AD NUMBER

ADB023230

NEW LIMITATION CHANGE

TO

Approved for public release, distribution unlimited

FROM

Distribution: DoD and DoD Contractors only: Others to British Ministry of Defence via the appropriate channel.

AUTHORITY

E-mail 25 Apr 2001, DERA-DRIC

TR 77058

BR58353

UNLIMITED

R 20/4/99 MA

ROYAL AIRCRAFT ESTABLISHMENT

*

Technical Report 77058

April 1977

BEHAVIOUR OF A VORTEX SHEET SEPARATING FROM A SMOOTH SURFACE

by

J.H.B. Smith

*

UNLIMITED

1. DRIC Reference (to be added by DRIC)	2. Originator's Reference RAE TR 77058	3. Agency Reference N/A	4. Report Security Classification/Marking UNCLASSIFIED

5. DRIC Code for Originator 850100	6. Originator (Corporate Author) Name and Location Royal Aircraft Establishment, Farnborough, Hants, UK

5a. Sponsoring Agency's Code N/A	6a. Sponsoring Agency (Contract Authority) Name and Location N/A

7. Title Behaviour of a vortex sheet separating from a smooth surface

7a. (For Translations) Title in Foreign Language

7b. (For Conference Papers) Title, Place and Date of Conference

8. Author 1. Surname. Initials Smith, J.H.B.	9a. Author 2	9b. Authors 3, 4	10. Date April 1977	Pages 64	Refs. 12

11. Contract Number N/A	12. Period N/A	13. Project	14. Other Reference Nos. Aero 3415

15. Distribution statement
    (a) Controlled by — ▮▮▮▮▮▮▮▮▮▮
    (b) Special limitations (if any) —

16. Descriptors (Keywords)   (Descriptors marked * are selected from TEST)

    Vortices*.   Boundary-layer separation*.   Slender bodies*.

17. Abstract

    The behaviour of a vortex sheet embedded in an irrotational steady flow and springing from a separation line on a smooth body is studied in the neighbourhood of the separation line. It is shown first that the vortex sheet is tangential to the wall along the separation line. Then, under the assumptions that the body is slender and the separation line is highly swept, it is shown that the exponent which defines how rapidly the sheet departs from the body must take one of a set of discrete values. The smallest of these, corresponding to the most rapid departure of the sheet, implies an infinitely large adverse pressure gradient on the upstream side of the separation line. The next largest exponent avoids this. The implications for modelling separated flows are briefly discussed.

RAE Form A143

UDC 532.527 : 532.526.5

ROYAL AIRCRAFT ESTABLISHMENT

Technical Report 77058

Received for printing 26 April 1977

BEHAVIOUR OF A VORTEX SHEET SEPARATING FROM A SMOOTH SURFACE

by

J. H. B. Smith

SUMMARY

The behaviour of a vortex sheet embedded in an irrotational steady flow and springing from a separation line on a smooth body is studied in the neighbourhood of the separation line. It is shown first that the vortex sheet is tangential to the wall along the separation line. Then, under the assumptions that the body is slender and the separation line is highly swept, it is shown that the exponent which defines how rapidly the sheet departs from the body must take one of a set of discrete values. The smallest of these, corresp   ng to the most rapid departure of the sheet, implies an infinitely large adverse pressure gradient on the upstream side of the separation line. The next largest exponent avoids this. The implications for modelling separated flows are br fly discussed.

Departmental Reference: Aero 3415

2

LIST OF CONTENTS

1   INTRODUCTION

Classical aerodynamics is based on regarding the flow past a streamlined body at a small angle of incidence as inviscid and irrotational outside a thin region which comprises the boundary layer, surrounding the body, and the wake, extending downstream of it. The fluid in the boundary layer and wake is characterised by its acquisition of rotation through the action of viscous forces and, at high Reynolds numbers, by its turbulence. In calculating a first approximation to the external flow, displacement effects of the boundary layer and wake are neglected, while the net rotation in the wake of a lifting surface is represented by a planar vortex sheet whose strength is determined by the Kutta-Joukowski condition of finite velocity at the trailing edge.

When the angle of incidence is no longer small, the boundary layer may leave the surface of the body well upstream of the trailing edge. This is known as 'separation', although the process is not essentially different from the departure of the boundary layer from the trailing edge in the classical aerodynamic model. In the separated flow which results from boundary layer separation, rotation is present not only in the boundary layer and wake downstream of the body but also in the external flow which surrounds it. Modelling such flows presents difficulties of two kinds: practical, computational difficulties of representing the rotation, and conceptual, physical difficulties of accounting for its origin. In general, these difficulties are compounded by a degree of flow unsteadiness very much greater than that of a turbulent boundary layer upstream of separation. This unsteadiness leads the aircraft designer to avoid such flows as far as possible, so there is some justification for their neglect by the aircraft aerodynamicist. However, if a separation line is highly swept, the separated flow associated with it is frequently steady, particularly when the position of the separation line is fixed by a salient edge on the body. The prime example is the flow over slender wings at incidence, with the formation of coiled vortex sheets. Such a flow is eminently applicable.

Considerable success has been achieved in modelling flows with highly-swept separation lines along salient edges by a straightforward extension of the classical aerodynamic model. The rotation is still represented by a vortex sheet, the displacement effects of the separated flow are ignored, and a Kutta condition is applied at the salient edge; on the other hand, the location and form of the vortex sheet can no longer be assumed *a priori*. Most work so far has made use of the framework of slender-body theory to reduce the computational complexity, but it has recently been demonstrated[1] that the approach can be

4

carried through more generally, in particular for fully three-dimensional incompressible flow.

Highly-swept separation lines also arise on smoothly curved bodies, for instance on the noses of aircraft at high incidence, on guided weapons and on up-swept rear fuselages of aircraft. Parts of the leading edges of Concorde are appreciably rounded and the leading edge of the US Space Shuttle is well rounded. Elsewhere on slender wings highly-swept separation lines are found on the upper surface beneath the primary vortices and may arise on the shoulder of a deflected leading-edge flap.

It is therefore of considerable interest to attempt to extend the coiled vortex-sheet model of separation from a salient edge to model steady separation from a smooth surface. If separation originates at a pointed apex, as for instance on an ogival nose at high incidence, the separated flow can be represented by a coiled vortex sheet which takes a conical form at the apex; but if the most upstream point at which the flow separates is an ordinary point on the surface, as on an ogival nose at lower incidence, its initial form is not clear. Apart from this initial problem, we may expect, based on experience in calculating flows with salient edges, that the boundary conditions on the vortex sheet (that it is a stream surface with no discontinuity of pressure across it) will define the sheet completely once the separation line from which it springs has been specified. (There may be more than one solution for a given body, onset flow and separation line, as found in Ref 2, but each is well-defined by the conditions.) The position of the separation line must be supplied to the inviscid model; it might be obtained from experiment or through an iteration between the calculation of the inviscid flow and the calculation of the boundary layer on the body upstream of separation. The background to these introductory remarks is given at greater length in a recent review[3].

It is in the immediate neighbourhood of the separation line itself that the main differences in behaviour are likely to arise between separation from a salient edge and separation from a smooth surface. The present analytical study was undertaken in order to shed some light on this behaviour and to provide a guide for choosing appropriate local representations in subsequent numerical work. The results obtained define the possible types of behaviour of the vortex sheet close to the separation line and, through the associated pressure fields, raise questions concerning the eventual matching of the inviscid model to a boundary layer calculation.

In section 2 it is shown, with some minor assumptions, that the vortex sheet leaves the surface tangentially, ie that at almost all points of the separation line the tangent plane to the sheet is also a tangent plane to the body. The argument is elementary, but is is set down to provide a basis for what follows and to make explicit the assumptions involved.

For the rest of the study, the assumptions of slender-body theory are made. This provides a manageable analytic framework and is likely to be appropriate in most situations involving highly-swept separation lines. The method of analysis employed is described in section 3 and applied in detail in section 4 to the problem of a conical vortex sheet leaving a semi-infinite plane wall. It is shown that, if the distance of the sheet from the wall increases like y^n, where y is the distance along the wall away from the separation line, then the only values of n not exceeding 3 for which the vortex sheet boundary conditions can be satisfied are 3/2 and 5/2. The former is associated with an adverse pressure gradient upstream of the separation line which becomes infinitely large as the line is approached. For $n = 5/2$ the pressure gradient is not dominated by the local flow behaviour. In section 5 it is shown that the restriction to conical flow can be removed without altering the conclusions and in section 6 it is shown that the results also hold for a vortex sheet leaving a smoothly curved body. Section 7 includes a brief discussion of the results as they relate to modelling separated flow, both steady flow in three dimensions and time-dependent plane flow.

2   THE VORTEX SHEET LEAVES THE BODY TANGENTIALLY

In this section it is shown that, if vorticity is being shed from an ordinary point of a separation line on a smooth body into a vortex sheet which is embedded in a steady, irrotational homentropic flow, then the vortex sheet is tangential to the body at that point.

By an 'ordinary point' is meant a point at which the separation line has a single, continuously varying, tangent; and by a 'smooth body' is meant a body which, on the separation line, has a single, continuously varying, tangent plane. The assumption of steady flow is primarily for simplicity. 'Homentropic' means that the entropy is the same throughout the flow field. The assumption of homentropic, irrotational flow excludes the study of vortex sheets in flows with strong shock waves, and of vortex sheets bounding closed bubbles of fluid which are not penetrated by fluid from the free stream. These assumptions are likely to be valid for separation from a highly-swept separation line on a stationary body producing small disturbances in a uniform stream, but it is not necessary

6

to assume that disturbances are small or that the separation line is highly swept.

As a consequence of the assumption of irrotationality and the elimination of closed bubbles, the circulation of the vortex sheet can be defined. The circulation along a path which begins at a point on one side of the vortex sheet and ends at the adjacent point on the opposite side of the sheet, lying otherwise in the irrotational fluid, depends only on the choice of point on the sheet. If the point lies on the separation line at the base of the sheet, the circulation along such a path is defined (with some sign convention) as the circulation of the sheet at that position on the separation line. The notion of the shedding of vorticity can now be made more precise. It is taken that vorticity is being shed from a point on the separation line if and only if the circulation is varying with position on the separation line at this point and the point is not a stagnation point of the mean (or convective) flow. An example in which vorticity is not being shed is provided by the inviscid model of the flow past a combination of a lifting wing and a cylindrical fuselage: the plane vortex sheet representing the wake from the wing lies along the body side without being fed from the body. The present results do not relate to such a situation.

Following this explanation, the proof is very simple. We assume first that the sheet is not tangential to the body and show that vorticity is not being shed.

Since the body and the separation line are continuously curved, there are unique normals to the body and to the vortex sheet at the point considered. These normals are distinct, since we assumed the sheet is not tangential to the body, and the component of the fluid velocity along each vanishes, since both the sheet and the body are stream surfaces of a steady flow. Hence the fluid velocity on both sides of the sheet is perpendicular to both normals, ie it is parallel to the tangent to the separation line.

The second boundary condition on the sheet is that the pressure is continuous across it. For steady, homentropic, irrotational flow, equality of pressure implies equality of fluid speed (neglecting body forces), so the fluid velocity on the two sides of the separation line is either (a) zero, (b) equal in magnitude and opposite in direction, or (c) non-zero and identical. Possibilities (a) and (b) both correspond to zero mean velocity, ie to a stagnation point of the mean flow, so that vorticity is not being shed. This leaves possibility (c), according to which the fluid velocity is the same on both sides of the separation line. Hence the fluid particles which at one instant lie on a path which surrounds the vortex sheet, beginning and ending at adjacent points on opposite

sides of the separation line, at the next instant also lie on a path which surrounds the sheet and terminates at adjacent points a short distance downstream along the separation line. However, the circulation along such a path does not change with time, so the circulation of the vortex sheet is not varying with position along the separation line, and so vorticity is not being shed.

This shows that vorticity is not being shed if the sheet is not tangential to the body. On the other hand we can see that vorticity can be shed if the sheet is tangential to the body. It is convenient to distinguish between the two sides of the separation line, and consequently of the vortex sheet, referring to the side on which the sheet lies close to the surface as the downstream side. Then on the downstream side the velocity vector must again lie parallel to the separation line, but on the opposite, upstream, side the only restriction imposed on the direction of the velocity is that it must be parallel to the common tangent plane of the sheet and the body. The magnitudes of the velocity vectors must again be equal, but this allows there to be both a non-zero component of the mean velocity normal to the separation line and a non-zero component of the vorticity parallel to the separation line. Vorticity can therefore be shed and the circulation of the sheet can vary along the separation line.

A sketch of the configuration envisaged is shown in Fig 1. The portions of body surface and vortex sheet lying between two parallel cross-flow planes are represented, with the vortex sheet touching the body along the separation line. Streamlines of the inviscid flow on the wall and on the sheet are shown by solid lines. On the downstream side of the separation line, these are all nearly parallel to the separation line, whether they lie on the wall or on the vortex sheet. On the upstream side, the inviscid streamlines on the wall cross the separation line at an angle and continue on the vortex sheet. The skin-friction lines, or limiting streamlines of the viscous flow, are shown on the wall upstream of the separation line as broken lines. These deviate inwards from the curved inviscid streamlines, approach the separation line, but do not cross it. On the downstream side, a nearly parallel flow is envisaged in which the skin-friction lines on the wall are essentially parallel to the inviscid streamlines. The direction of the skin friction thus varies continuously on the wall and the velocity varies continuously on each side of the sheet. The boundary layer on the downstream side is not obliged to separate in this concept of the flow, for which support is provided in subsection 4.4.

3   A BASIS FOR FURTHER ANALYSIS

The previous section showed that, under fairly general assumptions, (steady flow, an 'open' type of separation and an absence of strong shock waves) a vortex sheet into which vorticity is being shed from a smooth body is tangential to the body surface along the separation line. To find out more about the form of the vortex sheet and the flow field near the separation line a more complete analytic framework is needed. Since, as indicated in the Introduction, flows with highly-swept separation lines are of considerable interest, it is natural to choose the framework of slender-body theory. This means we are concerned with a body of slowly varying cross-section, whose lateral dimensions are small compared with its length, with its longitudinal axis at a small inclination to a uniform stream whose Mach number is not large compared with unity. On such bodies extensive flow separations of open type (ie which are penetrated by upstream fluid) are observed, for which a vortex-sheet model is likely to be appropriate. It is to such a model that the results obtained will apply.

According to slender-body theory, the flow is described by a disturbance velocity potential which is the sum of two terms. The first is a function of the streamwise coordinate only, with a form which depends on the Mach number of the free stream. This function only gives rise to a disturbance velocity in the streamwise direction, and this velocity is uniform in each cross-flow plane, so it emerges that it plays no part in the present analysis. The second term is a harmonic function (a solution of Laplace's equation) in the cross flow plane, and is therefore obtainable as the real part of a complex analytic function of the complex coordinate in this plane. For a wholly attached flow, which is completely irrotational, this function is determined (to within an additive constant) in each cross-flow plane by the boundary conditions on the body surface and at infinity. When vortex sheets are present further conditions are required. For instance, in a linearized approach, where the vortex sheet lies in a fixed position and represents the wake shed from the trailing edge of a wing, a condition of pressure continuity is used to determine its strength. For problems of separated flow, where the position as well as the strength of the vortex sheet is to be found, the two boundary conditions that the sheet is a stream surface and that the pressure is continuous across it are required. These conditions can be written as differential equations which express the variation of the position and strength of the vortex sheet in the streamwise direction in terms of the velocity field in the cross-flow plane (see, for example, the Appendix or Refs 4 and 5). For a sharp-edged body, for which the separation line is known, these equations can be integrated to trace the downstream development

of a vortex sheet specified at some station upstream[5]. It is to be expected that the same will be true for a smooth body, once the separation line is specified; leaving the specification of the separation line and the establishment of an initial vortex configuration upstream as the outstanding problems.

These problems do not enter the present analysis, since no attempt is made to determine the entire vortex sheet configuration. The more limited aim of further restricting the possible behaviour of a vortex sheet near a separation line can be achieved by arguments in a single cross-flow plane, like those used by Clapworthy and Mangler[6] to discuss the vortex she c leaving a salient edge in a conical flow. Suppose that we assume a shape for the section of the sheet by a cross-flow plane and also specify the variation with the streamwise coordinate, x , of that shape in the immediate neighbourhood of the cross-flow plane. Then the component of the velocity, in the cross-flow plane, which is normal to the curves in which the body and sheet intersect the plane is known (by equation (A-4), for instance) and so is the behaviour of the flow at infinity in this plane. Hence the entire flow in this plane is determined. It is helpful to set out one way of finding it explicitly.

Let us imagine that the entire region of the cross-flow plane, $Z = y + iz$, exterior to the body and the vortex sheet (or sheets) is mapped conformally on to the upper half of a transformed plane, Z^* , with the point at infinity in the Z^*-plane corresponding to the point at infinity in the Z-plane by an analytic function

$$Z = F(Z^*) .$$

In this mapping the surface of the body and both surfaces of the sheet(s) become part of the real axis in the Z^*-plane. We now construct the flow field in this transformed plane. A source distribution is introduced along the real axis whose strength per unit length is proportional to the product of the normal velocity in the cross-flow plane and the modulus of the mapping:

$$q(y^*) = 2v_n^*(y^*) = 2v_n(Z)\left|F'(y^*)\right| , \tag{1}$$

where Z is the point on the configuration corresponding to the point $Z^* = y^*$, real, ie $Z = F(y^*)$. Then the complex potential is given by

$$W = VZ^* + \frac{1}{2\pi} \int_{-\infty}^{\infty} q(y^*) \ln (Z^* - y^*)dy^* , \tag{2}$$

where V is the speed of the flow at infinity in the Z*-plane. The component of velocity, in the cross-flow plane, which is tangential to the curves in which the body and sheet intersect the plane is then given by

$$v_t(Z_0) = v_t^*(y_0^*)/|F'(y_0^*)| , \qquad (3)$$

where $Z_0 = F(y_0^*)$, y_0^* real, and where v_t^*, the corresponding velocity component in the transformed plane, is given by the derivative of (2):

$$v_t^*(y_0^*) = V + \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{q(y^*)dy^*}{y_0^* - y^*} . \qquad (4)$$

The integral in (4) is to be interpreted as a Cauchy principal value. From the values of v_t at points on the sheet, the right-hand side of equation (A-10),

$$\frac{\partial \Delta \Phi}{\partial x} = \Delta v_t \left(\frac{\partial y}{\partial x} \cos \psi + \frac{\partial z}{\partial x} \sin \psi - \frac{v_{t_m}}{U} \right) , \qquad (A-10)$$

expressing the condition of pressure continuity, can be found. So, too, can $\Delta \Phi$. If this determination can be carried through in terms of parameters whose variation with the streamwise coordinate, x , is known, the differentiation on the left of (A-10) can be performed and the consistency of the assumed shape of the sheet with the pressure condition can be checked.

This procedure is not a practical way to investigate the global solution, but it can be applied to obtain local information at points of the flow field at which the velocity is singular. To make this idea more precise, we recall that a complex function of a complex variable is regular at a point if, and only if, it can be expanded in a power series about that point, the power series having a non-zero radius of convergence. If a function is not regular at a point, it is singular there. The same definition can be applied to real functions of a real variable, with the power series being convergent in a non-vanishing interval centred on the point. (Note that a function which is singular in this sense may be bounded, like the Heaviside step function, and even differentiable, like $\exp(-1/x^2)$, both at x = 0 .) There is clearly a sense in which a singular function is unchanged when a regular function is added to it; we shall say that its singular behaviour is unchanged. On a more formal level we could define equivalence classes of functions, assigning functions to the same class if they differ by a regular function; but this refinement is not necessary. Note that

the singular behaviour of $\ln \sin x$ and $\ln x$ at $x = 0$ is the same, but that of $2 \ln x$ is different; nor is the singular behaviour of $(\ln \sin x)^2$ the same as that of $(\ln x)^2$.

We can now prove a *theorem* which is fundamental to the analysis which follows, viz: If v_ξ^* is given by (4), then its singular behaviour at a point P is uniquely determined by the singular behaviour of q (or v_η^*) at P. The proof follows.

We choose the origin at P and simplify the notation, writing (4) as

$$v(\xi) = V + \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{q(\eta)d\eta}{\xi - \eta} \ . \tag{4'}$$

We wish to prove that, if $q_1(\eta)$ and $q_2(\eta)$ have the same singular behaviour at $\eta = 0$ and $v_1(\xi)$ and $v_2(\xi)$ are related to them by (4'), then v_1 and v_2 have the same singular behaviour at $\xi = 0$. Let $q(\eta) = q_1(\eta) - q_2(\eta)$ and $v(\xi) = v_1(\xi) - v_2(\xi)$. Since (4') is linear, q and v are also related by it. Then, by the definition above of singular behaviour, we must prove that the regularity of q at $\eta = 0$ implies the regularity of v at $\xi = 0$. If q is regular at the origin we can write, for some $\epsilon > 0$,

$$q(\eta) = \sum_{n=0}^{\infty} a_n \eta^n \ ,$$

for $|\eta| \leqslant \epsilon$.

Now suppose $|\xi| < \epsilon$ and rewrite (4') as

$$2\pi v(\xi) = 2\pi V + \int_{-\infty}^{-\epsilon} \frac{q(\eta)d\eta}{\xi - \eta} + \int_{\epsilon}^{\infty} \frac{q(\eta)d\eta}{\xi - \eta} + \sum_{n=0}^{\infty} a_n I_n \ ,$$

where $I_n = \int_{-\epsilon}^{\epsilon} \frac{\eta^n d\eta}{\xi - \eta}$.

Writing $\eta^n = \xi\eta^{n-1} - (\xi - \eta)\eta^{n-1}$, we see that

$$I_n = \xi I_{n-1} - 2\epsilon^n/n \qquad\qquad \text{for } n > 1 \ ,$$

while

$$I_0 = \ln \frac{\epsilon + \xi}{\epsilon - \xi} \ .$$

12

Hence $I_n = \xi^n \ln \dfrac{\varepsilon + \xi}{\varepsilon - \xi}$ + a polynomial in ξ , and so I_n can be expanded as a power series in ξ for $|\xi| < \varepsilon$. In the integrals over infinite intervals in the expression for $2\pi v(\xi)$, we can expand $(\xi - \eta)^{-1}$ as a power series, for $|\xi| < \varepsilon < \eta$, giving

$$\int_{\varepsilon}^{\infty} \frac{q(\eta)d\eta}{\xi - \eta} = -\sum_{n=0}^{\infty} \xi^n \int_{\varepsilon}^{\infty} \frac{q(\eta)d\eta}{\eta^{n+1}}$$

and a similar expression. Hence $v(\xi)$ is regular at $\xi = 0$ and the proof is complete.

The significance of this theorem is that singular behaviour is likely to arise on the two sides of the separation line. This may come from the shape of the cross-section of the body and sheet, through the mapping function F in (1) and (3), or from the streamwise rate of change of the cross-section, through the normal velocity v_n in (1). Moreover, the behaviour of q or v_n^* is locally determined, through (1). Hence the theorem tells us that the singular behaviour of v_t^* on the two sides of the separation line is locally determined. This can be used to provide information on the local behaviour of v_t , through (3), and the consistency of this behaviour with equation (A-10) can be checked. This provides the main result, that a number of plausible forms for the vortex sheet must be rejected.

The last preliminary question to be discussed before embarking on the analysis concerns the mapping function, $F(Z*)$. The argument above only involves properties of F in the neighbourhoods of the two sides of the separation line. It is therefore sufficient to construct a function which maps the region outside the body and a truncated vortex sheet on to the upper half of the $Z*$ plane, provided that, near the separation line, the truncated sheet has the local behaviour which is to be investigated.

The next section begins with the construction of an appropriate mapping. The argument is then carried through for a conical sheet on a plane wall, a case which brings out the essential features without unnecessary complications. In sections 5 and 6 the limitations to conical flow and plane walls are removed. The reader who is prepared to take the details on trust may proceed to section 7.

4   THE CONICAL SHEET ON A PLANE WALL

4.1   The sheet shape and mapping function

In this section and the next, the body shape considered is a plane wall which extends laterally to infinity and is parallel to the undisturbed flow. It

does not, of itself, disturb this flow, so slender body theory is applicable, provided the vortex sheet is slender. Only highly-swept separation lines are considered and it is assumed that a vortex sheet springing from a highly-swept separation line is slender.

The shape of the sheet is described in a cross-flow plane which is normal to the separation line. In this section, the sheet is assumed to be conical, ie to be generated by straight lines through an apex, so that its entire shape is defined by its section by the cross-flow plane, together with the distance of that plane from the apex. In this case it is convenient to choose the origin, O , of the right-handed rectangular Cartesian system of coordinates at the apex, with Ox downstream along the separation line, Oy_2 to starboard and Oz_2 normal to the plane wall and into the fluid, as sketched in Fig 2. We suppose $y_2 < 0$ is the upstream side of the separation line on the wall, $z_2 = 0$. Then, near the separation line, ie for small values of y_2 and z_2 , the section of the sheet by the cross-flow plane $x = \text{const}$ is given by

$$z_2 \sim \mu y_2^n , \qquad\qquad n > 1 . \qquad\qquad (5)$$

In this Report the tilde is used to indicate asymptotic equality, ie

$$a(t) \sim b(t) \quad \text{if and only if} \quad \lim_{t \to t_0} \frac{a(t)}{b(t)} = 1 .$$

The appropriate variable, t , and limiting value, t_0 , is usually obvious from the context. When an indication of difference between the left and right hand sides is desired, the O notation is used:

$$a(t) = b(t) + O(f(t))$$

if and only if there exist A and ε such that

$$|a(t) - b(t)| \leqslant A|f(t)| \qquad\qquad \text{for } |t - t_0| < \varepsilon .$$

As explained in the previous section, it is sufficient to replace the sheet by any arc which has the same behaviour near the separation line. We obtain an arc in the plane of $Z_2 = y_2 + iz_2$ with the behaviour (5) by taking a circular arc in a plane Z_3 , related to Z_2 by

$$\frac{z_2}{a} = \left(\frac{z_3}{a}\right)^m , \qquad\qquad m > 0 , \qquad (6)$$

where a is the diameter of the circular arc. To obtain the required plane wall in the z_2 plane, we introduce a wall with a corner at the origin in the z_3 plane, as shown in Fig 3. In the z_3 plane, the fluid occupies the sector

$$0 \leqslant \theta_3 \leqslant \pi/m ,$$

where $z_3 = r_3 e^{i\theta_3}$.

m must be at least $\frac{1}{2}$ if the z_3 plane is not to be covered twice, a complication we choose to avoid. The half-lines $\theta_3 = 0$ and $\theta_3 = \pi/m$ represent the wall, and the vortex sheet is represented by a circular arc of radius a/2 , touching the wall $\theta_3 = 0$ at the origin. A general point, P , on the sheet is given by

$$r_3 = a \sin \theta_3 , \qquad\qquad 0 \leqslant \theta_3 \leqslant \theta_C \qquad (7)$$

where the suffix C denotes the value of θ_3 at the end of the arc. Suffixes ·u and d denote the upstream and downstream sides of the sheet at P . A is the point at infinity, B and D represent the upstream and downstream sides of the separation line.

Corresponding points are denoted by the same letters in the z_2 plane. Under (6) the sector becomes the upper half-plane $0 \leqslant \theta_2 \leqslant \pi$, where $z_2 = r_2 e^{i\theta_2}$. The wall is the real axis and the equation of the vortex sheet is, by (6) and (7)

$$r_2 = a\left(\frac{r_3}{a}\right)^m = a \sin^m\theta_3 = a \sin^m \frac{\theta_2}{m} . \qquad (8)$$

For small values of θ_2 , (8) yields

$$\frac{y_2}{a} = \frac{r_2}{a} \cos \theta_2 \sim \left(\frac{\theta_2}{m}\right)^m$$

$$\frac{z_2}{a} = \frac{r_2}{a} \sin \theta_2 \sim \frac{\theta_2^{m+1}}{m^m} \sim m\left(\frac{y_2}{a}\right)^{(1+m)/m} .$$

The required behaviour (5), is therefore obtained with

$$m = \frac{1}{n-1} \qquad \text{and} \qquad a = \left(\frac{m}{\mu}\right)^m . \qquad (9)$$

Thus the order of contact increases as m falls. The curvature of the sheet at the separation line is zero for $m < 1$, $n > 2$; finite for $m = 1$, $n = 2$; and infinite for $m > 1$, $1 < n < 2$.

The region occupied by the fluid in the Z_3 plane is bounded by straight lines and circular arcs, all with a common point. It can therefore be mapped on to a region bounded by straight lines by an inversion which carries the common point to infinity:

$$Z_4 = \frac{a^2}{Z_3} . \qquad (10)$$

The configuration in the Z_4 plane is shown in Fig 4. The half-lines BA and DA remain half-lines, though the origin is interchanged with the point at infinity. The circular arc becomes a third half-line, parallel to the real axis, since the arc touches the real axis at the origin. If Z_3 lies on the sheet, $Z_3 = a \sin \theta_3 e^{i\theta_3}$, by (7), and

$$Z_4 = a(\cot \theta_3 - i) ,$$

by (10). A convenient relation is obtained from (6) and (10):

$$\frac{Z_2}{a} = \left(\frac{Z_4}{a}\right)^{-m} . \qquad (11)$$

Since the region occupied by the fluid in the Z_4 plane is bounded by straight lines, it can be mapped on to the upper half of the Z_5 plane by a Schwarz-Christoffel transformation. If A maps into the point at infinity and B, C and D into $Z_5 = b$, c and d , where $b < c < d$, as in Fig 4, we find the mapping function is defined by

$$\frac{dZ_4}{dZ_5} = \frac{1}{m} \frac{Z_5 - c}{d - Z_5} \left(\frac{a}{Z_5 - b}\right)^{(m+1)/m} , \qquad (12)$$

where a disposable constant has been chosen so that $Z_2 \sim Z_5$ at infinity. The rules for constructing (12) are given in standard texts which deal with conformal mapping.

It is not easy to integrate (12) for general values of m. Fortunately, it is sufficient for our purpose to consider the behaviour of the transformation near B and near D. Let us first consider the neighbourhood of B and write:

$$Z_5 = b + a\zeta , \qquad\qquad \zeta \text{ small} . \qquad (13)$$

Introducing this into (12), we have

$$\frac{dZ_4}{d\zeta} = - \frac{ak_1}{m\zeta^{(m+1)/m}} (1 - k_2\zeta - k_3\zeta^2 + \ldots) , \qquad (14)$$

where the omitted terms are regular and $O(\zeta^3)$ and

$$k_1 = \frac{c - b}{d - b} , \qquad k_2 = \frac{a(d - c)}{(c - b)(d - b)} \qquad \text{and} \qquad k_3 = \frac{a^2(d - c)}{(c - b)(d - b)^2} .$$

$$\ldots\ldots (14a)$$

In integrating (14), it is necessary to consider separately the cases

$$m = \tfrac{1}{2} , \qquad \tfrac{1}{2} < m < 1 , \qquad m = 1 \qquad \text{and} \qquad m > 1 .$$

(a) $m = \tfrac{1}{2}$. Equation (14) becomes

$$\frac{dZ_4}{d\zeta} = 2ak_1\left(-\frac{1}{\zeta^3} + \frac{k_2}{\zeta^2} + \frac{k_3}{\zeta} + O(1)\right)$$

therefore $\quad Z_4 = ak_1\left(\frac{1}{\zeta^2} - \frac{2k_2}{\zeta} + 2k_3 \ln \zeta + \text{const} + O(\zeta)\right)$.

Hence, by equation (11),

$$\frac{Z_2}{a} = \frac{\zeta}{k_1^{\frac{1}{2}}} \left(1 + k_2\zeta - k_3\zeta^2 \ln \zeta + A\zeta^2 + O(\zeta^3 \ln \zeta)\right) , \qquad (15)$$

where A is a complex constant, so far arbitrary. Now, on BA, ζ is real and negative, so $\ln \zeta = \ln |\zeta| + i\pi$, while Z_2 must be real. Hence the imaginary part of A must cancel the imaginary part of the preceding term in (15):

$$A = k_3(A_r + i\pi) \quad , \tag{16}$$

where A_r is an arbitrary real constant. Also, on BC, ζ is real and positive, so, by (15) and (16)

$$\frac{z_2}{a} = \mathcal{J}\left\{\frac{Z_2}{a}\right\} \sim \frac{\pi k_3}{k_1^{\frac{3}{2}}} \zeta^3$$

and

$$\frac{y_2}{a} = \mathcal{R}\left\{\frac{Z_2}{a}\right\} \sim \frac{\zeta}{k_1^{\frac{2}{3}}} \quad .$$

Comparing these expressions with (5) and (9), we see we need

$$2\pi k_1 k_3 = 1 \quad \text{or} \quad 2\pi a^2(d - c) = (d - b)^3 \quad . \tag{17}$$

(b) $\frac{1}{2} < m < 1$. Equation (14) becomes

$$\frac{dZ_4}{d\zeta} = \frac{ak_1}{m}\left(- \frac{1}{\zeta^{(m+1)/m}} + \frac{k_2}{\zeta^{1/m}} + 0(\zeta^{(m-1)/m})\right)$$

therefore

$$Z_4 = \frac{ak_1}{\zeta^{1/m}}\left(1 - \frac{k_2\zeta}{1 - m} + A\zeta^{1/m} + 0(\zeta^2)\right)$$

therefore

$$\frac{Z_2}{a} = \frac{\zeta}{k_1^m}\left(1 + \frac{mk_2\zeta}{1 - m} - mA\zeta^{1/m} + 0(\zeta^2)\right) \quad . \tag{18}$$

Again, on BA we require Z_2 real, with $\zeta = |\zeta|e^{i\pi}$. Hence

$$A = |A|e^{-i\pi/m} \quad . \tag{19}$$

Also, for ζ real and positive, corresponding to BC, we require

$$\mathcal{J}\{Z_4\} = -a \quad ,$$

and so

$$-a = ak_1\mathcal{J}\{A\} \quad .$$

Hence, by (19),

$$A = \left(\cot\frac{\pi}{m} - i\right)\Big/k_1 \quad ,$$

18

and, by (18)

$$\frac{Z_2}{a} = \frac{\zeta}{k_1^m} \left(1 + \frac{mk_2\zeta}{1 - m} - m\left(\cot\frac{\pi}{m} - i\right)\zeta^{1/m}\Big/k_1 + O(\zeta^2) \right) . \tag{20}$$

(c) $m = 1$. Equation (14) becomes

$$\frac{dZ_4}{d\zeta} = ak_1\left(-\frac{1}{\zeta^2} + \frac{k_2}{\zeta} + O(1) \right)$$

therefore
$$Z_4 = \frac{ak_1}{\zeta}\left(1 + k_2\zeta \ln \zeta - A\zeta + O(\zeta^2) \right) ,$$

where A is an arbitrary complex constant. Hence, by (11),

$$\frac{Z_2}{a} = \frac{\zeta}{k_1}\left(1 - k_2\zeta \ln \zeta + A\zeta + O(\zeta^2 \ln \zeta) \right) . \tag{21}$$

As in (a) above, ζ real and negative corresponds to Z_2 real and so

$$A = k_2(A_r + i\pi) , \tag{22}$$

where A_r is real.

Also, for ζ real and positive,

$$\frac{Z_2}{a} \sim \frac{\pi k_2}{k_1} \zeta^2 \sim \pi k_1 k_2\left(\frac{y_2}{a}\right)^2 ,$$

and so, by comparison with (9):

$$\pi k_1 k_2 = 1 \qquad \text{or} \qquad \pi a(d - c) = (d - b)^2 . \tag{23}$$

(d) $m > 1$. The expression for Z_4 in (b) above becomes

$$Z_4 = \frac{ak_1}{\zeta^{1/m}}\left(1 + A\zeta^{1/m} + O(\zeta) \right)$$

therefore

$$\frac{z_2}{a} = \frac{\zeta}{k_1^m}\left(1 - mA\zeta^{1/m} + 0(\zeta, \zeta^{2/m})\right) \quad .$$

The same arguments as in (b) lead to the same value of A, so that

$$\frac{z_2}{a} = \frac{\zeta}{k_1^m}\left(1 - \frac{m}{k_1}\left(\cot\frac{\pi}{m} - i\right)\zeta^{1/m} + 0(\zeta, \zeta^{2/m})\right) \quad . \tag{24}$$

We have now, in equations (13) to (24), expressions describing the behaviour of the mapping near B, for all relevant values of m.

Near D the form of the expression does not depend on m. Let

$$z_5 = d + a\omega \quad , \qquad\qquad \omega \text{ small .} \tag{25}$$

Then, by equation (12)

$$\frac{dz_4}{d\omega} = -\left(\frac{a}{d-b}\right)^{(m+1)/m}\frac{d-c}{m\omega}\left(1 + \sum_{n=1}^{\infty} a_n\omega^n\right) \quad ,$$

where a_n is real.

Therefore

$$z_4 = -\left(\frac{a}{d-b}\right)^{(m+1)/m}\frac{d-c}{m}\left(\ln\omega + A + \sum_{n=1}^{\infty}\frac{a_n}{n}\omega^n\right) \quad . \tag{26}$$

Now, for ω real and positive, corresponding to DA, z_4 is real, so A is real. Let $B = e^A$, so now B is an arbitrary positive constant. For ω real and negative, corresponding to DC, $\mathscr{I}\{z_4\} = -a$, and so

$$-a = -\left(\frac{a}{d-b}\right)^{(m+1)/m}\frac{d-c}{m}\pi \quad . \tag{27}$$

Hence (26) becomes

$$z_4 = -\frac{a}{\pi}\ln B\omega\left(1 + 0\left(\frac{\omega}{\ln B\omega}\right)\right)$$

and, by (11)

$$\frac{Z_2}{a} = \left(\frac{\pi}{-\ln B\omega}\right)^m \left(1 + O\left(\frac{\omega}{\ln B\omega}\right)\right) \quad .$$

(28)

4.2 Local behaviour of velocity field

The first step along the line of approach described in section 3 is to obtain the normal velocity on the sheet in the cross-flow plane, Z_2 . In the Appendix, the boundary condition that the sheet is a stream surface is used to provide expressions for this normal velocity component. There, as here, the cross-flow plane is normal to the axis of x , which is inclined at a small angle to the undisturbed stream. If the sheet is conical, with apex at the origin, equation (A-5) becomes (see Fig 3 for notation)

$$\frac{v_n}{U} = -\frac{r}{x} \sin \phi \quad ,$$

where the sense of the normal is given by a positive rotation of $\pi/2$ from the direction of θ increasing along the curve, ie into the fluid on BC and into the sheet on DC. Note that r, θ, ϕ and v_n all refer to the Z_2-plane, but suffixes are omitted for simplicity. Equation (8) relates r to θ :

$$r = a \sin^m\left(\frac{\theta}{m}\right) \quad .$$

If σ is the arc length along the sheet in the Z_2-plane, measured from the separation line,

$$\left(\frac{d\sigma}{d\theta}\right)^2 = r^2 + \left(\frac{dr}{d\theta}\right)^2 = a^2 \sin^{2m-2}\left(\frac{\theta}{m}\right) \quad ,$$

and $\sin \phi = r d\theta/d\sigma$, so the boundary condition becomes

$$\frac{v_n}{U} = -\frac{a}{x} \sin^{m+1}\left(\frac{\theta}{m}\right) = -\frac{a}{x}\left(\frac{\sigma}{a}\right)^{(m+1)/m} \left(1 + O(\sigma^{2/m})\right) \quad .$$

(29)

To proceed to the next step, finding the normal velocity v_n^* (or the source strength q) in the Z_5-plane, through equation (1), we need the modulus of the mapping at points on the sheet. By (11) and (12)

$$\frac{dZ_2}{dZ_5} = \frac{dZ_4}{dZ_5} \bigg/ \frac{dZ_4}{dZ_2} = \frac{c - Z_5}{d - Z_5} \left(\frac{Z_2}{Z_5 - b}\right)^{(m+1)/m} . \qquad (30)$$

We need the local behaviour of the modulus of this on the real axis near B and D, ie for ζ and ω small and real. Since, on the sheet, $d\sigma/dZ_5 = |dZ_2/dZ_5|$, it is convenient to relate σ to ζ and ω at the same time. As in the previous sub-section, we deal first with the neighbourhood of B, for the different values of m. By (13) and (30)

$$\frac{dZ_2}{dZ_5} = \frac{c - b - a\zeta}{d - b - a\zeta} \left(\frac{Z_2}{a\zeta}\right)^{(m+1)/m} \qquad (31)$$

(a) $m = \frac{1}{2}$. Introducing (15) into (31), and taking the modulus for ζ real, we find

$$\left|\frac{dZ_2}{dZ_5}\right| = \frac{1}{k_1^{\frac{3}{2}}} \left(1 + 2k_2\zeta - 3k_3\zeta^2 \ln |\zeta| + O(\zeta^2)\right) . \qquad (32)$$

Integrating (32) for $\zeta > 0$ gives

$$\frac{\sigma}{a} = \frac{\zeta}{k_1^{\frac{3}{2}}} \left(1 + k_2\zeta - k_3\zeta^2 \ln \zeta + O(\zeta^2)\right) . \qquad (33)$$

(b) $\frac{1}{2} < m < 1$. Introducing (20) into (31) and taking the modulus for ζ real and positive, we have:

$$\left|\frac{dZ_2}{dZ_5}\right| = \frac{1}{k_1^m} \left(1 + \frac{2mk_2}{1-m}\zeta - \frac{1+m}{k_1} \cot \frac{\pi}{m} |\zeta^{1/m}| + O(\zeta^2)\right) . \qquad (34)$$

Integrating (34) for $\zeta > 0$ gives

$$\frac{\sigma}{a} = \frac{\zeta}{k_1^m} \left(1 + \frac{mk_2}{1-m}\zeta - \frac{m}{k_1} \cot \frac{\pi}{m} |\zeta^{1/m}| + O(\zeta^2)\right) . \qquad (35)$$

(c) $m = 1$. Introducing (21) into (31) and taking the modulus for ζ real, we find:

$$\left|\frac{dZ_2}{dZ_5}\right| = \frac{1}{k_1}\left(1 - 2k_2\zeta \ln|\zeta| + 0(\zeta)\right) \ . \tag{36}$$

Integrating (36) for $\zeta > 0$ gives

$$\frac{\sigma}{a} = \frac{\zeta}{k_1}\left(1 - k_2\zeta \ln \zeta + 0(\zeta)\right) \ . \tag{37}$$

(d) $m > 1$. Introducing (24) into (31) and taking the modulus for ζ real and positive, we have:

$$\left|\frac{dZ_2}{dZ_5}\right| = \frac{1}{k_1^m}\left(1 - \frac{m+1}{k_1} \cot \frac{\pi}{m} |\zeta^{1/m}| + 0(\zeta,\zeta^{2/m})\right) \ . \tag{38}$$

Integrating (38) for $\zeta > 0$ gives

$$\frac{\sigma}{a} = \frac{\zeta}{k_1^m}\left(1 - \frac{m}{k_1} \cot \frac{\pi}{m} |\zeta^{1/m}| + 0(\zeta,\zeta^{2/m})\right) \ . \tag{39}$$

The form of the leading terms in (32) to (39) is independent of m , but the higher order terms differ significantly.

For the neighbourhood of D, introducing (25) into (30) gives

$$\frac{dZ_2}{dZ_5} = \frac{c - d - a\omega}{- a\omega}\left(\frac{Z_2}{d - b - a\omega}\right)^{(m+1)/m} \ .$$

By (27) and (28) this reduces to

$$\frac{dZ_2}{dZ_5} = \frac{m}{\pi\omega}\left(- \frac{\pi}{\ln B\omega}\right)^{m+1}(1 + 0(\omega)) \ . \tag{40}$$

On the sheet near D, ω is real and negative, so that

$$\ln B\omega = \ln B|\omega| + i\pi = - \pi(1 - i\delta)/\delta \ , \tag{41}$$

where

$$\delta = - \frac{\pi}{\ln B|\omega|} \tag{42}$$

is a convenient small positive quantity. The modulus of (40) becomes

$$\left|\frac{dZ_2}{dZ_5}\right| = \frac{m\delta^{m+1}}{\pi|\omega|(1 + \delta^2)^{(m+1)/2}}(1 + 0(\omega)) \quad \text{for} \quad \omega < 0 \tag{43}$$

and

$$\left|\frac{dZ_2}{dZ_5}\right| = \frac{m\delta^{m+1}}{\pi\omega} (1 + 0(\omega)) \qquad \text{for } \omega > 0 . \qquad (44)$$

Note that, by (42), ω is exponentially small in relation to δ . Also by (42), $d\delta/d\omega = \delta^2/\pi\omega$ and so, for $\omega < 0$:

$$\frac{d\sigma}{d\delta} = \left|\frac{dZ_2}{dZ_5}\right|\left|\frac{dZ_5}{d\omega}\right|\left|\frac{d\omega}{d\delta}\right| = \frac{am\delta^{m-1}}{(1 + \delta^2)^{(m+1)/2}} (1 + 0(\omega))$$

and

$$\sigma = a\delta^m(1 + 0(\delta^2)) . \qquad (45)$$

We can now find the component v_n^* of the velocity normal to the real axis in the Z_5-plane, close to B and D. By (1) and (29) we have, on BC and DC,

$$\frac{v_n^*}{U} = -\frac{a}{x}\left(\frac{\sigma}{a}\right)^{(m+1)/m}\left|\frac{dZ_2}{dZ_5}\right|(1 + 0(\sigma^{2/m})) , \qquad (46)$$

with the normal into the fluid on BC and out of it on DC. On BA and DA, which correspond to the body, (A-5) and (1) show that $v_n^* = v_n = 0$. Introducing into (46) the leading terms from (32) to (39), (43) and (45), we find for the velocity component w^* normal to the real axis and into the fluid in the Z^*-plane:

on BA $\qquad w^* = v_n^* = 0$

on BC $\qquad w^* = v_n^* = -\frac{aU}{x}\frac{\zeta^{(m+1)/m}}{k_1^{2m+1}}\left(1 + 0(\zeta^{1/m}, \zeta \ln \zeta)\right)$

on DC $\qquad w^* = -v_n^* = \frac{maU}{\pi x|\omega|}\delta^{2m+2}(1 + 0(\delta^2))$ $\qquad\qquad$ (47)

on DA $\qquad w^* = v_n^* = 0$.

We now wish to find the tangential velocity, v^* , on the real axis in the Z_5-plane near B and D. The combination $v^* - iw^*$ is an analytic function of Z_5 . If $(v^* - iw^*)_S$ has the singular behaviour of $v^* - iw^*$ at B or D, then we can write

$$v^* - iw^* = (v^* - iw^*)_S + (v^* - iw^*)_R , \qquad (48)$$

where the second term is regular. This term is not determined by local arguments and we write it as

$$(v^* - iw^*)_R = \sum_{n=0}^{\infty} A_n \zeta^n \quad \text{(near B)} \qquad \text{or} \qquad \sum_{n=0}^{\infty} B_n \omega^n \quad \text{(near D)} \quad . \qquad (49)$$

The conditions (47) on BA and DA imply that A_n and B_n are real. The singular behaviour of $v - iw$ at B and D is determined by the singular behaviour of w^*, by the theorem proved in section 3. We therefore proceed by constructing analytic functions whose imaginary parts behave like w^* near B and D. As before, we start with B for the familiar four cases.

(a) $m = \tfrac{1}{2}$. By (47), for ζ real,

$$w^* \sim \begin{cases} 0 & \text{for } \zeta > 0 \\[2em] -\dfrac{aU}{x} \dfrac{\zeta^3}{k_1^2} & \text{for } \zeta > 0 \end{cases}$$

Consider the function $f(\zeta) = \zeta^3(i\pi - \ln \zeta)$. For ζ real and negative,

$$\mathscr{I}\{f\} = 0 \quad ,$$

and for ζ real and positive,

$$\mathscr{I}\{f\} = \pi\zeta^3 \quad .$$

Hence

$$(v^* - iw)_S \sim \frac{aU}{\pi x} \frac{\zeta^3}{k_1^2} (i\pi - \ln \zeta) \quad . \qquad (50)$$

(b) $\tfrac{1}{2} < m < 1$. By (47), for ζ real

$$w^* \sim \begin{cases} 0 & \text{for } \zeta < 0 \\[2em] -\dfrac{a\bar{U}}{x} \dfrac{\zeta^{\mu}}{k_1^{2m+1}} & \text{for } \zeta > 0 , \end{cases}$$

where $2 < \mu = (m+1)/m < 3$. For $\zeta = \rho e^{i\vartheta}$, define $\zeta^{\mu} = \rho^{\mu} e^{i\mu\vartheta}$, and consider the function

$$f(\zeta) = e^{-i\mu\pi}\zeta^{\mu}/\sin \mu\pi = \zeta^{\mu}(\cot \mu\pi - i) .$$

For ζ real and negative, $\vartheta = \pi$ and $\mathscr{I}\{f\} = 0$. For ζ real and positive, $\vartheta = 0$ and $\mathscr{I}\{f\} = -\zeta^{\mu}$, real. Hence

$$(v^* - iw^*)_S \sim \frac{aU}{x}\left(i - \cot \frac{\pi}{m}\right) \frac{\zeta^{(m+1)/m}}{k_1^{2m+1}} , \qquad (51)$$

since $\cot \mu\pi = \cot \frac{\pi}{m}$.

(c) $m = 1$. By (47), for ζ real

$$w^* \sim \begin{cases} 0 & \text{for } \zeta < 0 \\[2ex] -\dfrac{aU}{x}\dfrac{\zeta^2}{k_1^3} & \text{for } \zeta > 0 . \end{cases}$$

Proceeding as in (a) above, we find

$$(v^* - iw^*)_S \sim \frac{aU}{\pi x}\frac{\zeta^2}{k_1^3}(i\pi - \ln \zeta) . \qquad (52)$$

(d) $m > 1$. The argument in (b) applies, with $1 < \mu < 2$, leading again to (51).

Now in the neighbourhood of D, (47) gives, for ω real,

$$w^* = \begin{cases} -\dfrac{maU}{\pi x\omega}\delta^{2m+2}(1 + 0(\delta^2)) & \text{for } \omega < 0 \\[2ex] 0 & \text{for } \omega > 0 . \end{cases}$$

Consider the function $f_k(\omega) = (-\pi/\ln B\omega)^k/\omega$. For $\omega < 0$,

$$\ln B\omega = -\pi(1 - i\delta)/\delta$$

by (41), and so

$$f_k(\omega) = \delta^k (1 - i\delta)^{-k}/\omega$$

and

$$\mathscr{I}\{f_k\} = k\delta^{k+1}(1 + O(\delta^2))/\omega \quad .$$

For $\omega > 0$, $\mathscr{I}\{f_k\} = 0$. Hence

$$(v^* - iw^*)_S = \frac{maU}{(2m+1)\pi x\omega} \left(-\frac{\pi}{\ln B\omega}\right)^{2m+1} \left(1 + O\left(-\frac{\pi}{\ln B\omega}\right)^2\right) \quad . \tag{53}$$

Equations (48) and (49), with the appropriate equation from (50) to (53) give the complex conjugate velocity in the Z_5-plane near B and D.

We can now complete the calculation by using these expressions to find the tangential velocity, v_t, on the sheet in the Z_2-plane, using a form of equation (3):

$$v_t = v_t^* \left|\frac{dZ_5}{dZ_2}\right| \tag{54}$$

in which $v_t^* = v^*$ on BC and $v_t^* = -v^*$ on DC. Again we treat first the four cases which arise on BC, where ζ is real and positive.

(a) $m = \frac{1}{2}$. By (32), (48), (49) and (50)

$$v_t = \frac{A_0 + A_1\zeta + A_2\zeta^2 - \dfrac{aU}{\pi xk_1^2}\zeta^3 \ln \zeta + \dots}{\dfrac{1}{k_1^{\frac{1}{2}}}\left(1 + 2k_2\zeta - 3k_3\zeta^2 \ln \zeta + O(\zeta^2)\right)}$$

$$= k_1^{\frac{1}{2}}\left(A_0 + A_1'\zeta + 3A_0k_3\zeta^2 \ln \zeta + O(\zeta^2)\right) \quad ,$$

where A_1' is another real constant. Inverting (33) to give

$$\zeta = k_1^{\frac{1}{2}} \frac{\sigma}{a} (1 + O(\sigma)) \quad ,$$

and assuming for the moment that $A_0 \neq 0$, we can write, using (17),

$$v_t = k_1^{\frac{1}{2}}A_0\left(1 + A_1''\sigma + \frac{3}{2\pi}\left(\frac{\sigma}{a}\right)^2 \ln \sigma + O(\sigma^2)\right) \quad , \tag{55}$$

where A_1'' is another undetermined real constant.

(b) $\frac{1}{2} < m < 1$. By (34), (48), (49) and (51):

$$v_t = \frac{A_0 + A_1\zeta + A_2\zeta^2 - \frac{aU}{xk_1^{2m+1}} \cot \frac{\pi}{m} \zeta^{(m+1)/m} + \ldots}{\frac{1}{k_1^m}\left(1 + \frac{2mk_2}{1-m}\zeta - \frac{1+m}{k_1} \cot \frac{\pi}{m} \zeta^{1/m} + O(\zeta^2)\right)}$$

$$= k_1^m\left(A_0 + A_1'\zeta + A_0 \frac{1+m}{k_1} \cot \frac{\pi}{m} \zeta^{1/m} + O(\zeta^2)\right) .$$

Using the inverse of (35) and assuming $A_0 \neq 0$, we have

$$v_t = k_1^m A_0\left(1 + A_1''\sigma + (1+m) \cot \frac{\pi}{m} \left(\frac{\sigma}{a}\right)^{1/m} + O(\sigma^2)\right) , \tag{56}$$

where A_1' and A_1'' are again undetermined real constants.

(c) $m = 1$. By (36), (48), (49) and (52)

$$v_t = \frac{A_0 + A_1\zeta - \frac{aU}{\pi x k_1^3} \zeta^2 \ln \zeta + \ldots}{\frac{1}{k_1}\left(1 - 2k_2\zeta \ln \zeta + O(\zeta)\right)} .$$

Using the inverse of (37) with (23), and still assuming $A_0 \neq 0$, we have:

$$v_t = k_1 A_0\left(1 + \frac{2}{\pi} \frac{\sigma}{a} \ln \sigma + O(\sigma)\right) . \tag{57}$$

(d) $m > 1$. Using (38), (48), (49) and (51) we obtain a result like (56):

$$v_t = k_1^m A_0\left(1 + (1+m) \cot \frac{\pi}{m} \left(\frac{\sigma}{a}\right)^{1/m} + O(\sigma)\right) . \tag{58}$$

It is significant that the dominant terms in v_t arise in each case from the regular behaviour of v , for which the real axis in the Z_5-plane is a stream-line, ie the dominant terms in v_t near B are of the same form as they would be if the sheet were a streamline of the cross-flow.

We now consider the behaviour of v_t on DC, near D, where ω is real and negative. By (53) and (41)

$$v_S^* = \frac{maU\delta^{2m+1}}{(2m+1)\pi x \omega}(1 + O(\delta^2)) \quad .$$

By (48) and (49), v_R^* does not contribute to v^* to this order, and so, by (43) and (54):

$$v_t = -\frac{maU\delta^{2m+1}}{(2m+1)\pi x \omega}\frac{\pi|\omega|}{m\delta^{m+1}}(1 + O(\delta^2)) \quad .$$

We can write this, by (45), as

$$v_t = \frac{U\sigma}{(2m+1)x}(1 + O(\sigma^{2/m})) \quad . \tag{59}$$

In contrast to the situation near B, the dominant terms in v_t near D arise from the singular behaviour of v, so that they are uniquely determined by the shape of the vortex sheet near D.

Having determined v_t near D, we can justify the assumption that A_0 is non-zero. Equation (59) shows that v_t vanishes at D, while (55), (56), (57) and (58) show that v_t also vanishes at B if $A_0 = 0$. In terms of the discussion in section 2, this would mean that no vorticity is being shed from the separation line. In general this would mean that the point on the separation line was exceptional, but for the conical flow postulated in this section it would mean that no vorticity is being shed anywhere on the separation line. Hence $A_0 \neq 0$.

The convective velocity $\frac{1}{2}\left(v_t\big|_B + v_t\big|_D\right)$ at the separation line has the same sign as A_0, so we conclude that $A_0 > 0$ if vorticity is being shed.

4.3 Consistency of velocity field with continuity of pressure

For a conical field, the condition that the pressure is continuous across the vortex sheet, (A-10), becomes

$$\Delta\Phi = \Delta v_t \left(r \cos \phi - \frac{x v_{t_m}}{U} \right) , \tag{60}$$

since $\Delta\Phi$, y and z are proportional to x, $y = r \cos \theta$, $z = r \sin \theta$ and $\phi = \psi - \theta$. We wish to consider whether this relation is consistent with the tangential velocity components found in subsection 4.2 in the neighbourhood of the separation line. By steps like those leading to (29), we find

$$r \cos \phi = a \sin^m \frac{\theta}{m} \cos \frac{\theta}{m} = \sigma(1 + O(\sigma^{2/m})) \quad . \tag{61}$$

Examining equations (55) to (58), we see that the tangential velocity on the sheet near B can be written as

$$v_t\big|_{BC} = A_0'\left(1 + A_1''\sigma + (m + 1)\left(\frac{\sigma}{a}\right)^{1/m} f(\sigma) + o(\sigma^{1/m}f(\sigma))\right) , \tag{62}$$

where $A_0' = A_0(c - b)^m/(d - b)^m \neq 0$ and A_1'' are undetermined real constants;

$$f(\sigma) = \begin{cases} \dfrac{1}{\pi} \ln \sigma & \text{for } m = \tfrac{1}{2} \text{ or } 1 \\[2ex] \cot \dfrac{\pi}{m} & \text{for } m > \tfrac{1}{2} \text{ and } \neq 1 \ ; \end{cases} \tag{63}$$

and the notation $o(g(\sigma))$ stands for terms of higher order than $g(\sigma)$. On the sheet near D, the tangential velocity is given by (59) as

$$v_t\big|_{DC} = \frac{U\sigma}{(2m + 1)x} (1 + O(\sigma^{2/m})) \quad . \tag{64}$$

All the terms in (60) can now be represented for small values of σ . By (61), (62) and (63):

$$r \cos \phi - \frac{x}{U} v_{t_m} = r \cos \phi - \frac{x}{2U}\left(v_t\big|_{BC} + v_t\big|_{DC}\right)$$

$$= -\frac{xA_0'}{2U}\left(1 + C_1\sigma + (m + 1)\left(\frac{\sigma}{a}\right)^{1/m} f(\sigma) + o(\sigma^{1/m}f(\sigma))\right) , \tag{65}$$

where C_1 is another arbitrary real constant. Similarly

$$\Delta v_t = v_t\big|_{BC} - v_t\big|_{DC}$$

$$= A_0'\left(1 + C_2\sigma + (m + 1)\left(\frac{\sigma}{a}\right)^{1/m} f(\sigma) + o(\sigma^{1/m}f(\sigma))\right) , \tag{66}$$

where C_2 is a constant related to C_1 . Integrating (66) gives

$$\Delta\Phi - \Gamma = \int_0^\sigma \Delta v_t\, d\sigma$$

$$= A_0'\left(\sigma + \tfrac{1}{2}C_2\sigma^2 + ma\left(\frac{\sigma}{a}\right)^{(m+1)/m} f(\sigma) + o(\sigma^{(m+1)/m}f(\sigma))\right) , \tag{67}$$

where $\Gamma = \phi_B - \phi_D$ is the total circulation in the sheet.

When (65), (66) and (67) are introduced into (60), it becomes

$$\Gamma + A_0' \left(\sigma + \tfrac{1}{2} C_2 \sigma^2 + ma\left(\frac{\sigma}{a}\right)^{(m+1)/m} f(\sigma) + o\left(\sigma^{(m+1)/m} f(\sigma)\right) \right)$$
$$= -\frac{x A_0'^2}{2U} \left(1 + (C_1 + C_2)\sigma + 2(m+1)\left(\frac{\sigma}{a}\right)^{1/m} f(\sigma) + o(\sigma^{1/m} f(\sigma)) \right) \quad . \qquad (68)$$

There is no term on the left-hand side of (68) to balance the term

$$-\frac{x A_0'^2 (m+1)}{U} \left(\frac{\sigma}{a}\right)^{1/m} f(\sigma)$$

on the right, so this term must vanish. Hence, since $A_0' \neq 0$, $f(\sigma) = 0$. By (63) this is only possible if $\cot \pi/m = 0$, and $m > \tfrac{1}{2}$, ie

$$m = 2 \quad \text{or} \quad 2/3 \quad . \qquad (69)$$

The form of the condition makes it very likely that $m = 2/(2M + 1)$, $M = 0, 1, 2, \ldots$ is a complete set of possible values of m. However, in subsection 4.1 the possibility that $m < \tfrac{1}{2}$ was excluded in order to keep the mapping conceptually simple, so the present argument cannot supply a complete set. In fact the existence of the two possible values for m, corresponding by (9) to $n = 3/2$ and $5/2$ in (5), and the non-existence of intermediate values is enough for practical purposes. It is suggested in section 7.3 that larger values of n do not occur.

4.4 Local behaviour of velocity and pressure on the wall

We now consider the flow on the wall near the separation line, limiting the discussion to the two values of m, given by (69), for which a consistent solution seems to be possible. The velocity, v, normal to the separation line, is given by a form of (3):

$$v = v^* \left| \frac{dZ_5}{dZ_2} \right| \qquad (70)$$

on BA and DA, where v^* is the corresponding velocity component in the Z_5-plane and is given by (48) and (49), with (51) for BA or (53) for DA.

On BA, ζ is real and negative. For $m = 2/3$, $\zeta^{1/m} = -i|\zeta^{3/2}|$, and (20) gives

$$\frac{Z_2}{a} = \frac{\zeta}{k_1^{2/3}} \left(1 + 2k_2\zeta + \frac{2}{3k_1} |\zeta^{3/2}| + O(\zeta^2) \right) \; ;$$

and so by (31),

$$\left| \frac{dZ_2}{dZ_5} \right| = \frac{1}{k_1^{2/3}} \left(1 + 4k_2\zeta + \frac{5}{3k_1} |\zeta^{3/2}| + O(\zeta^2) \right) \; . \tag{71}$$

For $m = 2$, $\zeta^{1/m} = i|\zeta^{\frac{1}{2}}|$ and (24) gives

$$\frac{Z_2}{a} = \frac{\zeta}{k_1^2} \left(1 - \frac{2}{k_1} |\zeta^{\frac{1}{2}}| + O(\zeta) \right)$$

and so by (31),

$$\left| \frac{dZ_2}{dZ_5} \right| = \frac{1}{k_1^2} \left(1 - \frac{3}{k_1} |\zeta^{\frac{1}{2}}| + O(\zeta) \right) \; . \tag{72}$$

By (51), for ζ real and negative,

$$v_S^* \sim \frac{aU}{xk_1^{2m+1}} \; \mathrm{cosec} \, \frac{\pi}{m} |\zeta^{(m+1)/m}| \; .$$

Hence, by (48) and (49),

$$v^* = A_0 + A_1\zeta + \begin{cases} A_2\zeta^2 - \dfrac{aU}{xk_1^{7/3}} |\zeta^{5/2}| + \ldots & \text{for } m = 2/3 \\[3mm] \dfrac{aU}{xk_1^5} |\zeta^{3/2}| + \ldots & \text{for } m = 2 \; , \end{cases}$$

and so, by (70), (71) and (72), since $A_0 \neq 0$,

$$v = A_0' \left(1 + A_1'\zeta - \frac{5}{3k_1} |\zeta^{3/2}| + O(\zeta^2) \right) \qquad \text{for } m = 2/3$$

or

$$v = A_0' \left(1 + \frac{3}{k_1} |\zeta^{\frac{1}{2}}| + O(\zeta) \right) \qquad \text{for } m = 2 \; ,$$

where $A_0' \neq 0$ and A_1' are undetermined constants. Using the relations above to express ζ in terms of Z_2, and writing $Z_2 = y$, we finally find

$$
\left.
\begin{aligned}
v &= A_0'\left(1 + A_1''y - \frac{5}{3}\left(-\frac{y}{a}\right)^{3/2} + 0(y^2)\right) \qquad \text{for } m = 2/3 \\
v &= A_0'\left(1 + 3\left(-\frac{y}{a}\right)^{\frac{1}{2}} + 0(y)\right) \qquad\qquad \text{for } m = 2 \; .
\end{aligned}
\right\}
\tag{73}
$$

Thus we see that, for $m = 2$, as the flow approaches the separation line, $y = 0$, from upstream, the velocity component normal to the separation line falls, and its gradient normal to the separation line, v_y, becomes infinite there. For $m = 2/3$ no such behaviour arises.

On the downstream side, DA, where ω is positive, (53) with (42) gives:

$$
v_s^* = \frac{maU\delta^{2m+1}}{(2m + 1)\pi x \omega}\,(1 + 0(\delta^2)) \; .
$$

This dominates the regular part of v^*, given by (49). The modulus of the mapping is given by (44) and so, by (70)

$$
v = \frac{aU\delta^m}{(2m + 1)x}\,(1 + 0(\delta^2)) \; ;
$$

or, using (28) to relate δ to $Z_2 = y$, we have

$$
v = \frac{Uy}{(2m + 1)x}\,(1 + 0(y^{2/m})) \; .
\tag{74}
$$

As would be expected from the proximity of the sheet to the wall on the downstream side, this expression resembles (64) for the tangential velocity on the sheet. The velocity normal to the separation line on its downstream side is zero at the line itself and directed away from the line in its immediate neighbourhood. However, it is so small that the streamlines on the surface cross the conical rays towards the separation line, since $v/U < y/x$. Hence the velocity normal to the separation line decreases along a surface streamline, despite the appearance of (74). For $m = 2/3$ or 2 the velocity in the cross-flow plane is regular, to the order of the present analysis.

We now examine how far the behaviour of the pressure gradient near the separation line reflects that of the gradient of the velocity component normal to the separation line. For conical flow, the pressure gradient along the separation

line is zero. The gradient normal to the separation line, from the upstream to the downstream side, is $\partial p/\partial y$. The pressure coefficient is given, in the usual slender-body approximation, by

$$C_p = -\frac{2u}{U} - \frac{v^2 + w^2}{U^2} + \text{const.} \tag{75}$$

On the wall, $z = 0$, where $w = 0$,

$$\frac{\partial p}{\partial y} = \tfrac{1}{2}\rho U^2 \frac{\partial C_p}{\partial y} = -\rho(Uu_y + vv_y) .$$

To relate u_y to v_y , we recall that, for a conical field,

$$\phi = xF\left(\frac{y}{x},\frac{z}{x}\right) ,$$

and so

$$\phi = x\phi_x + y\phi_y + z\phi_z = xu + yv + zw .$$

Differentiating this with respect to y ,

$$v = \phi_y = xu_y + v + yv_y + zw_y ,$$

so we find, for $z = 0$,

$$xu_y = -yv_y ,$$

and the pressure gradient becomes

$$\frac{\partial p}{\partial y} = -\rho Uv_y\left(\frac{v}{U} - \frac{y}{x}\right) .$$

On the upstream side, where v is given by (73), it is clear that the pressure gradient behaves in the same way as the velocity gradient, ie there is an adverse gradient which becomes infinitely large if $m = 2$; while if $m = 2/3$ the gradient is not locally determined. On the downstream side, with v given by (74), the pressure gradient becomes

$$\frac{\partial p}{\partial y} \sim \frac{2\rho U^2 my}{(2m + 1)^2 x^2} .$$

Since the pressure itself is a function of y/x in a conical flow, it is given by

$$p - p_D \sim \frac{\rho U^2 m}{(2m + 1)^2} \left(\frac{y}{x}\right)^2 , \qquad (76)$$

where p_D is the pressure on the separation line. Hence, along a surface stream-line, on which as we have seen y/x decreases as x and y increase, the pressure falls and the gradient is favourable. With a small favourable pressure gradient and a slowly diverging external flow, there is no tendency for the boundary layer on the downstream side of the separation line to separate. This supports the concept of the flow advanced in section 2 and sketched in Fig 1, regardless of the value of m .

A marked difference in the behaviour of the limiting streamlines or skin-friction lines between the upstream and downstream sides of the separation line is frequently observed in surface oil-flow visualisations, particularly in approximately conical external flows with highly-swept separation lines. The limiting streamlines on the upstream side turn towards the direction of the separation line very late, with high curvature and strong convergence, so that their behaviour very close to the separation line is often masked by an accumulation of oil. On the downstream side, the limiting streamlines are almost straight and parallel, with very little trend towards the separation line. Examples are: primary separation on a slender circular cone at incidence, see for instance Nebbeling and Bannink[7]; secondary separation beneath the primary vortex on a slender delta wing, see for instance Lawford[8]; and both primary and secondary separations on an upswept rear fuselage, see Peake[9]. Pictures of glancing interactions (plane shock normal to the surface and oblique to the stream) by Oskam, Vas and Bogdonoff[10] show similar differences between the upstream and downstream sides of a region of streamline coalescence, although the authors do not regard the flow as separated.

Although the behaviour of the flow on the downstream side of the separation line is, in the present treatment, little affected by the value of m , on the upstream side the value of m has a marked effect. For $m = 2$, $n = 3/2$, the existence of an infinite adverse pressure gradient in the external flow makes it probable, though not certain, that the boundary layer, whether laminar or turbulent will separate before it reaches the postulated separation line. This difficulty does not arise if $m = 2/3$, $n = 5/2$. The implications of this for modelling the complete flow are discussed briefly in subsection 7.5.

5   EXTENSION TO NON-CONICAL SHEETS

We are still concerned with the behaviour of a sheet which leaves a plane wall tangentially, in accordance with the conclusion of section 2, from a separation line which is highly swept, ie inclined at a small angle λ to the undisturbed stream. As in section 4 we take the wall to be the plane $z = 0$, and choose the axis of x to be tangential to the separation line at the point considered, with x increasing downstream. We again study the configuration in the cross-flow plane, $x =$ const , and use slender-body theory. Near the separation line the cross-section of the sheet is again of the form

$$z = \mu y^n , \qquad\qquad n > 1 , \qquad (5)$$

but now μ and n are allowed to depend on x in unspecified ways. It will be assumed that the sheet is sufficiently smooth for μ and n to be differentiable twice with respect to x . The sequence of mappings used in subsection 4.1 still applies and, by (9), the variation of μ and n is equivalent to some equally smooth variation in a and m . It is again assumed that $m > \frac{1}{2}$.

The boundary conditions for this system of axes are obtained in the Appendix, where the normal component of velocity on the section of the sheet in the cross-flow plane is given by (A-5) as

$$v_n = - U \left.\frac{\partial r}{\partial x}\right|_{\theta \text{ const}} \sin \phi .$$

By (8), $r = a \sin^m(\theta/m)$ and so, with primes denoting derivatives,

$$\frac{1}{r}\frac{\partial r}{\partial x} = \frac{a'}{a} - m' + m' \ln \frac{\theta}{m} + 0(\theta^2) ; \qquad (77)$$

so that

$$\frac{v_n}{U} = - \left(\frac{a'}{a} - m' + m' \ln \frac{\theta}{m} + 0(\theta^2)\right) r \sin \phi .$$

By comparison with the first equation of subsection 4.2, we see that the normal velocity is a multiple

$$x\left(\frac{a'}{a} - m'\right) \qquad (78)$$

of the conical distribution, plus a contribution

$$v_n = - m'U \ln \left(\frac{\theta}{m}\right) r \sin \phi \quad , \tag{79}$$

plus a contribution of higher order. Recalling that the procedure followed in subsection 4.2 to obtain the tangential components from the normal components is linear, we see that the tangential velocities in the non-conical problem can be obtained as the sum of (78) times those found for conical flow, those corresponding to (79) and a contribution of higher order. The last will be neglected; and an analysis like that of subsection 4.2 will be pursued for the normal velocity given by (79). The tangential velocities so found will be used, as in subsection 4.3, to check the consistency of the condition of pressure continuity.

The same geometrical relations as were used in subsection 4.2 enable (79) to be rewritten as

$$v_n = - m'aU \sin^{m+1}\left(\frac{\theta}{m}\right) \ln \left(\frac{\theta}{m}\right)$$

$$= - \left(\frac{m'a}{m}\right) U\left(\frac{\sigma}{a}\right)^{(m+1)/m} \ln \left(\frac{\sigma}{a}\right)(1 + O(\sigma^{2/m})) \quad , \tag{80}$$

in place of (29). The geometrical relations (30) to (45) still apply, so we can obtain from (80) a set of equations like (47) for the component, w^*, of the velocity normal to the real axis in the Z_5-plane, near B and D, viz:

$$\left. \begin{array}{llll} \text{on BA} & w^* = v_n^* = 0 \\[2ex] \text{on BC} & w^* = v_n^* = - \dfrac{m'aU}{m} \dfrac{\zeta^{(m+1)/m}}{k_1^{2m+1}} \ln \zeta (1 + O(\zeta^{1/m}, \zeta \ln \zeta)) \\[2ex] \text{on DC} & w^* = - v_n^* = \dfrac{mm'aU}{\pi |\omega|} \delta^{2m+2} \ln \delta (1 + O(\delta^2)) \\[2ex] \text{on DA} & w^* = v_n^* = 0 \quad . \end{array} \right\} \tag{81}$$

As before, we regard the analytic function $v^* - iw^*$ as composed of a regular part and a singular part, and obtain the singular behaviour from an analytic function whose imaginary part behaves like w^*, as given by (81).

Starting with the neighbourhood of B, we deal first with the possibility that $(m + 1)/m$ is integral, ie, in the admissible range, $m = \frac{1}{2}$ or 1. Consider the function:

$$f(\zeta) = \zeta^{(m+1)/m}(i\pi - \ln \zeta)^2 \quad .$$

For ζ real and negative, $\mathscr{I}\{f\} = 0$, for ζ real and positive, $\mathscr{I}\{f\} = -2\pi\zeta^{(m+1)/m} \ln \zeta$. Hence

$$(v^* - iw^*)_S \sim -\frac{m'aU}{2\pi mk_1^{2m+1}} \zeta^{(m+1)/m}(i\pi - \ln \zeta)^2 \quad .$$

Adding a regular contribution, as given by (49), leads to

$$v^* - iw^* = \sum_{n=0}^{\infty} A_n\zeta^n - \frac{m'aU}{2\pi mk_1^{2m+1}} \zeta^{(m+1)/m}(i\pi - \ln \zeta)^2 + \dots \quad , \qquad (82)$$

near B, for $m = \frac{1}{2}$ or 1, where A_n is real and the terms omitted are of higher order than the last term on the right.

Still near B, but with $m > \frac{1}{2}$ and $\neq 1$, let $\mu = (m + 1)/m$ and consider

$$f(\zeta) = \frac{\zeta^\mu}{\sin \mu\pi} \left(\frac{\pi}{\sin \mu\pi} - e^{-i\mu\pi} \ln \zeta\right) \quad ,$$

where ζ^μ is defined as $\rho^\mu e^{i\mu\vartheta}$ when $\zeta = \rho e^{i\vartheta}$. For ζ real and negative, $\vartheta = \pi$ and $\mathscr{I}\{f\} = 0$. For ζ real and positive, $\mathscr{I}\{f\} = \zeta^\mu \ln \zeta$. Hence

$$(v^* - iw^*)_S \sim -\frac{m'aU}{mk_1^{2m+1}} \operatorname{cosec} \frac{\pi}{m} \zeta^{(m+1)/m}\left(e^{-i\pi/m} \ln \zeta - \pi \operatorname{cosec} \frac{\pi}{m}\right) \quad .$$

Adding a regular contribution, as before, leads to

$$v^* - iw^* = \sum_{n=0}^{\infty} A_n\zeta^n - \frac{m'aU}{mk_1^{2m+1}} \operatorname{cosec} \frac{\pi}{m} \zeta^{(m+1)/m}\left(e^{-i\pi/m} \ln \zeta - \pi \operatorname{cosec} \frac{\pi}{m}\right) + \dots \quad ,$$

$$\dots\dots (83)$$

near B, for $m > \frac{1}{2}$ and $\neq 1$, where A_n is real and the terms omitted are of higher order than the last term on the right.

Now, near D, consider the function

$$g_k(\omega) = \frac{1}{\omega} \left(-\frac{\pi}{\ln B\omega}\right)^k \left(\ln \left(-\frac{\pi}{\ln B\omega}\right) - \frac{1}{k}\right) \quad .$$

For ω real and negative,

$$-\frac{\pi}{\ln B\omega} = \frac{\delta}{1 - i\delta} \quad,$$

so that

$$g_k(\omega) = \left(\frac{\delta^{\frac{k}{\omega}}}{\omega}\right)\left(1 + ik\delta + 0(\delta^2)\right)\left(\ln \delta - \frac{1}{k} + i\delta + 0(\delta^2)\right)$$

and

$$\mathscr{I}\{g_k\} = \left(\frac{\delta^{k+1}}{\omega}\right)\left(k \ln \delta + 1 + 0(\delta^2 \ln \delta)\right) \quad.$$

For ω real and positive, $\mathscr{I}\{g_k\} = 0$. Hence, we have

$$\mathscr{I}\{g_k\} = \begin{cases} \left(\dfrac{k\delta^{k+1}}{\omega}\right) \ln \delta(1 + 0(\delta^2 \ln \delta)) & \text{for } \omega < 0 \\[2em] 0 & \text{for } \omega > 0 \quad. \end{cases}$$

With $k = 2m + 1$, this is the behaviour specified in (81), and so

$$(v^* - iw^*)_S \sim \frac{mm'aU}{\pi(2m + 1)\omega} \left(-\frac{\pi}{\ln B\omega}\right)^{2m+1} \left\{\ln\left(-\frac{\pi}{\ln B\omega}\right) - \frac{1}{2m + 1}\right\} \quad. \tag{84}$$

This dominates any regular contribution, so $v^* - iw^* \sim (v^* - iw^*)_S$.

The next step is to obtain the tangential velocities on the sheet. Consider first the upstream side BC, near B, where ζ is small and positive. (82) and (83) show that the corresponding component in the Z_S-plane is

$$v_t^* = v^* = \sum_{n=0}^{\infty} A_n\zeta^n - \begin{cases} \dfrac{m'aU}{2\pi m} \dfrac{\zeta^{(m+1)/m}}{k_1^{2m+1}} (\ln \zeta)^2 & \text{if } m = \frac{1}{2} \\ & \text{or } 1 \\[1.5em] \dfrac{m'aU}{m \sin \frac{\pi}{m}} \dfrac{\zeta^{(m+1)/m}}{k_1^{2m+1}} \left(\cos \frac{\pi}{m} \ln \zeta - \dfrac{\pi}{\sin \frac{\pi}{m}}\right) & \text{if } m > \frac{1}{2} \\ & \text{and } \neq 1 \end{cases} \quad.$$

$$\ldots\ldots (85)$$

From this v_t follows by (54). We can avoid the detailed argument, for which the various values of m have to be considered separately, by observing that,

although the singular terms in (85) are larger than those in the previous expressions (50) to (52), they are still of higher order than the singular terms which appear in $|dZ_5/dZ_2|$, as given by (32), (34), (36) and (38). Consequently the singular terms in (85) do not appear in the expressions for v_t, which are identical with (55) to (58). Again, the dominant terms in v_t near B are the same as they would be if the sheet were a streamline of the cross-flow.

On the downstream side DC, near D, on the other hand, the tangential component is not the same as in the case of conical flow. The corresponding velocity component in the Z_5-plane, v_t^*, is $-v^* = -v_S^*$, as given by the real part of (84) for ω real and negative:

$$v_t^* = -\frac{mn'aU}{\pi(2m + 1)\omega} \delta^{2m+1}\left(\ln \delta - \frac{1}{2m + 1}\right)(1 + 0(\delta^2)) \quad .$$

Hence, by (43) and (54)

$$v_t = \frac{m'aU}{2m + 1} \delta^m\left(\ln \delta - \frac{1}{2m + 1}\right)(1 + 0(\delta^2)) \quad ,$$

or, by (45),

$$v_t = \frac{m'U\sigma}{m(2m + 1)} \left(\ln \frac{\sigma}{a} - \frac{m}{2m + 1}\right) (1 + 0(\sigma^{2/m})) \quad . \tag{86}$$

This is of lower order than the corresponding result (59) for the conical sheet.

We can now write the tangential velocity components on the sheet near B and near D for the complete non-conical problem by combining (78) times the expressions for conical flow with those just obtained for the special problem defined by (79). Near B, both expressions are the same, so the result has still the form found for the conical flow:

$$v_t\big|_{BC} = A_0'\left(1 + A_1''\sigma + (m + 1)\left(\frac{\sigma}{a}\right)^{1/m} f(\sigma) + o(\sigma^{1/m}f(\sigma))\right) , \tag{62}$$

where $A_0' \neq 0$ and A_1'' are undetermined constants (not derivatives!) and

$$f(\sigma) = \begin{cases} \left(\frac{1}{\pi}\right) \ln \sigma & \text{for } m = \frac{1}{2} \text{ or } 1 \\ \\ \cot \left(\frac{\pi}{m}\right) & \text{for } m > \frac{1}{2} \text{ and } \neq 1 . \end{cases} \tag{63}$$

Near D, by (86) and (64), we have

$$v_t\big|_{DC} = \frac{U\sigma}{2m+1}\left(\frac{m'}{m}\ln\frac{\sigma}{a} + \frac{a'}{a} - \frac{2(m+1)m'}{2m+1}\right)(1 + 0(\sigma^{2\cdot n})) \quad . \qquad (87)$$

The condition of continuity of pressure across the sheet is given in the Appendix as

$$\frac{\partial\Delta\Phi}{\partial x}\bigg|_{\theta\ const} = \Delta v_t\left(\frac{\partial r}{\partial x}\bigg|_{\theta\ const}\cos\phi - \frac{v_{t_m}}{U}\right) \quad . \qquad (A-11)$$

From (62) and (87) we can write

$$\Delta v_t = A_0'\left(1 + C_1\sigma + (m+1)\left(\frac{\sigma}{a}\right)^{1/m}f(\sigma)\right) - \frac{Um'\sigma}{m(2m+1)}\ln\frac{\sigma}{a} + \ldots \quad , \qquad (88)$$

where C_1 is another undetermined constant and the omitted terms are of higher order than the smallest term included. Integrating (88) along the sheet gives

$$\Delta\Phi - \Gamma = A_0'\left(\sigma + C_2\sigma^2 + ma\left(\frac{\sigma}{a}\right)^{(m+1)/m}f(\sigma)\right) - \frac{Um'\sigma^2}{2m(2m+1)}\ln\frac{\sigma}{a} + \ldots \quad , \qquad (89)$$

where Γ is again the total circulation of the sheet, C_2 is a constant related to C_1 and the omitted terms are of higher order. This equation expresses $\Delta\Phi$ in terms of σ and a number of quantities, A_0', C_2, m and a, which are functions of x. It is therefore helpful to write:

$$\frac{\partial\Delta\Phi}{\partial x}\bigg|_{\theta\ const} = \frac{\partial\Delta\Phi}{\partial x}\bigg|_{\sigma\ const} + \frac{\partial\Delta\Phi}{\partial\sigma}\bigg|_{x\ const}\frac{\partial\sigma}{\partial x}\bigg|_{\theta\ const} \quad .$$

Introducing this into (A-11), we have

$$\frac{\partial\Delta\Phi}{\partial x}\bigg|_{\sigma\ const} = \Delta v_t\left(\frac{\partial r}{\partial x}\bigg|_{\theta\ const}\cos\phi - \frac{\partial\sigma}{\partial x}\bigg|_{\theta\ const} - \frac{v_{t_m}}{U}\right) \quad . \qquad (90)$$

Now $\sigma = a(\theta/m)^m(1 + 0(\theta^2))$, so

$$\frac{\partial\sigma}{\partial x}\bigg|_{\theta\ const} = \frac{m'}{m}\sigma\ln\frac{\sigma}{a} + \left(\frac{a'}{a} - m'\right)\sigma + 0(\sigma^{(m+2)/m}) \quad ;$$

while, by (61) and (77),

$$\frac{\partial r}{\partial x}\bigg|_{\theta \text{ const}} \cos \phi = \left(\frac{m'}{m} \sigma \ln \frac{\sigma}{a} + \left(\frac{a'}{a} - m'\right)\sigma\right)(1 + 0(\sigma^{2/m})) \quad .$$

Hence

$$\frac{\partial r}{\partial x}\bigg|_{\theta \text{ const}} \cos \phi - \frac{\partial \sigma}{\partial x}\bigg|_{\theta \text{ const}} = 0(\sigma^{(m+2)/m} \ln \sigma) \quad . \tag{91}$$

Hence, by (89) and (91), we can write (90) as

$$2 \frac{\partial}{\partial x}\left(\Gamma + A_0'\sigma + A_0'C_2\sigma^2 + A_0'ma\left(\frac{\sigma}{a}\right)^{(m+1)/m} f(\sigma) - \frac{Um'\sigma^2}{2m(2m+1)} \ln \frac{\sigma}{a} + \ldots\right)$$

$$= (v_t|_{DC})^2 - (v_t|_{BC})^2 + 0(\sigma^{(m+2)/m} \ln \sigma) \quad , \tag{92}$$

where the derivative is for constant σ and the velocities are given by (87) and (62).

Now, by (62), $(v_t|_{BC})^2$ contains a term

$$2A_0'^2(m+1)\left(\frac{\sigma}{a}\right)^{1/m} f(\sigma) \quad . \tag{93}$$

If $m > \frac{1}{2}$ and $\neq 1$, the exponent of σ in (93) is fractional and $f(\sigma) = \cot(\pi/m)$. No other term in (92) involves σ to a fractional power as small as $1/m$, so (93) must be zero. Since $A_0' \neq 0$, $f(\sigma) = \cot(\pi/m) = 0$, and so $m = 2/3$ or 2. Hence m cannot take values which are close to $\frac{1}{2}$ or 1; but m is continuous, so m can only take the values $\frac{1}{2}$ or 1 if $m' = 0$. If $m = \frac{1}{2}$, the term (93) is $3A_0'^2(\sigma/a)^2 \ln \sigma$. The only other terms in (92) of order $\sigma^2 \ln \sigma$ have m' as a factor, so there is nothing to balance (93). The term cannot vanish and so $m = \frac{1}{2}$ is excluded. If $m = 1$, (93) is $4A_0'^2(\sigma/a) \ln \sigma$ and there is no other term of this order in (92). Hence $m = 1$ is also excluded.

The only possible values for m are therefore 2/3 and 2, just as in the case of the conical vortex sheet.

Just as the dominant terms in the tangential velocity on the upstream side of the sheet are not affected by allowing the sheet to be non-conical, we can see that the dominant terms in the velocity and pressure on the wall upstream of

the separation line are also unchanged. There are again adverse gradients of cross-flow velocity and of pressure, becoming infinite at the separation line, if $m = 2$; but no locally determined gradient if $m = 2/3$. On the wall on the downstream side, a behaviour similar to that of the conical flow will be found again, since the additional, lower-order term in the velocity field, (84), vanishes for $m' = 0$. The details will be more complicated, owing to the extra freedom available in non-conical flow, and will not be explored here.

6   EXTENSION TO CURVED WALLS IN NON-CONICAL FLOW

In this section the restriction that the wall should be plane will be replaced by the weaker restriction that it should be smoothly curved. The argument is arranged so as to exploit the results obtained already and avoid repetition of the analysis.

The framework of slender-body theory is retained, so the curved wall is the surface of a slender body at a small angle of incidence to the uniform stream. The separation line on it is assumed to be smooth and highly swept, so that there is a tangent to it at the point considered and this tangent is inclined at a small angle to the uniform stream. We choose this tangent as the axis of x , with x increasing downstream. The body is also supposed to be smooth at this point, so it has a tangent plane, and we choose this as the plane $z_1 = 0$, with z_1 increasing away from the body. The y_1-axis is chosen so that (x, y_1, z_1) form a right-handed system. Then, according to slender-body theory, the flow in the plane $x = $ constant may be described in terms of analytic functions of $Z_1 = y_1 + iz_1$.

The argument is again a local one, so that the sheet and body surface can be replaced by other curves which have the same local behaviour near the point considered. The cross-section of the smoothly-curved wall can be replaced locally by a parabola:

$$z_1 = - hy_1^2 , \qquad h = h(x) . \tag{94}$$

Then the conformal mapping

$$Z_1 = Z_2 - ihZ_2^2 \tag{95}$$

maps the region above the parabola (94) in the Z_1-plane on to the upper half of the Z_2-plane. We regard this as the region occupied by the fluid, so that the body is locally convex if $h > 0$. The Z_1 and Z_2-planes are sketched in Fig 5.

On the same basis, the sheet is replaced by the same curve in the Z_2-plane that was used in sections 4 and 5 to represent the sheet in that plane, ie

$$Z_2 = r_2 e^{i\theta_2} , \qquad r_2 = a \sin^m\left(\frac{\theta_2}{m}\right) , \qquad 0 < \theta_2 < m\theta_c , \qquad (96)$$

as in equation (8). The same sequence of transformations as before maps the region exterior to the body and sheet on to the upper half of the Z_5-plane, but we shall not need to make explicit use of these.

We must first check that a general choice of the parameters a and m (and of their variation with x) provides the desired degree of generality of sheet shape in the cross-flow plane. If the distance of the sheet from the body, measured parallel to the z_1-axis, is denoted by $f(y_1)$, then by equations (94) to (96):

$$y_1 + if(y_1) = Z_1 + ihy_1^2 = Z_2 - ihZ_2^2 + ihy_1^2$$

$$= r_2 e^{i\theta_2} - ihr_2^2 e^{2i\theta_2} + ihy_1^2 .$$

Hence, for small values of θ_2 ,

$$y_1 = r_2 \cos \theta_2 + hr_2^2 \sin 2\theta_2 \sim a\left(\frac{\theta_2}{m}\right)^m$$

and

$$f(y_1) = r_2 \sin \theta_2 - hr_2^2 \cos 2\theta_2 + hy_1^2 \sim am\left(\frac{\theta_2}{m}\right)^{m+1}$$

$$\sim am\left(\frac{y_1}{a}\right)^{(m+1)/m} .$$

This corresponds to the general form expressed by equation (5) if

$$m = \frac{1}{n-1} \qquad \text{and} \qquad a = \left(\frac{m}{\mu}\right)^m ,$$

just as in (9). The form given by (96) is therefore sufficiently general.

The first stage of the argument is to relate the normal velocity on the sheet in the Z_1-plane which is required to make it the section of a stream surface of the three-dimensional flow to the normal velocity imposed on the corresponding curve in the Z_2-plane in the previous sections. At the same time we treat the normal velocity on the wall. The stream surface boundary condition

is expressed by equation (A-4), in which the parameter held constant in forming the x-derivatives may be chosen at will.

On the wall y_2 (which equals y_1, by (95)) is held constant, so that (A-4) becomes

$$\frac{v_{n1}}{U} = \left.\frac{\partial z_1}{\partial x}\right|_{y_2} \cos \psi_1 = \left.\frac{\partial z_1}{\partial x}\right|_{y_2} \frac{dy_1}{d\sigma_1} \ ,$$

from which

$$\frac{v_{n2}}{U} = \frac{v_{n1}}{U} \left|\frac{dZ_1}{dZ_2}\right| = \frac{v_{n1}}{U} \frac{d\sigma_1}{d\sigma_2} = \left.\frac{\partial z_1}{\partial x}\right|_{y_2} \frac{dy_1}{d\sigma_2} \ ,$$

where v_{n1} and v_{n2} are the normal velocities in the Z_1 and Z_2 planes. Now $\sigma_2 = y_2 = y_1$ on the wall and z_1 is given by (94), so

$$\frac{v_{n2}}{U} = - h'y_2^2 \ . \tag{97}$$

On the sheet θ_2 is held constant, so that (A-4) gives

$$\frac{v_{n2}}{U} = \frac{v_{n1}}{U} \frac{d\sigma_1}{d\sigma_2} = \left.\frac{\partial z_1}{\partial x}\right|_{\theta_2} \frac{dy_1}{d\sigma_2} - \left.\frac{\partial y_1}{\partial x}\right|_{\theta_2} \frac{dz_1}{d\sigma_2} \ .$$

Inserting (96) into (95) gives

$$\left.\begin{aligned} y_1 &= r_2 \cos \theta_2 + hr_2^2 \sin 2\theta_2 \\[2mm] z_1 &= r_2 \sin \theta_2 - hr_2^2 \cos 2\theta_2 \ , \end{aligned}\right\} \tag{98}$$

from which

$$\left.\begin{aligned} \frac{\partial y_1}{\partial x} &= (\cos \theta_2 + 2hr_2 \sin 2\theta_2) \frac{\partial r_2}{\partial x} + h'r_2^2 \sin 2\theta_2 \\[2mm] \frac{\partial z_1}{\partial x} &= (\sin \theta_2 - 2hr_2 \cos 2\theta_2) \frac{\partial r_2}{\partial x} - h'r_2^2 \cos 2\theta_2 \ . \end{aligned}\right\} \tag{99}$$

Also, remembering that $\cos \phi_2 = dr_2/d\sigma_2$ and $\sin \phi_2 = r_2 d\theta_2/d\sigma_2$, and that $\psi_2 = \theta_2 + \phi_2$, where ψ_2 is the inclination of the section of the sheet to the real axis, we have

$$\left.\begin{array}{l} \dfrac{dy_1}{d\sigma_2} = \cos \psi_2 + 2hr_2 \sin (\theta_2 + \psi_2) \\[4mm] \dfrac{dz_1}{d\sigma_2} = \sin \psi_2 - 2hr_2 \cos (\theta_2 + \psi_2) \end{array}\right\} \tag{100}$$

Introducing (99) and (100) into the boundary condition above yields

$$\frac{v_{n2}}{U} = - \left(1 + 4hr_2 \sin \theta_2 + 4h^2 r_2^2\right) \frac{\partial r_2}{\partial x} \sin \phi_2$$
$$- h'r_2^2 \left(\cos (\theta_2 - \phi_2) + 2hr_2 \sin \phi_2\right) . \tag{101}$$

To find the dominant terms in (101), all the quantities need to be expressed in terms of one, say θ_2. By (96)

$$\frac{d\sigma_2}{d\theta_2} = \left(r_2^2 + \left(\frac{dr_2}{d\theta_2}\right)^2\right)^{\frac{1}{2}} = a \sin^{m-1} \frac{\theta_2}{m} ,$$

so that $\sin \phi_2 = r_2 d\theta_2/d\sigma_2 = \sin (\theta_2/m)$,

$$\phi_2 = \frac{\theta_2}{m} \tag{102}$$

and

$$\sigma_2 \sim a\left(\frac{\theta_2}{m}\right)^m . \tag{103}$$

$\partial r_2/\partial x$ is given by (7) with the omitted suffix 2 restored and r_2 itself is given by (96). Neglecting $\theta_2^2 \sim m^2 (\sigma_2/a)^{2/m}$ in comparison with unity, as in the previous analysis (cf. (29) and (80)), we can write (101) as

$$-\frac{v_{n2}}{U} = \left(\frac{\theta_2}{m}\right)^{m+1} \left(\ln \frac{\theta_2}{m} + a' - am'\right) \left(1 + 4ham\left(\frac{\theta_2}{m}\right)^{m+1} + 4h^2 a^2\left(\frac{\theta_2}{m}\right)^{2m}\right)$$
$$+ h'a^2\left(\frac{\theta_2}{m}\right)^{2m}\left(1 + 2ha\left(\frac{\theta_2}{m}\right)^{m+1}\right) .$$

46

From this we can pick out the dominant terms in (101):

$$\frac{v_{n2}}{U} \sim \frac{\partial r_2}{\partial x} \sin \phi_2 - h'\sigma_2^2 \ . \tag{104}$$

Which of these two dominates depends on the value of m, so both are retained, using a small extension of the tilde notation.

The normal velocities specified by (97) and (104) are the sum of two contributions:

(a) $v_{n2} = 0$ on the wall and $\dfrac{v_{n2}}{U} = - \dfrac{\partial r_2}{\partial x} \sin \phi_2$ on the sheet, and

(b) $\dfrac{v_{n2}}{U} = -h'y_2^2$ on the wall and $\dfrac{v_{n2}}{U} = -h'\sigma_2^2$ on the sheet.

The linearity of the relation between the normal and tangential velocity distributions can be exploited, as it was in section 5, so that the complete tangential velocities can be obtained as the sum of those corresponding to distributions (a) and (b). Now distribution (a) is precisely that treated in sections 4 and 5, arising from the sheet on the plane wall. For distribution (b), we guess a complex potential

$$W_b = ih'UZ_2^3\big/3 \tag{105}$$

and verify that it gives the right behaviour on the wall and sheet. For the wall, it is enough to differentiate W_b and set $Z_2 = y_2$, real:

$$\frac{v_{n2}}{U} = \frac{w_2}{U} = -\mathscr{I}\left\{\frac{1}{U}\frac{dW_b}{dZ_2}\right\} = -\mathscr{I}\left\{ih'y_2^2\right\} = -h'y_2^2 \ .$$

This verifies the behaviour on the wall. For the contribution from W_b on the sheet, it is convenient to write

$$v_{t2} - iv_{n2} = \frac{dW_b}{d\sigma_2} = \frac{dW_b}{dZ_2}\frac{dZ_2}{d\sigma_2} \ .$$

Now $dZ_2/d\sigma_2 = e^{i\psi_2} = e^{i(m+1)\theta_2/m}$, by (102). Hence

$$v_{t2} - iv_{n2} = ih'Ur_2^2 e^{i(3m+1)\theta_2/m} \ , \tag{106}$$

so that

$$\frac{v_{n2}}{U} = - h' \sigma_2^2 \left(1 + 0(\sigma_2^{2/m})\right) \quad ,$$

and the behaviour on the sheet is verified, to the usual approximation. Hence, to leading order, the tangential velocities in the Z_2-plane are those found in the previous section, augmented by those derived from (105).

The final step is to consider the compatibility of these tangential velocities with the condition of continuity of pressure across the sheet. This condition can be written in the form (A-10), and it is again convenient to choose the parameter η to be θ_2 . Then:

$$\left.\frac{\partial \Delta\Phi}{\partial x}\right|_{\theta_2} = \Delta v_{t1}\left(\left.\frac{\partial y_1}{\partial x}\right|_{\theta_2} \cos \psi_1 + \left.\frac{\partial z_1}{\partial x}\right|_{\theta_2} \sin \psi_1 - \frac{v_{tm1}}{U}\right) , \qquad (107)$$

where the subscripts '1' indicate quantities in the Z_1-plane. In section 5 an expression, (89), was found for $\Delta\Phi$ in terms of σ . Since the extra velocity (106) is continuous across the sheet, this expression still holds, to the order considered, with σ replaced by σ_2 . It is therefore convenient to consider, as before, its derivative for constant σ_2 , and write

$$\left.\frac{\partial \Delta\Phi}{\partial x}\right|_{\theta_2} = \left.\frac{\partial \Delta\Phi}{\partial x}\right|_{\sigma_2} + \left.\frac{\partial \Delta\Phi}{\partial \sigma_2}\right|_x \left.\frac{\partial \sigma_2}{\partial x}\right|_{\theta_2} \quad .$$

The velocity components in the Z_1 and Z_2-planes are related by

$$\Delta v_{t1} = \Delta v_{t2}\left(\frac{d\sigma_2}{d\sigma_1}\right) \qquad \text{and} \qquad v_{tm1} = v_{tm2}\left(\frac{d\sigma_2}{d\sigma_1}\right) .$$

With these relations, (107) becomes:

$$\left.\frac{\partial \Delta\Phi}{\partial x}\right|_{\sigma_2} = \Delta v_{t2}\left(\frac{d\sigma_2}{d\sigma_1}\right)^2\left(\left.\frac{\partial y_1}{\partial x}\right|_{\theta_2} \frac{dy_1}{d\sigma_2} + \left.\frac{\partial z_1}{\partial x}\right|_{\theta_2} \frac{dz_1}{d\sigma_2} - \left(\frac{d\sigma_1}{d\sigma_2}\right)^2 \left.\frac{\partial \sigma_2}{\partial x}\right|_{\theta_2} - \frac{v_{tm2}}{U}\right) . \quad (108)$$

Using equations (99) and (100) and the relation

$$\left(\frac{d\sigma_1}{d\sigma_2}\right)^2 = 1 + 4hr_2 \sin \theta_2 + 4h^2 r_2^2 \quad , \qquad (109)$$

which holds on the sheet, we find

$$\frac{\partial y_1}{\partial x}\bigg|_{\theta_2} \frac{dy_1}{d\sigma_2} + \frac{\partial z_1}{\partial x}\bigg|_{\theta_2} \frac{dz_1}{d\sigma_2} = \left(\frac{d\sigma_1}{d\sigma_2}\right)^2 \frac{\partial r_2}{\partial x} \cos \phi_2 + h'r_2^2\Big(\sin (\theta_2 - \phi_2) + 2hr_2 \cos \phi_2\Big).$$

$$\ldots\ldots (110)$$

Introducing (110) into (108) and recalling (91), we have

$$\frac{\partial \Delta\Phi}{\partial x}\bigg|_{\sigma_2} = \Delta v_{t2}\left(\frac{d\sigma_2}{d\sigma_1}\right)^2 \left(h'r_2^2(\sin (\theta_2 - \phi_2) + 2hr_2 \cos \phi_2) - \frac{v_{tm2}}{U}\right.$$
$$\left. + 0\left(\sigma_2^{(m+2)/m} \ln \sigma\right)\right) \quad . \quad (111)$$

The additional tangential component of velocity deriving from W_b is given by (106), so that

$$\frac{v_{tm2}}{U} = \frac{v_{tm}}{U} - h'r_2^2 \sin \frac{3m + 1}{m} \theta_2 \qquad \text{and} \qquad \Delta v_{t2} = \Delta v_t ,$$

where v_{tn} and Δv_t are just as they were in section 5. Using (103) and (109) again, we find the condition of pressure continuity becomes

$$\frac{\partial \Delta\Phi}{\partial x}\bigg|_{\sigma_2} = \Delta v_t\left(1 + 4hr_2 \sin \theta_2 + 4h^2r_2^2\right)\left(-\frac{v_{tm}}{U} + h'r_2^2 \sin \frac{3m + 1}{m} \theta_2\right.$$
$$\left. + h'r_2^2 \sin \frac{m - 1}{m} \theta_2 + 2hh'r_2^3 \cos \frac{\theta_2}{m} + 0\left(\sigma_2^{(m+2)/m} \ln \sigma\right)\right) . \quad (112)$$

The corresponding equation in section 5 is (92), which can be written

$$\frac{\partial \Delta\Phi}{\partial x}\bigg|_{\sigma_2} = \Delta v_t\left(- \frac{v_{tm}}{U} + 0(\sigma_2^{(m+2)/m} \ln \sigma)\right) . \quad (113)$$

It was shown that this contains an unbalanced term of order $\sigma^{1/m}$ if $m > \frac{1}{2}$ and $m \neq 1$ or $\sigma^{1/m} \ln \sigma$ if $m = \frac{1}{2}$ or 1. Recalling that Δv_t and v_{tm} are both of order unity, we can see that the orders of the extra terms which appear in (112) and not in (113) are too high for them to balance this crucial term. It must therefore vanish, and the argument in section 5 which follows equation (92) again holds, showing that the only possible values of m in the range $\frac{1}{2} \leqslant m < \infty$ are still 2/3 and 2.

The extra freedom afforded by the smoothly curved wall is not therefore sufficient to allow a different behaviour of the sheet close to the separation line. Further, since the additional velocity field defined by (105) and (95) is regular, the singularity in the pressure gradient on the wall found previously on the upstream side of the separation line for $m = 2$ ($n = 3/2$) will still arise.

The shape of the sheet is described most simply in terms of its distance $f(y_1)$ from the wall, y_1 being the same, to the first approximation, as the distance along the wall. We have

$$f(y_1) \sim an\left(\frac{y_1}{a}\right)^n ,$$

where $n = 1 + 1/m$ is $3/2$ for $m = 2$ and $5/2$ for $m = 2/3$. In terms of the Cartesian coordinates y_1 and z_1 measured parallel and perpendicular to the tangent to the body cross-section (see Fig 5), the shape is given by equations (95) and (96) as

$$z_1 \sim 2a\left(\frac{y_1}{a}\right)^{3/2} \qquad \text{for } m = 2$$

$$z_1 + hy_1^2 \sim \frac{2}{3} a\left(\frac{y_1}{a}\right)^{5/2} \qquad \text{for } m = 2/3 .$$

7   RESULTS AND CONJECTURES

7.1   Main results

It has been shown that, under the assumptions of slender-body theory, the shape of a vortex sheet which contains circulation being shed from a highly-swept separation line on a smoothly curved wall in steady flow is restricted in form. If, in the vicinity of the separation line, the distance of the sheet from the wall increases like y^n, where y is the distance from the separation line, then the only possible values of n not exceeding 3 are $3/2$ and $5/2$. If $n = 3/2$, there is an adverse pressure gradient on the wall upstream of the separation line which becomes infinite at the separation line itself. If $n = 5/2$, the pressure gradient at the separation line remains finite. In both cases the strength of the vortex sheet (its circulation per unit length) varies linearly with the arc length along the sheet, close to the separation line.

For a conical sheet on a plane wall, the inviscid flow on the downstream side of the separation line is favourable to the growth of an attached boundary layer, so that it is only the boundary layer on the upstream side which separates.

These results are expected to be helpful in numerical treatments of vortex sheet models of separated flow and in understanding the mechanics of flow separation in three dimensions.

7.2 Equivalent results for unsteady plane flow

There is an exact analogy between slender-body theory for steady flow and the theory of the time-dependent, two-dimensional flow of an inviscid incompressible fluid. The time, t, in the plane flow is related to the downstream distance, x, in the steady three-dimensional flow of undisturbed speed U by

$$t = \frac{x}{U} . \tag{114}$$

The plane flow takes place in the cross-flow plane. Variations in the cross-section of the slender body correspond to changes with time in the shape of the body in the plane flow and variations in its inclination to the undisturbed stream correspond to changes in the relative velocity between the body and the fluid in the plane flow. The term $\partial\Phi/\partial t$ which arises in the time-dependent form of Bernoulli's equation corresponds to the contribution from the streamwise perturbation velocity $u = \partial\Phi/\partial x$ to the slender body form (75) of the three-dimensional Bernoulli relation. It is straightforward to verify that the boundary conditions on the vortex sheets correspond through equation (114).

The results of the present work therefore hold also for the behaviour of a vortex sheet representing an 'open' type of separated flow in plane, incompressible time-dependent flow, whether the time variations are in the imposed velocity, the body shape or the position of the separation point. In particular, the same two exponents 3/2 and 5/2 are the only ones (not greater than 3) which can occur in flows of this kind, and the smaller is inevitably associated with an infinite adverse pressure gradient.

7.3 Relation to previous results for plane flow

A considerable amount of work, dating back to Helmholtz and Kirchhoff, has been done on steady two-dimensional flows past a body from which spring a pair of constant-pressure streamlines. Since, in this context, constant pressure implies constant speed, hodograph methods can be used and, with their help, complete flowfields can be described analytically. A summary account, with

references to the original papers, is given by Thwaites[11]. The constant-pressure streamlines leave the body surface tangentially and, in some sense, enclose a finite or infinite region downstream of the body. The fluid inside this region must be at rest at the separation point itself, since it lies between the constant-pressure streamline and the downstream side of the body, which meet at a cusp. Since the static pressure is continuous, the constant pressure on the separation streamlines must be the stagnation or total pressure in the downstream region. The total pressure is therefore different in the downstream region from its value in the main flow and the separated flow cannot be of the 'open' type discussed in the present Report. There are two conventional interpretations of the downstream region: either the fluid there is the same as the mainstream fluid and it is at rest (a deadwater region), or, more plausibly, it is occupied by a fluid whose density is negligible compared to that of the mainstream fluid (a cavity in a liquid, filled by its vapour or another gas). Neither of these interpretations seems to have any aerodynamic significance in itself; but, since the constant-pressure streamlines are particular cases of vortex sheets springing from a smooth body, their behaviour near the separation point invites comparison with the present results.

In the vicinity of the separation point, equations (168) and (169) of Ref 11 describe the behaviour of the curvature, κ , of the streamline and the upstream pressure gradient on the wall. In condensed form:

$$\kappa = T(\Phi - \Phi_2)^{-\frac{1}{2}} + 0(\Phi - \Phi_2)^{\frac{1}{2}}$$

$$\frac{dC_p}{ds} = 2T(\Phi_2 - \Phi)^{-\frac{1}{2}} + 0(\Phi_2 - \Phi)^{\frac{1}{2}} \quad ,$$

where T is a constant, s is the distance along the body surface in the direction of the upstream flow and Φ is the velocity potential on the sheet or body, taking the value Φ_2 at the separation point. We can see that if $T > 0$ the curvature and pressure gradient behave in the same way as the present results predict for $n = 3/2$. If $T < 0$, the solution is meaningless, as the constant-pressure streamline lies initially within the body. If $T = 0$, both the curvature and pressure gradient vanish. The present results are that the curvature and pressure gradient are finite for $n = 5/2$; the curvature is that of the body and the pressure gradient is not locally determined. Hence the previous results for the neighbourhood of the separation point reduce almost, but not quite, to a special case of the present analysis. The difference concerns the curvature of the sheet in the case for which it is finite: the constant-pressure streamline

has zero curvature, the 'open' type of vortex sheet has the same curvature as the body. It is plausible that such a difference should arise, since the flow between the sheet and the body, on the downstream side of the separation point, does not enter the previous analysis.

The previous results also suggest the role played by the solutions for n = 3/2 and 5/2. To see this, we confine our attention to the so-called Kirchhoff flow past a circular cylinder, the flow for which the pressure on the constant-pressure streamline is the same as the pressure in the undisturbed stream, as illustrated in Fig IV.14 of Ref 11. There is a one-parameter family of these flows, corresponding to a variation in the position of the separation points on the cylinder. If the position of separation is specified by its angular displacement θ from the front stagnation point, then for $\theta > \theta_S = 56^O$, it is found that $T > 0$. As θ falls towards θ_S, T falls, and vanishes when $\theta = \theta_S$. Separation which occurs with $T = 0$ is called 'smooth'. Suppose, now, that we find a vortex sheet representation of the separated flow past a slender cone (of general cross-section) with an assumed inviscid separation line along a generator chosen at random; then it seems likely that the behaviour near the separation line will correspond to $n = 3/2$. The behaviour corresponding to $n = 5/2$ is not likely to arise for more than a finite number of positions of the separation line, though such positions may well be significant in relation to the real flow, as discussed in subsection 7.5. Vortex sheets which lie even closer to the surface, which might arise from values of n greater than 3, seem unlikely to occur at all.

7.4 Range of validity of present results

The detailed results of the present work have been obtained within the assumptions of slender-body theory. It is relevant to ask whether they hold more widely.

On the one hand, the proof in section 2 that a vortex sheet representing separation must be tangential to the wall depends crucially on the equality of the entropy on the two sides of the sheet. This clearly sets a limit to the extent to which the later results about the flow near the separation line can be generalized. On the other hand, since the arguments used in obtaining these results are local ones, it seems unlikely that the global limitations required for the application of slender-body theory are actually necessary. This impression is confirmed by the applicability of the present results to plane flow, a two-dimensional body being completely 'non-slender'. Further, the applicability

of the present results to unsteady plane flow suggests that the initial limitation to steady flow was unnecessary.

It seems that it might well be possible to extend the present results to unsteady incompressible flow past general bodies, and, perhaps, to compressible flows in small-disturbance approximations in which entropy changes are negligible.

7.5 Prediction of separation line

Studies of inviscid flow, like the present, are meant to lead to useful models of real flows. As suggested in section 1, vortex sheet models of flows separating at highly-swept separation lines are expected to be adequate if the position of the separation line can be determined. In Ref 3 it was suggested that this should be attacked by an iteration between a boundary-layer calculation which uses a given external velocity field to predict the location of separation and a calculation of an inviscid model of the separated flow which uses a given separation line to predict the external field. For turbulent boundary layers this still appears to be the way to proceed, but for laminar flows there may be an alternative procedure.

The resemblance between the present results for separation from highly-swept separation lines and earlier results for the constant-pressure streamline model of two-dimensional separation suggests that an analogue of Sychev's work[12] for plane flows may exist for highly-swept separation lines. Sychev seeks a local solution, valid near the separation point, to the Navier-Stokes equations for the steady flow of an incompressible, viscous fluid past a smooth body in two dimensions. He finds the structure of such a solution in the form of an asymptotic expansion in fractional powers of the inverse of the Reynolds number. As the Reynolds number tends to infinity, the solution tends towards an inviscid flow with a constant-pressure streamline which separates smoothly from the wall. For finite Reynolds number, the point of separation is further downstream, by a distance proportional to $Re^{-1/16}$. The constant of proportionality remains to be determined by a numerical calculation, which is required, in principle, to confirm that a solution with this structure actually exists.

If such an analogue of Sychev's work does exist, and preliminary work by N. Riley at the University of East Anglia indicates that it may do so for conical external flows, then the outcome of the iteration proposed above is predictable: since laminar boundary layer theory represents the limit of infinite Reynolds number, it is smooth separation which will emerge. This would be the best that could be achieved with a boundary-layer calculation and it could be achieved without it, by examining the behaviour of the inviscid model. Furthermore, if

the numerical solution to the analogue of Sychev's problem were known, the leading term (of order $Re^{-1/16}$) in the correction to this infinite Reynolds number solution could be found, and the inviscid model corresponding to this corrected position of separation could be calculated for large, but finite, Reynolds numbers.

This is all speculative at present, but already has a bearing on the construction of an inviscid model of the flow separating from highly-swept separation lines on smooth surfaces. If such a model is intended eventually to form part of a method which predicts the position of the separation line,

(a) it must represent the shed vorticity near the separation line in a continuous form, ie a vortex sheet representation is the simplest likely to be useful,

(b) the curvature of the sheet must be allowed to tend to infinity like the inverse square root of the distance from the separation line, and

(c) the coefficient of this singularity should be determined explicitly by the calculation procedure.

Finally, since the constant-pressure streamline model has figured so prominently in this section, its role should perhaps be clarified. The resemblances between the mathematical properties of the vortex sheet model of open separation (whether in steady flow from a highly-swept separation line or in unsteady two-dimensional flow) and the constant-pressure streamline model of steady two-dimensional separated or cavity flow have been noted and partially exploited. It is not suggested that the constant-pressure streamline model has the same relevance to closed separation in two-dimensional steady flow as the vortex sheet model is believed to have to open separation. Indeed, a more satisfactory approach to two-dimensional separation would be to recognize its essentially unsteady character and treat it as a time-dependent problem involving vortex sheets.

Acknowledgment

Appendix

BOUNDARY CONDITIONS

In this Appendix the boundary conditions on the body and on the steady vortex sheet are expressed in terms of quantities in a cross-flow plane, under the usual assumption of small disturbances. The expressions obtained are equivalent to those given by Smith[4] and Clarke[5]; but their derivation is a little more general, in that the axis system is no longer required to be exactly aligned with the undisturbed stream, and a little simpler, in that vector algebra is avoided.

The boundary conditions are that the body and the vortex sheet are stream surfaces of the three-dimensional flow and that the pressure is continuous across the vortex sheet. Since the sheet is assumed to be embedded in an irrotational flow, the continuity of pressure implies the continuity of speed across it.

Consider a right-handed system of rectangular Cartesian axes with Ox inclined at a small angle λ to the undisturbed stream of speed U. Denote the components of the velocity parallel to the axes Ox, Oy and Oz by $U \cos \lambda + u$, v and w, so that u, v and w are small compared with U by the assumptions of small disturbances and small inclination. Suppose the vortex sheet or body surface, Σ, is defined by $F(x,y,z) = 0$ and also, in parametric form, by $y = f(x,\eta), z = g(x,\eta)$, where η is a parameter which varies along the curve $\mathscr{C}$ in which the cross-flow plane $x = $ constant meets Σ. Then

$$F(x, f(x,\eta), g(x,\eta)) \equiv 0 ,$$

where the identity sign means that equality holds for all values of x and η in the ranges which define Σ. Hence the identity can be differentiated with respect to x and η to give:

$$F_x + F_y f_x + F_z g_x = 0 \tag{A-1}$$

$$F_y f_\eta + F_z g_\eta = 0 . \tag{A-2}$$

Since Σ is a stream surface of the three-dimensional flow:

$$(U \cos \lambda + u)F_x + vF_y + wF_z = 0 ,$$

or, since λ is small and u is small compared with U ,

$$UF_x + vF_y + wF_z = 0 . \tag{A-3}$$

The three equations (A-1) to (A-3) are linear and homogeneous in F_x, F_y and F_z, and they have a non-trivial solution, so the determinant of the coefficients vanishes. Expanding the determinant gives:

$$wf_\eta - vg_\eta = U(f_\eta g_x - g_\eta f_x) .$$

This provides an equation for the component v_n of the velocity in the cross-flow plane, normal to $\mathscr{C}$. If the sense of the normal is given by a positive (anticlockwise) rotation from the direction of increasing η,

$$v_n = w \cos \psi - v \sin \psi = (wf_\eta - vg_\eta)/(f_\eta^2 + g_\eta^2)^{\frac{1}{2}} ,$$

where ψ is the inclination of the tangent, in the sense of η increasing, to the axis Oy, and where the positive value of the radical is understood. Hence

$$\frac{v_n}{U} = \frac{f_\eta g_x - g_\eta f_x}{(f_\eta^2 + g_\eta^2)^{\frac{1}{2}}}$$

$$= \frac{\partial z}{\partial x} \cos \psi - \frac{\partial y}{\partial x} \sin \psi . \tag{A-4}$$

The parameter η does not appear explicitly in (A-4), but the same parameter must be held constant in the two partial derivatives. Particular cases of (A-4) are

$$\frac{v_n}{U} = \left.\frac{\partial z}{\partial x}\right|_y \cos \psi = -\left.\frac{\partial y}{\partial x}\right|_z \sin \psi = -\left.\frac{\partial r}{\partial x}\right|_\theta \sin \phi , \tag{A-5}$$

where (r,θ) are polar coordinates and ϕ is the inclination of the tangent to the radius vector, as in Fig 3.

It is sometimes convenient to work with the stream function, Ψ, in the cross-flow plane. If the velocity potential Φ is the real part of the complex potential $W(Z)$, where $Z = y + iz$, then Ψ is the imaginary part of W. Hence, on $\mathscr{C}$

$$\frac{\partial \Psi}{\partial \sigma} = -\frac{\partial \Phi}{\partial n} = -v_n .$$

Hence, by (A-4):

$$\Psi = U\int\left(\frac{\partial y}{\partial x}\sin\psi - \frac{\partial z}{\partial x}\cos\psi\right)d\sigma = U\int_{\mathscr{C}}\left(\frac{\partial y}{\partial x}dz - \frac{\partial z}{\partial x}dy\right) . \tag{A-6}$$

Equations (A-4) to (A-6) are equivalent expressions of the condition that Σ is a stream surface.

The second boundary condition applying when Σ is a vortex sheet is that the fluid speed is continuous across Σ, ie

$$\Delta\left((U\cos\lambda + u)^2 + v^2 + w^2\right) = 0 ,$$

where Δ is a difference operator across Σ. To be definite, Δ is chosen so that $\Delta A = A_1 - A_2$, where the suffix 1 denotes the side of Σ towards which the normal points and 2 the opposite side. Because λ and the disturbances are small, this condition reduces to

$$2U\Delta u + \Delta(v^2 + w^2) = 0 . \tag{A-7}$$

Now, if v_t is the component of velocity in the cross-flow plane, tangential to $\mathscr{C}$, in the sense of η increasing,

$$v^2 + w^2 = v_t^2 + v_n^2 ;$$

while, to the order of accuracy of this analysis, (A-4) shows that v_n is continuous across Σ. Hence (A-7) reduces to

$$U\Delta u + v_{t_m}\Delta v_t = 0 , \tag{A-8}$$

where v_{t_m} is the mean value of v_t across Σ. Now, if the velocity potential in the fluid on the two sides of Σ is $\Phi_1(x,y,z)$ and $\Phi_2(x,y,z)$,

$$\Delta u = \Phi_{1_x} - \Phi_{2_x} = \Delta\Phi_x\big|_{y,z} .$$

On the other hand, the jump in Φ across Σ is

$$\Delta\Phi = \Phi_1\big(x, f(x,\eta), g(x,\eta)\big) - \Phi_2\big(x, f(x,\eta), g(x,\eta)\big)$$

therefore
$$\left.\frac{\partial \Delta \Phi}{\partial x}\right|_{\eta} = \Delta \Phi_x + f_x \Delta \Phi_y + g_x \Delta \Phi_z$$

$$= \Delta u + f_x \Delta v + g_x \Delta w \quad . \tag{A-9}$$

Since v_n is continuous across Σ,

$$\Delta v = \Delta v_t \cos \psi \qquad \text{and} \qquad \Delta w = \Delta v_t \sin \psi \quad ,$$

so that, by (A-9),

$$\Delta u = \frac{\partial \Delta \Phi}{\partial x} - \Delta v_t (f_x \cos \psi + g_x \sin \psi) \quad .$$

Hence, the condition of continuity of pressure, (A-8), becomes

$$\frac{\partial \Delta \Phi}{\partial x} = \Delta v_t \left(\frac{\partial y}{\partial x} \cos \psi + \frac{\partial z}{\partial x} \sin \psi - \frac{v_{t_m}}{U} \right) \quad . \tag{A-10}$$

As in (A-5), the parameter η does not appear explicitly in the boundary condition, but the partial derivatives on both sides of (A-10) must be calculated for fixed values of the same parameter. If, for example, the polar angle θ is fixed, (A-10) becomes

$$\left.\frac{\partial \Delta \Phi}{\partial x}\right|_{\theta} = \Delta v_t \left(\left.\frac{\partial r}{\partial x}\right|_{\theta} \cos \phi - \frac{v_{t_m}}{U} \right) \quad . \tag{A-11}$$

For conical flow, with $s = Kx$, this reduces to the familiar form

$$\Delta \Phi = \Delta v_t \left(r \cos \phi - \frac{s v_{t_m}}{KU} \right) \quad .$$

Similarly, Clark's equations, (A-18) and (A-19) of Ref 5, are obtained from (A-5) and (A-10) when the sheet is defined in terms of his parameters.

LIST OF SYMBOLS

a	diameter of circular arc in Z_3-plane
A_0, A_1, A_1' ...	undetermined real constants
b, c, d	coordinates of points B, C, D in Z_5-plane
B	arbitary positive constant
C_p	pressure coefficient, $(p - p_\infty)/\frac{1}{2}\rho U^2$.
$\mathscr{C}$	curve in which Σ meets the cross-flow plane
F	mapping function
h	parameter describing local shape of wall, see (94)
k_1, k_2, k_3	constants defined by (14a)
m	$= 1/(n - 1)$, exponent, see (6)
n	exponent defining order of contact, see (5); also normal to $\mathscr{C}$
p	pressure
q	source strength
r	polar distance in cross-flow plane
t	time
U	speed of undisturbed stream
u, v, w	disturbance velocity components parallel to Ox, y, z
v_n, v_t	normal and tangential components of cross-flow velocity
v_{t_m}	mean value of v_t across vortex sheet
W	complex potential
x, y, z	right-handed, rectangular Cartesian axes
Z	$y + iz$, cross-flow plane
Z^*	final transformed plane
Γ	total circulation of sheet
δ	small positive quantity, see (42)
Δ	difference operator across vortex sheet
ζ	small complex quantity, see (13)
η	parameter on $\mathscr{C}$ in Appendix; dummy variable on real axis
θ	polar angle in cross-flow plane
λ	inclination of x-axis to undisturbed stream
ξ	dummy variable on real axis
ρ	density; $|\zeta|$
σ	arc length on $\mathscr{C}$
Σ	surface of body or vortex sheet
ϕ	inclination of tangent to $\mathscr{C}$ to radius vector
Φ	velocity potential

LIST OF SYMBOLS (concluded)

ψ inclination of tangent to $\mathscr{C}$ to Oy

Ψ stream function in cross-flow plane

ω small complex quantity, see (25)

affix * denotes a quantity in the transformed plane Z^* or Z_5

suffixes 1, 2, 3, 4, 5 denote quantities in successive transformed planes

suffixes S, R denote singular and regular contributions

REFERENCES

No.	Author	Title, etc
1	J.A. Weber G.W. Brune F.T. Johnson P. Lu P.E. Rubbert	Three-dimensional solution of flows over wings with leading-edge vortex separation. AIAA Journal 14, 4, 519-25 (1976)
2	E.S. Levinsky M.H.Y. Wei	Nonlinear lift and pressure distribution on slender conical bodies with strakes at low speeds. NASA CR 1202 (1968)
3	J.H.B. Smith	A review of separation in steady, three-dimensional flow. AGARD-CP-168, paper 31 (1975) RAE Technical Memorandum Aero 1620 (1975)
4	J.H.B. Smith	Boundary conditions on a vortex sheet. RAE Technical Memorandum Aero 1368 (1971)
5	R.W. Clark	Non-conical flow past slender wings with leading-edge vortex sheets. RAE Technical Report 76037 (1976)
6	G.J. Clapworthy K.W. Mangler	The behaviour of a conical vortex sheet on a slender wing near the leading edge. RAE Technical Report 74150 (ARC R & M 3790) (1974)
7	C. Nebbeling W.J. Bannink	Experimental investigation of the supersonic flow field about a slender cone at high incidences. Delft Univ. of Techn., Dept. of Aero. Eng., Report LR-233 (1976)
8	J.A. Lawford	Low-speed wind-tunnel experiments on a series of sharp-edged delta wings. Part II: surface flow patterns and boundary layer transition measurements. RAE Technical Note Aero 2954 (ARC R & M 3424) (1964)
9	D.J. Peake	Three-dimensional swept shock/turbulent boundary-layer separations with control by air injection. NAE, Canada, Report LR-592 (1976)
10	B. Oskam I.E. Vas S.M. Bogdonoff	An exploratory study of a three-dimensional shock-wave boundary layer interaction at Mach 3. AGARD-CP-168, paper 41 (1975)

62

REFERENCES (concluded)

No.	Author	Title, etc
11	B. Thwaites (ed.)	Incompressible aerodynamics, Chapter IV. Oxford University Press (1960)
12	V.V. Sychev	On laminar separation. Izv. Akad. Nauk, Mekh. Zhid. i Gaza, 3, 47-59 (1972) Translated in Fluid Dynamics, Plenum Pub. Corp., 407-417, March-April 1974
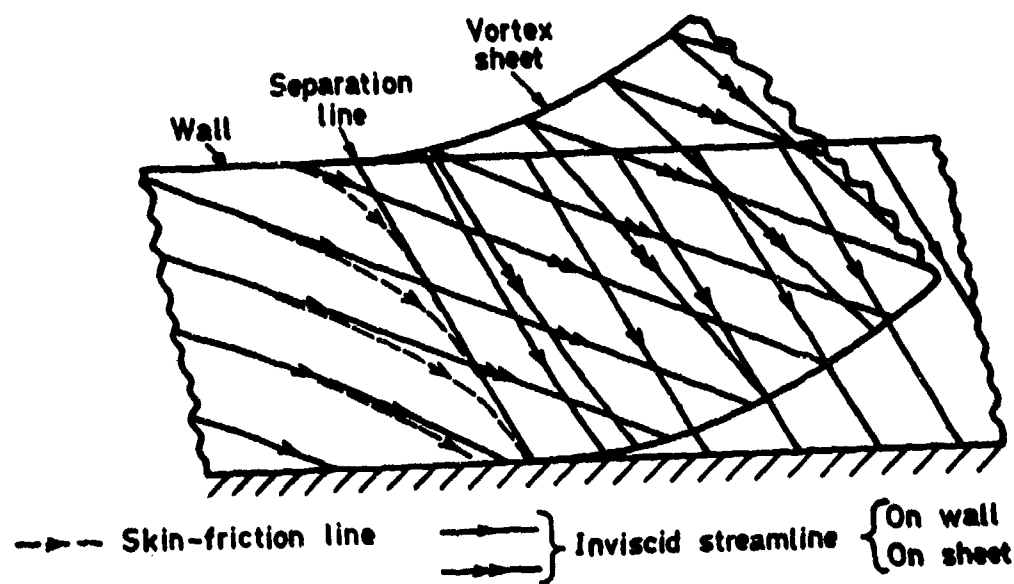
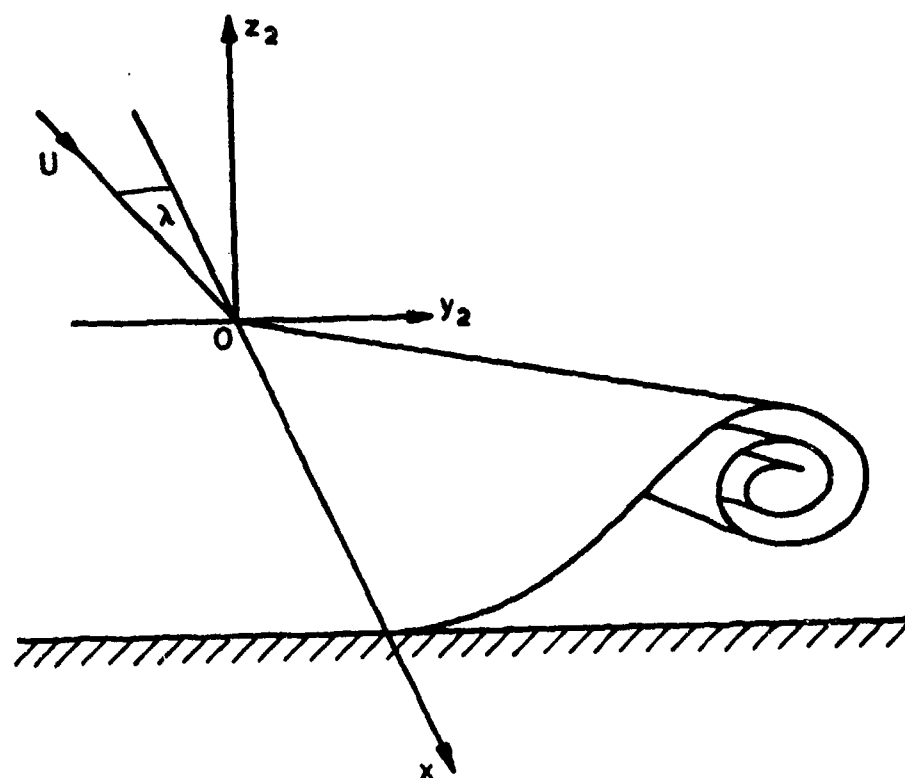Fig 1   Section of wall, separation line and vortex sheet, showing surface streamlines

Fig 2   Conical sheet on plane wall, with coordinates
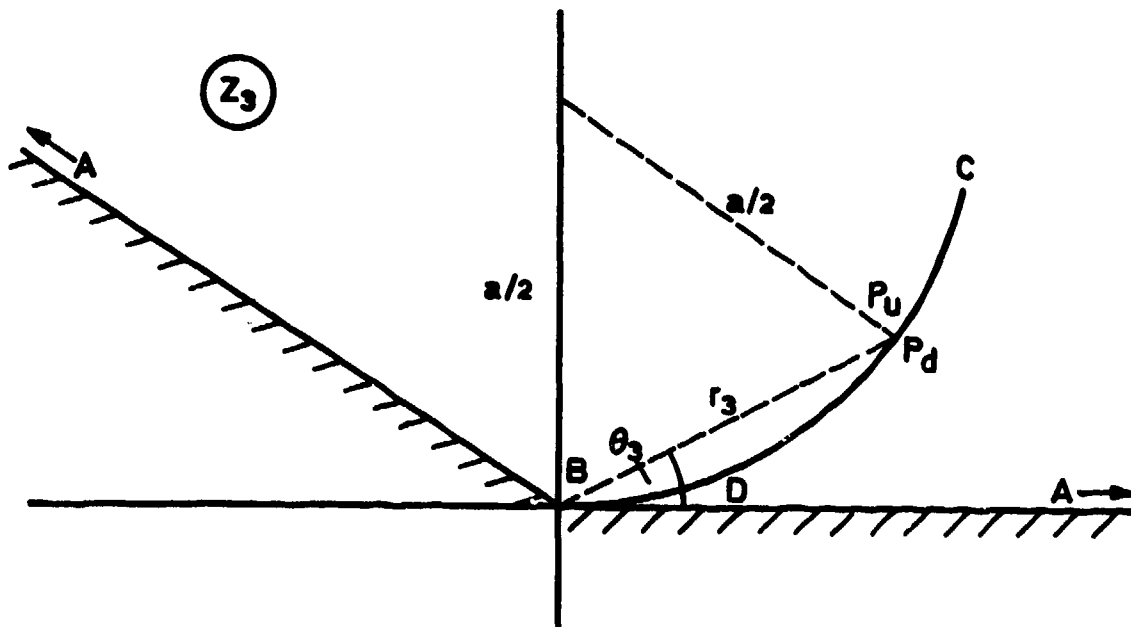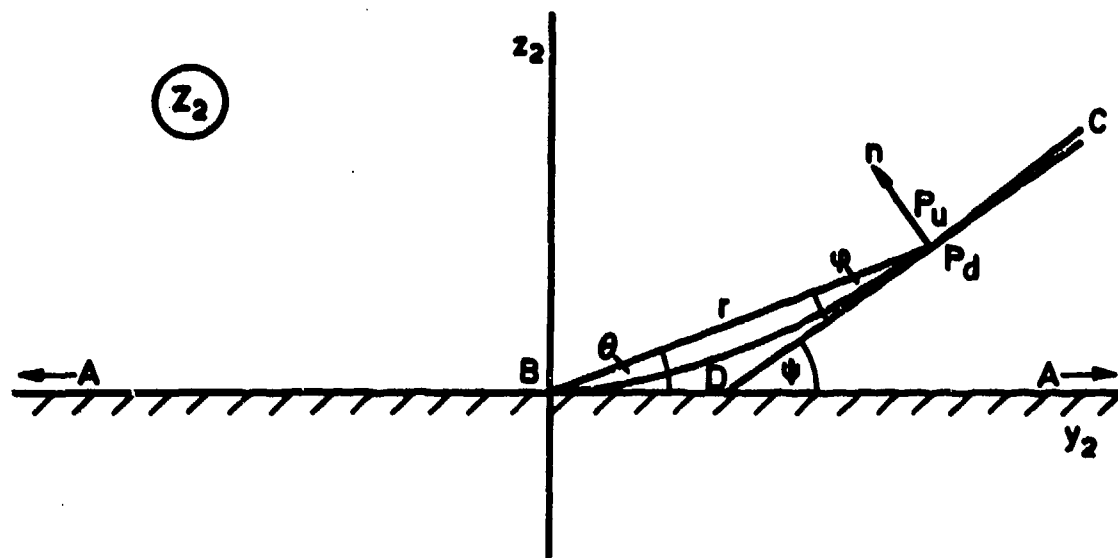
Fig 3   The cross-flow plane, Z_2, and its first transform, Z_3

Fig 4
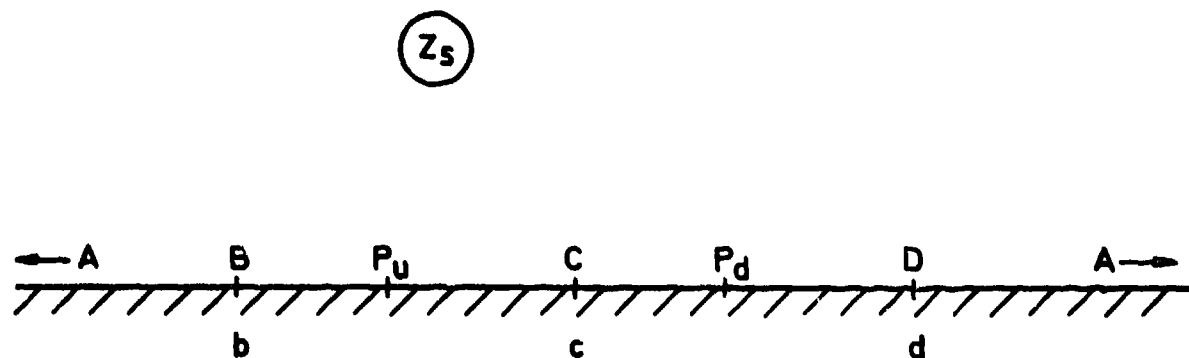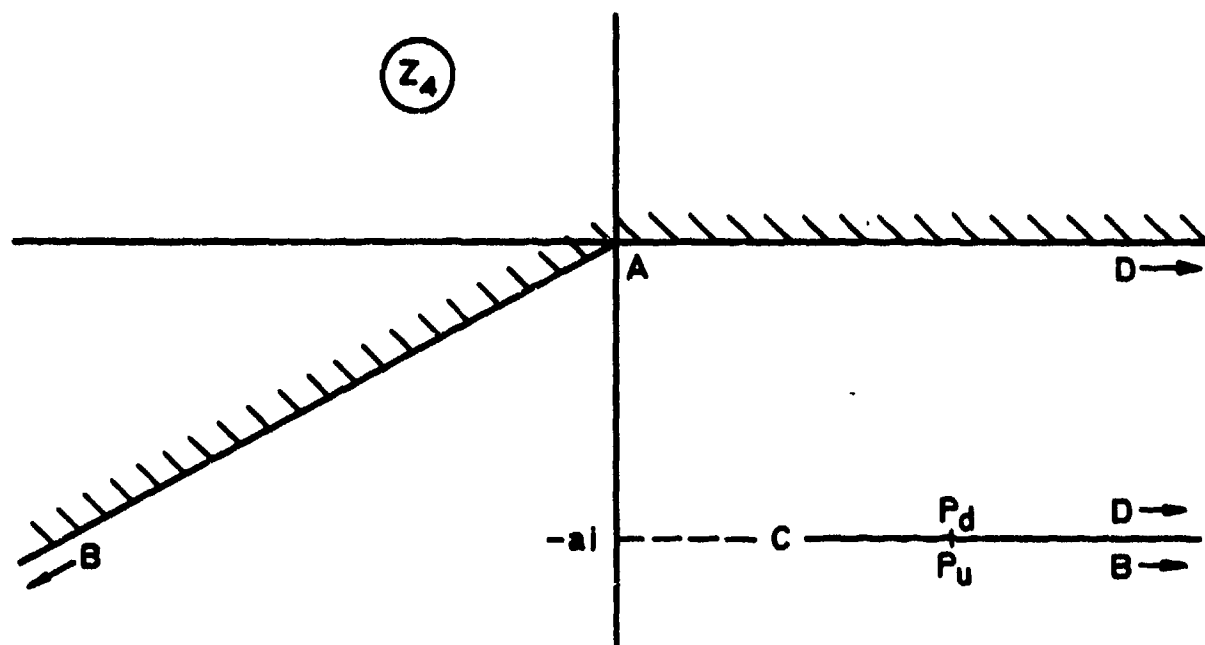
Fig 4 The intermediate and final transformed planes, Z_4 and Z_5

Fig 5
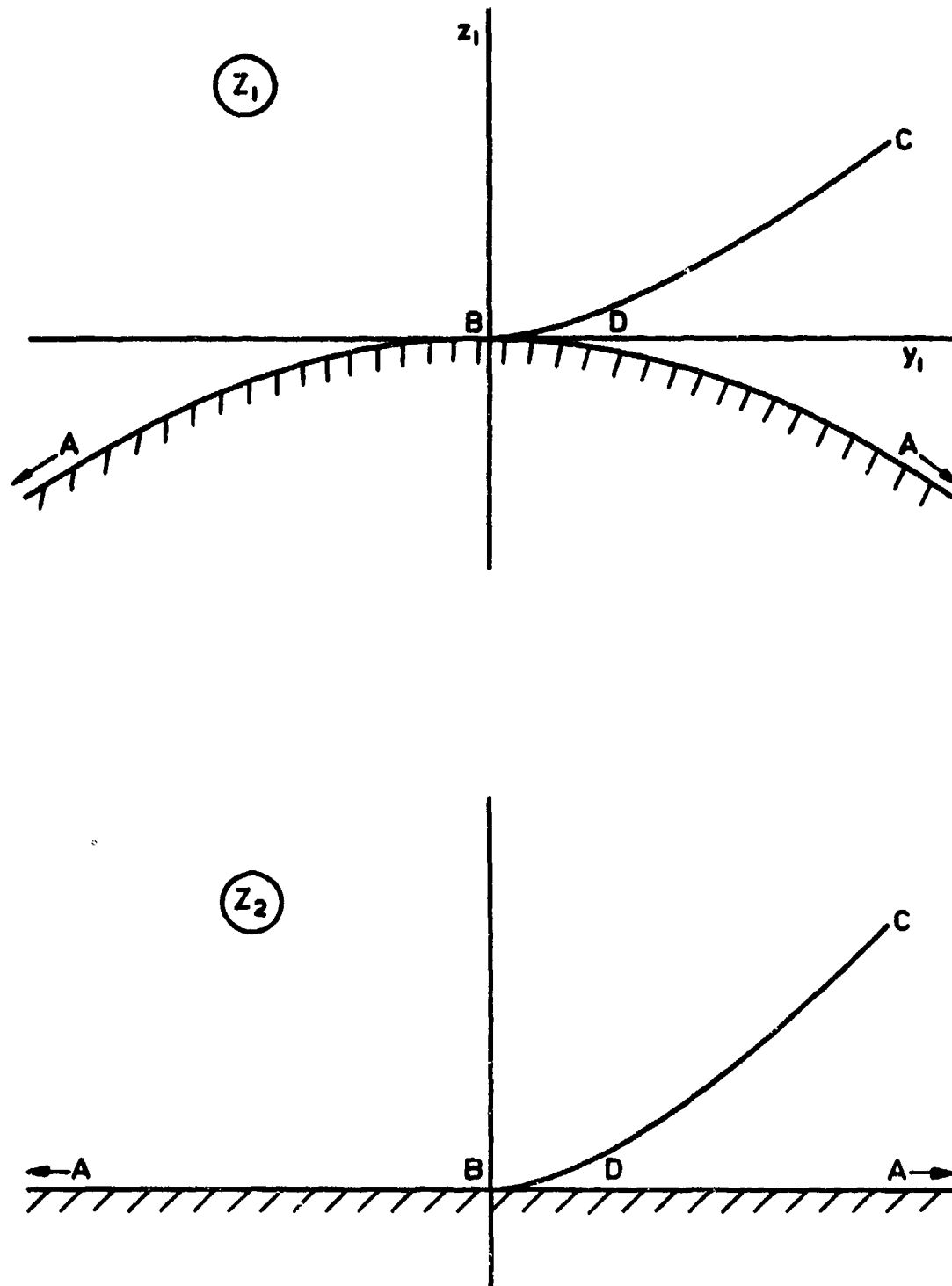
Fig 5   Curved wall: cross-flow plane Z_1 and first transformed plane Z_2

Larry,

former limitation —

see markings on document cover
& DRIC email dated 25 April 2001

Bonnie

I've added this clip.

Klein, Bonnie

From: Jackson-Morris Karen [KJMORRIS@dera.gov.uk]
Sent: Wednesday, April 25, 2001 8:58 AM
To: 'Klein, Bonnie'
Subject: DRIC Document Distribution Statements

Hi Bonnie,

See my replies to your queries below.

1. "Unlimited" means releaseable / available to the general public.

2. The "Uncontrolled copy" stamp means that once the requester has been supplied with the document, DRIC won't be issuing any future amendments. (The onus is on the requester to contact DRIC to check if any amendments had been issued). No, it does not change the distribution audience.

3. I need to investigate this a little further. I am out of the office this afternoon until Monday (30/4/01). Please could you hold this document while I sort this out for you. Thanks.

Could you give me Chris Cupp's e-mail address?

Thanks.
Regards,

Karen Jackson-Morris
Team Leader, Customer Services
DERA-DRIC
Kentigern House, Room 0229
65 Brown Street,
Glasgow,
G2 8EX
Tel. 0141 224 2456
Fax. 0141 224 2470
KJMORRIS@dera.gov.uk
> WWW: http://www.dera.gov.uk
>

-----Original Message-----
From: Klein, Bonnie [mailto:BKlein@DTIC.MIL]
Sent: 24 April 2001 18:55
To: Jackson-Morris Karen
Subject: Document Distribution Statements Inquiries

Karen,

   I need your help resolving conflicting distribution statements for tthree documents.

1) RAE-TR-77058
   Behavoir of a Vortex Sheet Separating from a Smooth Surface
   J.H.B. Smith
   April 1977
   AD-B023230. Unclassified, Dist. D: DoD and DoD Contractors.

DTIC has this documents and sent a Form 55 on behalf of NASA. The approval was accompanied by a copy of the document. However, the cover carries the

stamp UNLIMITED. What does "Unlimited" equate to? In the U.S., it means
the document would be releaseable to the general public. Please define.

2) A similar question, but different term.
   Tech. Memo AERO 1003
   Investigation of the Parameters Controlling the Behavior of Laminar
Separation Bubbles
   D.S. Woodward
   August 1967
   AD-907318. Unclassified, Dist. E: DoD Only.
   Marking pasted on document: "Uncontrolled copy. Any future amendment
will be issued only on request (DERA-DRIC Kentigern House Glasgow).

   Again DTIC has this document. What does "Uncontrolled copy" mean; does
it change the distribution audience?

3) This may be for Chrissie Stewart's attention. Let me know if you want
me to fax copies of the dueling RDPs.

DERA/CIS/CIS1/CR990854
Evaluation of Modern HF Ray Tracing (Final Report)
Neil Rogers
November 1999

   The U.S. European Office of Aerospace Research & Development(EOARD)
http://www.ehis.navy.mil/eoardwebsite2/ sent a copy of the above contracted
report. EOARD completed a RDP which states that the report is "Public
Release, Distribution Unlimited." However, DERA marked every page "UK
Unclassified - US for Official Use Only." EOARD is adamant that it and not
DERA is the controlling office and that DERA does not have the right or
authority to mark the document on behalf of the U.S. Air Force.
   EOARD should probably have handled this by asking DERA to produce a copy
of the report without the markings. However, they didn't, and I have the
document here waiting to be processed.
   Can I get an email or statement from DRIC or DERA stating that this
document done under U.S. Air Force Contract F61775-98-W-WE133 is Public
Release as determined by EOARD?

Bonnie Klein
Program Manager for Foreign Acquistions
Defense Technical Information Center
Ft. Belvoir, VA
(703) 767-8037

--