane $E_o : x_4 = x_5 = x_6 = 0$. In general, all lines through an arbitrary point in P^3 possess γ-images which lie in a 2-dimensional subspace of M_2^4. The same holds for lines contained in an arbitrary plane in P^3. Thus, M_2^4 contains two 3-parametric families of 2-dimensional subspaces.

We emphasize that $\mathcal{L}$ and $\bar{\mathcal{L}}$ are *not* Euclidean, affine or projective spaces.

Local coordinates of lines

We have seen that $\mathcal{L}$ is isomorphic to $M_2^4 - E_\omega$, where E_ω consists of image points of all lines at infinity. Let T be the tangent hyperplane of M_2^4 at a point Z and let $\tau = M_2^4 \cap T$. It is known that τ is the γ-image of all lines intersecting the line $L = Z\gamma^{-1}$.

Lemma 1. *$M_2^4 - \tau = A^4$ is an affine space.*

Proof: This lemma is a result of classical geometry, and is proved by stereographic projection. Let Q be a regular quadric in P^n. Let Z be a point in Q and T its tangent hyperplane, see Figure 1. Further, consider E to be a hyperplane in P^n, not incident with Z. The intersection $\tau = Q \cap T$ is a quadratic cone with vertex Z. The intersection $e = E \cap T$ is a hyperplane

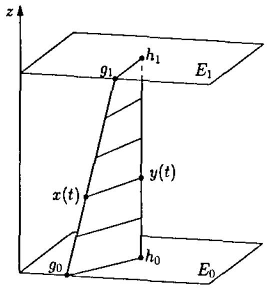

Fig. 2. Local coordinates, distance function.

in E. This says that $E - e$ is an affine space. The stereographic projection $\sigma : Q - \tau \to E - e$ with center Z is bijective and maps points $P \in Q - \tau$ to points P' in affine space $E - e$. $\Box$

Figure 1 shows a low dimensional example. Q is a hyperboloid, and τ is a pair of lines. Planes E and T are parallel such that e is at infinity.

We come back to line geometry and the Klein quadric M_2^4. Let $Z = (0,0,0,0,0,1)\mathbb{R}$ be the center of a stereographic projection. It is the γ-image of the line at infinity which is determined by horizontal planes $z =$const. with normal vector $(0,0,1)$. The tangent hyperplane T at Z with respect to M_2^4 is given by the equation $x_3 = 0$. The exceptional set $\tau = M_2^4 \cap T$ consists of γ-images $(l_1, l_2, 0; \ldots)\mathbb{R}$ of all horizontal lines. Lemma 1 says that all non-horizontal lines form an affine space A^4.

Consider two horizontal planes $E_0 : z = 0$ and $E_1 : z = 1$. The intersection points $\mathbf{g}_0 = (g_1, g_2, 0)$ and $\mathbf{g}_1 = (g_3, g_4, 1)$ of a line G and planes E_i (Figure 2) define a parametrization of all non-horizontal lines by

$$\begin{aligned} \mathbb{R}^4 = \mathbb{R}^2 \times \mathbb{R}^2 &\to \mathcal{L} \\ (g_1, g_2, g_3, g_4) &\mapsto G. \end{aligned} \tag{5}$$

Plücker coodinates of G are $\mathbf{G} = (g_3 - g_1, g_4 - g_2, 1; g_2, -g_1, g_1 g_4 - g_2 g_3)$. The stereographic projection with center Z onto $x_6 = 0$ gives

$$\mathbf{G}' = (g_3 - g_1, g_4 - g_2, 1; g_2, -g_1, 0).$$

This equals (5) up to a linear mapping. Hence, the mapping (5) from non-horizontal lines to points in $\mathbb{R}^4$ is geometrically equivalent to a stereographic projection of $M_2^4 - \tau$.

Distance function of lines

For practical purposes, it is sufficient to consider distances of lines within a *domain of interest.* To specify this domain, we will consider only lines which enclose an angle $\leq \phi_0$ with a fixed unit vector $\mathbf{z}$. The unit direction vector $\mathbf{g}$ of such a line G satisfies

$$\mathbf{g} \cdot \mathbf{z} \geq \cos \phi_0.$$

We have chosen a Cartesian coordinate system with $\mathbf{z}$ as third axis. Further, we will consider only segments of lines between two planes E_0, E_1, bounding the domain of interest. This is motivated by the fact that we are interested in particular in distances between points lying between those planes. Let $\mathbf{g}_i, \mathbf{h}_i$ be intersection points of lines G, H with E_i, and consider points $\mathbf{x}, \mathbf{y}$ on G and H, respectively,

$$\begin{aligned}\mathbf{x}(t) &= (1-t)\mathbf{g}_0 + t\mathbf{g}_1, \\ \mathbf{y}(t) &= (1-t)\mathbf{h}_0 + t\mathbf{h}_1.\end{aligned} \tag{6}$$

The square of a useful distance between lines G, H within the above domain of interest is defined by

$$\begin{aligned}d(G,H)^2 &= \int_0^1 \|\mathbf{x}(t) - \mathbf{y}(t)\|^2 dt \\ &= (\mathbf{g}_0 - \mathbf{h}_0)^2 + (\mathbf{g}_1 - \mathbf{h}_1)^2 + (\mathbf{g}_0 - \mathbf{h}_0)\cdot(\mathbf{g}_1 - \mathbf{h}_1).\end{aligned} \tag{7}$$

It measures horizontal distances between corresponding points $\mathbf{x}, \mathbf{y}$ of G, H. We will not distinguish between a line X and its coordinate vector $X = (x_1, x_2, x_3, x_4)$ in $\mathbb{R}^4$ according to parametrization (5). Formula (7) is a positive definite quadratic form in $\mathbb{R}^4$ with the following coordinate representation

$$\langle X, X\rangle = x_1^2 + x_2^2 + x_3^2 + x_4^2 + x_1x_3 + x_2x_4.$$

Remark 2. *These distances differ from orthogonal distances (from a point to a line G) only by a factor $\leq \cos\phi_0$. So, taking ϕ_0 relatively small will control the difference between these distances and the Euclidean distances in E^3.*

Summary 3. *The restriction to specific subsets $\mathcal{L}_0$ of line space allows parametrizations $\mathbb{R}^4 \to \mathcal{L}_0$. A positive definite quadratic form in $\mathbb{R}^4$ serves to define distances between lines in a useful manner.*

Choice of local coordinates

Distance d is not invariant under motions in E^3, but depends on the choice of $\mathbf{z}$ and planes E_0, E_1. Consider oriented lines $L_i, i = 1, \ldots, N$ with unit direction vectors $\mathbf{l}_i$. Assume that $\mathbf{l}_j \cdot \mathbf{l}_k < C$. This expresses that the angle between any two lines is bounded by $\arccos(C)$. A good choice for the vector $\mathbf{z}$ can be computed as solution of a regression problem. Assuming $\|\mathbf{l}_i\| = 1$, we want to maximize

$$\sum_{i=1}^{N} (\mathbf{l}_i \cdot \mathbf{z})^2 \tag{8}$$

over all unit vectors $\mathbf{z}$. Maximizing the quadratic form (8) under the quadratic side condition $\mathbf{z}\cdot\mathbf{z} = 1$ leads to an eigenvalue problem in $\mathbb{R}^3$. Thus, we found a possibility to construct $\mathbf{z}$ with respect to a set of lines L_i. Planes E_0, E_1 perpendicular to $\mathbf{z}$ bounding the domain of interest have to be chosen depending on the problem. In this sense we can say that the coordinate system is connected with the problem in an invariant way.

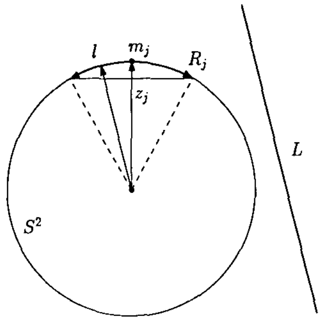

Fig. 3. Definition of domains.

If direction vectors $\mathbf{l}_i$ of lines L_i are distributed over a whole hemisphere or more, we have to split the set of lines into subsets and perform the construction of coordinate systems for the subsets. Remark 2 gives information about the deviation of distances compared to usual distances in E^3.

§3. Representation of Functions on $\mathcal{L}$

Given N lines L_i with corresponding function values f_i, we would like to compute a function $F : \mathcal{L} \to \mathbb{R}$ with $F(L_i) = f_i$. This is a scattered data interpolation problem on $\mathcal{L}$ (or M_2^4). With help of local parametrizations we obtain scattered data interpolation problems on $\mathbb{R}^4$. The given algorithm consists of three steps.

1) Find a covering $\{U_j, j = 1, \ldots, M\}$ of $\mathcal{L}$ with domains U_j which are parametrized over $\mathbb{R}^4$. Decide the membership of lines and domains.
2) Compute partial solutions F_j of the interpolation problem for all domains U_j.
3) Merge all partial solutions F_j in a global solution F with required continuity.

First of all we want to find a covering of lines L_i by domains U_j with $1 < j < M$. We choose M unit vectors $\mathbf{z}_j$ and real numbers R_j which serve as centers and spherical radii of caps of the unit sphere S^2. These caps determine domains U_j in the following way. A line L belongs to U_j if and only if

$$\mathbf{l} \cdot \mathbf{z}_j \geq \cos R_j$$

holds for its direction vector, see Figure 3. Clearly, L can be contained in more than one domain. We determine the membership of all lines L_i for domains U_j.

In a second step we compute partial solutions F_j of the interpolation problem for each domain U_j. This is done by letting

$$F_j(X) = \sum_{k=1}^{N_j} a_{jk} B_k(X),$$

where N_j shall be the number of lines L_i belonging to domain U_j. $X = (x_1, x_2, x_3, x_4) \in \mathbb{R}^4$ is coordinate vector of a line X according to parametrization (5). $B_k(X)$ are (for instance) radial basis functions and depend only on the distance $d(X, L_k)$. The coefficients a_{jk} are solutions of linear systems. The problem of regularity of such systems dependent on the type of basis function is solved in [5]. So we get partial solutions F_j valid in domains U_j.

In the last step we have to merge all partial solutions to a unique one. This can be done by forming a weighted sum

$$F(X) = \sum_{j=1}^{M} w_j(X) F_j(X).$$

The weights can be chosen as

$$w_j(X) = \frac{(1 - \arccos(\mathbf{x} \cdot \mathbf{m}_j)/R_j)_+^r}{\sum_{l=1}^{M}(1 - \arccos(\mathbf{x} \cdot \mathbf{m}_l)/R_l)_+^r},$$

where $\mathbf{m}_j$ and R_j are center and radius of the spherical cap which defines U_j and $\mathbf{x}$ denotes the normalized direction vector of the line X. The notation $(q)_+^r$ expresses that $w_j(X)$ is positive in the interior of U_j and is zero outside. This says that $(q)_+^r = q^r$ for positive q, and $(q)_+^r = 0$ otherwise.

Weights $w_j(X)$ are in the differentiability class C^{r-1}. If partial solutions F_j possess the same smoothness, then also F is in C^{r-1}.

§4. Visualization of Functions on Lines

Since the dimension of $\mathcal{L}$ is four, visualization of function values is an advanced topic. In general, displaying functions on low dimensional subsets seems to be promising. We decided to choose several bundles of lines for evaluation and want to describe two methods of visualization.

We choose an appropriate number of points $\mathbf{v}_i$ within the domain of interest, and evaluate F at sufficiently many lines passing through vertices $\mathbf{v}_i$. Let F_{max} be an (existing!) upper bound of the absolute function values. Consider lines L_{ij} with function values $F(L_{ij}) = f_{ij}$ passing through vertex $\mathbf{v}_i$. Assume that L_{ij} are oriented lines. Displaying the star-shaped surfaces

$$\mathbf{p}_i = \mathbf{v}_i + (1 + \frac{f_{ij}}{F_{max}})\mathbf{l}_{ij}$$

for all chosen vertices $\mathbf{v}_i$ is one possibility to visualize function values. If function values for L and $-L$ are equal, the $\mathbf{p}_i$ will be centrally symmetric surfaces. For functions on nonoriented lines, we will use both direction vectors $\mathbf{l}_{ij}$ and $-\mathbf{l}_{ij}$ for the definition of $\mathbf{p}_i$, and assign the same function value f_{ij} to them. Thus we always get centrally symmetric surfaces. Figure 4 shows an interpolant. The test function is a function of the distances between lines L_i and points (not displayed).

For the second method we use spheres S_i, centered at vertices $\mathbf{v}_i$. All lines L_{ij} of the bundle $\mathbf{v}_i$ with constant function values form a cone C with vertex $\mathbf{v}_i$. Intersecting these cones $C(c_i)$ for several constants c_i gives level curves on spheres S_i (not displayed).

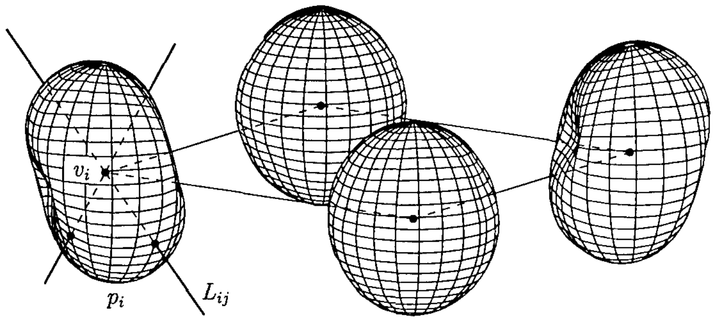

Fig. 4. Visualization of functions on lines.

Acknowledgments. This work has been supported by project P13648-MAT of the Austrian Science Foundation.

References

1. Chen, H-Y. and H. Pottmann, Approximation by ruled surfaces, J. Comput. Appl. Math. **102** (1999), 143–156.
2. Fasshauer, G. E. and L. L. Schumaker, Scattered data fitting on the sphere, *Mathematical Methods for Curves and Surfaces II*, M. Dæhlen, T. Lyche, and L. L. Schumaker (eds.), Vanderbilt University Press, Nashville, 1998, 117–166.
3. Hoschek, J. and D. Lasser, *Fundamentals of Computer Aided Geometric Design*, AK Peters, Wellesley, MA, 1993.
4. Levoy, M. and P. Hanrahan, Light field rendering, *SIGGRAPH 96, Annual Conference Series*, 1996, 31–41.
5. Micchelli, C. A., Interpolation of scattered data: distance matrices and conditionally positive definite functions, Constructive Approximation **2** (1986), 11–22.
6. Powell, M. J. D., The theory of radial basis function approximation in 1990, in *Advances in Numerical Analysis Vol.2*, W. Light (ed), Claredon Press, Oxford, 1992, 105–210.
7. Wendland, H., Konstruktion und Untersuchung radialer Basisfunktionen mit kompaktem Träger, Dissertation, Universität Göttingen, 1996.

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