TITLE: Analytical Solutions for the Unsteady Compressible Flow Equations Serving as Test Cases for the Verification of Numerical Schemes

TITLE: Verification and Validation Data for Computational Unsteady Aerodynamics [Donnees de verification et de valadation pour l'aerodynamique instationnaire numerique]

2. ANALYTICAL SOLUTIONS FOR THE UNSTEADY COMPRESSIBLE FLOW EQUATIONS SERVING AS TEST CASES FOR THE VERIFICATION OF NUMERICAL SCHEMES

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INTRODUCTION

The verification of numerical schemes for solving the equations of inviscid and viscous compressible unsteady flow equations is limited to a small number of analytical solutions of the equations governing the one-dimensional unsteady flow including moving discontinuities. Among them the most important were given first by B. Riemann (1859-1860) and later by W.J.M. Rankine (1870), P.H. Hugoniot (1887), Lord Rayleigh (1910) and G.I. Taylor (1910). The scope of the present chapter is to overview the analytical solutions, serving as test case for the accuracy of the numerical schemes. It is worth noting that the analytical solutions are of importance for Euler and Navier-Stokes equations for laminar flow and does not give any indication for the behaviour of the numerical schemes in the prediction of turbulent flows. For each analytical solution a corresponding FORTRAN program is attached.

LINEAR ADVECTION EQUATION

The linear advection equation is used as a simple model for contact discontinuities in fluid dynamics, and is the simplest model equation for the representation of wave propagation. The linear advection equation is:

$$u_t + c_0 u_x = 0 \tag{1}$$

where c_0 is a positive constant ($c_0 > 0$) called the velocity of the wave. The general solution of equation (1) is

$$u(x,t) = f(x - c_0 t) \tag{2}$$

$f(x)$ is an arbitrary function defined by the initial conditions of the problem

$$u(x,0) = f(x) \tag{3}$$

The solution (3) apparently describes a wave motion to the positive x-axis, since the initial profile $f(x)$ is translated unchanged in shape a distance $c_0 t$ to the right at time t [8].

Oscillatory solution of linear advection equation with a discontinuity in the derivative (case LADV-1)

The problem of oscillatory solution of linear advection equation is approached using the initial conditions [8]:

$$u(x,0) = f(x;\alpha,k) = \begin{cases} -\alpha \sin(k\pi x), x < 0 \\ x \ , \ x \geq 0 \end{cases} \quad \text{with } \alpha = 0.1 \text{ and } k = 6 \tag{4}$$

At x=-1 the boundary condition

$$u(-1,t) = 0 \tag{4a}$$

is imposed. The solution in accordance with the above consideration, is presented in Figure 1 at time instances t=0 and t=0.5

This test case gives information about the capability of a computational method to capture an oscillating solution with a discontinuity in the derivative.

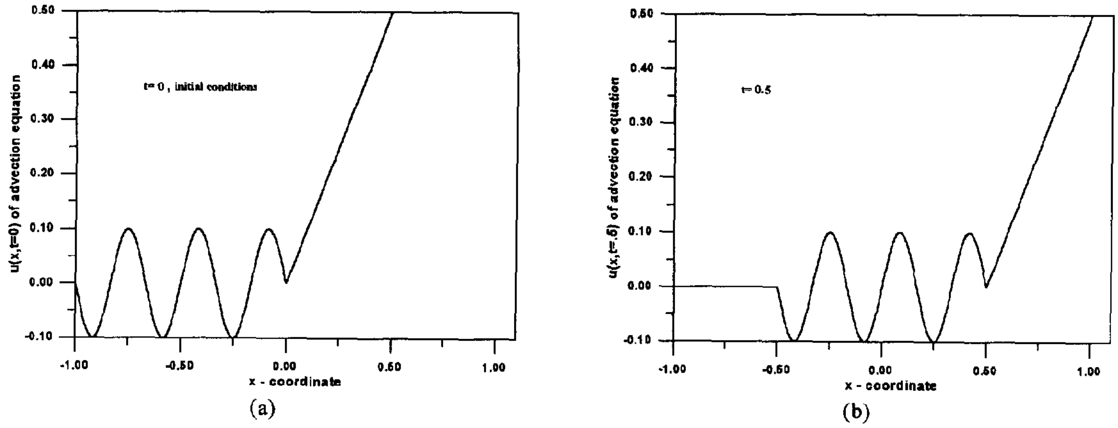


Figure 1: Oscillatory solution of linear advection equation with a discontinuity in the derivative

Simulation of discontinuities (case LADV-2)

To simulate discontinuities we again solve the linear advection equation, this time using piecewice continuous initial data:

$$u(x,0) = f(x;u_L,u_R) = \begin{cases} u_L, x < 0 \\ u_R, x \geq 0 \end{cases} \text{ with } u_L = 1 \text{ and } u_R = 0 \tag{5}$$

At x=-1 the boundary condition

$$u(-1,t) = 0 \tag{5a}$$

is imposed. The resultant solution is a square wave travelling with speed 1 to the right, as it is depicted in Figure 2 .

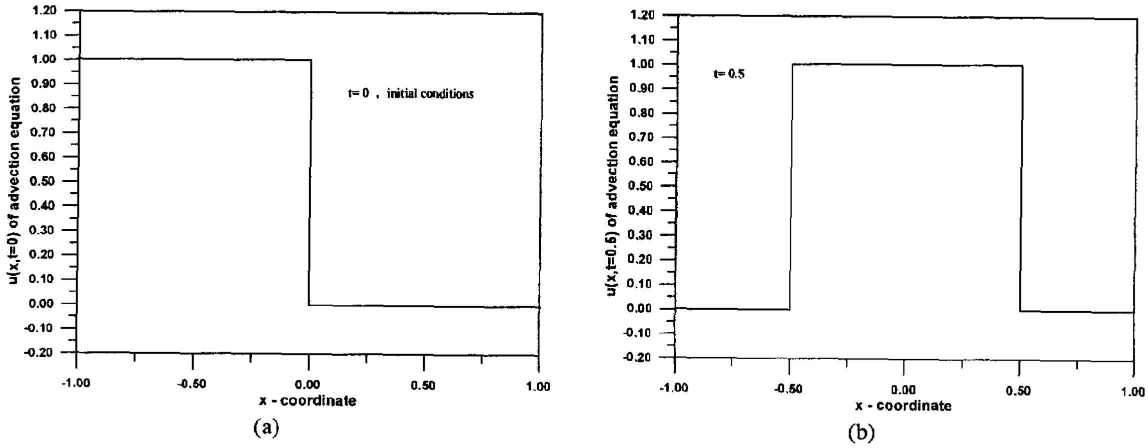


Figure 2: Contact discontinuity, advection equation.

BURGERS' EQUATION

Inviscid Burgers' equation.

The non linear first order equation:

$$u_t + uu_x = 0 \quad \text{or} \quad u_t + \left(\frac{u^2}{2}\right)_x = 0 \tag{6}$$

with Initial Conditions

$$u(x,0) = f(x) \quad \text{for} \quad -\infty < x < \infty \tag{6a}$$

is the x-momentum equation without pressure gradient or other external forces and it is so called inviscid Burgers' equation. The general solution of the above equation is given by:

$$du = 0 \text{ along the characteristic } \frac{dx}{dt} = u \tag{7}$$

expressing that u remains constant on the characteristic.

For the above initial distribution of u(x,0)=f(x) the general solution is concluded in an implicit form, as follows [4]:

$$u(x,t) = f(x - ut) \tag{8}$$

The characteristics have a slope proportional to $1/f(x_0)$ in the (x,t) plane, where x_0 is a position at initial state, and if $f'(x_0)$ is positive , which is typical for an expansion profile, they will never intersect. On the other hand for a decreasing initial distribution of u, that means $f'(x_0) < 0$, the characteristics will intersect as for a typical compression profile. An initial profile with decreasing intensities will lead to a breakdown of a continuous solution and to the appearance of a shock discontinuity. The shock will appear at the time instance t_s, when the tangent to u(x) profiles becomes vertical:

$$t_s = \frac{-1}{\max|f'(x_0)|} \tag{9}$$

The shock wave velocity , u_s, satisfies the Rankine-Hugoniot relations and is equal to:

$$u_s = \frac{1}{2}(u_2 + u_1) \tag{10}$$

where u_1, u_2 are the values upstream and downstream of the shock.

At this point the following three types of initial conditions are proposed for the analyses of non oscillatory shock capturing methods.

Initial shock discontinuity (case IB-1)

For this case, the Riemann problem for Burgers' equation is solved. The test case is provided by an initial discontinuous distribution :

$$u(x,0) = f(x; u_L, u_R) = \begin{cases} u_L, x < 0 \\ u_R, x \geq 0 \end{cases} \quad \text{with } u_L > u_R \tag{11}$$

At the left boundary the condition

$$u(x_L, t) = u_L \tag{11a}$$

is imposed. The solution of Burgers' equation gives a shock propagating at speed $(u_L + u_R)/2$ with unmodified intensity $[u] = u_L - u_R$, as shown in Figure 3. If $u_R = -u_L$, the shock is stationary and it is used as a non-linear test case for steady-state methods.

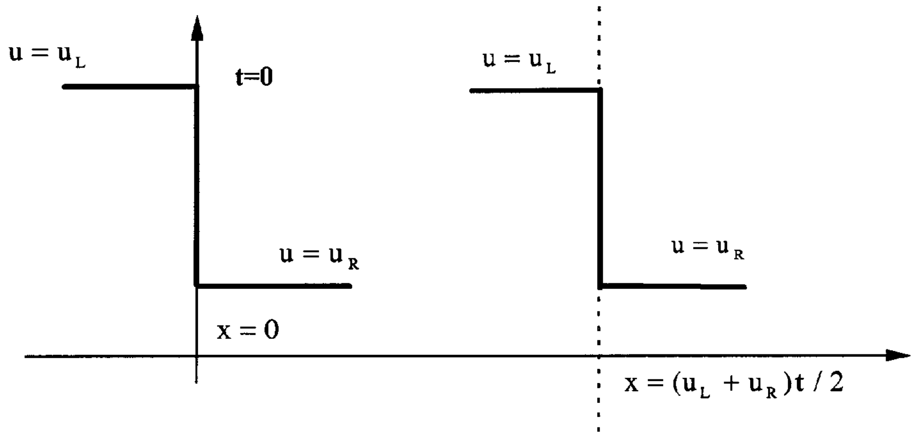


Figure 3 : Burgers' solution for a propagating discontinuity.

Initial linear discontinuity (case IB-2)

A different initial distribution with $f'(x_0) < 0$ leads to the same shock structure. The initial linear distribution is:

$$u(x,0) = f(x;u_L,u_R) = \begin{cases} u_L, & x < 0 \\ u_L\left(1 - \frac{x}{L}\right) + u_R \frac{x}{L}, & 0 \le x \le L \\ u_R, & x > L \end{cases} \quad \text{with } u_L > u_R \tag{12}$$

and at the left boundary the condition

$$u(x_L,t) = u_L \tag{12a}$$

is imposed. A shock is formed at time instance

$$t_s = \frac{L}{u_L - u_R} \tag{13}$$

and at position $x_s = t_s u_L = L + t_s u_R$. The solution of Figure 4 for $t > t_s$ is :

$$u(x,t) = \begin{cases} u_L \text{ for } x < \frac{u_L + u_R}{2} t \\ u_R \text{ for } x > \frac{u_L + u_R}{2} t \end{cases} \tag{14}$$

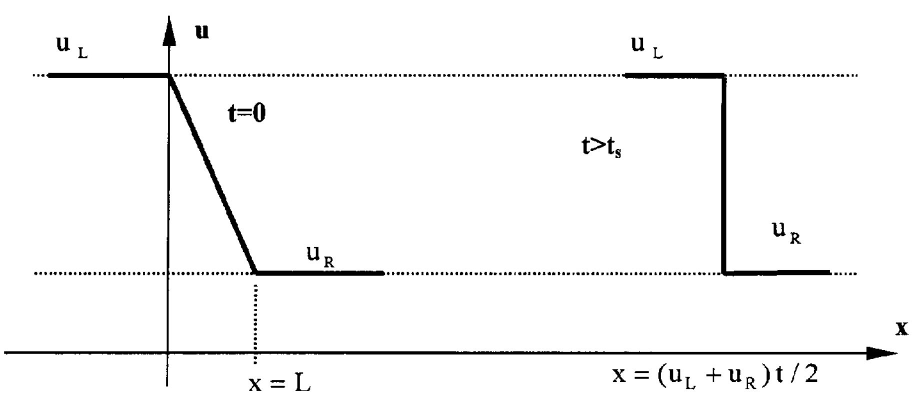


Figure 4 : Shock formation for an initial linear distribution

Burgers' equation for a rarefaction wave (case IB-3)

Burgers' equation with the following initial conditions gives a propagating rarefaction wave.

$$u(x,t=0)=f(x;u_L,u_R)=\begin{cases}u_L & x<0\\ u_R & x>0\end{cases} \quad \text{with } u_L<u_R \tag{15}$$

Between points $u_L t < x < u_R t$ the solution is not determined by the intersection of characteristics. So, a continuous solution is possible in the following form (Figure 5):

$$u(x,t)=\begin{cases}u_L & x/t<u_L\\ x/t & u_L<x/t<u_R\\ u_R & x/t>u_R\end{cases} \tag{16}$$

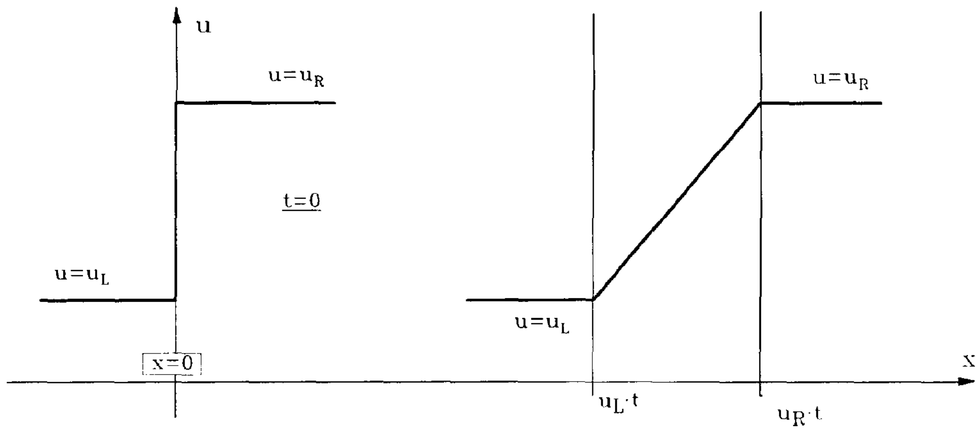


Figure 5 : Initial state and time evolution of a propagating rarefaction wave as a solution of Burgers' equation.

Viscous Burgers' equation (case VB-1)

The complete nonlinear Burgers' equation adding a viscous term is :

$$u_t + uu_x = \nu u_{xx} \quad \text{or} \quad u_t + \left(\frac{u^2}{2}\right)_x = \nu u_{xx} \tag{17}$$

with Initial Conditions

$$u(x,0)=f(x) \text{ for } -\infty<x<\infty \tag{17a}$$

The "Viscous" Burgers' equation serves as a model equation for the boundary-layer equation, the "parabolized" Navier-Stokes equations and the complete Navier-Stokes equations.
The problem with the following initial values:

$$u(x,0)=f(x;u_L,u_R)=\begin{cases}u_L, x<0\\ u_R, x\geq 0\end{cases} \quad \text{with } u_L>u_R \tag{18}$$

and boundary conditions

$$u(x=-\infty,t)=u_L,\ u(x=\infty,t)=u_R \tag{18a}$$

has a solution of the form:

$$u = u_R + \frac{u_L - u_R}{1 + h\exp\frac{u_L - u_R}{2\nu}(x - Ut)}\ , \quad U = \frac{u_L + u_R}{2}$$

where:

$$h = \frac{\int_{-(x-u_R t)/\sqrt{4\nu t}}^{\infty} e^{-\zeta^2} d\zeta}{\int_{(x-u_L t)/\sqrt{4\nu t}}^{\infty} e^{-\zeta^2} d\zeta} \tag{19}$$

The diffusing shock still propagates with the "inviscid" velocity equal to U. Due to viscosity effects the inviscid discontinuities are transformed into continuous shaped "steps", as it is shown for the test case of Figure 6.

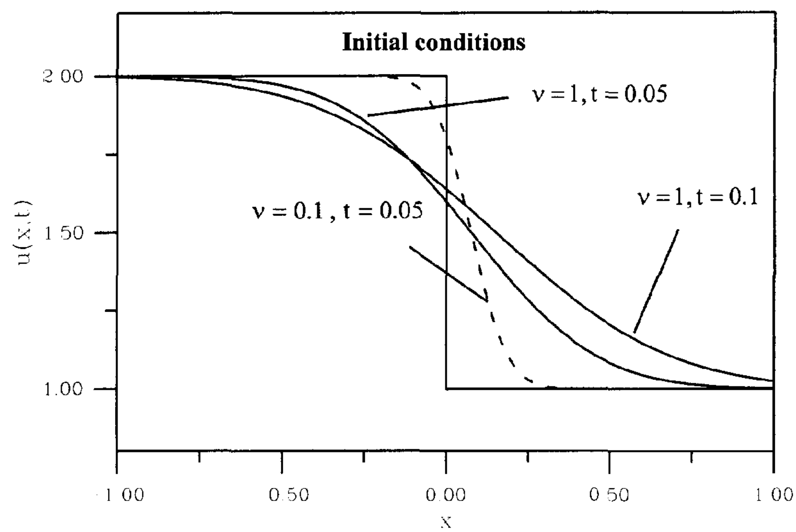


Figure 6 : Shock wave solution of viscous unsteady Burgers' equation ($u_L = 2$, $u_R = 1$).

UNSTEADY EULER EQUATIONS

Reflection of a moving shock on a closed boundary (case RMS-1)

The general discontinuity equations for the moving shock are shown in the literature reference [1] to be:

$$\frac{\hat{\rho}}{\rho} = \frac{u}{\hat{u}} \tag{20}$$

$$\frac{\hat{u}}{u} = 1 - \frac{2}{\gamma+1}\left(1 - \frac{c^2}{u^2}\right) \tag{21}$$

$$\frac{\hat{p}}{p} = 1 + \frac{2\gamma}{\gamma+1}\left(\frac{u^2}{c^2} - 1\right) \tag{22}$$

where u, $\hat{u}$ denotes the relative velocities in front and behind the moving shock

$$u = \upsilon - w, \qquad \hat{u} = \hat{\upsilon} - w \tag{23}$$

while $\upsilon, \hat{\upsilon}$ are the absolute velocities and w the velocity of the shock front. The velocity of sound c in front of the moving shock wave is given by:

$$c = \sqrt{\gamma \frac{p}{\rho}} \tag{24}$$

The test problem proposed here is the reflection of a shock wave moving with constant velocity towards the closed boundary of a tube (figure 7). The fluid behind the shock wave moving to the left, has a velocity with the absolute value υ_0 ($\upsilon = -\upsilon_0$, $\upsilon_0 > 0$), pressure and density p_0, ρ_0 ($p = p_0, \rho = \rho_0$) respectively, so that the velocity of sound be $c_0 = \sqrt{\gamma \frac{p_0}{\rho_0}}$. The given data are p_0, ρ_0, υ_0, which are the initial conditions of the problem at starting point. The reflection condition, which has to be satisfied is $\hat{\upsilon} = 0$.

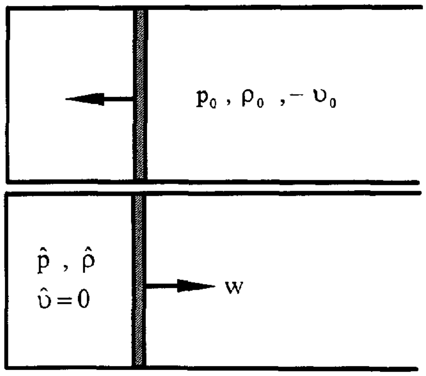


Figure 7 : Reflection of a moving on the left with constant velocity shock wave on the left closed boundary of a tube.

The velocity of the shock front after the reflection, the pressure and the density behind the reflected shock wave, are given by the following relationships [2]:

$$w = \frac{\gamma - 3}{4}\upsilon_0 + \sqrt{\left(\frac{\gamma + 1}{4}\upsilon_0\right)^2 + \gamma \frac{p_0}{\rho_0}} \tag{25}$$

$$\hat{p} = p_0\left[1 + \frac{2\gamma}{\gamma + 1}\left(\frac{(\upsilon_0 + w)^2}{\gamma p_0}\rho_0 - 1\right)\right] \tag{26}$$

$$\hat{\rho} = \frac{\upsilon_0 + w}{w}\rho_0 \tag{27}$$

Analytical solutions for the unsteady inviscid, non-conducting fluid conservation equations

By neglecting viscous, heat-conduction effects and field forces, the unsteady compressible conservation equations of mass, momentum and energy, in one dimensional conservative form, which will be referred to as Euler equations, have the following form:

Continuity equation:

$$\frac{\partial \rho}{\partial t} + \frac{\partial(\rho u)}{\partial x} = 0 \tag{28}$$

Momentum equation:

$$\frac{\partial(\rho u)}{\partial t} + \frac{\partial(\rho u^2 + p)}{\partial x} = 0 \tag{29}$$

Energy equation:

$$\frac{\partial\left[\rho\left(e+\frac{1}{2}u^2\right)\right]}{\partial t}+\frac{\partial\left[\rho u\left(e+\frac{1}{2}u^2+\frac{p}{\rho}\right)\right]}{\partial x}=0 \tag{30}$$

The above system is closed by the constitutive equation, in the form of ***Gibbs relation***, namely:

$$Tds = de + p d\left(\frac{1}{\rho}\right) \tag{31}$$

Which leads to the entropy relation:

$$\frac{\partial(\rho s)}{\partial t}+\frac{\partial(\rho u s)}{\partial x}=0 \tag{32}$$

In the space - time plane the transformation of independent (x,t) to the new variables (ξ,η) (Figure 8) is introduced:

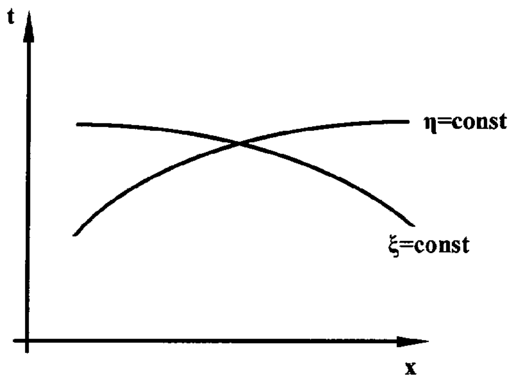


Figure 8: Sketch of the two families of characteristics in the (x,t) plane.

$$\xi=\xi(x,t),\ \eta=\eta(x,t) \tag{33}$$

so that:

$$\xi=\text{const}: \frac{dx}{dt}=u-c,\quad \eta=\text{const}: \frac{dx}{dt}=u+c \tag{34}$$

where c is the isentropic velocity of sound:

$$c^2=\left(\frac{\partial p}{\partial \rho}\right)_s \tag{35}$$

Then it can be shown that the system of governing equations in terms of (ξ,η) takes the form:

$$\begin{aligned}\eta=\text{const}&: \frac{dx}{dt}=u+c,\ u+\omega=\text{const}\\ \xi=\text{const}&: \frac{dx}{dt}=u-c,\ u-\omega=\text{const}\end{aligned} \tag{36}$$

where:

$$\omega=\int\frac{c(\rho)}{\rho}d\rho \tag{37}$$

The $\xi = \text{const.}, \eta = \text{const.}$ are the two families of characteristics which are wave fronts of the kinematic discontinuities. The kinematic discontinuities in the considered one-dimensional case correspond to lines across which the first derivative of the flow quantities are discontinuous , while the flow quantities are continuous.

The last equation for ideal gas of constant coefficients of specific heat of ratio γ reduces to the form:

$$\omega = \frac{2}{\gamma-1}c, \qquad c^2 = \gamma\frac{p}{\rho} \tag{38}$$

The reduced system of equations can be solved analytically in certain problems such as **moving piston** case (two sub-cases: expansion and compression) and the so called **Riemann or shock tube problem** which includes a shock wave, a contact discontinuity and an expansion wave [1], [3].

(a) Expansion flow behind a moving piston (case MP-1)

A piston is considered to move towards the negative x-direction, figure 9. This results to an expansion of the gas behind the piston.

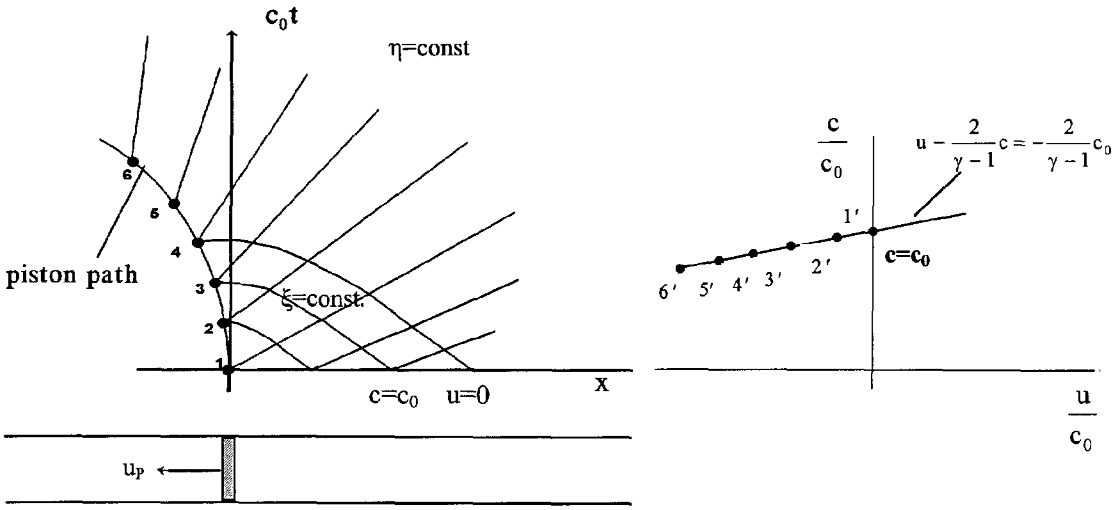


Figure 9: Expansion flow on the right of a left moving piston

The flow is studied in the plane x-t where the path of the piston $u=-u_0$ is shown. As an initial value of the problem a gas with zero velocity and a constant temperature (constant velocity of sound) will be considered.

$$u(x,0) = 0,\ c(x,0) = c_0 \quad x \ge 0 \tag{39}$$

As a boundary condition the equality of the gas velocity to the velocity of the piston is taken into account:

$$u(x_p,t) = u_p(t) = \dot{x}_p \tag{40}$$

Starting from the initial values $c = c_0, u = 0$ (for $t = 0$) we could easily observe that in the open region between the positive x-axis and the positively inclined characteristic ($\eta = \text{const.}$) originating from point 1 both families of characteristics are straight lines since they originate from the positive part of the x-axis where the constant initial conditions are valid.
The movement of the piston affects the flow field left of the characteristic originating from point 1. Since all the $\xi = \text{const}$ characteristics originate from the positive x-axis where the initial conditions are valid, across all the negatively inclined characteristics the following relation is valid:

$$\xi = \text{const},\ u - \frac{2}{\gamma-1}c = -\frac{2}{\gamma-1}c_0 \tag{41}$$

All the $\xi = \text{const}$ characteristics end on the piston path, so that

$$\xi = \text{const} \quad u_p - \frac{2}{\gamma - 1}c_p = -\frac{2}{\gamma - 1}c_0 \tag{42}$$

while for the $\eta = \text{const}$ characteristics the following relation is valid:

$$\eta = \text{const} \quad u + \frac{2}{\gamma - 1}c = u_p + \frac{2}{\gamma - 1}c_p \tag{43}$$

By substracting the previous two equations for the region between the piston path and the positively inclined characteristic originating from point 1, the following relation along the positively inclined characteristics is valid :

$$\eta = \text{const} \quad u + \frac{2}{\gamma - 1}c = 2u_p + \frac{2}{\gamma - 1}c_0 \tag{44}$$

From the relations (42) and (43) we conclude that:

$$\eta = \text{const}, \quad u = u_p, c = c_p \tag{45}$$

Thus the values of u and c on each $\eta = \text{const}$ keep constant and this family of characteristics are straight lines with the following inclination :

$$\eta = \text{const}, \quad \frac{dx}{dt} = u_p + c_p = c_0 + \frac{\gamma + 1}{2}u_p \tag{46}$$

Note: From equation results a limit maximum piston velocity.

$$(u_p)_{max} = -\frac{2}{\gamma - 1}c_0 \quad c_p = 0 \tag{47}$$

With a piston velocity increasing further, a cavitation zone is developed behind the moving piston (where pressure vanishes). The cavitation area is located in the region between the piston path and the positively inclined characteristic with inclination w :

$$w = \frac{dx}{dt} = c_0 + \frac{\gamma + 1}{2}\left(u_p\right)_{max} = -\frac{2}{\gamma - 1}c_0 \tag{48}$$

This limit velocity $w \approx -5c_0$ is quite higher than the maximum isentropic steady flow velocity in vacuum ($u_{max} = \sqrt{5}c_0$). This shows the basic differences between the steady and unsteady flows. Of course, we should note that these regions are on the limit of continuums mechanics validity.

The whole phenomenon can be considered as simple wave. This is the case of the wave when one family of the characteristics are straight lines.

Special case I:
In the special case of a piston moving with constant velocity, so that the piston path is a straight line. The expansion region is limited to an angle around the axis origin, so that all the gas particles hold the constant velocity of the piston. The wave is called central simple expansion wave.

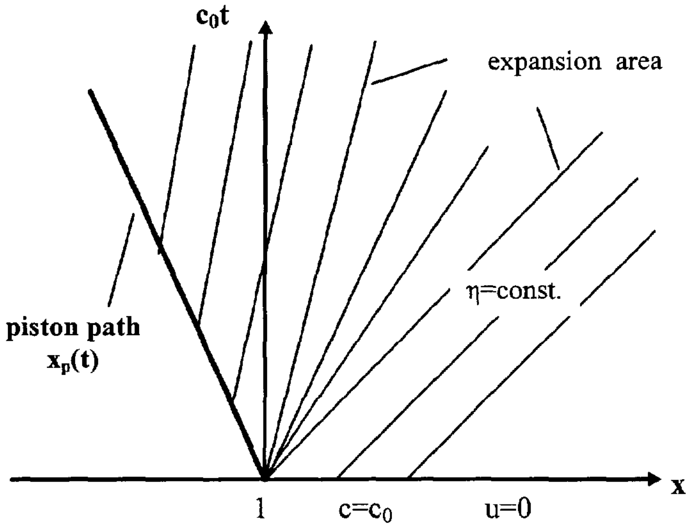


Figure 10 : Central expansion wave behind a piston moving with constant velocity

Special case II:

For the special case of the constant accelerating piston, for which the velocity argument is linearly increasing with time (U and t_0 are constants):

$$u_p = -U\frac{t}{t_0} \tag{49}$$

leads to a fully analytical expression of the flow velocity and the isentropic velocity of sound in the expansion area, which are given by the relations:

$$u(x,t) = -\frac{1}{\gamma}\left(c_0 + \frac{\gamma+1}{2}U\frac{t}{t_0}\right) + \left(\frac{1}{\gamma^2}\left(c_0 + \frac{\gamma+1}{2}U\frac{t}{t_0}\right)^2 + \frac{2}{\gamma}\frac{U}{t_0}(x - c_0 t)\right)^{\frac{1}{2}} \tag{50}$$

$$c(x,t) = c_0 + \frac{\gamma-1}{2}u \tag{51}$$

The distribution of velocity as a function of the space variable x is shown in figure 11 for various time levels.

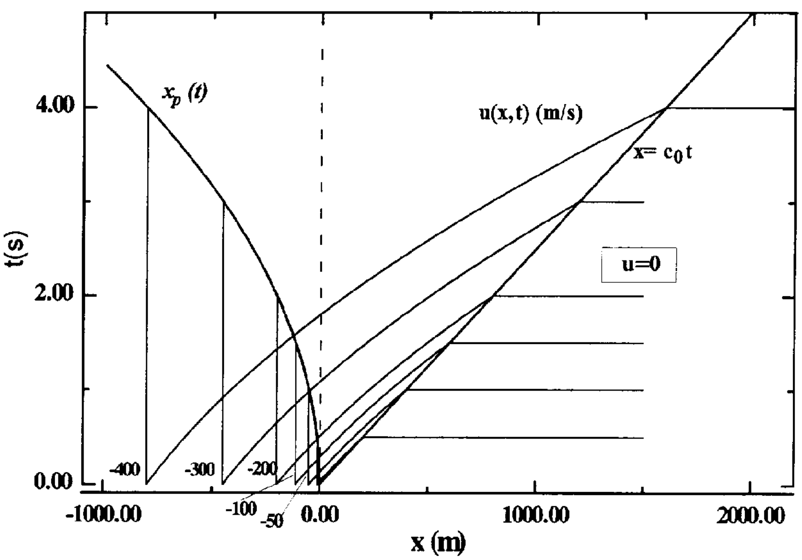


Figure 11: Velocity distribution as function of x for various times for the case of gas expansion due to the movement of a piston with a constant acceleration ($U = 100\,m/s, t_0 = 1\,s, c_0 = 400\,m/s, \gamma = 1.4$).

(b) Compression flow in front of a piston moving in a non-moving gas (MP-2)

The theory of isentropic flow for the compression flow in front of a piston obeys the same analysis as the expansion one (eqs (38)-(47)). One should remark in this case that the $\eta = \text{const}$ characteristics converge and form an envelope (Figure 12). The envelope can be in general shown that appears at earliest time at the point (x_c, t_c), that is defined as:

$$x_c = \frac{2c_0^2}{(\gamma+1)\dot{u}_p(0)}, \quad t_c = \frac{2c_0}{(\gamma+1)\dot{u}_p(0)} \tag{52}$$

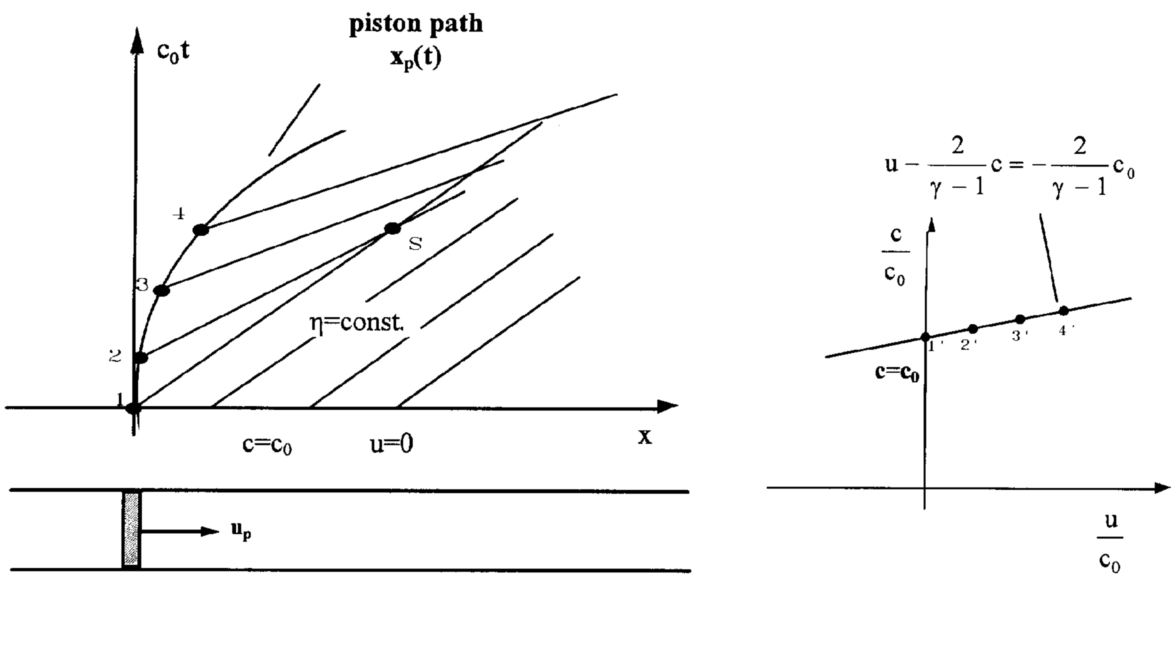


Figure 12 : Compression flow in front of a piston moving in a non-moving gas

After this point and for ($t > t_c$) a moving shock wave appears, which propagates in the gas at rest in the same direction with the piston and the flow is anisentropic. An analytic description of the flow field succeeds for the following two cases:

Special case III

In the case that the piston moves with constant velocity in a gas at rest:

$$u_p = U , \qquad U = \text{const} \tag{53}$$

so that the piston path is a straight line, figure 13, the $\eta = \text{const}$ characteristics are parallel to each other and a shock forms, which propagates with constant velocity u_s higher than that of the piston:

$$u_s = \frac{dx_s}{dt} = \frac{\gamma+1}{4}U + \sqrt{\left(\frac{\gamma+1}{4}U\right)^2 + c_0^2} \tag{54}$$

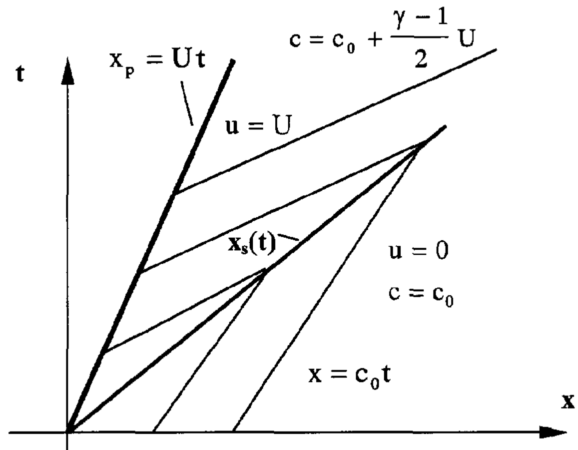


Figure 13 : Compression flow of a gas in front of a moving piston with a constant velocity

Special case IV

For the case of the compression flow of a gas in front of a moving piston with a velocity linearly increasing with time (constant accelerating piston):

$$u_p = U\frac{t}{t_0} \tag{55}$$

(U, t_0 constants), we lead as in the case II to a fully analytic expression for the flow field velocity in the isentropic region, that is before the appearance of the shock wave. The expression of the velocity in the isentropic compression region, can be shown to be similar to the expression of the expansion flow:

$$u(x,t) = -\frac{1}{\gamma}\left(c_0 - \frac{\gamma+1}{2}U\frac{t}{t_0}\right) + \sqrt{\frac{1}{\gamma^2}\left(c_0 - \frac{\gamma+1}{2}U\frac{t}{t_0}\right)^2 - \frac{2}{\gamma}\frac{U}{t_0}(x - c_0 t)} \tag{56}$$

$$c(x,t) = c_0 + \frac{\gamma-1}{2}u \tag{57}$$

and the point (x_c, t_c) is calculated as:

$$x_c = \frac{2c_0^2 t_0}{(\gamma+1)U}, \qquad t_c = \frac{2c_0 t_0}{(\gamma+1)U} \tag{58}$$

The distribution of velocity as function of the space variable x for various times t is shown in figure 14.

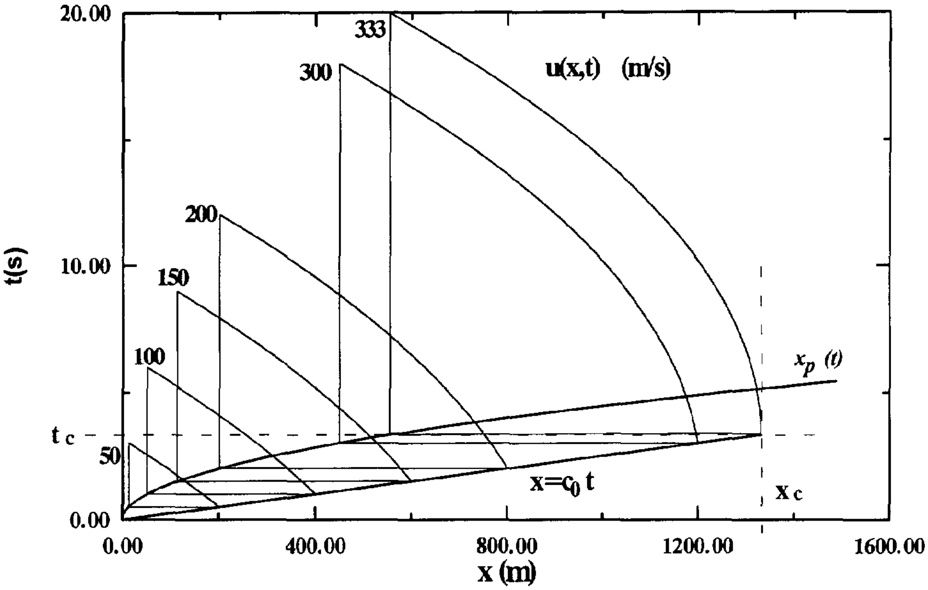


Figure 14 : Distribution of the velocity as a function of x for various t in the isentropic compression area of a gas in front of a constant accelerating piston ($U = 100\,m/s, t_0 = 1\,s, c_0 = 400\,m/s, \gamma = 1.4$).

(c) Moving shock in a shock tube or Riemann-problem (case ST-1)

The shock tube problem, often called Riemann problem, is the flow owing to the abrupt removal of the valve which separates a high pressure gas from a low pressure gas in the shock tube [1], [6]. The resulting wave effect of the propagation of the discontinuity and the relating nomenclature are shown in figure 15. The diaphragm located in position $x = 0$ at time $t = 0$ separates the space of high pressure $p_H \equiv p_4$ from the space of lowpressure $p_L \equiv p_1$. Thus, the basic parameter of the flow effect is the pressure ratio:

$$\frac{p_H}{p_L} = \frac{p_4}{p_1} \tag{59}$$

The two parts of the tube may have also different temperatures (T_4, T_1) and different gases (R_4, R_1).

At initial time the pressure distribution is a step distribution, figure 15a. This causes the separation of the problem in two problems, the propagation of a shock wave in the low pressure gas p_L and the propagation of an expansion wave in the high pressure gas p_H. The state behind the moving shock wave is indicated by the index 2 and the state behind the expansion wave with the index 3. The interface between the states 2 and 3 is a contact discontinuity. This is the contact point of the two gases, initially separated by the diaphragm, and have different temperatures and densities. On the other hand they should have the same velocities and pressures. The basic problem is how the flow quantities can be calculated with a given initial pressure ratio.

Introducing the following expressions:

$$P = \frac{p_2}{p_L}, \qquad \alpha = \frac{\gamma+1}{\gamma-1} \tag{60}$$

we have firstly the relations connecting the quantities on both sides of the moving shock wave:

Moving shock wave relations:

$$\frac{\rho_2}{\rho_L} = \frac{1+\alpha P}{\alpha + P} \text{ (Hugoniot relation)} \tag{61}$$

$$\frac{\upsilon_2 - \upsilon_L}{c_L} = \left(\frac{2}{\gamma(\gamma-1)}\right)^{\frac{1}{2}} \frac{P-1}{(1+\alpha P)^{\frac{1}{2}}} \tag{62}$$

$$\frac{c_2}{c_L} = \left(P \frac{\alpha + P}{1 + \alpha P} \right)^{\frac{1}{2}} \tag{63}$$

$$\frac{w - c_L}{c_L} = \left(\frac{\gamma - 1}{2\gamma} \right)^{\frac{1}{2}} (1 + \alpha P)^{\frac{1}{2}} \tag{64}$$

The pressure and velocity on both sides of the contact discontinuity are equal.

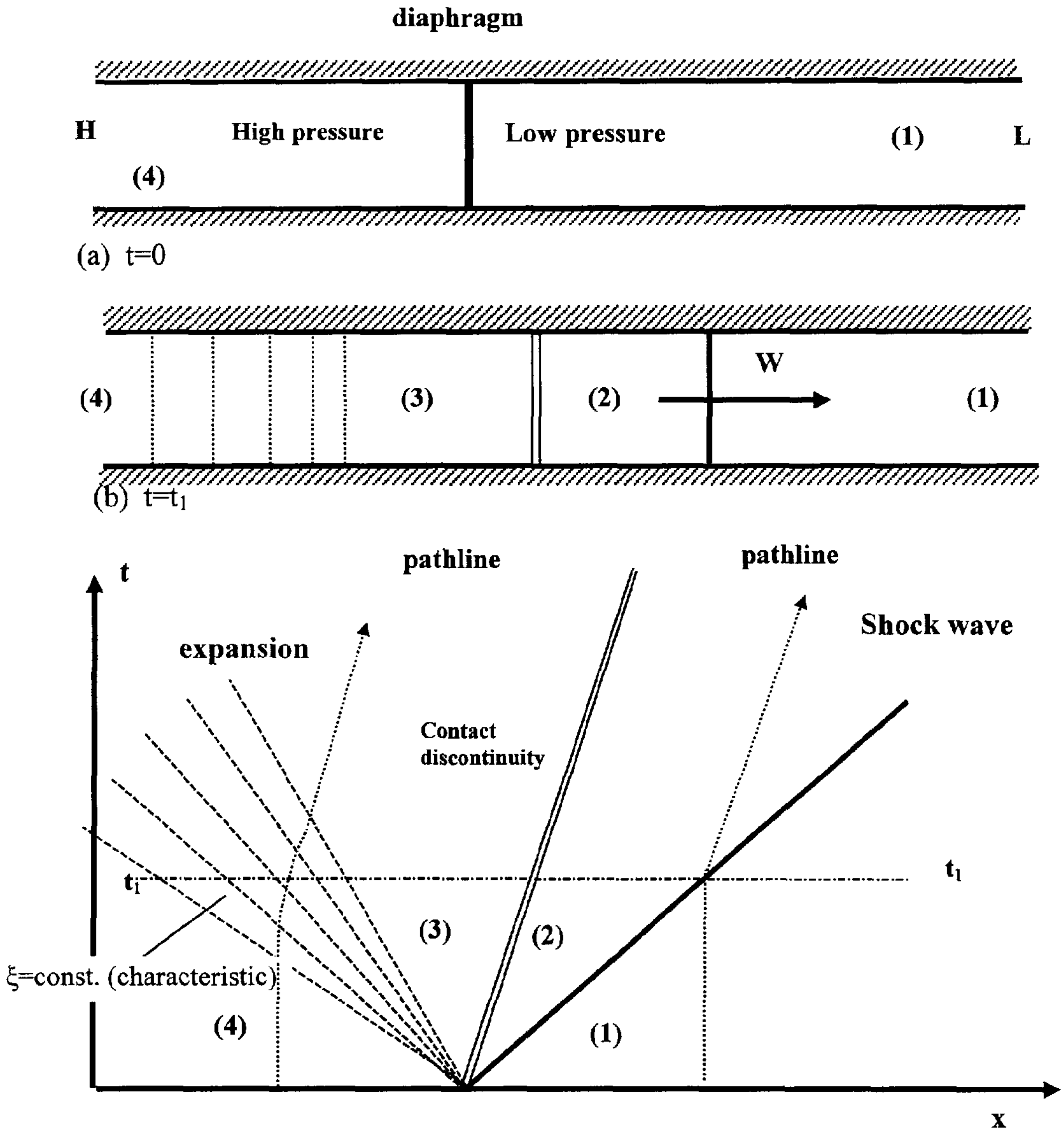


Figure 15 : Flow effects in a shock tube

Equations of the contact discontinuity:

$$\begin{aligned} p_3 &= p_2 \\ \upsilon_3 &= \upsilon_2 = v \end{aligned} \tag{65}$$

v is the velocity of the contact discontinuity.

Pressure and velocity in the regions 3 and $H \equiv 4$ are related with Riemann conditions on positive inclined characteristics $(\eta = \text{const})$.

Riemann equations across positive inclined characteristics:

$$\upsilon_3 - \upsilon_H = c_H \frac{2}{\gamma - 1}\left[1 - \left(\frac{p_3}{p_H}\right)^{\frac{\gamma-1}{2\gamma}}\right] \tag{66}$$

After eliminating unknowns from equations (61) , (64), (65) the following equation, which has as unknown the pressure relation P, is obtained:

$$\left(\frac{2}{\gamma(\gamma-1)}\right)^{\frac{1}{2}} \frac{P-1}{(1+\alpha P)^{\frac{1}{2}}} = \frac{c_H}{c_L}\frac{2}{\gamma-1}\left[1 - \left(\frac{p_L}{p_H}P\right)^{\frac{\gamma-1}{2\gamma}}\right] + \frac{\upsilon_H - \upsilon_L}{c_L} \tag{67}$$

The solution of the implicit algebraic equation (66) is accomplished by numerical integration.

Example:

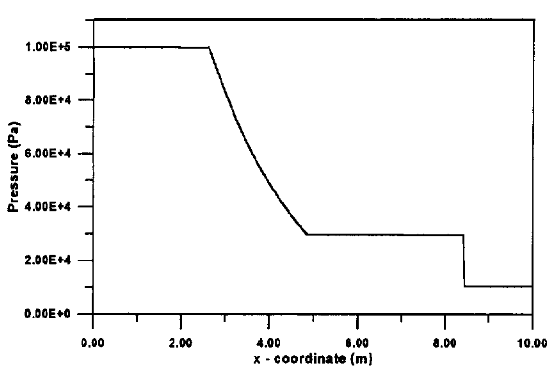
For the following values of the variables:

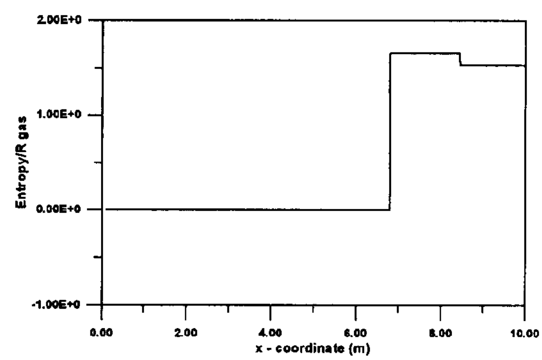
$p_H = 10^5, \rho_H = 1, \qquad \upsilon_H = 0, \qquad p_L = 10^4, \qquad \rho_L = 0.125, \qquad \upsilon_L = 0, \gamma = 1.4$

the solution of equation (66) gives the following values of the unknown parameters:

$$P = 3.0313, \qquad p_2 = 30313, \qquad \upsilon_2 = v = 203, \qquad w = 544$$

In figure 16 the variation of flow quantities are shown for the time instant $t = 6{,}110^{-3}$.





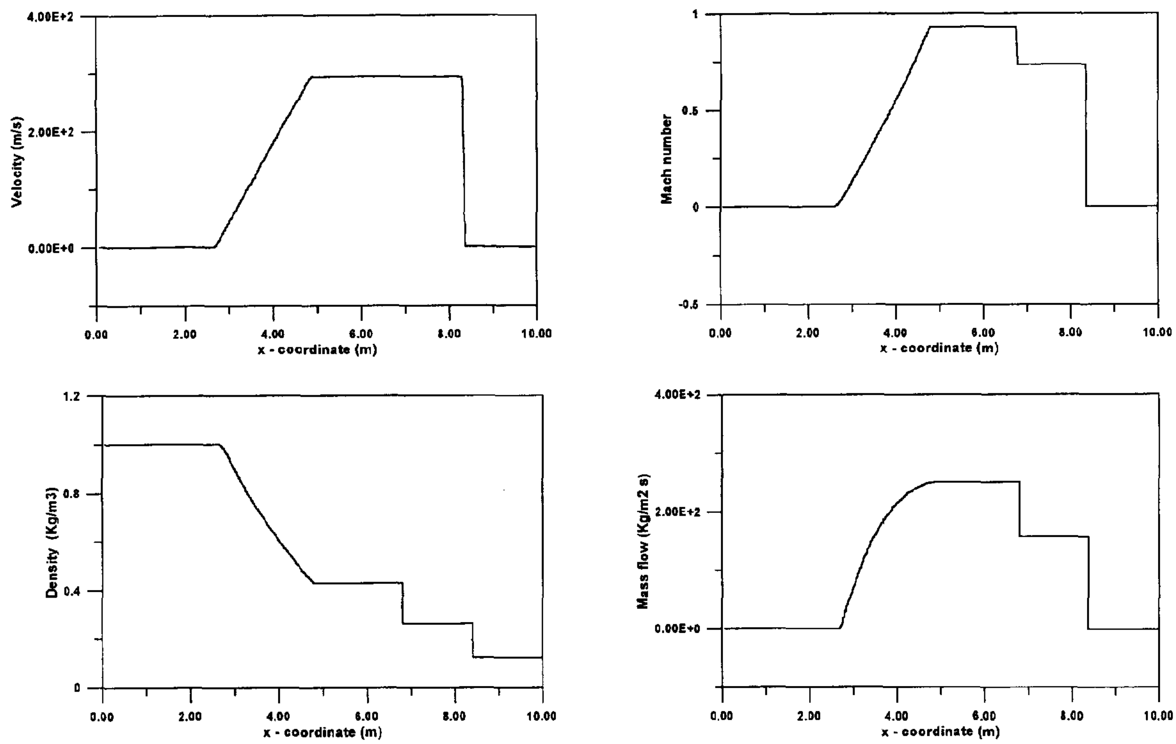


Figure 16 : Distribution of flow quantities at t=6,1 ms for $p_H = 10^5, \rho_H = 1, \; \upsilon_H = 0, \; p_L = 10^4, \rho_L = 0.125, \; \upsilon_L = 0, \gamma = 1.4$

ANALYTICAL SOLUTIONS FOR THE UNSTEADY COMPRESSIBLE LAMINAR FLOW FOR HEAT CONDUCTING FLUID

The existing analytical solutions for the unsteady, compressible laminar flow for a heat conducting fluid concern only the Lighthill's approximation of finite amplitude sound waves, for the case of one-dimension. For the derivation of this theory we refer to the review of M.J.Lighthill [5]. This theory leads to the following equation for the velocity u derived from the equations of continuity, momentum and energy after simplifications coming from the assumption of sound waves of finite amplitude for a perfect gas of constant γ :

$$\frac{\partial u}{\partial t} + \left(c_0 + \frac{\gamma + 1}{2} u \right) \frac{\partial u}{\partial x} = \frac{\delta}{2} \frac{\partial^2 u}{\partial x^2} \tag{68}$$

δ is the Lighthill's diffusion coefficient for the sound waves of finite amplitudes propagating to the positive - x direction:

$$\delta = \frac{\bar{\mu}_0}{\rho_0} + \frac{(\gamma - 1)\lambda_0}{\rho_0 c_p} \tag{69}$$

$\bar{\mu}_0 = 2\mu + \mu'$, where μ μ' are the dynamic and volumetric viscosities of the gas , λ_0 is the coefficient of heat conductivity, c_p is the heat coefficient for constant pressure and ρ_0 is the undisturbed density.

$$u_0 = 0, \quad c = c_0 \tag{70}$$

Through the transformation:

$$X = x - c_0 t, \quad \bar{u} = c_0 + \frac{\gamma + 1}{4} u \tag{71}$$

the differential equation can be transformed to the following non-linear **Burgers'** equation (1940), which is the simplest non linear equation describing convective effects combined with diffusive one :

$$\frac{\partial \bar{u}}{\partial t}+\bar{u}\frac{\partial \bar{u}}{\partial X}=\frac{\delta}{2}\frac{\partial^2 \bar{u}}{\partial X^2} \tag{72}$$

The general solution of the Burgers' equation is defined by the introduction of a new dependent variable, the function ϕ defined by the following relations, satisfying the Burgers' equation:

$$\frac{\partial \phi}{\partial X}=-\bar{u},\quad \frac{\partial \phi}{\partial t}=\frac{1}{2}\bar{u}^2-\frac{\delta}{2}\frac{\partial \bar{u}}{\partial X} \tag{73}$$

The differential equation for ϕ leads by elimination of $\bar{u}$ from the previous equations:

$$\frac{\partial \phi}{\partial t}=\frac{1}{2}\left(\frac{\partial \phi}{\partial X}\right)^2+\frac{\delta}{2}\frac{\partial^2 \phi}{\partial X^2} \tag{74}$$

By introducing again a new function ψ :

$$\phi=\delta \ln \psi \tag{75}$$

the equation which satisfies ψ is the standard linear equation for heat transfer by conduction:

$$\frac{\partial \psi}{\partial t}=\frac{1}{2}\delta\frac{\partial^2 \psi}{\partial X^2} \tag{76}$$

So, all the known solutions of the linear heat conduction equation are at the same time solution of the Burgers' equation, of course in the transformed variables.

The relations connecting ψ and $\bar{u}$ are:

$$\bar{u}=-\frac{\delta}{\psi}\frac{\partial \psi}{\partial X},\qquad \psi=\exp\left(\frac{1}{\delta}\int_X^{\infty}\bar{u}\,dX\right) \tag{77}$$

As a first example we refer to the initial value problem owing to Laplace. When the initial value of the wave form is given by $\bar{u}(X,0)$, then the solution is defined by the integral:

$$\psi(X,t)=\frac{1}{\sqrt{2\pi\delta t}}\int_{-\infty}^{\infty}\psi(Y,0)\exp\left[-\frac{(X-Y)^2}{2\delta t}\right]dY \tag{78}$$

$$\bar{u}(X,t)=\frac{\displaystyle\int_{-\infty}^{\infty}\frac{X-Y}{t}\exp\frac{1}{\delta}\left\{\int_Y^{\infty}\bar{u}(Y,0)dY-\frac{(X-Y)^2}{2t}\right\}dY}{\displaystyle\int_{-\infty}^{\infty}\exp\frac{1}{\delta}\left\{\int_Y^{\infty}\bar{u}(Y,0)dY-\frac{(X-Y)^2}{2t}\right\}dY} \tag{79}$$

INDICATIONS FOR USE OF THE EXAMINED TEST CASES

The above test cases are frequently used for testing various properties of the examined numerical schemes. Specifically:

- The test case of linear advection equation (LADV-1) that results in oscillatory solution and contains discontinuity in the derivative, can be used to test the diffusion and dispersion properties of schemes and to define the accuracy of the scheme on smooth functions of the wave number k.
- The linear advection equation of propagating discontinuity with the velocity α (LADV-2) is important with regard to properties of the schemes at handling propagating discontinuities. If the discontinuity is an expansion shock, the numerical scheme can propagate and dump the expansion through an introduced entropy condition or any other form of dissipative mechanism.

- The behaviour of invisvid Burgers' equation against non linearities is representative of the examination of Euler equations behaviour due to the non linear term of Burgers' equation. Inviscid Burgers' equation (IB-1) with initial shock discontinuity gives information about the capability of the numerical scheme on shock capturing with the correct shock propagating speed (time accurate scheme). The shock capturing without the presentation of non physical oscillations in the vicinity of the discontinuity ensures the monotonicity of the numerical scheme or its property for total variations diminishing. . Inviscid Burgers' equation (IB-2) with initial linear discontinuity gives information about the diffusion and dispersion properties of the numerical scheme. IB-2, also ensures, as IB-1 for the shock capturing capability of the scheme and its characteristics about monotonicity. Inviscid Burgers' equation (IB-3) for a rarefaction wave is used to test the additional entropy condition that is imposed on the numerical schemes in order to capture expansion shocks for inviscid flow equation. Viscous Burgers' equation (VB-3) serves as a model equation for the boundary-layer equation or the "parabolized" Navier-Stokes equations and examines the capturing of diffusing shock.
- The problem of shock tube presents an exact solution to the full system of one-dimensional Euler equations containing simultaneously a shock wave, a contact discontinuity and an expansion fan. Consequently, it can be used for the testing of all the above properties of numerical schemes.

SOFTWARE

The analytical solutions for the above cases are also presented in the corresponding FORTRAN 90 programs that have been attached to the present paper. These programs are:

FORTRAN program	Input Data file	Test case
advection.for	*advect.int*	Linear advection equation
inv_Burgers.for	*inv_burg.int*	Inviscid Burgers' equation
vis_Burgers.for	*vis_burg.int*	Viscous Burgers' equation
expansion.for	*expansion.int*	Expansion flow behind a moving piston
compresion.for	*compression.int*	Compression flow in front of a moving piston in a non-moving gas
shock_tube.for	*shock.int*	Moving shock in a shock tube or Riemann problem

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