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APPARATUS AND METHOD FOR SYNCHRONIZING NONLINEAR SYSTEMS USING FILTERED SIGNALS

Cross Reference To Related Patents And Applications
This application is related to commonly assigned U.S. Pat. Nos. 5,245,660 (having Navy Case No. 72,593); 5,379,346 (having Navy Case No. 74,222); and 5,402,334 (having Navy Case No. 73,912). This application is also related to commonly assigned U.S. Patent Application Serial No. 08/267,696 filed June 29, 1994 (having Navy Case No. 75,496). U.S. Pat. Nos. 5,245,660 and 5,379,346 and U.S. Patent Application Serial No. 08/267,696 are all incorporated herein by reference.

1. Field Of The Invention
The present invention relates generally to physical systems with dynamical characteristics which involve synchronization of a transmitter system and a receiver system and, more particularly, to a system which allows the synchronization of a nonlinear system when the driving signal has been filtered.

2. Description of the Related Art
The design of most man-made mechanical and electrical systems assumes that the systems exhibit linear behavior (stationary) or simple nonlinear behavior (cyclic). In recent years there has been an increasing understanding of a more complex form
of behavior, known as chaos, which is now recognized to be generic to most nonlinear systems. Systems evolving chaoticly (chaotic systems) display a sensitivity to initial conditions, such that two substantially identical chaotic systems started with slightly different initial conditions (state variable values) will quickly evolve to values which are vastly different and become totally uncorrelated, even though the overall patterns of behavior will remain the same. This makes chaotic systems nonperiodic (there are no cycles of repetition whatsoever), unpredictable over long times, and thus such systems are impos-

cusses the relationship between synchronization and chaotic systems in which selected parameters are outside some range required for synchronization.

Summary Of The Invention

It is therefore an object of the invention to provide sys-
tems for producing synchronized signals, and particularly nonlinear dynamical systems.

Another object of the invention is provide communication systems for encryption utilizing synchronized nonlinear sending and receiving circuits.
Another object of the invention is to provide improved
control devices which rely on wide-frequency band synchronized
signals.

Another object of the invention is to provide physical
systems with dynamical characteristics which involve
synchronization of a transmitter system and a receiver system
and, more particularly, to a system which allows the
synchronization of a nonlinear system when the driving signal has
been filtered.

Another object of the invention is to provide synchronizable
systems in which only a single drive signal is transmitted
between systems, multiple synchronized signals can be produced
and in which the drive signal can be reproduced to confirm
synchronization.

Another object of the invention is to provide synchronizable
systems in which the total dimension of the synchronized systems
or the number of elements can be the same.

Another object of the invention is to provide synchronizable
chaotic systems when the drive signal is filtered.

Another object of the invention is to transmit information
using cascaded synchronizable chaotic systems with a filtered
drive signal.

A further object of the invention is to transmit information
using cascaded synchronizable chaotic systems using parameter
changes to transmit information when the drive signal is filtered.

A cascaded synchronized nonlinear system includes a nonlinear drive system having stable first and second subparts. The first subpart produces a first drive signal for driving the second subpart and the second subpart produces a second drive signal for driving the first subpart. The nonlinear transmitter transmits the second drive signal to a nonlinear cascaded response system. The response system, being for producing an output signal in synchronization with the second drive signal, comprises a first stage (a duplicate of the first subpart) responsive to the second drive signal for producing a first response signal. The response system further comprises a second stage (a duplicate of the second subpart) responsive to the first response signal for producing the output signal.

A cascaded synchronized nonlinear system with a filtered drive signal includes a nonlinear drive system having stable first and second subparts. The first subpart produces a first drive signal for driving the the second subpart and the second subpart produces a second drive signal for driving the first subpart. The second drive signal is passed through the transmitter filter and the transmitter filter output is subtracted from the second drive signal to produce the broadcast signal. The broadcast signal is transmitted to a nonlinear
receiver with a cascaded section.  

The receiver, being for producing an output signal in  
synchronization with the second drive signal, consists of a  
cascaded nonlinear response system and a filter section. The  
cascaded system comprises a first stage (a duplicate of the first  
subpart) responsive to the receiver driving signal produced by  
the filter section. The first stage produces a first response  
signal. The cascaded section further comprises a second stage (a  
duplicate of the second subpart) responsive to the first response  
signal for producing the output signal.  

The filter section is comprised of a receiver filter  
identical to the transmitter filter and an adding circuit. The  
receiver filter is responsive to the receiver output signal, and  
produces the filter output signal. The adding circuit adds the  
filter output signal to the broadcast signal to produce the  
receiver driving signal.  

The filtered cascaded synchronized nonlinear system can be  
used in an information transfer system. The transmitter  
responsive to an information signal produces a broadcast signal  
for transmission to the receiver. An error detector compares the  
receiver drive signal described above and the output signal  
produced by the receiver to produce an error signal indicative of  
the information contained in the information signal.  

These and other objects, features and advantages of the present  

invention are described in or apparent from the following
detailed description of preferred embodiments.

Brief Description Of The Drawings

The preferred embodiments will be described with reference
to the drawings, in which like elements have been denoted
throughout by like reference numerals, and wherein:

FIG. 1 is a general block diagram of a nonlinear dynamical
physical system of the prior art;

FIG. 2 is a general block diagram of an synchronized chaotic
system of the prior art;

FIG. 3 is a schematic circuit diagram of a synchronized
chaotic circuit system of the prior art;

FIG. 4 illustrates a cascaded synchronized system of the
prior art with two stages;

FIG. 5 shows an embodiment of a two stage cascaded
synchronization system of the prior art.

FIGS. 6 - 9 illustrate details of the prior art system of
FIG. 5.

FIG. 10 is a block diagram of the present invention as
applied to an autonomous nonlinear dynamical system;

FIG. 11 is a block diagram of a second embodiment of the
present invention as applied to a non-autonomous nonlinear
dynamical system;
FIG. 12-18 show details of the second embodiment of FIG. 11;

FIG. 19 shows a modification of FIG. 14 in which FIG. 19 replaces FIG. 16 to produce a third embodiment of the invention comprised of FIGS. 12-15 and 17-19;

FIG. 20(a) illustrates the power spectrum of the x signal from the drive system circuit 3000 (FIG. 12) described by equations 26-28;

FIG. 20(b) illustrates the power spectrum of the broadcast signal s, from the transmitter filter circuit (FIG. 19) described by equations 37-39;

FIG. 21 shows the receiver forcing signal F'' (2240) vs. the transmitter forcing signal F (2130) from the circuit implementation of the present invention when the band-stop filter arrangement of FIG. 19 is used;

FIG. 22 illustrates a dotted line showing the value of a parameter A in the drive system circuit 3000 of FIG. 12 that was varied and the solid line shows the value of the error signal A from the phase control circuit 5200 when a set of nonautonomous circuits were synchronized according to the block diagram of FIG. 11;

FIG. 23 shows a solid line representing a time series of the numerical output of the y signal from a 4-th order Runge-Kutta integration routine executed on a digital computer to simulate the system of equations 43, and a dotted line
representing the signal broadcast signal $y_t$ from the filter of
equations 44 and 45;

FIG. 24 shows a power spectrum of the numerical $y$ signal
from equations 43;

FIG. 25 shows a power spectrum of the broadcast signal $y_t$
from equation 45;

FIG. 26 shows the numerical response system output signal
$y''$ from equation 48 vs the drive system output signal $y$ from
equation 43; and

FIG. 27 illustrates a block diagram of a fourth embodiment
of the present invention to correct for channel filtering
effects.

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Detailed Description Of The Preferred Embodiments

All physical systems can be described by state variables.
For example, a billiard game can be described by the position and
the velocity of a ball at any instant of time; and an electronic
circuit can be described by all of its currents and voltages at a
particular time. This invention is a tangible system which can
be of any form. The state variables and associated signals can
be, as further examples, pressure or other force, temperature,
concentration, population, or electro-magnetic field components.
The evolution of a physical system depends on the dynamical
relations between the state variables, which are usually ex-
pressed as functional relations between the rates of change of
the variables. Thus, most but not all physical systems are
describable in terms of ordinary differential equations (ODEs).
Mathematical models of chaotic systems often involve two types of
systems: flows and iterative maps. The former evolve as solu-
tions of differential equations, and the latter evolve in dis-
crete steps, such as by difference equations. For example,
seasonal measurements of populations can be modeled as iterative
maps. Cf., Eckmann et al., Rev. Mod. Phys., Vol. 57, pp.617-618,
619 (1985). Some iterative maps could be considered as numerical
solutions to differential equations. Solution or approximate
solution of these equations, such as approximate, numerical, or
analytical solution, provides information about the qualitative
and quantitative behavior of the system defined by the equations.

As used herein, the synchronization of two or more evolving
state variables of a physical system means the process by which
the variables converge toward the same or linearly related but
changing set of values. Thus, if one synchronized variable
changes by a certain amount, the change of the other synchronized
variable will also approach a linear function of the same amount.
Graphically, the plot of the synchronized variables against each
other as they evolve over time would approach a straight line.

Referring to FIG 1, an n-dimensional autonomous
nonlinear dynamical drive system 9 can be arbitrarily divided, as
shown, into first and second parts or subsystems 12 and 14, each
of which subsystems is also a nonlinear dynamical system. Drive
system 9 and, more specifically, subsystem 14, has output signal
S_o.

The following discussion involves mathematical modeling of
the system 10 in terms of solutions to differential equations and
provides theoretical support for the invention. However, it is
not necessary in practicing this invention that system 10 be
susceptible to such modeling. For example, as stated earlier,
some iterated maps cannot be modeled as solutions to differential
equations, and yet this invention encompasses systems evolving
according to iterated maps. As a further example, it is imprac-
tical to accurately model an ideal gas by individually consider-
ing the position and momentum of every molecule because of the
vast number of molecules and variables involved.

This discussion about mathematical modeling is in two parts
to correspond to two sources of difficulty in synchronizing
signals: instability within a single system (chaos) and instabil-
ity between two systems (structural instability). It is under-
stood that both discussions apply to this invention and neither
part should be read separately as limiting the practice of this
invention.

A system with extreme sensitivity to initial conditions is
considered chaotic. The same chaotic system started at infinite-
Similarly different initial conditions may reach significantly
different states after a period of time. As known to persons
skilled in the art and discussed further, for example, in Wolf et
al., *Determining Lyapunov Exponents from a Time Series*, Physica,
Vol. 16D, p. 285 et seq. (1985), Lyapunov exponents (also known
in the art as "characteristic exponents") measure this diver-
gence. A system will have a complete set (or spectrum) of
Lyapunov exponents, each of which is the average exponential rate
of convergence (if negative) or divergence (if positive) of
nearby orbits in phase space as expressed in terms of appropriate
variables and components. If all the Lyapunov exponents are
negative, then the same system started with slightly different
initial conditions will converge (exponentially) over time to the
same values, which values may vary over time. On the other hand,
if at least one of the Lyapunov exponents is positive, then the
same system started with slightly different initial conditions
will not converge, and the system behaves chaotically. It is
also known by persons skilled in the art that "in almost all real
systems there exist ranges of parameters or initial conditions
for which the system turns out to be a system with chaos... ."
Chernikov et al., *Chaos: How Regular Can It Be?*, 27 Phys. Today
27, 29 (Nov. 1988).

Drive system 9 can be described by the ODE

\[ \frac{du(t)}{dt} = f(u(t)) \quad \text{or} \quad \dot{u} = f(u) \]  

(1)
where \( u(t) \) are the \( n \)-dimensional state variables.

Defined in terms of the state variables \( v \) and \( w \) for subsystems 12 and 14, respectively, where \( u = (v, w) \), the ODEs for subsystems 12 and 14 are, respectively:

\[
\begin{align*}
\dot{v} &= g(v, w) \\
\dot{w} &= h(v, w)
\end{align*}
\]

where \( v \) and \( w \) are \( m \) and \( n-m \) dimensional, respectively, that is, where \( v = (u_1, \ldots, u_m) \), \( g = (f_1(u), \ldots, f_m(u)) \), \( w = (u_{m+1}, \ldots, u_n) \) and \( h = (f_{m+1}(u), \ldots, f_n(u)) \).

The division of drive system 9 into subsystems 12 and 14 is truly arbitrary since the reordering of the \( u_i \) variables before assigning them to \( v, w, g \) and \( h \) is allowed.

If a new subsystem 16 identical to subsystem 14 is added to drive system 9, thereby forming system 10, then substituting the variables \( v \) for the corresponding variables in the function \( h \) augments equations (2) for the new three-subsystem system 10 as follows:

\[
\begin{align*}
\dot{v} &= g(v, w) \\
\dot{w} &= h(v, w) \\
\dot{w}' &= h(v, w').
\end{align*}
\]

Subsystem 16 has output signal \( S_o' \).

The \( w \) and \( w' \) subsystems (subsystems 14 and 16) will only synchronize if \( \Delta w \to 0 \) as \( T \to \infty \), where \( \Delta w = w' - w \).

The rate of change of \( \Delta w \) (for small \( \Delta w \)) is:
\[
\Delta \hat{w} = d\Delta w/dt = h(v, w') - h(v, w) = D_v h(v, w) \Delta w + W; \tag{4}
\]

where \(D_v h(v, w)\) is the Jacobian of the \(v\) subsystem vector field with respect to \(w\) only, that is: an \((n-m) \times (n-m)\) linear operator (matrix)

\[
(D_v h)_{ij} = \partial h_i / \partial w_j \tag{5}
\]

for \((m+1) \leq i \leq n\) and \(1 \leq j \leq (n-m)\), and where \(W\) is a nonlinear operator. When Equation 4 is divided by \(|\Delta w(0)|\), and \(\xi = \Delta w(t) / \Delta w(0)\), an equation for the rate of change (the growth or shrinkage) of the unit displacement \((n-m)\) dimensional vector, \(\xi\), is obtained. In the infinitesimal limit, the nonlinear operator vanishes and this leads to the variational equation for the subsystem

\[
d\xi/dt = D_v h(v(t), w(t)) \xi. \tag{6}
\]

The behavior of this equation or its matrix version, using the usual fundamental matrix approach, depends on the Lyapunov exponents of the \(w\) subsystem. These are hereinafter referred to as sub-Lyapunov exponents to distinguish them from the full Lyapunov spectrum of the \((v, w) = (u)\) system. Since the \(w\) subsystem 14 is driven by the \(v\) subsystem 12, the sub-Lyapunov exponents of the \(w\) subsystem 14 are dependent on the \(m\) dimensional \(v\) variable. If at least one of the sub-Lyapunov exponents is positive, the unit displacement vector \(\xi\) will grow without bounds and synchronization will not take place. Accordingly, the sub-systems 14 and 16 (\(w\) and \(w'\)) will synchronize only if the sub-Lyapunov exponents are all negative. This principle provides
a criterion in terms of computable quantities (the sub-Lyapunov
exponents) that is used to design synchronizing systems in
accordance with the present invention.

The \( v = (v_1, \ldots, v_n) \) components (subsystem 12) can be viewed
more broadly as driving variables and the \( w' = (w'_1, \ldots, w'_n) \)
components (subsystem 16) as responding variables. The drive
system 9 \((v, w)\) can be viewed as generating at least one drive
signal \( S_d \), in the formula \( v(t) \), which is applied to the response
systems \( w \) and \( w' \) (subsystems 14 and 16, respectively) to synch-
ronize the drive system and the response system outputs. This is
the approach taken in accordance with the present invention to
provide synchronized nonlinear dynamical systems.

In practicing this invention, the above discussion applies
to identical subsystems 14 and 16. This might be achievable, for
example, in digital systems. In such systems 10, the signals \( S_o \)
and \( S_o' \) may each be chaotic because the system 9 might be cha-
otic. They may differ because of different initial conditions in
subsystems 14 and 16. However, they will approach each other \((\Delta w \rightarrow 0)\) because systems 14 and 16 are stable (that is, with all
negative sub-Lyapunov exponents) when considered as driven by the
same at least one drive signal \( S_d \).

In most physical systems, subsystems 14 and 16 are not
identical. For example, two electrical components with the same
specifications typically do not have identical characteristics.
The following explanation based on mathematical modeling shows that the signals $S_o$ and $S_o'$ will nevertheless be synchronized if both subsystems 14 and 16 have negative sub-Lyapunov exponents. According to this mathematical model, the synchronization is affected by differences in parameters between the $w$ and $w'$ systems which are found in real-life applications. Let $\mu$ be a vector of the parameters of the $w$ subsystem (subsystem 14) and $\mu'$ of the $w'$ subsystem (subsystem 16), so that $h = h(v,w,\mu)$, for example. If the $w$ subsystem were one-dimensional, then for small $\Delta w$ and small $\Delta \mu = \mu' - \mu$:

$$\Delta \dot{w} \approx h_w \Delta w + h_\mu \Delta \mu$$ (7)

where $h_w$ and $h_\mu$ are the partial derivatives of $h$ with respect to $w$ and $\mu$, respectively. Roughly, if $h_w$ and $h_\mu$ are nearly constant in time, the solution of this equation will follow the formula

$$\Delta w(t) = [\Delta w(0) + \frac{h_\mu}{h_w} \Delta \mu \frac{e^{h_\mu t}}{h_w}] - \frac{h_\mu}{h_w} \Delta \mu$$ (8)

If $h_w < 0$, the difference between $w$ and $w'$ will level off at some constant value and the systems will be synchronized. Although this is a simple one-dimensional approximation, it turns out to be the case for all systems that have been investigated numerically, even when the differences in parameters are rather large (~10-20%). This is also the case in the exemplary electronic synchronizing circuit described in more detail.
hereinbelow. Furthermore, it can be established on a mathematical basis that the small changes in parameters only lead to proportionally small degradations of synchronization, which approach a constant value. See Pecora et al., "Driving systems with chaotic signals", Physical Review A, Vol. 44, No. 4, August 15, 1991, pages 2374-2383.

Since m-dimensional variable v may be dependent on (n-m)-dimensional variable w, there may be feedback from subsystem 14 to subsystem 12. As shown in FIG 2, a response part 15 of subsystem 14 may produce a feedback signal S_r responsive to the m-dimensional driving variable v, and a drive part 17 of subsystem 12 may respond to the feedback signal S_r to produce the at least one drive signal S_d.

As shown in FIG 2, subsystems 14 and 16 need not be driven by the same at least one drive signal S_d but could be driven by at least one input signals S_i and S'_i responsive to the at least one drive signal S_d. System 10 could have primary and secondary means 18 and 19, respectively, coupled to subsystem 12 and responsive to the at least one drive signal S_d for generating input signals S_i and S'_i, respectively. If these primary and secondary means 18 and 19, respectively are linearly responsive to the at least one drive signal S_d, then the above mathematical analysis would continue to apply since linear transformations do not affect the signs of the sub-Lyapunov exponents.
In accordance with the prior art, in order to develop electrical circuits, for example, which have chaotic dynamics, but which will synchronize, a nonlinear dynamical circuit (the drive subsystem) is duplicated (to form a response subsystem). A selected portion of the response circuit is removed, and all broken connections are connected to voltages produced at their counterparts in the drive circuit. These driving voltages constitute the at least one drive signal $S_d$ shown in Figure 1, and advantageously are connected to the response circuit via a buffer amplifier to ensure that the drive circuit is not affected by the connection to the response circuit, i.e., it remains autonomous.

As a specific example, FIG. 3 shows an electrical circuit system 20 constructed in accordance with the prior art which has two synchronized nonlinear dynamical subsystems, a drive circuit 22 and a response circuit 16. Circuits 22 and 16 correspond to the u and w' systems, respectively, discussed above.

Drive circuit 22 comprises a hysteretic circuit formed by a differential amplifier 30, resistors 42, 44, 46, 48, 50 and 52; potentiometer 74; capacitor 76; and diodes 82 and 84 connected as shown; and an unstable oscillator circuit formed by differential amplifiers 32, 34, 36, 38 and 40; resistors 58, 60, 62, 64, 66, 68, 70 and 72; and capacitors 78 and 80 connected as shown. In an experimental implementation of circuit system 20 which has
been successfully tested, amplifiers 30 - 40 were MM741 operational amplifiers, and diodes 82 and 84 were 1N4739A diodes. Component values for the resistors and capacitors which were used are set forth in the following table:

Resistor 42 = 10kΩ  
Resistor 44 = 10kΩ  
Resistor 46 = 10kΩ  
Resistor 48 = 20kΩ  
Resistor 50 = 100kΩ  
Resistor 52 = 50kΩ  
Resistor 54 = 3kΩ  
Resistor 56 = 20kΩ  
Resistor 58 = 100kΩ  
Resistor 60 = 100kΩ  
Resistor 62 = 220kΩ  
Resistor 64 = 150kΩ  
Resistor 66 = 150kΩ  
Resistor 68 = 330kΩ  
Resistor 70 = 100kΩ  
Resistor 72 = 100kΩ  
Potentiometer 74 = 10kΩ  
Capacitor 76 = 0.01μF  
Capacitor 78 = 0.01μF  
Capacitor 80 = 0.001μF

Drive circuit 22 can be subdivided into two subparts 14 and 12. Although the illustrative subparts 14 and 12 shown in FIG. 3 correspond to the two circuits forming drive circuit 22, this is not necessary, and the division of a given drive circuit into subparts in order to determine the proper configuration for a synchronized response circuit is made in accordance with the analysis described herein. Subpart 14 corresponds to the w subsystem (subsystem 14 in Figure 1), subpart 12 corresponds to
the v subsystem described above. Those parts of subpart 14 which affect the signal at X4 and those parts of subpart 12 responsive thereto, respectively, constitute response part 15 (FIG. 2) and drive part 17 (FIG. 2), to provide feedback. Response circuit 16 is substantially a duplicate of subpart 14 of drive circuit 22 (the specifications for primed components, such as resistor 50', are the same as the specification for unprimed components, such as resistor 50) and corresponds to subsystem w' (subsystem 16) described hereinafore. Signals \( X_1, X_2, X_3, \) and \( X_4 \) are characteristic voltages of drive circuit 22. The signal \( X_4 \) is connected as drive signal \( S_d \) through a buffer amplifier 25, which ideally is an operational amplifier having linear characteristics such as an AD381 manufactured by Analog Devices, to response circuit 16 at the junction in circuit 16 corresponding to the junction in circuit 22 at which the signal \( X_4 \) is generated. Signal \( X_4 \) replaces the circuitry (subpart 12) of drive circuit 22 which is missing in response circuit 16. The subsystem of buffer amplifier 25 is the secondary means 19.

Drive circuit 22 is an autonomous system and behaves chaotically. It can be modeled by the following equations of motion for the three voltages \( X_1, X_2 \) and \( X_3 \) shown in FIG. 3.

\[
\begin{align*}
\dot{X}_1 &= X_2 + \gamma X_1 + cX_3 \\
\dot{X}_2 &= -\omega X_1 + \delta X_2 \\
\epsilon X_3 &= (1 - X_3) (sX_1 + r + X_3) - \delta X_3, 
\end{align*}
\]

(9)
where $\gamma = 0.12$, $C = 2.2$, $\omega_2 = 10.0$, $\delta_2 = \delta_3 = 0.001$, $\epsilon = 0.001$, $s = 1/6$, and $r = 0.0$.

An analysis of the sub-Lyapunov exponents for the response circuit 16 requires a transformation of the equations of motion from the $(X_1, X_2, X_3)$ system to the $(X_1, X_2, X_4)$ system. This is done by analyzing the circuit, and finding that $X_3 = aX_4 - \beta X_1$

where $a = 6.6$ and $\beta = 7.9$. This gives the following equations of motion:

$$\dot{X}_1 = X_2 + \gamma X_1 + c(aX_4 - \beta X_1)$$
$$\dot{X}_2 = -\omega_2 X_1 - \delta_2 X_2$$
$$\epsilon \dot{X}_4 = (1/a)((1 - (aX_4 - \beta X_1)^2)(sX_1 - r + aX_4 - \beta X_1) - \delta_3 aX_4 - \beta X_1 - \beta X_2 - \beta \gamma X_1 - \beta c(aX_4 - \beta X_1))$$

The equations of motion for the response are just the $X_1$ and $X_2$ equations. The sub-Lyapunov exponents are calculated directly from the Jacobian of the $X_1$ and $X_2$ equations, which is a constant in this case. It will be appreciated that conventional methods for calculating Lyapunov exponents, as analytical, measurement, numerical and otherwise can be used, such as, for example, those described by Eckmann et al., Rev. Mod. Phys., Vol. 57, p.617 et seq. (1985); Lichtenberg et al., Regular and Stochastic Motion, Springer-Verlag, New York (1983); Rashband, Chaotic Dynamics of Nonlinear Systems, John Wiley and Sons, New York (1990); and Wolf et al., Physica, Vol. 16D, p. 285 et seq.
(1985). The sub-Lyapunov exponents in this case are -16.587 and -0.603, implying that synchronization of the two electrical circuits 22 and 16 will occur. \( X_4 \) is the drive signal \( S_4 \) for the response subsystems and \( (X_1', X_2') \) and \( (X_1, X_2) \) are the synchronized signals \( S_o \) and \( S_o' \).

Circuit 22 itself runs in the realm of a few hundred Hz. Response circuit 16 synchronizes with drive circuit 22 within about two milliseconds. It has been observed experimentally that small changes (~10%) of the circuit parameters do not affect synchronization greatly, in that the response voltages still remain close to their counterparts in drive circuit 22; but larger changes (~50%) do. Even though the sub-Lyapunov exponents for the larger changes both remain negative, the response voltages no longer remain close to their drive counterparts.

The circuit of FIG. 3 has been used to transmit a pure frequency signal hidden in a chaotic signal as follows. With circuits 22 and 16 operating in a synchronized mode, a sine wave of a few hundred Hz was added to the \( X_2 \) signal from the drive circuit and sent to the response circuit. The \( X_2' \) signal produced by response circuit 16 was then subtracted from the sum of the \( X_2 \) signal and the sine wave, thereby extracting the sine wave from the chaotic signal. Spectral analysis of the \( (X_2 + \text{sine wave}) \) combination signal showed that the sine wave could not be detected in the chaos of the \( X_2 \) signal. The smallest sine wave
that could be extracted this way was approximately 40 millivolts
peak to peak compared to a two volt peak to peak $X_2$ signal, or a
50:1 ratio of chaotic signal to sine wave.

Many other possible choices for the drive circuit are
possible and may require transformation of the circuit equations
to model them. This can be determined as described hereinabove
for nonlinear circuits by analyzing the circuit dynamics in terms
of the sub-Lyapunov exponents to determine which signal(s) to
choose as a drive signal or signals, and which subcircuit is to
be used as a model for the response circuit.

It will also be appreciated that the prior art as described
previously is applicable to any system which requires
synchronization of remote signals and/or their low correlation
with each other. For example, the prior art is particularly
suited for use in control devices relying on wide-frequency-band
synchronized signals.

Similar principles as discussed previously can be applied to
cascaded subsystems which allow the multiple signals to be
synchronized. In the following discussion the previously
discussed design of synchronized subsystems is built on by cas-
cading two or more subsystem responses.

The objective here is to get a synchronization of the
response with its counterpart in the drive system but to build a
response setup which produces signals in synchronization with one
or more of the original input drive signals. The new
synchronization signal may be used to process the original input
drive, to detect parameter changes between the drive system and
the responses, and to detect other information transmitted along
with the output of the drive system.

FIG. 4 illustrates a cascaded system having a drive system
1400 and a response system 1500. The drive system 1400 includes
two subsystems A and B which are interdependent and may or may
not overlap. Subsystem A drives subsystem B with signal $S_A$ and
subsystem B drives subsystem A with signal $S_B$. The response
system 1500 produces a signal $S_B^*$ which is to be synchronized
with a signal $S_A$ produced in the drive system 1400. The sub-
ystem B of drive system 1400 transmits a drive signal $S_B$ to the
response system 1500. The response system 1500 includes two
subsystems A' and B" that are cascade connected. As in the
single stage subsystems discussed earlier (see FIG. 1),
subsystems A' and B" are duplicates of subsystems A and B,
respectively, which have all-negative sub-Lyapunov exponents.

The subsystem A' receives the drive signal $S_B$ and provides a
response signal $S_A'$ to the subsystem B". The subsystem B" in
turn produces signal $S_B^*$ in synchronization with signal $S_B$.
Unlike the single stage synchronization systems discussed earlier
(see FIG. 1), in the cascade system shown in FIG. 4, the same
signal $S_B$ which the response system 1500 synchronizes with
respect to is also used to drive the response system 1500. In
the single stage synchronization system 10 of FIG. 1, the
synchronized signal $S_0$ may be different than the drive signal $S_d$.

The response system 1500 with the cascaded subsystems $A'$ and
$B''$ is not only capable of producing the signal $S_{a''}$ synchronized
with the signal $S_b$ but is also capable of producing the signal $S_{a'}$
which is in synchronization with the signal $S_{a'}$. Because the
signal $S_{a''}$ can be compared to the signal $S_b$, the fact of syn-
chronization can be clearly determined allowing those on the
response system side to rely on the synchronization of the $S_b$ and
$S_{a''}$ signals in concluding that signal $S_{a'}$ is in synchronization
with signal $S_{a'}$.

Because of the nature of nonlinear dynamical systems driven
in the chaotic regime, properties of one chaotic system do not
necessarily carry over to another chaotic system. Nevertheless,
the prior art applies to any chaotic system in general, so long
as the chaotic system includes at least two stable subsystems.

The two response signals or outputs $S_{a'}$ and $S_{a''}$ are produced
as follows. The first subsystem $A'$ accepts the input signal $S_b$
and produces its response signal $S_{a'}$ in synchronization with its
counterpart ($S_{a'}$) in the drive system 1400. The second subsystem
$B''$ is driven by signal $S_{a'}$ from the first subsystem $A'$. The
second subsystem response $S_{a''}$ produces signal $S_{a''}$ in synchroniza-
tion with its counterpart $S_b$ in the drive system 1400, which in
this case is the original drive signal $S_b$ coming from the element B. The subsystems A' and B" are selected so that all of the essential elements of the drive system 1400 that are not present in the first subsystem A' are present in the second subsystem B" and vice-versa. In other words, the logical union of subsystems A' and B" includes all of the essential elements of the drive system 1400.

It is to be noted that each subsystem A' and B" in the response system 1500 is driven by a signal which supplies information in the drive system 1400 which is lacking in the drive subsystem. Thus, subsystem A' in the response system 1500 is driven by the same signal $S_b$ that drives subsystem A in the drive system 1400. Subsystem B" in the response system 1500 is driven by signal $S_A$, produced by subsystem A', just as subsystem B in the drive system 1400 is driven by signal $S_A$ produced by subsystem A.

As discussed earlier, subsystems A, A', B and B" must have all-negative sub-Lyapunov exponents. In other words, subsystems A, A', B and B" are stable subsystems.

The same principles discussed above concerning cascaded systems with 2 subsystems apply equally well to cascaded systems with more than 2 subsystems. In particular, each of the cascaded subsystems in the response system 1500 is a duplicate of a stable subsystem in the drive system 1400. Each subsystem in the
response system 1500 is driven by a signal which supplies
information from the complete system that is lacking in the drive
subsystem, in particular, by a signal corresponding to the signal
which drives the corresponding subsystem in the drive system
1400.

To understand the theory behind the cascaded system of FIG.
4 it is necessary to build on the previous discussion of
equations 1-3. Once the first subsystem of the response system
1500 is created a second system is created, say modeled by the
set of differential equations r=a(r,s) and s=b(r,s), where r and
s are subsets of variables of u in the same way that v and w are
subsets of variables of u. The r variables are the drives for
the second subsystem just as the v variables were for the first
subsystem. The functions a and b are the corresponding vector
field components. If this second subsystem is a stable subsystem
(See Pecora et al., Synchronization in Chaotic Systems, Physical
Review Letters, Vol. 64, No. 8, February 1990 and Pecora et al.,
44, No. 4, August 1991, both incorporated by reference herein,
for a discussion of how to determine whether stability exists),
the s variables synchronize with their corresponding variables in
the first system and with the drive signal. This then provides a
signal in synchronization with the input drive (one or more of
the variables).
For any two dynamical systems to become synchronized, they must start in the same basin of attraction. That is, their starting points (initial conditions) must be in the same set of points which will converge to the same attractor. Since many dynamical systems can have more than one attractor, it is possible for two such systems to start in different basins.

If the response system 1500 has somewhat different parameters than the drive system 1400, the synchronized signals will not be exactly equal and in general will have a difference which at small parameter changes will be proportional to the derivative of the vector fields with respect to the parameters. As discussed below, this effect along with others in the dynamical system allows communication using signals from nonlinear systems, including chaotic ones.

The details of cascaded synchronized systems and the circuit design, construction, and operation thereof will now be discussed.

FIG. 5 functionally illustrates an example of a cascaded system. It includes a drive system 800 which includes elements $X_1, X_2, X_3,$ and $X_4$ characterized by state variables $x_1, x_2, x_3$ and $x_4,$ respectively, (FIG. 6). Element $X_4$ constitutes subsystem $A,$ and elements $X_1, X_2, X_3$ constitutes subsystem $B.$ Both subsystems $A$ and $B$ are stable, that is they have all negative sub-Lyapunov exponents. Subsystem $A$ drives subsystem $B$ with signal $S_A$ and
subsystem B drives subsystem A with signal $S_b$. The response system 900 is a cascade of two subsystems $A'$ and $B''$ where the first subsystem system $A'$ includes a single element $X_4$, and the second subsystem $B''$ includes three elements $X_1$, $X_2$, and $X_3$. The first subsystem $A'$ is a duplicate of the subsystem $A$ in the drive system 800 and the second subsystem $B''$ is a duplicate of the subsystem $B$ in the drive system 800. The drive system 800 drives the first subsystem $A'$ with signal $S_b$ and the first subsystem $A'$ drives the second subsystem $B''$ with signal $S_A'$. The second subsystem $B''$ produces an output signal $S_{b''}$ in synchronization with drive signal $S_b$.

The operation of the elements in this example is modeled by the following equations:

\[
dx_1/dt = -\alpha_1[\beta_1A_1x_1 - \gamma_1x_2 + x_3 - x_4 + g_1(x_4) + \delta x_1], \tag{11}\n\]
\[
dx_2/dt = -\alpha_2(x_1 + \delta x_2), \tag{12}\n\]
\[
dx_3/dt = -\alpha_3(x_2 + \delta x_3), \tag{13}\n\]
\[
dx_4/dt = -\alpha_4((-\beta_4/R_4)x_1 + \gamma_4A_3x_4 + g_2(x_4)), \tag{14}\n\]
\[
dx''_1/dt = -\alpha_1[\beta_1A_1'x''_1 - \gamma_1x''_2 + x''_3 - x''_4 + g_1(x''_4) + \delta x''_1], \tag{15}\n\]
\[
dx''_2/dt = -\alpha_2(x''_1 + \delta x''_2), \tag{16}\n\]
\[
dx''_3/dt = -\alpha_3(x''_2 + \delta x''_3), \tag{17}\n\]
\[
dx''_4/dt = -\alpha_4((-\beta_4/R_4)x_1 + \gamma_4A_3'x''_4 + g_2(x''_4)), \tag{18}\n\]

where the $g_1$ and $g_2$ functions are defined as:

\[
g_1(x) = \beta_5(|x-2.5|-|x+2.5|), \tag{19}\n\]
\[
g_2(x) = \beta_6x + \gamma_6(|x-1.3|-|x+1.3|) + \varepsilon(|x-2.6|-|x+2.6|) \tag{20}\n\]
and the constants are $\alpha_1 = 1098$, $\alpha_2 = 10980$, $\alpha_3 = 4972$, $\alpha_4 = 10980$,
$\beta_1 = 1.466$, $\gamma_1 = 2.466$, $\beta_4 = 10^5$, $\gamma_4 = 0.5$, $\beta_5 = 0.5$, $\beta_6 = 0.5$, $\gamma_6 = 0.164$, and
$\varepsilon = 0.361$. The constant $\delta$, set at 0.2, is a phenomenological
damping constant used to account for leakage current in the
capacitors. Its value was set to make the stability of eqns.
(11)-(20) match the stability of the actual circuit. $A_1$ and $A_4$
are variable parameters normally set at 1.0.

As $R_v$ is decreased from 50,000 ohms to 46,000 ohms, the
circuit goes from a limit cycle through a period doubling to a
one-well chaotic attractor to a two-well chaotic attractor. With
$R_v$ held constant the drive system 800 and response system 900 can
produce a number of synchronized signals with the output $S_a$ of
the element $B^n$ being used to confirm synchronicity as previously
discussed. If $R_v$ is varied information can be transferred.

FIGS. 6-9 illustrate the circuit details of an example of a
system of FIG. 5 where multiple synchronized signals can be
produced and synchronization verified. FIG. 6 depicts the
details of the drive system 800. This circuit 800 includes the
following particular circuit elements:

Resistor $R_1 = 100k\Omega$  Resistor $R_{11} = 221k\Omega$
Resistor $R_2 = 100k\Omega$  Resistor $R_{12} = R_v$
Resistor $R_3 = 100k\Omega$  Resistor $R_{13} = 100k\Omega$
Resistor $R_4 = 100k\Omega$  Resistor $R_{14} = 200k\Omega$
Resistor $R_5 = 68.2k\Omega$  Resistor $R_{15} = 100k\Omega$
Resistor R6 = 100kΩ
Resistor R7 = 100kΩ
Resistor R8 = 68.2kΩ
Resistor R9 = 1MΩ
Resistor R10 = 100kΩ
Capacitor C1 = 910pf
Capacitor C2 = 910pf
Capacitor C3 = 910pf
Capacitor C4 = 910pf.

R_v is selected from among 47.8kΩ and 46.9kΩ with 47.8kΩ preferable. Resistor tolerances are preferably 1% and all capacitors are preferably 5% mica capacitors. The system also includes operational amplifiers 01, 02, 03, 04, 05, 06 and 07 all of which are 741 type amplifiers and diode DO which is an IN485B type. The circuit details of the functions g1(x) (eqn. 19) and g2(x) (eqn. 20) are depicted in the circuit diagrams of FIGS. 7 and 8, respectively.

Returning now to the example shown in FIG. 6, if one cuts the circuit at points a and b, the resulting systems A and B are stable. Subsystem B consisting of X_1, X_2, and X_3 can be driven with the S_a signal from the full system. Subsystem A consisting of X_4 may be driven with the S_b signal from the full circuit. When driving the B subsystem including elements x_1, x_2, and x_3, it does not actually matter whether the S_a driving signal is coming from the full circuit or from an A (or A') subsystem synchronized to the full circuit. Conversely, when driving the A subsystem,
it does not actually matter whether the $S_i$ driving signal is
coming from the full circuit or from a B or B" subsystem
synchronized to the full circuit. This arrangement, in which the
stable subsystems are driven by signals from subsystems and not
necessarily the full circuit, is called "cascaded synch-
ronization".

FIG. 7 depicts a circuit with response $g_1(x)$ (eqn. 19). In
this circuit the resistors $R=10\, k\Omega$, the operational amplifiers O8,
O9, O10 and O11 are 741 types and the diodes D1, D2, D3 and D4
are preferably type IN485B.

FIG. 8 depicts a circuit with response $g_2(x)$ (eqn. 20) where
operational amplifiers O12 and O13 are 741 type amplifiers and
Resistors $R_{21} = 27.4\, k\Omega$  
Resistors $R_{29} = 50.1\, \Omega$
Resistors $R_{22} = 27.4\, k\Omega$  
Resistors $R_{30} = 50.1\, \Omega$
Resistors $R_{23} = 49.9\, k\Omega$  
Resistors $R_{31} = 50.1\, \Omega$
Resistors $R_{24} = 49.9\, k\Omega$  
Resistors $R_{32} = 50.1\, \Omega$
Resistors $R_{25} = 200\, k\Omega$  
Resistors $R_{33} = 20\, k\Omega$
Resistors $R_{26} = 200\, k\Omega$  
Resistors $R_{34} = 178\, k\Omega$
Resistors $R_{27} = 825\, k\Omega$  
Resistors $R_{35} = 156.2\, k\Omega$
Resistors $R_{28} = 825\, k\Omega$  
Resistors $R_{36} = 100\, k\Omega$
Diodes D5, D6, D7 and D8 are type IN485B.

FIG. 9 depicts the circuit details of the response system
900 of FIG. 5. In this circuit 900 the resistor, capacitor,
amplifier and function components are the same as previously
discussed regarding FIGS. 6-8.

In FIG. 6 any of the nodes can be used as the source of the signals to be synchronized. However, the drive signal must come from a particular cut point as discussed above.

The above discussion of separated system synchronization is performed with electronic hardware components or other equivalent devices. Using the same concepts, systems can also be synchronized using software.

FIG. 10 shows a filtered cascaded synchronized nonlinear system having a transmitter 1100 and a receiver 1200. The transmitter 1100 includes subsystems A and B which are independent and may or may not overlap. Subsystems A and B each contain 1 or more variables, that is, each subsystem is at least 1-dimensional but may contain more than 1 dimension. Neither subsystem A nor subsystem B is contained within the other subsystem. At least part of subsystem A is external to subsystem B and at least part of subsystem B is external to subsystem A. Subsystem A drives subsystem B with signal $S_A$ and subsystem B drives subsystem A with signal $S_B$. The signal $S_B$ is the input to filter 1110, and the output of filter 1110 is the signal $S_f$. The subtractor 1120 subtracts the filter output signal $S_f$ from $S_B$ to produce the broadcast signal $S_t$, which is transmitted to the receiver 1200.

The receiver 1200 is responsive to the broadcast signal $S_t$. 
and produces an output signal \( S_{g''} \) in synchronization with signal \( S_g \). The receiver 1200 consists of subsystem \( A' \) which is a duplicate of \( A \) and \( B'' \) which is a duplicate of \( B \). The broadcast signal \( S_t \) is used as one input to the adder 1220. The output of the adder 1220 is the signal \( S_d \). The signal \( S_d \) is used to drive the subsystem \( A' \), and the output signal \( S_{A'} \) from subsystem \( A' \) is used to drive the subsystem \( B'' \). Subsystem \( B'' \) does not directly drive subsystem \( A' \), and the sub-Lyapunov exponents (as defined in US patents 5,379,346 and 5,245,660) for subsystems \( A' \) and \( B'' \) are all negative.

The signal \( S_{g''} \) from subsystem \( B'' \) is used as an input for the filter 1210, which is identical to the filter 1110 in the transmitter. The filter 1210 produces an output signal \( S_g \) which is used as an input for the adder 1220.

When the receiver 1200 is synchronized to the transmitter 1100, then the signals in subsystem \( A' \) reproduce the signals in subsystem \( A \) and the signals in subsystem \( B'' \) reproduce the signals in subsystem \( A \). If the subsystems \( A' \) and \( B'' \) and the filter 1210 are not exact replicas of the subsystems \( A,B \) and the filter 1100 (which will be the case in an electronic circuit implementation of the present invention), then the signals in \( A' \) and \( B'' \) can be made arbitrarily close to the signals in \( A \) and \( B \) by making the differences between \( A \) and \( A' \), \( B \) and \( B'' \), and filters 1110 and 1210 arbitrarily small.
In order to determine if the receiver 1200 will synchronize to the transmitter 1100, it is necessary to determine the stability of the receiver 1200 in the synchronized state. Techniques for determining the stability of such a system are well known; see, for example, J. M. T Thompson and H. B. Stewart, "Nonlinear Dynamics and Chaos", (Wiley, New York, 1986) or F. C. Moon, "Chaotic Vibrations", (Wiley, New York, 1987).

The synchronization of the receiver 1200 to the transmitter 1100 may be confirmed by comparing the receiver output signal $S_g$, to the receiver driving signal $S_d$. When the receiver is synchronized to the transmitter, then signal $S_g$ will match signal $S_d$.

FIG. 11 shows a filtered cascaded synchronized nonlinear system having a transmitter 2100 and a receiver 2200 when the nonlinear systems are nonautonomous, that is, they have a periodic forcing part $F$ (2130). The description of the transmitter 2100 is the same as the description of the transmitter 1100 in FIG. 10 except that the periodic forcing source $F$ provides periodic forcing signals $F_A$ and $F_B$ to subsystems $A$ and $B$. Either $F_A$ or $F_B$ may be zero, but they may not both be zero.

The receiver 2200 in FIG. 11 contains a periodic forcing source $F'$ (2240) which provides the periodic forcing signal $F'_A$ to subsystem $A'$ and periodic forcing signal $F'_B$ to subsystem $B'$. 
If $F_A$ in transmitter 2100 is zero, then $F'_A$ in receiver 2200 is zero, and if $F_b$ in the transmitter 2100 is zero, then $F'_b$ in receiver 2200 is zero.

The receiver 2200 in FIG. 11 operates in the same manner as the receiver 1200 in FIG. 10, except that it is necessary to match the phase of the periodic forcing source $F'$ (2240) in the receiver 2200 to the phase of the periodic forcing source $F$ (2130) in the transmitter 2100. The receiver 2200 contains a phase control system 2230 responsive to signals $S_b$, and $S_d$. The phase control system 2230 generates an error signal $\Delta$ proportional to the phase difference between $F$ and $F'$. The periodic forcing source $F'$ uses the error signal $\Delta$ to match the phase of $F'$ to the phase of $F$. The procedures for producing the error signal $\Delta$ are described in U.S. Patent Application Serial No. 08/267,696 (Navy Case No. 75,496), entitled: "SYNCHRONIZATION OF NONAUTONOMOUS CHAOTIC SYSTEMS", filed June 29, 1994, Inventors: Thomas L. Carroll et al.

The systems in FIGS. 10 and 11 may be any nonlinear dynamical system or combination of systems, provided that they may be subdivided into stable subsystems. The systems may be electronic circuits, they may be sets of differential equations or recursion relations (maps) to be solved on a computer, they may be implemented in digital signal processing systems or other physical or electronic systems, or they may be any other physical
system that can be broken into stable subsystems. The filters, 
adders and subtracters may also be electronic devices or they may 
be implemented as computer algorithms. It is also possible for 
part of the system to be of one type (such as a computer 
algorithm) and the other part of the system to be of some other 
type (such as an electronic circuit).

Figure 12-18 show how the present invention may be built as 
an electronic circuit.

The circuit details of an electronic example of a chaotic 
circuit 3000 are shown in FIGS. 12, 13 and 14. This circuit 3000 
includes the following particular circuit elements:

13  Resistor R1 = 10kΩ  Resistor R2 = 39.2kΩ
14  Resistor R3 = 10kΩ  Resistor R4 = 10kΩ
15  Resistor R5 = 10kΩ  Resistor R6 = 10kΩ
16  Resistor R7 = 100kΩ Resistor R8 = 1MΩ
17  Resistor R9 = 1MΩ  Resistor R10 = 100kΩ
18  Resistor R11 = 1MΩ  Resistor R12 = 100kΩ
19  Resistor R13 = 100kΩ Resistor R14 = 100kΩ
20  Resistor R15 = 5.2kΩ  Resistor R16 = 100kΩ
21  Resistor R17 = 100kΩ Resistor R18 = 1MΩ
22  Capacitor C1 = 1nF  Capacitor C2 = 1nF
23  Capacitor C3 = 1nF
Resistor tolerances are preferably 1% or better and all capacitors are preferably 5% mica capacitors. The system also includes operational amplifiers Op1, Op2, Op3, Op4, Op5, Op6, Op7, Op8 and Op9, all of which are 741 type amplifiers.

The circuit details of the circuit g of FIG. 12 is depicted in the circuit diagram of Figure 13, having the following particular elements:

Resistor R19 = 100kΩ  Resistor R20 = 100kΩ Resistor
R21 = 100kΩ  Resistor R22 = 100kΩ Resistor
R23 = 680kΩ  Resistor R24 = 2MΩ Resistor
R25 = 680kΩ  Resistor R26 = 2MΩ Resistor
R27 = 100kΩ
Potentiometer P1 = 20kΩ  Potentiometer P2 = 50kΩ
Potentiometer P3 = 20kΩ  Potentiometer P4 = 50kΩ
Potentiometer P5 = 20kΩ in parallel with a 100Ω
resistor (not shown)  parallel with a 100Ω resistor (not shown)
Potentiometer P6 = 20kΩ in parallel with a 100Ω
Potentiometer P7 = 20kΩ in parallel with a 100Ω
resistor (not shown)  resistor (not shown)

Resistor tolerances are preferably 1% or better. The system
also includes operational amplifier Op10 which is a type 741, and
diodes D1, D2, D3, and D4 which are of type 1N485B. As explained
further below, the potentiometers $P_1$-$P_8$ are used to match
different circuits $g$ to each other.

The circuit details of the circuit $f$ of FIG. 12 are depicted
in the circuit diagram of FIG. 14, having the following
particular elements:

Resistor $R_{28} = 10k\Omega$  
Resistor $R_{29} = 10k\Omega$

Resistor $R_{30} = 490k\Omega$  
Resistor $R_{31} = 490k\Omega$

Resistor $R_{32} = 50k\Omega$  
Resistor $R_{33} = 50k\Omega$

Resistor $R_{34} = 20k\Omega$  
Resistor $R_{35} = 100k\Omega$

Resistor $R_{36} = 100k\Omega$  
Resistor $R_{37} = 100k\Omega$.

Resistor tolerances are preferably 1% or better. The system
also includes operational amplifiers Op11 and Op12 which are of
type 741, and diodes D5, and D6 which are of type 1N485B.

The circuit shown in FIGS. 12-14 is modeled by the following
equations:

$$\frac{dx}{dt} = B[y-z] \quad (21)$$

$$\frac{dy}{dt} = B[-\Gamma_y y - g(x) + a \cdot \cos(\omega_1 t)] \quad (22)$$
\[ \frac{dz}{dt} = \beta [f(x) - \Gamma_{x} \cdot z] \]
\[ g(x) = -3.8 + 0.5 \times (|x+2.6| + |x-2.6| + |x+1.2| + |x-1.2|) \]
\[ f(x) = 0.5 \times x + |x-1| + |x+1|, \]

where \( \alpha = 2.0, \Gamma_{y} = 0.2, \Gamma_{z} = 0.1 \), the time factor \( \beta = 10^{4} \text{/sec} \), and the angular frequency \( \omega = 2\pi f_{t} \), where the transmitter forcing frequency \( f_{t} = 769 \text{Hz} \). The cosine term in Equation (22) is provided by a signal \( S_{t} \) supplied by an HP 3300A function generator 2130. The functions \( g(x) \) (equation (24)) and \( f(x) \) (equation (25)) are piecewise linear functions produced by the circuits shown in FIGS. 13 and 14, respectively. Equations (21-25) model the B subsystem (FIG. 11) of the transmitter 2100, and equation (23) models the A subsystem (FIG. 11) of the transmitter 2100.

This circuit is designed so that it is possible to create a synchronizing subsystem. Equations (21-22) (with \( z \) treated as a parameter) constitute the well known 1-well Duffing equations. For the parameter settings used here, the behavior of such a subsystem is periodic, indicating that the largest Lyapunov exponent for this subsystem is zero. Equation (23) was added to the Duffing system of Equations 21-22 to provide an instability for certain values of \( x \), thereby leading to chaos. If the feedback loop between equations (21) and (23) were not completed, i.e. if the subsystem of equations (21-22) were not dependent on
the z-variable produced by the subsystem of equation (23), or if
the subsystem of equation (23) were not dependent on the x-
variable produced by the subsystem of equations (21-22), then the
system of variables x, y and z would be periodic. In other words
variables x, y and z would each be periodic or a fixed point.
The largest conditional Lyapunov exponent with respect to the
signal $S_1$ would be less than or equal to zero. The feedback loop
between equations (21) and (23) can be disconnected by cutting
the system at node $T_1$ and grounding the input (x) to the circuit
f, or by cutting the system at node $T_2$. Such disconnection would
remove the dependence of equation (23) on the variable x, or the
dependence of equation (21) on the variable z, respectively.

The conditional Lyapunov exponents for the transmitter
system of Figures 12-14 calculated from equations (21-25) with
the above parameters are $284 \text{s}^{-1}$, $-1433 \text{s}^{-1}$ and $-1854 \text{s}^{-1}$. The
sinusoidal forcing term $\cos(\omega_1 t)$ of equation (22) is treated as
a parameter in this calculation, so the zero exponent attribu-
table to signal $S_1$ does not show up here. Since one of the
conditional Lyapunov exponents is positive, therefore the system
modeled by equations (21-25) and shown in FIGS. 12-14 operates in
the chaotic regime.

In FIG. 15, $R_{38}=10,000 \text{ ohms}$, $R_{39}=10,000 \text{ ohms}$, $R_{40}=10,000$
$\text{ ohms}$, $R_{41}=10,000 \text{ ohms}$, $R_{42}=10,000 \text{ ohms}$, $R_{43}=5000 \text{ ohms}$, $R_{44}$
$=10,000 \text{ ohms}$, $R_{45}=10,000 \text{ ohms}$ and $R_{46}=10,000 \text{ ohms}$. In FIG. 16,
resistor R47=31,380 ohms and C4 = 10^{-6} F. All operational amplifiers are type 741.

The transmitter 6000 in FIG. 15 may be described by the following differential equations when the filter 4000 of FIG. 16 is used (equations 26-30):

\[
\frac{dx}{dt} = \beta(y-z) \tag{26}
\]

\[
\frac{dy}{dt} = \beta\left( -\Gamma_y y - g(x) + \alpha \cos(\omega t) + A \right) \tag{27}
\]

\[
\frac{dz}{dt} = \beta\left( f(x) - \Gamma_z z \right) \tag{28}
\]

\[
\frac{du}{dt} = \frac{dx}{dt} - u/RC \tag{29}
\]

\[
s_t = x - u \tag{30}
\]

\[
\frac{dv}{dt} = \frac{dx''}{dt} - v/RC \tag{31}
\]

\[
s_d = s_t + v \tag{32}
\]

\[
\frac{dz'}{dt} = \beta\left( f(s_d) - \Gamma_z z' \right) \tag{33}
\]

\[
\frac{dx''}{dt} = \beta\left( y'' - z' \right) \tag{34}
\]

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\[
dy''/dt = \beta \left( -\Gamma_y y'' - g(x') + \alpha \cos(\omega t + \phi_r) + A \right)
\]

(35)

\[
\alpha = 1.9, \quad \Gamma_y = 0.2, \quad \Gamma_z = 0.1, \quad A = 0, \quad \beta = 10^4 \text{s}^{-1}, \quad \omega = 2\pi \times 780 \text{ Hz}.
\]

The receiver 7000 (FIG. 15) is shown in FIGS. 17 and 18. The receiver 7000 consists of a cascaded response circuit 5100 (FIG. 17) and a phase control circuit 5200 (FIG. 18), along with the filter 4000 (FIG. 16) and the adder formed by operational amplifier op14 (FIG. 15). The cascaded response circuit 5100 is described by equations 33-35. In FIG. 17, R51=10,000 ohms, R52=39,200 ohms, R53=10,000 ohms, R54=10,000 ohms, R55=10,000 ohms, R56=10,000 ohms, R57=100,000 ohms, R58=1,000,000 ohms, R59=1,000,000 ohms, R60=100,000 ohms, R61=1,000,000 ohms, R62=100,000 ohms, R63=100,000 ohms, R64=100,000 ohms, R65=5,200 ohms, R66=100,000 ohms, R67=100,000 ohms, R68=1,000,000 ohms, C6=1nF, C7=1 nF, C8=1 nF. The phase control circuit 5200 is shown in FIG. 18.

Referring now to FIG. 18, the details of a phase-detector/controller 5200 is shown. This phase-detector/controller 5200 is responsive to the receiver drive signal $S_d$ and to the receiver output signal $S_b$, for producing a correction signal $\Delta$ responsive to the phase difference between the transmitter forcing signal $F_1$ and the receiver forcing signal $F'$. 

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The strobe input signal $S_{g}$ generated by the response system 5100 is applied to an amplifier 5220 with a high gain such as a 741 type amplifier with a gain of -100. The output of the amplifier 5220 is applied to a conventional comparator 5221, such as an AD 790. The comparator 5221 produces an output when the input signal $S_{g}$ is less than zero. The positive-going signal from the comparator 5221 triggers a conventional Schmitt trigger circuit 5222, such as an SN 74121 monostable multi-vibrator. As a result, the Schmitt trigger circuit 5222 produces a pulse of about 1 microsecond (μs) duration when the strobe input signal $S_{g}$ crosses 0 in the negative direction. A difference device 5225, such as a 741 operational amplifier, generates the difference signal $S_{d}-S_{g}$ between the receiver drive signal $S_{d}$ and the strobe signal $S_{g}$. The difference signal $S_{d}-S_{g}$ produced by the difference device 5225 is applied to the signal input of a conventional sample and hold circuit 5226, such as an LM 398, and the output of the Schmitt trigger circuit 5222 is applied to the logic input of the sample and hold circuit 5226. In other words, the difference $S_{d}-S_{g}$ between the receiver drive signal $S_{d}$ and the strobe signal $S_{g}$ is applied to the sample and hold circuit 5226, which holds the difference seen when the strobe signal $S_{g}$ passes through 0 going negative.

The sampled signal produced by the sample and hold circuit 5226 is applied to the negative terminal of a 741 type amplifier.
5228, and the correction signal $\Delta$ is applied to the positive
terminal of the amplifier 5228 thereby providing negative feed-
back. The amplifier 5228 thus accumulates the sampled difference
signal and the correction signal $\Delta$. The correction signal $\Delta$ is
produced by a conventional integrator 5229, having a long time
constant preferably of about 10 seconds (s), such as type 741
amplifier with a mica capacitor used for feedback, that averages
the output of the amplifier 5228. In other words, the output of
the sample and hold circuit 5226 is applied to an integrator to
produce a correction signal $\Delta$ proportional to the average phase
difference transmitter forcing signal $F$ and receiver forcing
signal $F'$.

Referring back to FIG. 11, a signal generator 2130 respon-
sive to the correction signal $\Delta$ produced by the phase-
detector/controller 2230 of FIG. 18 which is itself responsive to
a receive drive signal $S_4$ produced by the circuit shown in FIG.
15 preferably utilizes an HP 8116A function generator (not
shown). Such a signal generator 2130 multiplies the correction
signal $\Delta$ produced by the phase-detector/controller 5200 of FIG.
18 by a factor of 1/100 and uses the resulting signal to modulate
the frequency of the HP8116A function generator.

The transmitter 2100 of eqs. 26-30 and the response system
2200 of eqs. 31-35 are not identical. The transmitter 2100 and
receiver 2200 are effectively identical when they are
synchronized. It is necessary for the synchronized state to be stable. A Lyapunov exponent calculation from the equations shows that the largest Lyapunov exponent in the response system is -319 s⁻¹, indicating that the response system is stable.

An alternate filter 4000 is shown in FIG. 19. The resistor values were given by RAᵢᵢ=:

<table>
<thead>
<tr>
<th>i=1</th>
<th>j=1</th>
<th>j=2</th>
<th>j=3</th>
</tr>
</thead>
<tbody>
<tr>
<td>10</td>
<td>204,000 Ω</td>
<td>408,000 Ω</td>
<td>1026 Ω</td>
</tr>
<tr>
<td>11</td>
<td>102,000 Ω</td>
<td>204,000 Ω</td>
<td>513 Ω</td>
</tr>
<tr>
<td>12</td>
<td>68,000 Ω</td>
<td>136,000 Ω</td>
<td>342 Ω</td>
</tr>
<tr>
<td>13</td>
<td>51,000 Ω</td>
<td>102,000 Ω</td>
<td>256 Ω</td>
</tr>
<tr>
<td>14</td>
<td>40,800 Ω</td>
<td>82,000 Ω</td>
<td>205 Ω</td>
</tr>
</tbody>
</table>

and the capacitor CA was 10⁻⁸F. This filter was described by the equations:

\[ w = \frac{dx}{dt} \]  
\[ \frac{u_i}{dt} = -(2/R_{i2}C)u_i - (1/R_{i2}C)(1/(R_{i3}C +1/R_{i1}C)v_i(1/R_{i1}C) \]  
\[ \frac{dv_i}{dt} = u_i \]  
\[ s_t = x + \Sigma v_i \]  
\[ s_d = s_t - \Sigma r_i \]  
\[ \frac{dq_i}{dt} = -(2/R_{i2}C)q_i - (1/R_{i2}C)(1/(R_{i3}C) + 1/(R_{i1}C))r_i \]  
\[ - (1/R_{i1}C)\frac{dx''}{dt} \]  
\[ \frac{dr_i}{dt} = q_i \]
where the resistor values are given by the above table. The largest Lyapunov exponent for the response system when alternate filter 4000 was used was found to be \(-10 \, \text{s}^{-1}\), indicating that the response system was stable. The transmitter filter output signal \(v\) was added to the drive output signal \(x\) in equation 39 because the filter of equations 37 and 38 inverted the input signal. For the same reason, the receiver filter output signal was subtracted from the transmitted signal in equation 40.

FIG. 20(a) shows the power spectrum of the drive system output signal \(x\) from equation 26, while FIG. 20(b) shows the power spectrum of the transmitted signal \(s_t\) described in equation 30, demonstrating the change in the power spectrum caused by the filtering.

It may be shown that this technique also allows phase synchronization of the periodic forcing parts of nonautonomous synchronized nonlinear systems. The controller 5200 of FIG. 18 was used to control the phase of the response circuit periodic forcing to match that of the drive circuit. The controller 5200 generated a series of voltages that corresponded to the value of the response system output signal \(x^n\) when the input signal \(s_d\) crossed zero. If the drive and response circuits were synchronized, these voltages would all be zero. An integrator 5229 with a time constant of 1 s averaged the series of voltages to produce an error signal \(\Delta\), which was used to vary the
frequency of the response periodic forcing 2240 to bring the
phase into sync with the drive periodic forcing 2130.

FIG. 21 shows the periodic forcing $F''$ for the response
vs. the periodic forcing $F$ for the drive. There is some
fluctuation of the response phase and a constant phase offset
which is an artifact of the control circuit, but the basic
principle works. This demonstrates that the nonperiodic part of
the chaotic signal carries information about the phase of the
periodic part. Most of the phase fluctuation is believed to be
caused by component mismatch between the two circuits. There is
also a phase flip caused by a sign change in the filters.

Several authors have demonstrated communication between
cascaded chaotic circuits via parameter switching in the sending
Shang, Transmission of Digital Signals by Chaotic
Synchronization, International Journal of Bifurcations and Chaos,
vol. 2, p. 973 (1992)]. Parameter switching may also be used with
the filtered nonautonomous chaotic circuits. The forcing offset $A$
in eq. 27 was switched between $\pm 1.0$ V, and the parameter
switching was detected by monitoring the error signal $\Delta$
generated by the controller 5200.

FIG. 22 shows the offset signal $A$ as a time series and
the resulting error signal $\Delta$ coming from the response system
controller. As may be seen by the sharp edges on the error
signal transitions, the offset signal could be switched up to a factor of about 4 faster. The switching speed is limited by the time constant of the integrator that produces $A$, about 1 s for this system.

Autonomous nonlinear systems may also be synchronized when the driving signal is filtered. The Piecewise Linear Rossler (PLR) system is a nonlinear system that may be synchronized in a cascaded fashion (T. L. Carroll, "A simple circuit for demonstrating regular and synchronized chaos", American Journal of Physics, vol 63, #4, pp. 377-379, April 1995). A bandpass filter was used to isolate a large periodic component in the output of the PLR circuit and reduce its presence in the transmitted signal. Reducing the size of the periodic component reduced the power contained in the transmitted signal by a large amount, so that the transmitted signal could be sent with less power. The PLR system and the filter were described by the equations:

\[ \begin{align*}
    \frac{dx}{dt} &= -500(x + 10y + 20z) \\
    \frac{dy}{dt} &= -10^4(-x - 0.13y + 0.02y) \\
    \frac{dz}{dt} &= -10^4(z - g(x)) \\
    \frac{du}{dt} &= -800u - 5 \times 10^7v - 400(dy/dx) \\
    \frac{dv}{dt} &= u
\end{align*} \]
\[ Y_t = y + 1.5v \]  
\[ \frac{dw}{dt} = -800w - 5 \times 10^7 r - 400(dy'/dt) \]  
\[ \frac{dr}{dt} = w \]  
\[ Y_d = Y_t - 1.5r \]  
\[ \frac{dx'}{dt} = -500(x' + 10y' + 20z') \]  
\[ \frac{dy'}{dt} = -10^4(-x' - 0.13y_d + 0.02y') \]  
\[ \frac{dz}{dt} = -10^4(z' - g(x')) \]  
\[ g(x) = \begin{cases} 0 & \text{if } x < 3, \\ 15(x - 3) & \text{otherwise} \end{cases} \]

Equations 43 are the drive system and equations 44 and 45 are the drive system filter. Equations 48 are the response system and equations 46 and 47 are the response system filter. The transmitter filter output signal \( v \) was added to the drive output signal \( y \) in equation 45 because the filter of equations 44 inverted the input signal. For the same reason, the receiver filter output signal was subtracted from the transmitted signal in equation 47. Equation 49 is the nonlinear function \( g(x) \). Equations 43-49 form an embodiment of the present invention as a complete algorithm.

FIG. 23 shows the output signal (the \( y \) signal) from the
drive system of equations 43 (the solid line in FIG. 23). This
signal has a large periodic component, as may be seen in the
power spectrum of the y signal in FIG. 24. The bandpass filter
of equations 44 is tuned to this periodic component. The filter
output signal v is then subtracted from the drive system output
signal y to give the transmitted signal y_t. The transmitted
signal y_t is shown as a dotted line in FIG. 23. The power
spectrum of the transmitted signal y_t from equation 45 is shown
in FIG. 25. The numerical integration routine generated 20,000
point output time series of y and y_t were squared and integrated
to give an estimate of the power in each signal. The power in the
y signal was 155,371 (arbitrary units), while the power in the y_t
signal was 15,941 (arbitrary units). Filtering of the y signal to
produce the y_t signal reduced the power contained in the signal
by a factor of approximately 10, reducing the power that must be
transmitted.

The receiver is composed of equations 46, 47, 48 and 49.
The filter output signal r is subtracted from the transmitted
signal y_t to produce the receiver driving signal y_d (equation
47). The driving signal y_d is then used to drive the cascaded
response system of equations 48 to produce the response system
output signal y'. The derivative of the response system output
signal y' is used to drive the receiver filter of equations 46 to
produce the filter output signal r.
FIG. 26 shows the response system output signal $y'$ vs. the drive system output signal $y$ to demonstrate that the drive and response systems are indeed synchronized.

The filtered synchronized communications system may also be used to correct for the effects of filtering by the communications channel. FIG. 27 shows a transmitter 6100 which sends a signal to a receiver 6300 through a communications channel 6200. If the effect of the communications channel is to filter the signal $S_\phi$ with a filter of the form $(1-F)$ to produce a signal $S_t$, then the effect of the communications channel filtering may be removed by using a filter $F$ for filter 6320 in receiver 6300.

Therefore, what has been described in a preferred embodiment is a filtered cascaded synchronized nonlinear system which includes a nonlinear transmitter having stable first and second subsystems. The first subsystem produces a first transmitter signal for driving the second subsystem, and the second subsystem produces a second transmitter signal for driving the first subsystem. A first filter filters the second transmitter signal to produce a filter output signal. A subtractor subtracts the filter output signal from the second transmitter signal to produce a transmitter output signal which is transmitted to a nonlinear cascaded receiver. The receiver includes an adder for summing the received transmitter output signal with a receiver.
filter output signal to restore frequencies that were subtracted
from the second transmitter signal in order to produce a first
receiver drive signal. The receiver includes cascaded third and
fourth subsystems that are respective duplicates of the first and
second subsystems. The third subsystem is driven by the first
receiver drive signal to produce a first receiver signal in
synchronization with the first transmitter signal. The fourth
subsystem is driven by the first receiver signal to produce a
second receiver signal in synchronization with the second
transmitter signal. A second filter filters the second receiver
signal to produce the receiver filter output signal.

It should therefore readily be understood that many
modifications and variations of the present invention are
possible.
ABSTRACT

A filtered cascaded synchronized nonlinear system includes a nonlinear transmitter having stable first and second subsystems. The first subsystem produces a first transmitter signal for driving the second subsystem, and the second subsystem produces a second transmitter signal for driving the first subsystem. A first filter filters the second transmitter signal to produce a filter output signal. A subtractor subtracts the filter output signal from the second transmitter signal to produce a transmitter output signal which is transmitted to a nonlinear cascaded receiver. The receiver includes an adder for summing the received transmitter output signal with a receiver filter output signal to restore frequencies that were subtracted from the second transmitter signal in order to produce a first receiver drive signal. The receiver includes cascaded third and fourth subsystems that are respective duplicates of the first and second subsystems. The third subsystem is driven by the first receiver drive signal to produce a first receiver signal in synchronization with the first transmitter signal. The fourth subsystem is driven by the first receiver signal to produce a second receiver signal in synchronization with the second transmitter signal. A second filter filters the second receiver signal to produce the receiver filter output signal.
NONLINEAR DYNAMICAL SYSTEM

PRIOR ART

FIG. 1
PRIOR ART

FIG. 3
PRIOR ART

FIG. 8
FIG. 16

R47

C4

4000
FIG. 19

Alternate circuit 4000

$S_A$ in

RA1

R13

RA21

RA23

RA31

RA33

RA41

RA43

RA51

RA53

RA12

RA22

RA32

RA42

RA52