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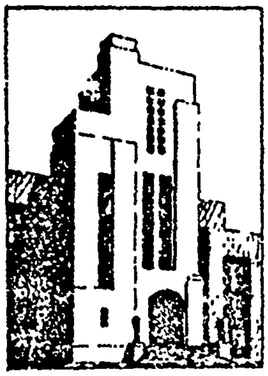
MATHEMATICAL FORMULATION OF BODIES OF REVOLUTION

by

L. Landweber and M. Gertler



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MATHEMATICAL FORMULATION OF BODIES OF REVOLUTION

by

L. Landweber and M. Gertler

ABSTRACT

Various methods of defining bodies of revolution are considered with the conclusion that the most satisfactory method is one which defines the sectional-area curve by means of a polynomial. The polynomial form possesses certain advantages in ease of computation and ready application to hydrodynamical problems, such as the computation of theoretically derived pressure distributions.

The degree of the polynomial fixes the number of parameters that may be prescribed to determine a form. In order to generate the sixth-degree polynomial forms the dimensionless parameters chosen are the nose and tail radii, r_0 and r_1, the prismatic coefficient, C_p, and the position of the maximum section at $x = m$. It is shown that the polynomial expression for the sectional-area curve is a linear combination of r_0, r_1, and C_p, with polynomials of the sixth degree as coefficients. Formulas and tables for these polynomial coefficients are provided, so that when r_0, r_1, C_p, and m are given, the offsets of a form may be rapidly computed.

Not all combinations of these parameters give practical or desirable forms. The range of useable forms may be limited by imposing the restrictions that the sectional-area curve have no maximum or minimum other than at $x = m$, or that the body have no inflection points. These criteria are formulated mathematically and a method of computing boundary curves delineating permissible ranges of parameters is developed.

Formulas for generating seventh-degree polynomial forms are also derived and applied to compare sixth- and seventh-degree forms with the same values of r_0, r_1, C_p, and m. It is found that practical seventh-degree forms with the same values of those parameters may differ appreciably from the sixth-degree form. Thus these parameters do not suffice to fix a form, although they serve to develop the entire class of sixth-degree polynomial forms.

Bodies of revolution with useful application derived from polynomials not of the sixth degree may be fitted (by the method of least squares) very closely by means of sixth-degree forms. From this point of view the usefulness of a series of sixth-degree polynomial forms is greatly enhanced.

INTRODUCTION

When the David Taylor Model Basin became interested in making studies of the hydrodynamical behavior of streamline bodies of revolution, it was decided that such work could be most satisfactorily accomplished with families of bodies of revolution for which certain parameters could be systematically varied. Accordingly, prior to the testing, a program to establish a procedure for the development of such families was initiated.

It was determined that the best approach would be to define these families by a general mathematical equation. The main advantages of the use of a mathematical expression over the empirical or "fairing by eye" method are: The geometry of the body can be precisely defined, fairness between given offsets is assured, and the geometrical parameters can be directly and accurately varied.

A search of the literature reveals that various methods for obtaining mathematical definition of forms have been tried but generally only for application to single forms rather than to families of forms. Among these have been combinations of known analytical curves such as an ellipse with a parabola, an ellipse with a hyperbola, etc.,[1,2] polynomials of various degrees,[3,4] and trigonometric series.

The polynomial method was selected as the basis for the development described herein since it appeared to have distinct advantages in ease of handling and furthermore because of its ready application to hydrodynamical problems such as computations of theoretically derived pressure distributions. It provides a simple method for evaluating the constants in the general equation once a given set of parameters has been selected, and supplies data for readily computing the offsets of a wide variety of forms.

THE GEOMETRIC PARAMETERS

Of the various geometric properties that may be employed to characterize the shape of an elongated body of revolution, it has been convenient to choose, for practical reasons, the following primary quantities to define the body:

l is the length.

d is the maximum diameter.

X_m is the distance of the maximum section from the nose.

R_o is the radius of curvature at the nose.

[1]References are listed on page 64.

R_1 is the radius of curvature at the tail.

$\forall$ is the volume.

Other characteristics, such as the surface area, the position of the center of gravity, the radii of gyration, etc., are of interest for various purposes. These are considered as derived quantities in the present report, and are included in Appendix 5.

It is convenient to employ the following dimensionless combinations of the primary geometric quantities:

$$\lambda = \frac{l}{d}, \quad m = \frac{X_m}{l}, \quad r_0 = \frac{R_0 l}{d^2}, \quad r_1 = \frac{R_1 l}{d^2},$$

and the prismatic coefficient $C_p = \frac{4\forall}{\pi d^2 l}$.

The question as to how well the foregoing parameters define the shape of a body is discussed in a subsequent section.

CHOICE OF MATHEMATICAL FORM

For both mathematical and physical reasons the development has been based on the sectional-area curve of a form, rather than a meridian section of the form itself. Thus it will be shown that the slopes of the sectional-area curve at the ends of a body are proportional to the radii of curvature at the ends, a relation which greatly simplifies the determination of the equation for a body. The physical reason is that the sectional-area curve is proportional to an axial doublet distribution which, to a good approximation, generates the desired body in a uniform stream.[5] Consequently it is desirable to have simple mathematical expressions for the sectional-area curve of a body for the purpose of computing the potential-flow field about it, and its pressure distribution.

The question remains as to the most convenient mathematical form in which the equation of a sectional-area curve can be expressed. Let us consider for a moment the converse of the present problem; i.e., the determination of the geometrical characteristics of a given body, rather than the development of an equation for a body of given characteristics. The geometrical characteristics can be computed directly from the equation for a body. To obtain its equation, a given body may be curve-fitted with any complete set of orthogonal functions, each of which can give a "best" fit in the least-square sense.[6] Practically, however, it is convenient to employ, for this purpose, either the trigonometric functions or Legendre polynomials. The former fit the equation of given form by means of a finite number of terms of the Fourier expansion; the latter by means of a polynomial. In either case it is possible, for the direct problem, to solve for the coefficients of the expansion for a body of prescribed geometrical characteristics by means of sets of linear equations. This is illustrated for the polynomial in the following section. The direct application of the method of linear equations is tedious, however, and another method—in which the equation of the sectional-area curve is given directly as a linear combination of tabulated functions—is developed. As will be shown, the determination of the latter functions is simplified in the case of the polynomial form because of the property that its zeros appear as factors. Furthermore, the polynomial form appears to be more suitable for the purpose of computing pressure distributions. Because of its advantages, the polynomial representation is used in the succeeding developments.

EQUATIONS FOR POLYNOMIAL COEFFICIENTS

The equation of a meridian section of a body of revolution will now be expressed in terms of rectangular coordinates (X,Y) with the X-axis taken along the axis of the body and the origin at one end (the nose) of the body. Assuming a polynomial for the equation of the sectional-area curve, then

$$\pi Y^2 = A_1 X + A_2 X^2 + \dots + A_n X^n \qquad [1]$$

It will be convenient to operate with this equation in dimensionless form. For this purpose put $x = X/l$, $y = Y/d$. Then Equation [1] may be written a

$$y^2 = a_1 x + a_2 x^2 + \dots + a_n x^n \qquad [2]$$

where

$$a_s = A_s \frac{l^s}{\pi d^2}, \quad s = 1, 2, \dots n \qquad [3]$$

Sketches of a sectional-area curve in dimensional and dimensionless forms are shown in Figures 1 and 2.

The coefficients a_1, a_2, ... are to be determined in terms of prescribed values of the geometrical parameters m, r_0,r_1 and C_p. In the dimensionless form the length and maximum diameter are unity so that λ is eliminated as a parameter. The length and diameter conditions are then that $y = 0$ when $x = 1$, $y = \frac{1}{2}$ when $x = m$, and $\frac{dy}{dx} = 0$ when $x = m$. These respectively give the equations:

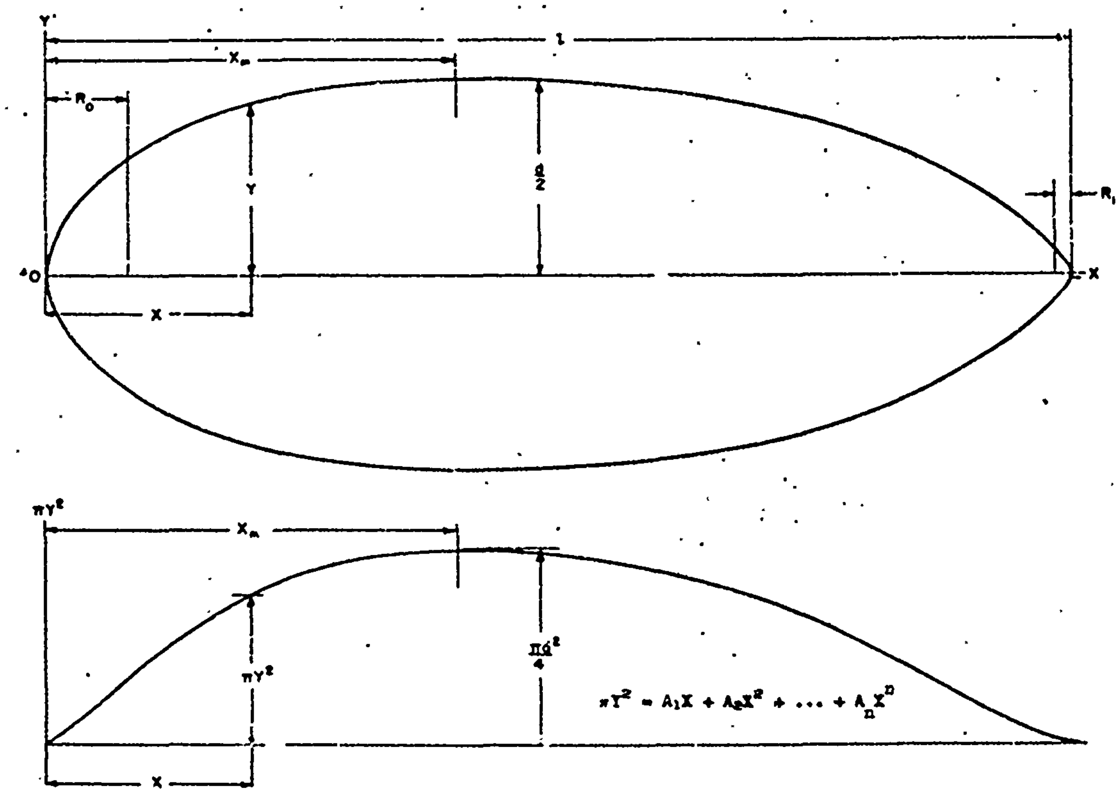


Figure 1 - Sketch Showing Principal Dimensions of a Body of Revolution and Its Sectional-Area Curve

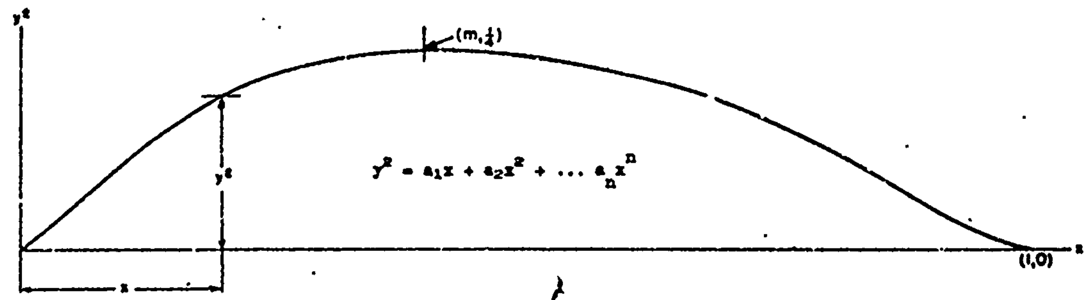


Figure 2 - Nondimensional Representation of a Sectional-Area Curve

$$a_1 + a_2 + a_3 + \ldots + a_n = 0 \qquad [4]$$

$$a_1 m + a_2 m^2 + \ldots + a_n m^n = \frac{1}{4} \qquad [5]$$

$$a_1 + 2a_2 m + \ldots + na_n m^{n-1} = 0 \qquad [6]$$

The radius of curvature R may be evaluated from the formula

$$R = \pm \frac{1}{\frac{d^2X}{dY^2}} \left[1 + \left(\frac{dX}{dY}\right)^2\right]^{3/2}$$

which may be written in the dimensionless form

$$r = \pm \frac{1}{\frac{d^2x}{dy^2}} \left[1 + \frac{l^2}{d^2}\left(\frac{dx}{dy}\right)^2\right]^{3/2} \qquad [7]$$

But differentiating Equation [2] successively with respect to y gives

$$2y = (a_1 + 2a_2 x + \ldots + na_n x^{n-1}) \frac{dx}{dy} \qquad [8]$$

and

$$2 = (a_1 + 2a_2 x + \ldots + na_n x^{n-1}) \frac{d^2x}{dy^2} + \left[2a_2 + \ldots + n(n-1)a_n x^{n-2}\right]\left(\frac{dx}{dy}\right)^2 \qquad [9]$$

If $a_1 \neq 0$, it is seen from Equation [8] that when $x = 0$, $\frac{dx}{dy} = 0$, and hence, from Equation [9], that $\frac{d^2x}{dy^2} = \frac{2}{a_1}$. Consequently, substituting these values into Equation [7], we obtain

$$a_1 = 2r_o \qquad [10]$$

If, on the other hand, $a_1 = 0$, the body has a pointed nose and $r_o = 0$. Hence Equation [10] is valid for both cases.

Similarly when $x = 1$, $y = 0$ and from Equation [8] $\frac{dx}{dy} = 0$, unless

$$a_1 + 2a_2 + \ldots + na_n = 0 \qquad [11]$$

Hence Equations [7] and [9] give

$$a_1 + 2a_2 + \ldots + na_n = -2r_1 \qquad [12]$$

The positive sign is taken in [10] and the negative in [12] because r_0 and r_1 are taken intrinsically positive; but a_1 is the slope of the sectional-area curve at $x = 0$, and hence is positive, and $a_1 + 2a_2 + \ldots + na_n$ is the slope of the sectional-area curve at $x = 1$; and hence is negative. If [11] is satisfied, the body has a pointed tail and $r_1 = 0$, so that [12] is valid for both cases.

The volume of the body may be expressed as

$$\forall = \int_0^l \pi Y^2 dX = \pi d^2 l \int_0^1 y^2 dx,$$

or, substituting for y^2 from Equation [2],

$$\frac{1}{2} a_1 + \frac{1}{3} a_2 + \ldots + \frac{1}{n+1} a_n = \frac{1}{4} C_p \qquad [13]$$

For convenience, the foregoing linear equations in the a_n's are assembled here:

$$a_1 + a_2 + \ldots + a_n = 0 \qquad [4]$$

$$a_1 m + a_2 m^2 + \ldots + a_n m^n = \frac{1}{4} \qquad [5]$$

$$a_1 + 2a_2 m + \ldots + na_n m^{n-1} = 0 \qquad [6]$$

$$a_1 = 2r_0 \qquad [10]$$

$$a_1 + 2a_2 + \ldots + na_n = -2r_1 \qquad [12]$$

$$\frac{1}{2} a_1 + \frac{1}{3} a_2 + \ldots + \frac{1}{n+1} a_n = \frac{1}{4} C_p \qquad [13]$$

SOLUTION OF EQUATIONS FOR POLYNOMIALS OF SIXTH DEGREE

Corresponding to the parameters m_1, r_0, r_1, and C_p there are the six equations [4], [5], [6], [10], [12], and [13]. Consequently a polynomial of the sixth degree is, in general, determinable so that we choose $n = 6$ in these equations. The solution of these equations by the determinant rule is tedious and consequently an alternative procedure is developed.

The form of the solution by the determinant rule shows that the a_n's are linear functions of r_0, r_1, and C_p. Hence y^2 is also a linear function of r_0, r_1, and C_p and may be written in the form

$$y^2 = 2r_0R_0(x) + 2r_1R_1(x) + C_pP(x) + Q(x) \qquad [14]$$

where $R_0(x)$, $R_1(x)$, $P(x)$, and $Q(x)$ are polynomials of the sixth degree in x. Corresponding to Equations [4], [5], [6], [10], [12] and [13] the polynomial $y^2(x)$ satisfies the following conditions identically in r_0, r_1, and C_p:

(a) $y^2(0) = 0$

(b) $\frac{d}{dx} y^2(0) = 2r_0$

(c) $y^2(1) = 0$

(d) $\frac{d}{dx} y^2(1) = -2r_1$

(e) $y^2(m) = \frac{1}{4}$

(f) $\frac{d}{dx} y^2(m) = 0$

and

(g) $\int_0^1 y^2(x)dx = \frac{1}{4} C_p.$

Since conditions (a) through (g) are satisfied identically in r_0, r_1, and C_p, their application to [14] has the following consequences:
Since $y^2(0) = 0$, regardless of the values of r_0, r_1, and C_p, we must have

$$R_0(0) = R_1(0) = P(0) = Q(0) = 0 \qquad [15]$$

Similarly we obtain the following equations:

From Condition (b),

$$R_0'(0) = 1, \quad R_1'(0) = P'(0) = Q'(0) = 0 \qquad [16]$$

where the prime denotes differentiation with respect to x. From Condition (c)

$$R_0(1) = R_1(1) = P(1) = Q(1) = 0 \qquad [17]$$

from Condition (d),

$$R_1'(1) = -1, \ R_0'(1) = P'(1) = Q'(1) = 0 \qquad [18]$$

From Condition (e),

$$R_0(m) = R_1(m) = P(m) = 0, \quad Q(m) = \frac{1}{4} \qquad [19]$$

From Condition (f),

$$R_0'(m) = R_1'(m) = P'(m) = Q'(m) = 0 \qquad [20]$$

and from Condition (g),

$$\int_0^1 P(x)dx = \frac{1}{4}, \quad \int_0^1 R_0(x)dx = \int_0^1 R_1(x)dx = \int_0^1 Q(x)dx = 0 \qquad [21]$$

The values of $R_0(x)$, $R_1(x)$, $P(x)$, and $Q(x)$ will now be derived on the basis of the relations in Equations [15] through [21]:

EVALUATION OF $R_0(x)$

Since $R_0(0) = R_0(1) = R_0'(1) = R_0(m) = R_0'(m) = 0$, and since $R_0(x)$ is a polynomial of the sixth degree, it may be written factorially in the form

$$R_0(x) = (\alpha_0 + \alpha_1 x)\, x\, (x-1)^2(x-m)^2 \qquad [22]$$

The coefficients α_0 and α_1 may be evaluated as follows: From Equation [16] $R_0'(0) = 1$; whence, from [22],

$$\alpha_0 = \frac{1}{m^2} \qquad [23]$$

Equation [22] may be rewritten as

$$\begin{aligned} R_0(x) = {} & \alpha_0\left[x^5 - 2x^4(1+m) + x^3(1+4m+m^2) - 2mx^2(1+m) + m^2x\right] \\ & + \alpha_1\left[x^6 - 2x^5(1+m) + x^4(1+4m+m^2) - 2mx^3(1+m) + m^2x^2\right] \end{aligned}$$

Hence, from [21],

$$\begin{aligned} \int_0^1 R_0(x)dx = {} & \alpha_0\left[\frac{1}{6} - \frac{2}{5}(1+m) + \frac{1}{4}(1+4m+m^2) - \frac{2}{3}m(1+m) + \frac{1}{2}m^2\right] \\ & + \alpha_1\left[\frac{1}{7} - \frac{1}{3}(1+m) + \frac{1}{5}(1+4m+m^2) - \frac{1}{2}m(1+m) + \frac{1}{3}m^2\right] = 0 \end{aligned}$$

Then, simplifying, substituting $\alpha_0 = \frac{1}{m^2}$, and solving for α_1, we obtain

$$\alpha_1 = -\frac{7(1 - 4m + 5m^2)}{2m^2(2-7m + 7m^2)} \qquad [24]$$

EVALUATION OF $R_1(x)$

Since $R_1(0) = R_1'(0) = R_1(1) = R_1(m) = R_1'(m) = 0$, and since $R_1(x)$ is a polynomial of the sixth degree, it may be written factorially in the form

$$R_1(x) = (\beta_0 + \beta_1 x)\, x^2\, (x - 1)(x - m)^2 \qquad [25]$$

The coefficients β_0 and β_1 are evaluated as follows:

From [16], $R_1'(1) = -1$; whence, from [25],

$$(\beta_0 + \beta_1)(1 - m)^2 = -1 \qquad [26]$$

Equation [25] may be rewritten as:

$$R_1(x) = \beta_0\left[x^5 - x^4(1 + 2m) + x^3(m^2 + 2m) - m^2x^2\right]$$

$$+ \beta_1\left[x^6 - x^5(1 + 2m) + x^4(m^2 + 2m) - m^2x^3\right]$$

But, from [21], $\int_0^1 R_1(x)dx = 0$. Hence, integrating the above expression for $R_1(x)$ and simplifying, gives

$$7\beta_0(2 - 6m + 5m^2) + \beta_1(10 - 28m + 21m^2) = 0 \qquad [27]$$

Solving for [26] and [27] simultaneously, we obtain

$$\beta_0 = \frac{10 - 28m + 21m^2}{2(1 - m)^2(2 - 7m + 7m^2)} \qquad [28]$$

$$\beta_1 = -\frac{7(2 - 6m + 5m^2)}{2(1 - m)^2(2 - 7m + 7m^2)} \qquad [29]$$

EVALUATION OF P(x)

Since $P(0) = P'(0) = P(1) = P'(1) = P(m) = P'(m) = 0$ and since $P(x)$ is a polynomial of the sixth degree, it may be written factorially in the form

$$P(x) = \gamma x^2(x - 1)^2(x - m)^2 \qquad [30]$$

To evaluate the coefficient γ, we have, from [21], $\int_0^1 P(x)dx = \frac{1}{4}$. Hence, writing

$$P(x) = \gamma\left[x^6 - 2x^5(1 + m) + x^4(1 + 4m + m^2) - 2x^3(m + m^2) + m^2x^2\right]$$

and integrating, we obtain

$$\gamma\left[\frac{1}{7} - \frac{1}{3}(1 + m) + \frac{1}{5}(1 + 4m + m^2) - \frac{1}{2}(m + m^2) + \frac{1}{3}m^2\right] = \frac{1}{4}$$

or

$$\gamma = \frac{105}{2(2 - 7m + 7m^2)} \qquad [31]$$

EVALUATION OF Q(x)

Since $Q(0) = Q'(0) = Q(1) = Q'(1) = 0$ and $Q(x)$ is a polynomial of the sixth degree, it may be written in the form

$$Q(x) = (\delta_0 + \delta_1 x + \delta_2 x^2)x^2(x - 1)^2 \qquad [32]$$

The undetermined coefficients are evaluated as follows: From [19], $Q(m) = \frac{1}{4}$, and hence

$$\delta_0 + m\delta_1 + m^2\delta_2 = \frac{1}{4m^2(1 - m)^2} \qquad [33]$$

From [20], $Q'(m) = 0$, and hence

$$(\delta_0 + \delta_1 m + \delta_2 m^2)\, 2m(m - 1)(2m - 1) + (\delta_1 + 2m\delta_2)\, m^2(m - 1)^2 = 0$$

or, dividing by $m(m - 1)$ and simplifying, we obtain

$$\delta_0(4m - 2) + \delta_1 m(5m - 3) + \delta_2 m^2(6m - 4) = 0 \qquad [34]$$

From [21], $\int_0^1 Q(x)dx = 0$. Hence, writing

$$Q(x) = (\delta_0 + \delta_1 x + \delta_2 x^2)(x^4 - 2x^3 + x^2)$$

and integrating, we obtain

$$\delta_0\left(\frac{1}{5} - \frac{2}{4} + \frac{1}{3}\right) + \delta_1\left(\frac{1}{6} - \frac{2}{5} + \frac{1}{4}\right) + \delta_2\left(\frac{1}{7} - \frac{2}{6} + \frac{1}{5}\right) = 0$$

or

$$14\delta_0 + 7\delta_1 + 4\delta_2 = 0 \qquad [35]$$

Solving Equations [33], [34], and [35] simultaneously for δ_0, δ_1, and δ_2 gives

$$\delta_0 = \frac{3(7m^2 - 8m + 2)}{4m^2(1 - m)^3(7m^2 - 7m + 2)} \qquad [36]$$

$$\delta_1 = -\frac{21m^3 - 14m^2 - 4m + 2}{2m^3(1 - m)^3(7m^2 - 7m + 2)} \qquad [37]$$

$$\delta_2 = \frac{7(5m^2 - 5m + 1)}{4m^3(1 - m)^3(7m^2 - 7m + 2)} \qquad [38]$$

The results of the present section will now be summarized. It has been shown that a body of revolution whose nose and tail radii, position of maximum section and prismatic coefficient are prescribed may be represented by a polynomial of the sixth degree in the form

$$y^2 = 2r_0 R_0(x) + 2r_1 R_1(x) + C_p P(x) + Q(x) \qquad [14]$$

where

$$\left.\begin{aligned} R_0(x) &= x(x-1)^2(x-m)^2(\alpha_0 + \alpha_1 x) \\ R_1(x) &= x^2(x-1)(x-m)^2(\beta_0 + \beta_1 x) \\ P(x) &= x^2(x-1)^2(x-m)^2\,\gamma \\ Q(x) &= x^2(x-1)^2(\delta_0 + \delta_1 x + \delta_2 x^2) \end{aligned}\right\} \qquad [39]$$

and

$$
\left.
\begin{aligned}
\alpha_0 &= \frac{1}{m^3} \\
\alpha_1 &= -\frac{7(1 - 4m + 5m^2)}{2m^3(2 - 7m + 7m^2)} \\
\beta_0 &= \frac{10 - 28m + 21m^2}{2(1 - m)^2(2 - 7m + 7m^2)} \\
\beta_1 &= -\frac{7(2 - 6m + 5m^2)}{2(1 - m)^2(2 - 7m + 7m^2)} \\
\gamma &= \frac{105}{2(2 - 7m + 7m^2)} \\
\delta_0 &= \frac{3(2 - 8m + 7m^2)}{4m^3(1 - m)^3(2 - 7m + 7m^2)} \\
\delta_1 &= -\frac{2 - 4m - 14m^2 + 21m^3}{2m^3(1 - m)^3(2 - 7m + 7m^2)} \\
\delta_2 &= \frac{7(1 - 5m + 5m^2)}{4m^3(1 - m)^3(2 - 7m + 7m^2)}
\end{aligned}
\right\} \qquad [40]
$$

The coefficients α_0 through δ_2 are tabulated for values of m from 0.10 to 0.50, in intervals of 0.02, in Appendix 1. Appendix 2 contains tables of the polynomials $R_0(x)$, $R_1(x)$, $P(x)$, and $Q(x)$ for values of x from 0 to 1, in intervals of 0.02, and for the same values of m. Graphs of these functions, for selected values of m, are shown in Appendix 3.

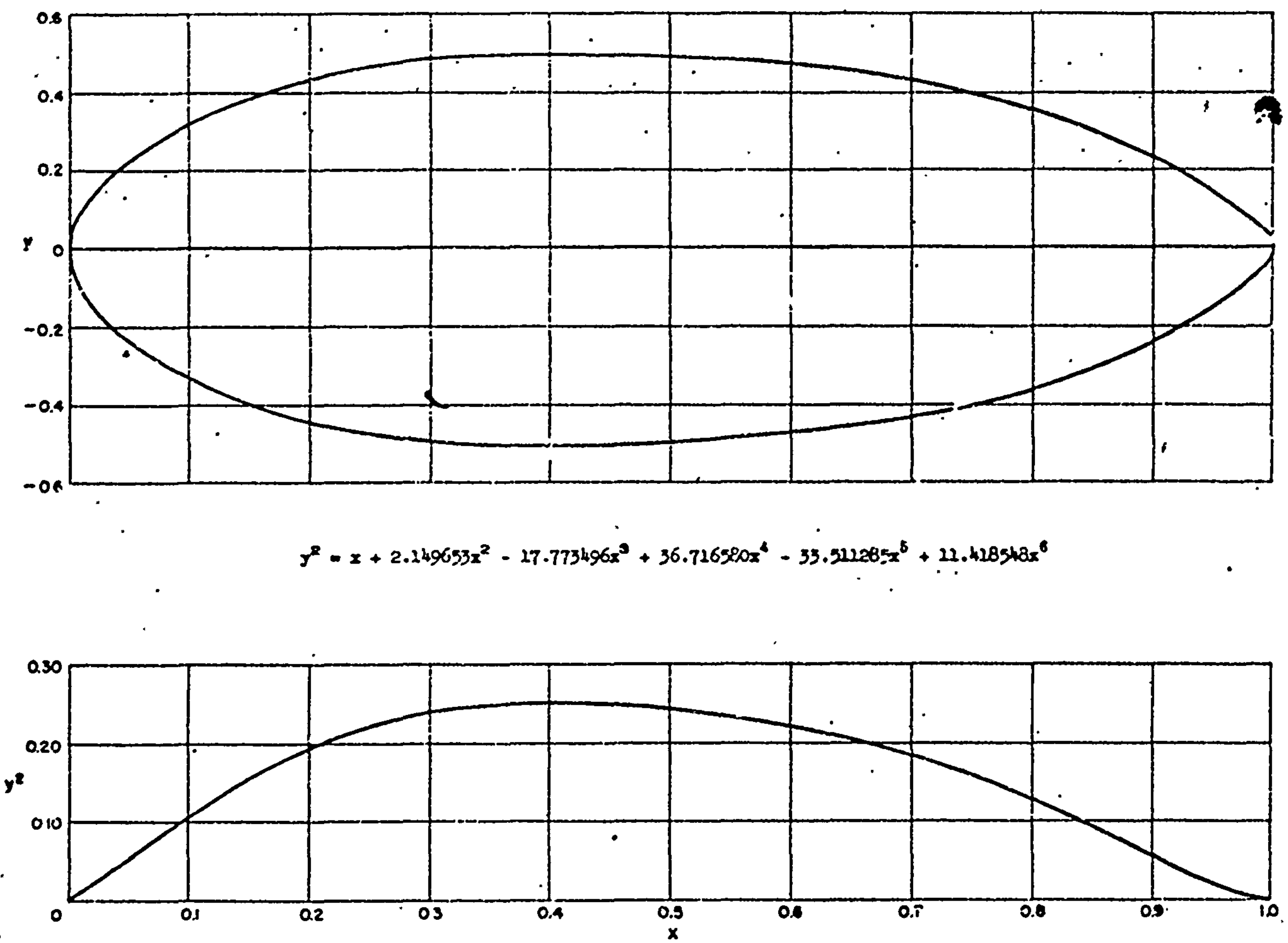
The numerical example in Table 1 illustrates how the tables of Appendix 2 can be used to calculate the offsets of a given form. The figures apply to a body whose geometric parameters are $r_0 = 0.50$, $r_1 = 0.10$, $C_p = 0.65$, and $m = 0.40$. The calculations for y^2 and y, shown in Table 1, are based directly upon the tables for $R_0(x)$, $R_1(x)$, $P(x)$, and $Q(x)$ corresponding to $m = 0.40$. The resulting body and sectional-area curves, in dimensionless form, are shown in Figure 3.

The graphs of the basic polynomials given in Appendix 3 are useful in that they provide a visual means of showing how each geometrical parameter affects the shape of the body. Thus, if a body with certain prescribed parameters is not suitable for the intended purpose, the graphs will indicate the parameter changes that are necessary to produce a desired volumetric distribution or contour.

TABLE 1

Calculations of y^2 and y for a Typical Si[illegible]h Degree Polynomial Form

x	$2r_0R_0(x)$	$2r_1R_1(x)$	$C_pP(x)$	$Q(x)$	y^2	y
0	0	0	0	0	0	0
0.02	0.01658	-0.00010	0.00592	-0.00167	0.02073	0.1439
0.04	.02725	.00035	.02038	.00488	.04240	.2059
.06	.03328	.00068	.03921	.00746	.06435	.2537
.08	.03575	.00101	.05915	.00784	.08605	.2934
.10	.03560	.00132	.07774	-0.00494	.10708	.3272
.12	.03358	.00157	.09323	+0.00182	.12706	.3565
.14	.03035	.00175	.10450	.01265	.14575	.3818
.16	.02642	.00184	.11095	.02744	.16297	.4037
.18	.02220	.00185	.11245	.04580	.17860	.4226
.20	.01800	.00178	.10920	.06713	.19255	.4388
.22	.01406	.00164	.10174	.09066	.20482	.4526
.24	.01054	.00145	.09083	.11552	.21544	.4641
.26	.00752	.00122	.07737	.14075	.22442	.4737
.28	.00506	.00097	.06241	.16538	.23188	.4815
.30	.00316	.00072	.04703	.18842	.23789	.4878
.32	.00178	.00049	.03232	.20894	.24255	.4925
.34	.00085	.00029	.01933	.22607	.24596	.4959
.36	.00031	.00013	.00906	.23904	.24828	.4982
.38	.00006	.00003	.00237	.24719	.24959	.4996
.40	0	0	0	.25000	.25000	.5000
.42	.00003	.00004	.00253	.24709	.24961	.4997
.44	0.00005	.00014	.01036	.23826	.24853	.4986
.46	-0.00002	.00031	.02369	.22345	.24681	.4968
.48	.00026	.00054	.04252	.20280	.24452	.4944
.50	.00073	.00082	.06665	.17661	.24171	.4917
.52	.00148	.00114	.09567	.14534	.23839	.4882
.54	.00254	.00148	.12897	.10963	.23458	.4844
.56	.00390	.00182	.16575	.07027	.23030	.4799
.58	.00557	.00213	.20503	+0.02817	.22550	.4749
.60	.00750	.00240	.24570	-0.01562	.22018	.4692
.62	.00965	.00260	.28650	.05998	.21427	.4629
.64	.01194	.00271	.32607	.10368	.20774	.4557
.66	.01430	.00271	.36301	.14548	.20052	.4478
.68	.01663	.00258	.39587	.18410	.19256	.4388
.70	.01883	.00230	.42326	.21832	.18381	.4287
.72	.02077	.00186	.44382	.24696	.17423	.4174
.74	.02236	.00126	.45634	.26895	.16377	.4046
.76	.02349	-0.00050	.45981	.28339	.15243	.3905
.78	.02406	+0.00040	.45344	.28956	.14022	.3744
.80	.02400	.00142	.43680	.28704	.12718	.3566
.82	.02325	.00252	.40982	.27569	.11340	.3368
.84	.02179	.00364	.37293	.25578	.09900	.3146
.86	.01965	.00472	.32711	.22801	.08417	.2901
.88	.01688	.00565	.27399	.19360	.06916	.2630
.90	.01362	.00633	.21595	.15436	.05430	.2330
.92	.01008	.00661	.15620	.11276	.03997	.2000
.94	.00651	.00633	.09892	.07201	.02673	.1635
.96	.00331	.00530	.04931	.03616	.01514	.1230
.98	-0.00094	+0.00328	0.01378	-0.01017	0.00595	0.0771
1.00	0	0	0	0	0	00



$$y^2 = x + 2.149653x^2 - 17.773496x^3 + 36.716580x^4 - 33.511285x^5 + 11.418548x^6$$

Figure 3 - Graph of a Typical Body and Sectional-Area Curve Computed from a Sixth-Degree Polynomial

LIMITATION OF RANGES OF PARAMETERS

Not all combinations of the parameters r_0, r_1, C_p, and m give desirable forms. For example, since for arbitrary values of the parameter, the right member of [14] may be considered as an arbitrary polynomial of the sixth degree, (subject to the restriction that $y^2 = \frac{1}{4}$ at x = m) it could very well become negative (which would be meaningless) or assume several maxima and minima and inflection points in the interval $0 \leq x \leq 1$. Indeed maxima greater than the desired one at x = m may occur.

The term "desirable" remains to be defined, but for the development of forms which may be considered as streamlined bodies, simple geometrical restrictions may be imposed. The simplest conditions for ensuring a desirable form appear to be that the polynomial remain positive in the range $0 \leq x \leq 1$ and that the body have no inflection points. Since the form has a maximum of $y = \frac{1}{2}$ at x = m, and crosses the x-axis at x = 0 and 1, these conditions alone preclude the occurrence of minima or of more than one maximum in the form of the body.

There may be occasions, however, where bodies with inflection points near the tail are permitted or desired. For example, such forms have been proposed as laminar-flow forms. In this case a suitable condition is that the sectional-area curve have no stationary value (maximum or minimum) other than at x = m.

The aforementioned conditions may be formulated mathematically and employed to determine permissible ranges of the parameters. First, let us consider the maximum or minimum condition.

MAXIMUM OR MINIMUM CONDITION

Put $y^2 = f(x)$. The derivative $f'(x)$ is known to be divisible by $(x - m)$ since $f'(x) = 0$ when x = m. Removal of this factor leaves a polynomial, $\bar{f}(x)$ for example, which must not pass through zero over the interval in order to avoid a minimum of $f(x)$. Indeed, since $f'(x)$ is positive near the nose, when x - m is negative, it is seen that $\bar{f}(x)$ must be negative. Hence the condition may be expressed in the form

$$\bar{f}(x) = \frac{f'(x)}{x - m} < 0 \qquad [41]$$

for all values of x in the range $0 \leq x \leq 1$. Put

$$\bar{R}_0 = \frac{R_0'(x)}{x - m}, \ \bar{R}_1 = \frac{R_1'(x)}{x - m}, \ \bar{P}(x) = \frac{P'(x)}{x - m}, \ \bar{Q}(x) = \frac{Q'(x)}{x - m} \qquad [42]$$

Then, also,

$$\bar{T}(x) = 2r_o\bar{R}_o(x) + 2r_1\bar{R}_1(x) + C_p\bar{P}(x) + \bar{Q}(x) \qquad [43]$$

It is seen that $\bar{T}(x)$ is a function of five variables, i.e., explicitly,

$$\bar{T}(x) = \bar{T}(x,m;\ r_o,\ r_1,\ C_p) < 0$$

Now consider the equation

$$\bar{T}(x,m;\ r_o,\ r_1,\ C_p) = 0 \qquad [44]$$

For fixed x and m, it is seen from [43] that [44] is the equation of a plane in an (r_o, r_1, C_p) rectangular coordinate system. This plane divides the (r_o, r_1, C_p) space into two parts, in one of which $\bar{T}(x) > 0$, in the other $\bar{T}(x) < 0$. Hence Condition [41] is satisfied on only one side of the plane. We may now consider x and m as parameters which define a two-parameter family of planes in the (r_o, r_1, C_p) space and an envelope surface. As x and m are varied, the region of permissible values of r_o, r_1, and C_p becomes more restricted as the successive planes intersect and reduce the space in which Condition [41] is satisfied. Indeed, one can readily convince himself, by sketching an element of an uninflected curve and drawing a succession of tangent lines to it (so that the curve element is the envelope of these tangent lines), that the side of the envelope curve towards the center of curvature of the element remains on the same side (plus or minus) of all the tangent lines to the element. Similarly, in the three-dimensional (r_o, r_1, C_p) space, the envelope surface delineates the region in which Condition [41] is satisfied. Since the envelope surface may have inflection points and may intersect itself, the permissible region may become further restricted. The equations of the envelope surface, in parametric form, are

$$\left.\begin{aligned} \bar{T}(x,\ m;\ r_o,\ r_1,\ C_p) &= 0 \\ \frac{\partial \bar{T}}{\partial x} &= 0 \\ \frac{\partial \bar{T}}{\partial m} &= 0 \end{aligned}\right\} \qquad [45]$$

For each x and m Equations [45] are three linear equations which may be solved simultaneously for r_o, r_1, C_p, giving a point in the (r_o, r_1, C_p) space corresponding to each pair of values (x, m).

Because of the graphical difficulty of presenting a surface in three dimensions the foregoing analysis will be illustrated in a two-dimensional case. We will suppose that both m and C_p are prescribed. Then, considering x as a parameter, Equation [44] represents a one-parameter family of straight lines in an (r_o, r_1) plane. The envelope of this family of lines has the equations

$$\begin{aligned} \bar{f}(x; r_o, r_1) &= 0 \\ \frac{\partial \bar{f}}{\partial x} &= 0 \end{aligned} \qquad [46]$$

For each x, Equations [46] are linear equations which may be solved simultaneously for r_o and r_1, giving a point in the (r_o, r_1) plane. By computing the pairs of values of (r_o, r_1) for a sufficient number of values of x in the range $0 \leq x \leq 1$, the envelope curve of r_1 against r_o may be graphed. Figure 4 shows two envelope curves of r_1 against r_o for $m = 0.40$ and $C_p = 0.55$ and $m = 0.40$ and $C_p = 0.65$. The corresponding values of x are marked along the curves. Proceeding along the curve in the direction of increasing x, the negative or permissible side of the tangent line at each x (representing $\bar{f}(x) = 0$) is on the right, so that the right side of the envelope curve is the permissible side. Consider the curve for $C_p = 0.65$. For a point (r_o, r_1) to be permissible it must be on the right side of all possible tangent lines that can be drawn to the envelope curve for $0 \leq x \leq 1$. It is seen in Figure 4 that tangent lines for $x > 0.68$ begin to eliminate regions which previously were on the right side of the envelope curve. Furthermore only points (r_o, r_1) in the first quadrant need be considered since negative radii are meaningless. In this way the permissible regions for r_o, r_1, bounded by segments of the r_o and r_1 axes and by arcs of the envelope curves, shown shaded in Figure 4, are obtained.

INFLECTION POINT CONDITION

If the body is to have no inflection points, its slope must be monotonically decreasing as x increases from 0 to 1, i.e.,

$$\frac{d^2y}{dx^2} < 0 \qquad [47]$$

for all values of x in the range $0 \leq x \leq 1$. Again put $y^2 = f(x)$. Then, differentiating successively with respect to x, we get

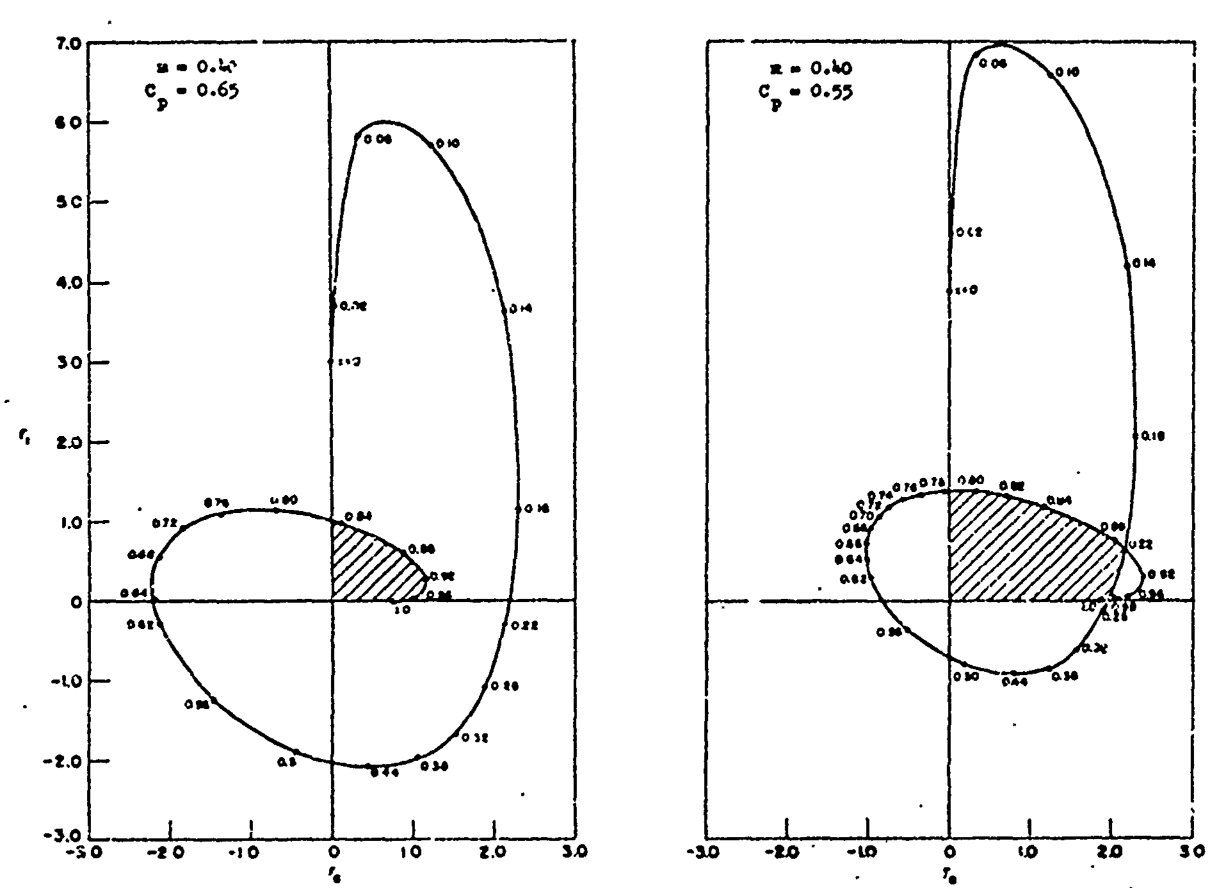


Figure 4 - Typical Envelope Curves of r_1 and r_0 Defined by the Maximum-Minimum Condition

The curves are for values of m = 0.40 and C_p = 0.55, and m = 0.40 and C_p = 0.65, for sixth-degree polynomials. The shaded areas indicate regions of permissible values of r_0 and r_1.

$$2yy' = f'(x) \qquad [48]$$

$$2y'^2 + 2yy'' = f''(x) \qquad [49]$$

or, eliminating y' between [48] and [49], we obtain

$$y'' = \frac{2ff'' - f'^2}{4f^{3/2}} \qquad [50]$$

We must also suppose that $y^2 > 0$, or

$$f(x) > 0 \text{ for } 0 < x < 1 \qquad [51]$$

Then also, from [47] and [50],

$$2ff'' - f'^2 < 0 \qquad [52]$$

Both Conditions [51] and [52] can be shown to determine envelope surfaces bounding regions in an (r_0, r_1, C_p) space, and the region which is common to both is the permissible one.

In cases where all the parameters but one are prescribed and it is desired to determine the permissible range of variation of that parameter, a procedure somewhat simpler than the envelope method may be employed. For example, suppose r_0, r_1 and m are prescribed and the limits of permissible variation of C_p are to be found. Then the equation $f(x) = 0$ may be solved for C_p as a function of x and plotted against x. The resulting curve is the boundary separating positive from negative values of f(x). A value of C_p for which Condition [51] is not satisfied for any x must be discarded. Hence the permissible range of C_p would be defined by the ordinates of the horizontal tangent lines, at the maxima and minima of the curve of C_p against x, which do not otherwise intersect the curve. Similarly the equation $2ff'' - f'^2 = 0$ can be solved for C_p plotted as a function of x, and the ordinates of these tangent lines, at the maxima and minima, which do not again intersect the curve, obtained. The range of C_p which is common to the ranges satisfying [51] and [52] separately then gives the desired range which satisfies the conditions simultaneously. This procedure will be illustrated in a later section in the case of a seventh-degree polynomial, where the parameters m, r_0, r_1, and C_p are prescribed, and a new parameter a_2, to be defined, is varied.

SEVENTH-DEGREE POLYNOMIAL FORMS

After the work in connection with the sixth-degree polynomial had been completed, it was of interest to determine how well a form was defined by the parameters m, r_o, r_1, and C_p. For this purpose it was decided to introduce an additional degree of freedom and to develop the equations for the class of forms represented by polynomials of the seventh degree. For then, by holding the original parameters constant, allowing the new parameter to vary, and comparing the form determined by the polynomial of the sixth degree with those of the seventh, an answer to the above question could be obtained.

DERIVATION OF BASIC POLYNOMIALS

The new parameter was chosen to be a_2, the coefficient of x^2 in Equation [2]. Then y^2 is a linear function of r_o, r_1, a_2, and C_p and may be written in the form

$$y^2 = 2r_oS_o(x) + 2r_1S_1(x) + a_2U(x) + C_pV(x) + W(x) \qquad [53]$$

where $S_o(x)$, $S_1(x)$, $U(x)$, $V(x)$, and $Q(x)$ are polynomials of the seventh degree in x. To determine $y^2(x)$ we have the same conditions for these polynomials as (a) through (g), tabulated on page 9 and, in addition, the equation

$$\text{(h)} \qquad \frac{d^2}{dx^2}\, y^2(0) = 2a_2$$

On the basis that these conditions must be satisfied identically in r_o, r_1, a_2, and C_p, we obtain, as for the polynomials of the sixth degree, the relations

$$S_o(0) = S_o''(0) = S_o(1) = S_o'(1) = S_o(m) = S_o'(m) = \int_0^1 S_o(x)dx = 0$$

$$S_o'(0) = 1$$

$$S_1(0) = S_1'(0) = S_1''(0) = S_1(1) = S_1(m) = S_1'(m) = \int_0^1 S_1(x)dx = 0$$

$$S_1'(1) = -1$$

$$U(0) = U'(0) = U(1) = U'(1) = U(m) = U'(m) = \int_0^1 U(x)dx = 0$$

$$U''(0) = 2$$

$$V(0) = V'(0) = V''(0) = V(1) = V'(1) = V(m) = V'(m) = 0$$

$$\int_0^1 V(x)dx = \frac{1}{4}$$

$$W(0) = W'(0) = W''(0) = W(1) = W'(1) = W'(m) = \int_0^1 W(x)dx = 0$$

$$W(m) = \frac{1}{4}$$

The procedure for determining these basic polynomials is similar to that used for the sixth degree. Omitting the details of the calculations, we have

$$y^2 = 2r_0S_0(x) + 2r_1S_1(x) + a_2U(x) + C_pV(x) + W(x) \qquad [53]$$

where

$$\left.\begin{aligned} S_0(x) &= (\alpha_0 + \alpha_1 x + \alpha_2 x^2)x^3(x - 1)^2(x - m)^2 \\ S_1(x) &= (\beta_0 + \beta_1 x)x^3(x - 1)(x - m)^2 \\ U(x) &= (\gamma_0 + \gamma_1 x)x^2(x - 1)^2(x - m)^2 \\ V(x) &= \delta x^3(x - 1)^2(x - m)^2 \\ W(x) &= (\epsilon_0 + \epsilon_1 x + \epsilon_2 x^2)x^3(x - 1)^2 \end{aligned}\right\} \qquad [54]$$

and

$$\left.\begin{aligned} \alpha_0 &= \frac{1}{m^2} \\ \alpha_1 &= \frac{2(1 + m)}{m^3} \\ \alpha_2 &= -\frac{2(8 - 13m - 28m^2 + 63m^3)}{m^3D} \\ \beta_0 &= \frac{15 - 40m + 28m^2}{(1 - m)^2D} \\ \beta_1 &= -\frac{2(10 - 28m + 21m^2)}{(1 - m)^2D} \\ \gamma_0 &= \frac{1}{m^2} \end{aligned}\right\} \qquad [55]$$

$$\left.\begin{aligned}
\gamma_1 &= -\frac{4(2 - 7m + 7m^2)}{m^2 D} \\
\delta &= \frac{210}{D} \\
\epsilon_0 &= -\frac{10 - 35m + 28m^2}{2m^3(m-1)^3 D} \\
\epsilon_1 &= \frac{15 - 25m - 70m^2 + 98m^3}{4m^4(m-1)^3 D} \\
\epsilon_2 &= -\frac{3(2 - 8m + 7m^2)}{m^4(m-1)^3 D}
\end{aligned}\right\} \quad [55] \text{ (Contd.)}$$

where

$$D = 5 - 16m + 14m^2$$

LIMITATION OF RANGE OF a_2 BY THE "NO-INFLECTION POINT" CONDITION

Let us again assume the values of the geometrical parameters which were used to illustrate the sixth-degree polynomial form, viz. $m = 0.40$, $r_0 = 0.50$, $r_1 = 0.10$, and $C_p = 0.65$. When m is substituted into the Formulas [55] for the coefficients in the seventh-degree polynomials [54], the following values are obtained:

$\alpha_0 = 6.25$	$\alpha_1 = 43.75$	$\alpha_2 = -87.5$	$\beta_0 = 11.5079$
$\beta_1 = -14.2857$	$\gamma_0 = 6.25$	$\gamma_1 = -9.5238$	$\delta = 250$
$\epsilon_0 = 20.668$	$\epsilon_1 = -3.8752$	$\epsilon_2 = -51.67$	$D = 0.84$

Put $y^2 = f(x)$, and

$$g(x) = 2r_0S_0(x) + 2r_1S_1(x) + C_pV(x) + W(x)$$

Then

$$f(x) = a_2U(x) + g(x) \qquad [56]$$

where U(x) and g(x) are given polynomials.

The border curve for Condition [51], $f(x) = 0$, can now be expressed as

$$a_2 = -\frac{g(x)}{U(x)} \qquad [57]$$

and graphed with a_2 as ordinate and x as abscissa, for $0 \leqq x \leqq 1$. The curve is shown in Figure 5, where the nonpermissible side of the curve is cross-hatched. The values of a_2 for which the lines a_2 = constant intersect the curve [56] must be excluded since, for these values, Condition [51] is not satisfied for all values of x_0. In this way it is seen from Figure 5 that only values of a_2 in the range $-26 \leqq a_2 \leqq 11$ satisfy Condition [51].

Similarly Condition [52] defines the boundary curve

$$2\left[a_2U(x) + g(x)\right]\left[a_2U''(x) + g''(x)\right] - \left[a_2U'(x) + g'(x)\right]^2 = 0$$

or

$$a_2^2(2UU'' - U'^2) + 2a_2(gU'' - g'U' + g''U) + 2gg'' - g'^2 = 0 \qquad [58]$$

from which a_2 may be graphed as ordinate against x as abscissa, for $0 \leqq x \leqq 1$. This curve is also shown in Figure 5, and its nonpermissible side is also indicated by shading. The curve shows that the limits of a_2 to satisfy Condition [52] are $-\infty < a_2 \leqq 158$, and $-2 \leqq a_2 \leqq b$. Hence Condition [51] and [52] are satisfied simultaneously only by values of a_2 in the range $-2 \leqq a_2 \leqq 6$.

COMPARISON OF SIXTH- AND SEVENTH-DEGREE POLYNOMIAL FORMS

The class of bodies represented by polynomials of the seventh degree, $y^2 = a_1x + a_2x^2 + \ldots + a_7x^7$, includes as a subclass, when $a_7 = 0$, those of the sixth degree. As a check on the formulas for the coefficients occurring in the basic polynomials [54], the example shown in Figure 3 to illustrate the polynomial of the sixth degree was also applied to the seventh degree. The form was assumed to have the same primary geometrical characteristics as before, viz, $m = 0.40$, $r_o = 0.50$, $r_1 = 0.10$, $C_p = 0.65$, and a_2 was chosen to have the value in the resulting sixth-degree polynomial, $a_2 = 2.1497$, as given in Figure 3. Now if these values of the parameters and the values of the coefficients listed in the preceding section are substituted into Equation [53], the resulting polynomial obtained is found to be identical with the original sixth-degree polynomial, with a_7 reducing to zero.

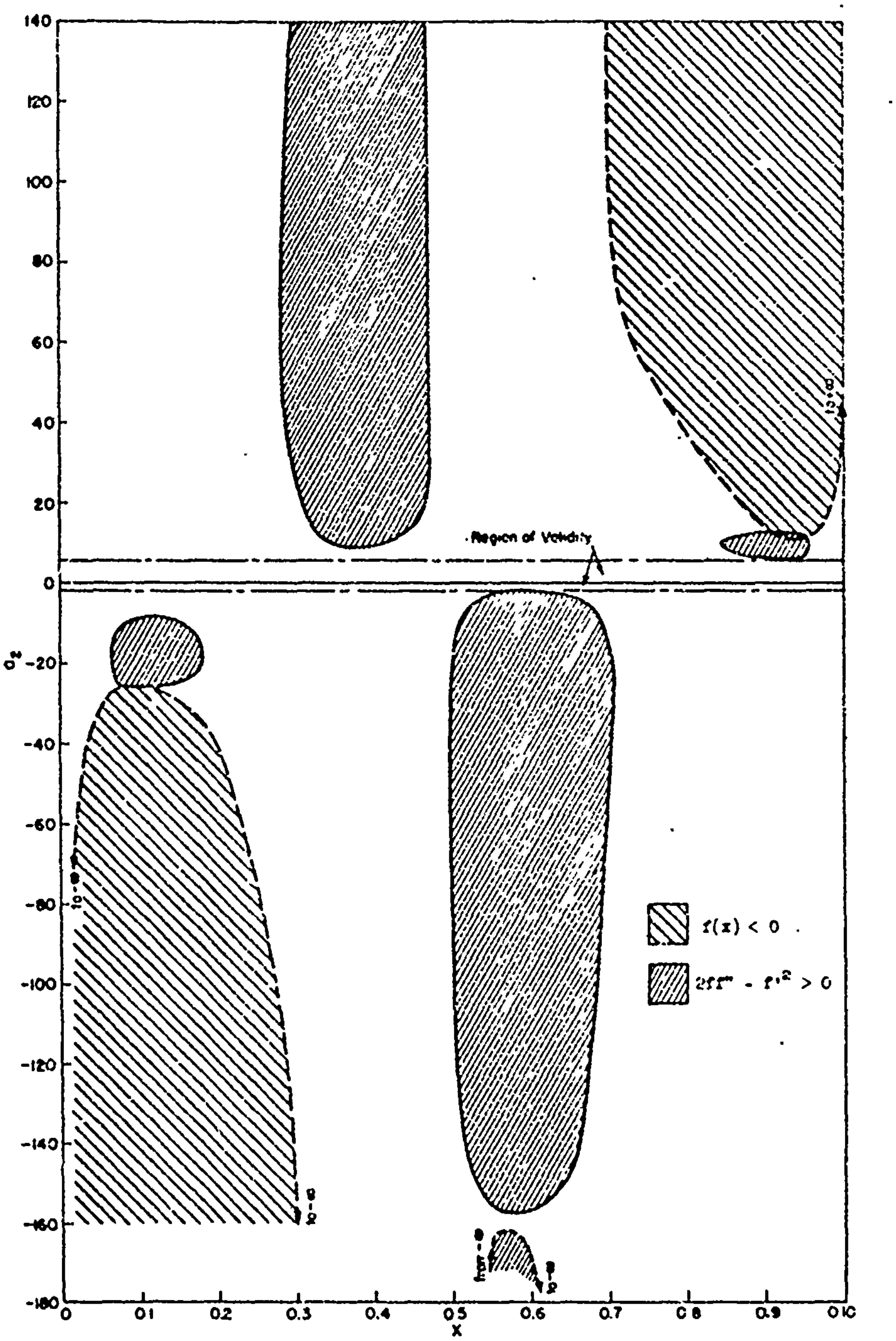


Figure 5 - Boundary Curves Defined by the No-Inflection-Point Condition

Useable values of a_2 are those for which the horizontal lines a_2 = constant do not intersect the shaded areas.

By assuming other values for a_2 it is now possible to obtain new forms having the same values of the primary geometrical quantities m, r_o, r_1, and C_p. The range of choice of a_2 is limited, however, when inflection points on the body are not permitted. With the values of m, r_o, r_1, and C_p as above, it was shown in the previous section that a_2 is restricted to the range $-2 \leq a_2 \leq 6$. Forms corresponding to values of a_2 near the limits of this range, $a_2 = -1$, and $a_2 = 5$, are shown in Figure 6, together with the original sixth-degree polynomial. It is seen that there is appreciable variation between these forms. It must be concluded that the geometrical parameters m, r_o, r_1, and C_p alone do not suffice to fix a form.

From a practical point of view, however, the useable forms generated by polynomials of the seventh degree can be fitted so closely by polynomials of the sixth degree that no essentially new forms are introduced by the increase in the number of degrees of freedom. As an illustration of this the seventh-degree polynomials shown in Figure 6 have been fitted by sixth-degree polynomials by the method of least squares, and, as is shown in Figures 7 and 8, the fits obtained are excellent. The method of curve fitting employed here is described in Appendix 4.

It is further believed that practically all streamlined forms satisfying the no-inflection-point condition can be well-fitted by sixth-degree polynomial forms. From this point of view, although the parameters m, r_o, r_1, and C_p alone do not suffice to fix a form, by their variation they determine all the desirable forms generated by sixth-degree polynomials, and thus, in the above sense, all the desirable stream forms.

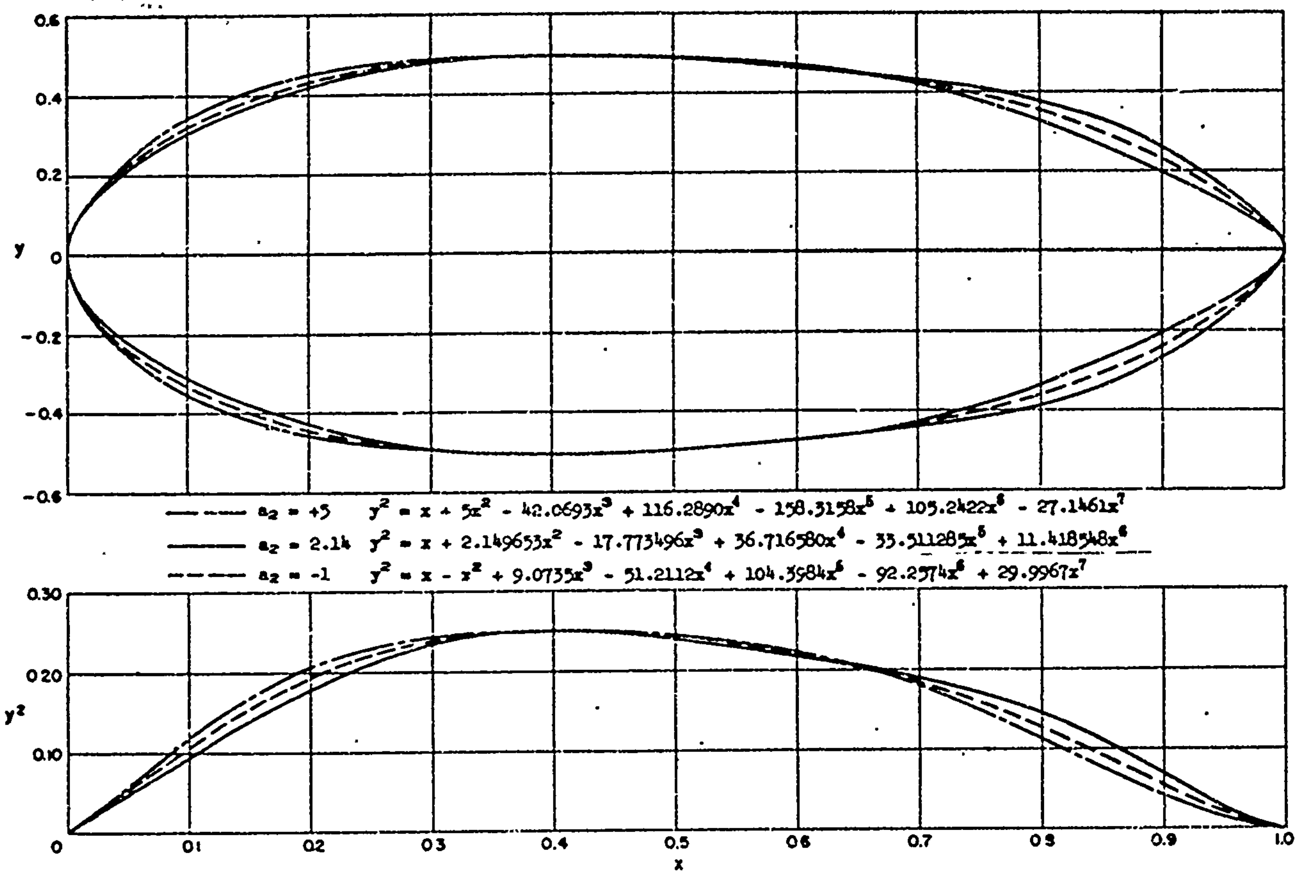


Figure 6 - Forms Derived from Seventh-Degree Polynomials Showing the Effect of Varying the Parameter a_2

The form corresponding to $a_2 = 2.1497$ is the same as the sixth-degree polynomial form shown in Figure 3. These forms have the same values of the geometrical parameters m, r_o, r_1 and C_p.

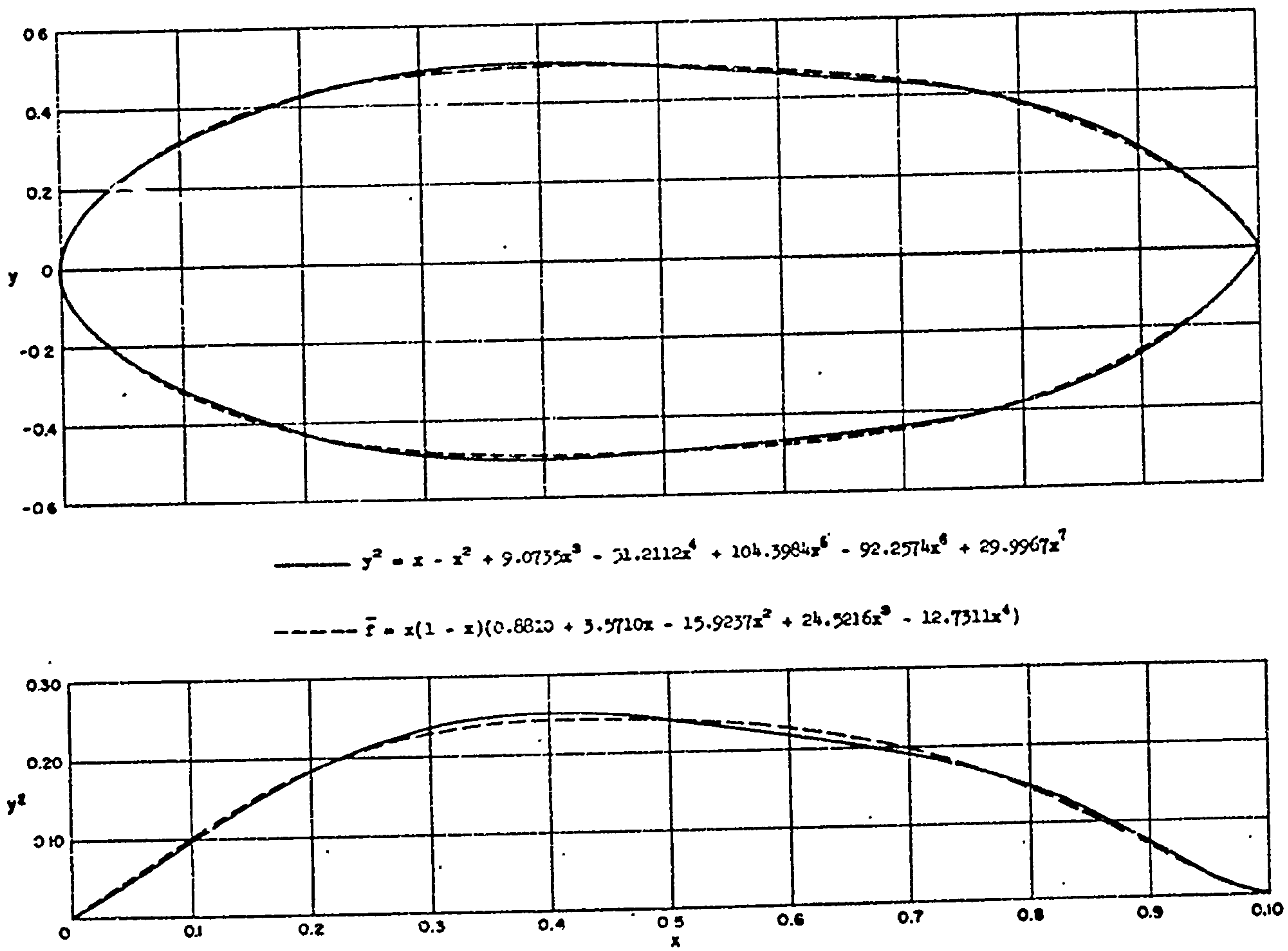


Figure 7 - Least Square Fit of a Seventh- by a Sixth-Degree Polynomial Form for $a_2 = -1.0$

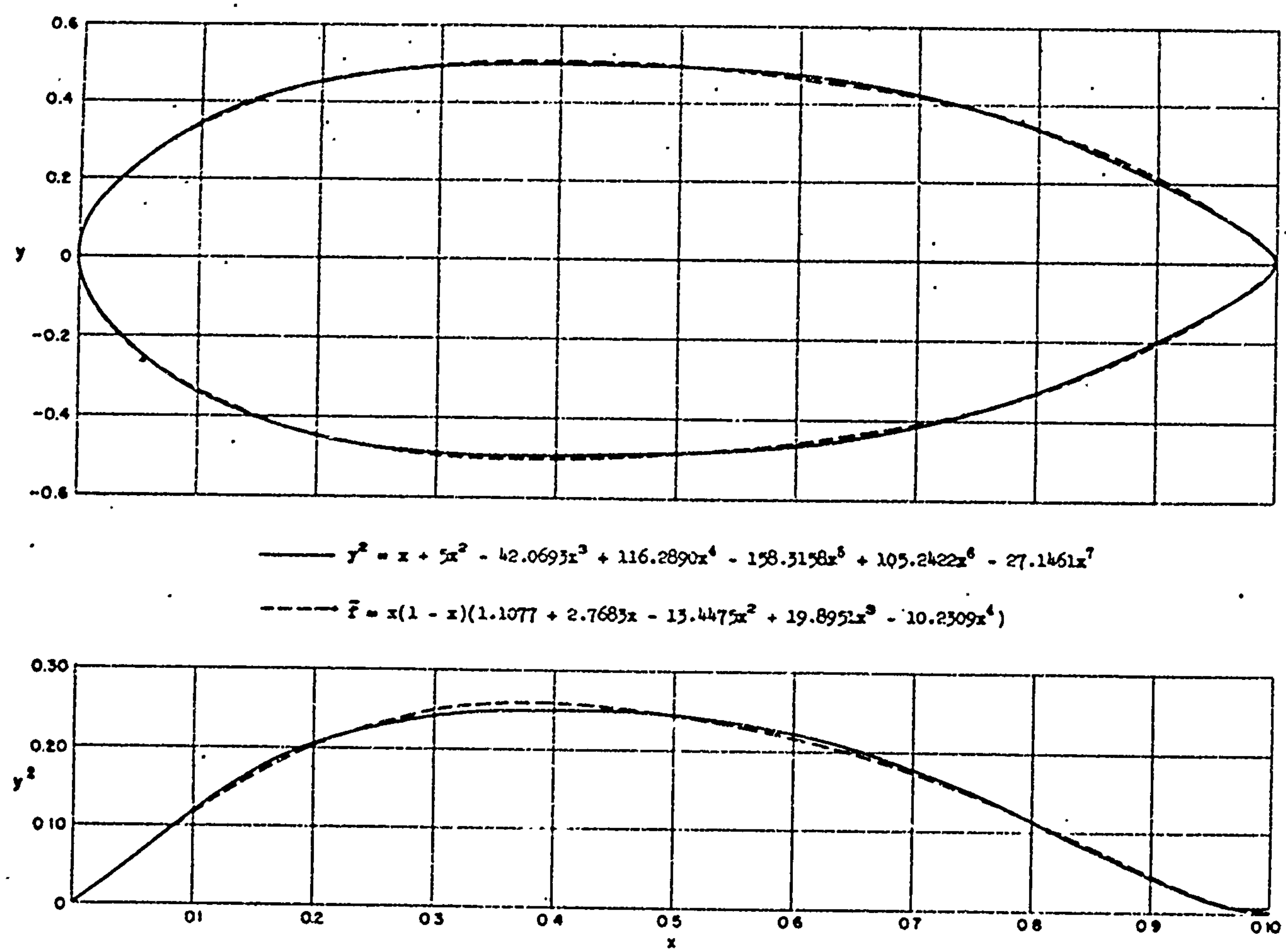


Figure 8 - Least Square Fit of a Seventh- by a Sixth-Degree Polynomial Form for $a_2 = 5.0$

APPENDIX 1

TABLE OF COEFFICIENTS FOR BASIC SIXTH-DEGREE POLYNOMIALS

The coefficients for basic sixth-degree polynomials occur in the following formulas:

$$y^2 = 2r_0R_0(x) + 2r_1R_1(x) + C_pP(x) + Q(x) \qquad [14]$$

where

$$\left.\begin{aligned} R_0(x) &= x(x-1)^2(x-m)^2(\alpha_0 + \alpha_1 x) \\ R_1(x) &= x^2(x-1)(x-m)^2(\beta_0 + \beta_1 x) \\ P(x) &= x^2(x-1)^2(x-m)^2\gamma \\ Q(x) &= x^2(x-1)^2(\delta_0 + \delta_1 x + \delta_2 x^2) \end{aligned}\right\} \qquad [39]$$

and

$$\left.\begin{aligned} \alpha_0 &= \frac{1}{m^2} \\ \alpha_1 &= -\frac{7(1-4m+5m^2)}{2m^2(2-7m+7m^2)} \\ \beta_0 &= \frac{10-28m+21m^2}{2(1-m)^2(2-7m+7m^2)} \\ \beta_1 &= -\frac{7(2-6m+5m^2)}{2(1-m)^2(2-7m+7m^2)} \\ \gamma &= \frac{105}{2(2-7m+7m^2)} \\ \delta_0 &= \frac{3(2-8m+7m^2)}{4m^2(1-m)^3(2-7m+7m^2)} \\ \delta_1 &= -\frac{2-4m-14m^2+21m^3}{2m^3(1-m)^3(2-7m+7m^2)} \\ \delta_2 &= \frac{7(1-5m+5m^2)}{4m^3(1-m)^3(2-7m+7m^2)} \end{aligned}\right\} \qquad [40]$$

The coefficients are tabulated for values of m from 0.10 to 0.50 in intervals of 0.02.

m	α_0	α_1	β_0	β_1	γ	δ_0	δ_1	δ_2
0.10	100.00000	-166.05800	3.33874	-4.57331	38.32130	95.37100	-741.44300	963.72500
0.12	69.44440	-114.12500	3.55523	-4.84656	41.64030	69.15330	-456.21500	556.34300
0.14	51.02040	-83.02060	3.79242	-5.14450	45.36830	52.88200	-302.82300	344.85300
0.16	39.06250	-62.98980	4.05261	-5.46985	49.56580	41.96250	-211.48030	223.22200
0.18	30.86420	-49.38660	4.33826	-5.82547	54.30280	34.16680	-152.61000	147.48400
0.20	25.00000	-39.77270	4.65199	-6.21449	59.65900	28.29800	-112.08300	97.10150
0.22	20.66120	-32.77130	4.99640	-6.64006	65.72350	23.66040	-82.50100	61.56530
0.24	17.36110	-27.55880	5.37393	-7.10523	72.59400	19.81830	-59.68830	35.09030
0.26	14.79290	-23.62060	5.78654	-7.61269	80.37350	16.48110	-41.13030	14.29420
0.28	12.75510	-20.62310	6.23520	-8.16420	89.16450	13.44180	-25.22530	-2.90195
0.30	11.11110	-18.34380	6.71929	-8.76011	99.05650	10.54330	-10.89990	-17.82680
0.32	9.76563	-16.63110	7.23538	-9.39801	110.10900	7.66028	2.59270	-31.34780
0.34	8.65052	-15.37830	7.77601	-10.07170	122.32100	4.69005	15.77530	-44.02200
0.36	7.71605	-14.507[illegible]	8.32803	-10.76940	135.58900	1.55078	28.99530	-56.16930
0.38	6.92521	-13.9[illegible]700	8.87076	-11.47220	149.65800	-1.81402	42.42830	-67.90050
0.40	6.25000	-13.67180	9.37500	-12.15280	164.06300	-5.42535	56.06200	-79.11980
0.42	5.66893	-13.59540	9.80330	-12.77590	178.08700	-9.25455	69.66550	-89.52300
0.44	5.16529	-13.66400	10.11320	-13.30200	190.77000	-13.20990	82.77300	-98.61780
0.46	4.72590	-13.80500	10.26440	-13.69380	200.99600	-17.13200	94.71280	-105.78500
0.48	4.34028	-13.94110	10.22860	-13.92690	207.67400	-20.80660	104.70000	-110.40300
0.50	4.00000	-14.00000	10.00000	-14.00000	210.00000	-24.00000	112.00000	-112.00000

APPENDIX 2

TABLES OF THE BASIC SIXTH-DEGREE POLYNOMIALS

The polynomials are defined by:

$$\left.\begin{aligned} R_0(x) &= x(x-1)^2(x-m)^2(\alpha_0+\alpha_1 x) \\ R_1(x) &= x^2(x-1)(x-m)^2(\beta_0+\beta_1 x) \\ P(x) &= x^2(x-1)^2(x-m)^2\gamma \\ Q(x) &= x^2(x-1)^2(\delta_0+\delta_1 x+\delta_2 x^2) \end{aligned}\right\} \quad [39]$$

m = 0.10

x	$R_0(x)$	$R_1(x)$	$P(x)$	$Q(x)$
0	0	0 -	0	0
0.02	0.011385	-0.000008	0.000094	0.031089
.04	.012389	.000017	.000204	0.099172
.06	.007638	.000017	.000195	0.172898
.08	.002348	.000007	.000083	0.228723
.10	. 0	0	0	0.249999
.12	.002976	.000014	.000171	0.226100
.14	.012715	.000073	.000889	0.151602
.16	.029844	.000202	.002492	0.025497
.18	.054307	.000428	.005343	-0.149536
.20	.085489	.000776	.009810	0.367834
.22	.122327	.001268	.016249	0.621383
.24	.163419	.001923	.024989	0.900435
.26	.207117	.002753	.036315	1.194054
.28	.251624	.003764	.050462	1.490641
.30	.295074	.004956	.067599	1.778399
.32	.335605	.006320	.087822	2.045754
.34	.371433	.007839	.111150	2.281745
.36	.400905	.009489	.137516	2.476352
.38	.422557	.011237	.166766	2.620794
.40	.435155	.013041	.198657	2.707776
.42	.437735	.014856	.232860	2.731687
.44	.429631	.016625	.268955	2.688768
.46	.410495	.018289	.306442	2.577214
.48	.380315	.019784	.344743	2.397257
.50	.339420	.021042	.383213	2.151188
.52	.288477	.021994	.421141	1.843328
.54	.228486	.022571	.457771	1.479982
.56	.160758	.022706	.492308	1.069316
.58	.086897	.022338	.523934	0.621214
.60	0.008765	.021411	.551826	-0.147082
.62	-0.071559	.019879	.575171	+0.340402
.64	.151821	.017708	.593188	0.827558
.66	.229653	.014879	.605148	1.300052
.68	.302628	.011393	.610399	1.743255
.70	.368337	.007272	.608388	2.142665
.72	.424463	-0.002564	.598693	2.484376
.74	.468867	+0.002654	.581045	2.755578
.76	.499681	.008271	.555364	2.945114
.78	.515408	.014139	.521786	3.044072
.80	.515032	.020065	.480702	3.046426
.82	.498130	.025811	.432790	2.949725
.84	.465003	.031087	.379055	2.755812
.86	.416798	.035544	.320863	2.471616
.88	.355654	.038772	.259990	2.109952
.90	.284845	.040292	.198657	1.690389
.92	.208934	.039552	.139580	1.240163
.94	.133940	.035918	.086012	0.795121
.96	.067498	.028672	.041793	0.400719
0.98	-0.019044	+0.017003	0.011400	+0.113066
1.00	0	0	0	0

m = 0.12

x	$R_0(x)$	$R_1(x)$	$P(x)$	$Q(x)$
0	0	0	0	0
0.02	0.012900	-0.000014	0.000160	0.023146
.04	.015307	.000033	.000393	0.076375
.06	.011947	.000040	.000477	0.139273
.08	.006534	.000030	.000367	0.196184
.10	.001880	.000011	.000135	0.235671
.12	0	0	0	0.250000
.14	.002215	.000019	.000242	0.234656
.16	.009245	.000096	.001204	0.187881
.18	.021308	.000257	.003266	0.110237
.20	.038191	.000530	.006822	0.004197
.22	.059344	.000940	.012262	-0.126240
.24	.083949	.001508	.019949	0.275907
.26	.110986	.002250	.030212	0.438805
.28	.139306	.003177	.043325	0.608407
.30	.167683	.004289	.059497	0.777943
.32	.194870	.005583	.078866	0.940654
.34	.219648	.007044	.101486	1.090025
.36	.240870	.008650	.127321	1.219988
.38	.257496	.010370	.156247	1.325105
.40	.268629	.012167	.188041	1.400714
.42	.273544	.013993	.222388	1.443067
.44	.271703	.015795	.258878	1.449428
.46	.262781	.017513	.297013	1.418146
.48	.246670	.019081	.336208	1.348714
.50	.223493	.020432	.375803	1.241789
.52	.193598	.021494	.415071	1.099196
.54	.157559	.022197	.453226	0.923896
.56	.116163	.022470	.489441	0.719944
.58	.070401	.022250	.522858	0.492401
.60	0.021442	.021476	.552610	-0.247245
.62	-0.029390	.020098	.577835	+0.008770
.64	.080642	.018079	.597703	0.268264
.66	.130775	.015396	.611430	0.523401
.68	.178200	.012045	.618311	0.766111
.70	.221323	.008043	.617743	0.988306
.72	.258600	-0.003433	.609252	1.182146
.74	.288594	+0.001709	.592525	1.340313
.76	.310032	.007277	.567444	1.456326
.78	.321875	.013124	.534116	1.524870
.80	.323393	.019059	.492914	1.542156
.82	.314238	.024846	.444511	1.506304
.84	.294528	.030192	.389922	1.417756
.86	.264940	.034747	.330545	1.279711
.88	.226794	.038096	.268206	1.098584
.90	.182163	.039753	.205205	0.884497
.92	.133966	.039158	.144361	0.651789
.94	.086085	.035667	.089064	0.419554
.96	.043477	.028546	.043325	0.212207
0.98	-0.012292	+0.016968	0.011831	+0.060073
1.00	0	0	0	0

m = 0.14

x	$R_0(x)$	$R_1(x)$	P(x)	Q(x)
0	0	0	0	0
.02	0.013653	-0.000021	0.000251	0.018042
.04	.017584	.000055	.000669	.060930
.06	.015621	.000075	.000924	.114369
.08	.010818	.000072	.000885	.167185
.10	.005536	.000047	.000588	.210991
.12	.001526	.000016	.000202	.239857
.14	0	0	0	.250000
.16	.001704	.000026	.000328	.239495
.18	.006986	.000122	.001582	.207993
.20	.015859	.000318	.004181	.156457
.22	.028060	.000643	.008550	.086923
.24	.043106	.001120	.015094	0.002266
.26	.060348	.00173[illegible]	.024104	-0.094019
.28	.079018	.002602	.036140	0.198006
.30	.098273	.003627	.051219	0.305526
.32	.117235	.004842	.069601	0.412336
.34	.135032	.006236	.091381	0.514264
.36	.150823	.007790	.116563	0.607343
.38	.163837	.009476	.145052	0.687950
.40	.173391	.011257	.176653	0.752867
.42	.178913	.013089	.211068	0.799410
.44	.179962	.014918	.247900	0.825481
.46	.176239	.016685	.286652	0.829625
.48	.167602	.018324	.326738	0.811072
.50	.154064	.019767	.367483	0.769758
.52	.135802	.020940	.408139	0.706333
.54	.113154	.021771	.447894	0.622156
.56	.086513	.022186	.485884	0.519269
.58	.056817	.022118	.521209	0.400354
.60	0.024540	.021504	.552956	0.268681
.62	-0.009331	.020288	.580210	-0.128036
.64	.043811	.018430	.602084	+0.017368
.66	.077843	.015901	.617738	0.162995
.68	.110328	.012692	.626408	0.304104
.70	.140155	.008817	.627432	0.435883
.72	.166239	-0.004316	.620282	0 553606
.74	.187557	+0.000744	.604595	0.652793
.76	.203196	.006256	.580210	0.729396
.78	.212398	.012077	.547201	0.780001
.80	.214609	.018019	.505918	0.802038
.82	.209540	.023845	.457028	0.794012
.84	.197219	.029262	.401557	0.755753
.86	.178060	.033916	.340934	0.688677
.88	.152924	.037389	.277040	0.596062
.90	.123192	.039189	.212258	0.483346
.92	.090841	.038746	.149519	0.358432
.94	.058517	.035403	.092361	0.232021
.96	.029620	.028414	.044982	0.117948
0.98	-0.008392	+0.016930	0.012298	+0.033542
1.00	0	0	0	0

$m = 0.16$

x	$R_0(x)$	$R_1(x)$	$P(x)$	$Q(x)$
0	0	0	0	0
0.02	0.014232	-0.000030	0.000373	0.014530
.04	.019398	.000085	.001052	.049929
.06	.018706	.000126	.001577	.095675
.08	.014744	.000136	.001719	.143402
.10	.009554	.000114	.001445	.186678
.12	.004684	.000069	.000884	.220789
.14	.001253	.000022	.000288	.242527
.16	0	0	0	.249999
.18	.001342	.000033	.000432	.242441
.20	.005420	.000151	.002030	.220040
.22	.012145	.000387	.005255	.183769
.24	.021244	.000768	.010554	.135235
.26	.032298	.001316	.018348	.076531
.28	.044783	.002049	.029008	0.010104
.30	.058101	.002978	.042843	-0.061372
.32	.071615	.004104	.060081	.135103
.34	.084675	.005421	.080867	.208289
.36	.096650	.006912	.105247	.278209
.38	.106942	.008554	.133161	.342304
.40	.115015	.010311	.164447	.398255
.42	.120408	.012140	.198831	.444044
.44	.122751	.013990	.235928	.478001
.46	.121775	.015801	.275250	.498850
.48	.117322	.017508	.316207	.505740
.50	.109352	.019041	.358113	.498266
.52	.097942	.020325	.400199	.476480
.54	.083291	.021285	.441624	.440891
.56	.065712	.021845	.481485	.392461
.58	.045633	.021935	.518842	.332566
.60	.023578	.021486	.552726	.262996
.62	0.000167	.020440	.582168	.185889
.64	-0.023906	.018750	.606218	.103691
.66	.047890	.016384	.623975	-0.019095
.68	.070994	.013328	.634609	+0.065037
.70	.092412	.009590	.637394	.145744
.72	.111549	.005204	.631741	.220069
.74	.127051	-0.000236	.617231	.285156
.76	.138836	+0.005214	.593655	.338362
.78	.146128	.011004	.561048	.377383
.80	.148496	.016949	.519735	.400385
.82	.145694	.022811	.470373	.406148
.84	.137706	.028297	.413998	.394219
.86	.124790	.033053	.352072	.365073
.88	.107527	.036653	.286533	.320296
.90	.086879	.038601	.219852	.262762
.92	.064237	.038315	.155083	.196827
.94	.041481	.035127	.095925	.128541
.96	.021045	.028275	.046776	.065857
0.98	-0.005975	+0.016891	0.012803	+0.018860
1.00	0	0	0	0

m = 0.18

x	$R_0(x)$	$R_1(x)$	$P(x)$	$Q(x)$
0	0	0	0	0
0.02	0.014691	-0.000042	0.000534	0.011976
.04	.020873	.000124	.001569	.041728
.06	.021300	.000194	.002488	.081245
.08	.018224	.000228	.002942	.124059
.10	.013440	.000216	.002815	.165083
.1[illegible]	.008343	.000166	.002180	.200472
.1[illegible]	.003968	.000095	.001259	.227475
.16	.001037	.000029	.000393	.244303
.18	0	0	0	.250003
.20	.001075	.000041	.000556	.244332
.22	.004283	.000185	.002558	.227647
.24	.009[illegible]88	.000463	.006504	.200801
.26	.016423	.000904	.012865	.165031
.28	.024728	.001528	.022070	.121879
.30	.033971	.002350	.034485	.073093
.32	.043678	.003377	.050396	0.020550
.34	.053356	.004605	.070002	-0.033807
.36	.062515	.006023	.093397	.088057
.38	.070683	.007608	.120568	.140335
.40	.077429	.009330	.151388	.188911
.42	.082373	.011147	.185609	.232210
.44	.085200	.013009	.222869	.268875
.46	.085669	.014858	.262689	.297784
.48	.083622	.016627	.304476	.318080
.50	.078988	.018247	.347538	.329195
.52	.071786	.019640	.391083	.330854
.54	.06[illegible]129	.020731	.434240	.323086
.56	.050217	.021439	.476070	.306221
.58	.036341	.021690	.515580	.280879
.60	.020867	.021413	.551751	.247962
.62	0.004238	.020544	.583549	.208625
.64	-0.013044	.019032	.609962	.164246
.66	.030428	.016838	.630014	.116407
.68	.047327	.013944	.642806	.066826
.70	.063140	.010352	.647541	-0.017330
.72	.077267	.006092	.643562	+0.030212
.74	.089135	-0.001224	.630388	.073951
.76	.098218	+0.004155	.607754	.112119
.78	.104069	.009907	.575651	.143119
.80	.106341	.015849	.534374	.165596
.82	.104827	.021745	.484568	.178531
.84	.099484	.027300	.427276	.181340
.86	.090477	.032157	.363993	.173974
.88	.078212	.035888	.296720	.157029
.90	.063376	.037987	.228020	.131867
.92	.046982	.037864	.161080	.100734
.94	.030412	.034838	.099772	.066897
.96	.015463	.028129	.048716	.034774
0.98	-0.004399	+0.016850	0.013351	+0.010085
1.00	0	0	0	0

m = 0.20

x	$R_0(x)$	$R_1(x)$	$P(x)$	$Q(x)$
0	0	0	0	0
0.02	0.015063	-0.000058	0.000743	0.010025
.04	.022092	.000173	.002252	.035345
.06	.023498	.000284	.003720	.069735
.08	.021274	.000352	.004654	.108084
.10	.017028	.000363	.004833	.146292
.12	.012030	.000317	.004258	.181169
.14	.007243	.000229	.003114	.210334
.16	.003366	.000126	.001724	.232124
.18	.000864	.000038	.000520	.245508
.20	0	0	0	.249997
.22	.000870	.000050	.000703	.245570
.24	.003428	.000221	.003176	.232596
.26	.007514	.000547	.007950	.211761
.28	.012879	.001052	.015518	.184012
.30	.019210	.001756	.026310	.150483
.32	.026150	.002671	.040678	.112443
.34	.033317	.003797	.058882	.071243
.36	.040323	.005127	.081074	0.028270
.38	.046790	.006644	.107293	-0.015098
.40	.052364	.008318	.137454	.057[illegible]4
.42	.056727	.010111	.171347	.097797
.44	.059609	.011975	.208632	.134756
.46	.060794	.013852	.248843	.167424
.48	.060129	.015677	.291394	.194972
.50	.057529	.017378	.335582	.216750
.52	.052977	.018879	.380597	.232303
.54	.046532	.020098	.425537	.241373
.56	.038320	.020956	.469421	.243904
.58	.028541	.021373	.511208	.240037
.60	.017455	.021273	.549817	.230118
.62	0.005384	.020588	.584151	.214673
.64	-0.007299	.019261	.613121	.194404
.66	.020180	.017250	.6356[illegible]8	.170161
.68	.032815	.014528	.650843	.142928
.70	.044744	.011093	.657741	.113787
.72	.055503	.006969	.655638	.083892
.74	.064647	-0.002211	.643982	.054433
.76	.071760	+0.003088	.622446	.026585
.78	.076487	.008794	.590971	-0.001472
.80	.078545	.014727	.549817	+0.019885
.82	.077756	.020652	.499611	.036626
.84	.074067	.026274	.441402	.048109
.86	.067584	.031233	.376719	.053971
.88	.058595	.035096	.307625	.054202
.90	.047608	.037350	.236787	.049214
.92	.035379	.037395	.167532	.039915
.94	.022953	.034537	.103920	.027799
.96	.011695	.027977	.050812	.015022
0.98	-0.003333	+0.016307	0.013944	+0.004500
1.00	0	0	0	0

$m = 0.22$

x	$R_0(x)$	$R_1(x)$	P(x)	Q(x)
0	0	0	0	0
0.02	0.015371	-0.000076	0.001010	0.008465
.04	.023112	.000235	.003140	.030168
.06	.025373	.000398	.005352	.060222
.08	.023941	.000515	.006978	.094550
.10	.020277	.000561	.007666	.129811
.12	.015546	.000532	.007329	.163332
.14	.010651	.000439	.006098	.193045
.16	.006266	.000305	.004274	.217417
.18	.002859	.000162	.002291	.235395
.20	.000722	.000047	.000673	.246344
.22	0	0	0	.250000
.24	.000710	.000060	.000875	.246407
.26	.002766	.000262	.003893	.235876
.28	.006002	.000638	.009616	.218937
.30	.010189	.001211	.018550	.196289
.32	.015055	.002000	.031120	.168772
.34	.020301	.003009	.047658	.137320
.36	.025617	.004237	.068382	.102926
.38	.030694	.005668	.093393	.066613
.40	.035238	.007280	.122656	0.029403
.42	.038980	.009034	.156004	-0.007709
.44	.041686	.010887	.193129	.043774
.46	.043162	.012781	.233586	.077920
.48	.043264	.014653	.276794	.109363
.50	.041900	.016428	.322045	.137422
.52	.039035	.018031	.368512	.161535
.54	.034689	.019378	.415264	.181264
.56	.028942	.020385	.461275	.196302
.58	.021929	.020969	.505453	.206483
.60	.013840	.021051	.546651	.211777
.62	0.004913	.020557	.583702	.212289
.64	-0.004571	.019424	.615438	.208255
.66	.014296	.017604	.640725	.200032
.68	.023918	.015065	.658408	.188096
.70	.033076	.011798	.667793	.173018
.72	.041407	.007822	.667793	.155453
.74	.048554	-0.003186	.657866	.136122
.76	.054188	+0.002023	.637614	.115786
.78	.058016	.007675	.606920	.095232
.80	.059808	.013592	.566000	.075229
.82	.059408	.019540	.515461	.056511
.84	.056761	.025225	.456355	.039739
.86	.051934	.030284	.390242	.025467
.88	.045139	.034230	.319254	.014099
.90	.036759	.036692	.246164	.005853
.92	.027375	.036910	.174451	-0.000713
.94	.017795	.034224	.108379	+0.001618
.96	.009083	.027818	.053070	.001767
0.98	-0.002594	+0.016762	0.014584	+0.000744
1.00	0	0	0	0

$m = 0.24$

x	$R_0(x)$	$R_1(x)$	P(x)	Q(x)
0	0	0	0	0
0.02	0.015628	-0.000099	0.001350	0.007160
.04	.023975	.000313	.004282	.025786
.06	.026981	.000542	.007482	.052051
.08	.026273	.000724	.010067	.082705
.10	.023187	.000823	.011525	.115023
.12	.018807	.000825	.011657	.146763
.14	.013981	.000738	.010523	.176124
.16	.009358	.000583	.008393	.201704
.18	.005403	.000392	.005694	.222462
.20	.002427	.000202	.002974	.237677
.22	.000605	.000058	.000855	.246917
.24	0	0	0	.250000
.26	.000581	.000071	.001075	.246962
.28	.002240	.000306	.004721	.238030
.30	.004812	.000735	.011525	.223585
.32	.008090	.001382	.021999	.204139
.34	.011835	.002257	.036555	.180309
.36	.015798	.003363	.055492	.152788
.38	.019723	.004692	.078978	.122327
.40	.023363	.006222	.107044	.089709
.42	.026489	.007922	.139573	.055730
.44	.028895	.009747	.176296	0.021186
.46	.030410	.011644	.216795	-0.013155
.48	.030897	.013549	.260503	.046554
.50	.030265	.015390	.306710	.078328
.52	.028466	.017087	.354573	.107854
.54	.025497	.018556	.403131	.134575
.56	.021406	.019711	.451318	.158025
.58	.016286	.020463	.497981	.177817
.60	.010275	.020730	.541911	.193660
.62	0.003551	.020432	.581860	.205362
.64	-0.003670	.019502	.616575	.212825
.66	.011140	.017882	.644830	.216056
.68	.018588	.015537	.665463	.215156
.70	.025729	.012450	.677415	.210324
.72	.032270	.008634	.679774	.201850
.74	.037923	-0.004131	.671817	.190108
.76	.042419	+0.000976	.653066	.175553
.78	.045517	.006563	.623337	.158706
.80	.047024	.012454	.582796	.140149
.82	.046807	.018418	.532022	.120506
.84	.044810	.024161	.472065	.100435
.86	.041076	.029317	.404517	.080611
.88	.035766	.033445	.331580	.061706
.90	.029175	.036017	.256138	.044372
.92	.021762	.036410	.181834	.029222
.94	.014168	.033901	.113150	.016805
.96	.007242	.027654	.055492	.007584
0.98	-0.002071	+0.016716	0.015272	-0.001912
1.00	0	0	0	0

$m = 0.26$

x	$R_0(x)$	$R_1(x)$	$P(x)$	$Q(x)$
0	0	0	0	0
0.02	0.015844	-0.000127	0.001779	0.006018
.04	.024708	.000408	.005736	.021910
.06	.028365	.000721	.010227	.044740
.08	.028308	.000988	.014106	.071949
.10	.025777	.001158	.016666	.101340
.12	.021781	.001210	.017567	.131043
.14	.017126	.001146	.016777	.159502
.16	.012434	.000982	.014518	.185442
.18	.008165	.000751	.011207	.207854
.20	.004640	.000491	.007407	.225967
.22	.002055	.000248	.003787	.239234
.24	.000506	.000069	.001070	.247302
.26	0	0	0	.250001
.28	.000475	.000083	.001307	.247323
.30	.001813	.000353	.005671	.239398
.32	.003854	.000840	.013701	.226482
.34	.006409	.001562	.025902	.208937
.36	.009274	.002526	.042666	.187215
.38	.012236	.003731	.064243	.161843
.40	.015085	.005158	.090739	.133407
.42	.017623	.006782	.122008	.102537
.44	.019670	.008560	.158103	.069891
.46	.021072	.010442	.198370	.036140
.48	.021704	.012365	.242353	0.001994
.50	.021475	.014257	.289345	-0.031903
.52	.020330	.016038	.338492	.064883
.54	.018255	.017622	.388805	.096318
.56	.015274	.018919	.439175	.125624
.58	.011451	.019838	.488391	.152260
.60	.006887	.020291	.535172	.175743
.62	0.001719	.020193	.578187	.195659
.64	-0.003884	.019470	.616091	.211663
.66	.009726	.018061	.647558	.223491
.68	.015589	.015920	.671320	.230965
.70	.021241	.013025	.686210	.233989
.72	.026444	.009380	.691210	.232578
.74	.030951	.005024	.685497	.226835
.76	.034569	-0.000031	.668502	.216970
.78	.037067	+0.005478	.639962	.203295
.80	.038291	.011352	.599985	.186234
.82	.038126	.017303	.549113	.166318
.84	.036520	.023095	.488391	.144187
.86	.033501	.028344	.419439	.120592
.88	.029194	.032600	.344528	.096393
.90	.023835	.035330	.266660	.072558
.92	.017795	.035900	.189652	.050162
.94	.011596	.033570	.118220	.030381
.96	.005933	.027486	.058072	.014496
0.98	-0.001698	+0.016668	0.016007	-0.003880
1.00	0	0	0	0

m = 0.28

x	$R_0(x)$	$R_1(x)$	$P(x)$	$Q(x)$
0	0	0	0	0
0.02	0.016026	-0.000161	0.002316	0.004970
.04	.025332	.000523	.007573	.018326
.06	.029554	.000941	.013728	.037910
.08	.030078	.001315	.019320	.061782
.10	.028062	.001580	.023400	.088211
.12	.024456	.001705	.025454	.115673
.14	.020026	.001682	.025334	.142837
.16	.015372	.001526	.023193	.168557
.18	.010945	.001266	.019426	.191872
.20	.007070	.000943	.014609	.211985
.22	.003960	.000603	.009452	.228264
.24	.001731	.000299	.004746	.240228
.26	.000421	.000082	.001321	.247540
.28	0	0	0	.250001
.30	.000386	.000095	.001573	.247536
.32	.001457	.000404	.006755	.240185
.34	.003062	.000950	.016164	.228097
.36	.005031	.001750	.030293	.211519
.38	.007184	.002805	.049493	.190787
.40	.009343	.004105	.073957	.166314
.42	.011336	.005627	.103706	.138579
.44	.013003	.007335	.138584	.108122
.46	.014205	.009180	.178254	.075532
.48	.014827	.011101	.222199	.041434
.50	.014783	.013026	.269723	0.006480
.52	.014016	.014876	.319966	-0.028658
.54	.012503	.016562	.371913	.063308
.56	.010252	.017993	.424414	.096801
.58	.007308	.019073	.476200	.128478
.60	0.003748	.019710	.525914	.157712
.62	-0.000323	.019814	.572138	.183915
.64	.004769	.019304	.613426	.206551
.66	.009432	.018110	.648343	.225138
.68	.014134	.016183	.675507	.239269
.70	.018682	.013491	.693632	.248628
.72	.022879	.010032	.701582	.252090
.74	.026526	.005835	.698422	.252238
.76	.029436	-0.000971	.683477	.246382
.78	.031437	+0.004446	.656397	.235556
.80	.032391	.010250	.617218	.220049
.82	.032106	.016215	.566438	.200302
.84	.030807	.022047	.505088	.176931
.86	.028243	.027379	.434812	.150734
.88	.024603	.031758	.357950	.122706
.90	.020085	.034642	.277626	.094053
.92	.014996	.035386	.197837	.066204
.94	.009774	.033235	.123549	.040825
.96	.005002	.027315	.060796	.019631
0.98	-0.001432	+0.016619	0.016784	-0.005404
1.00	0	0	0	0

m = 0.30

x	$R_0(x)$	$R_1(x)$	P(x)	Q(x)
0	0	0	0	0
0.02	0.016180	-0.000201	0.002984	0.003964
.04	.025860	.000661	.009574	.014862
.06	.030569	.001207	.018149	.031253
.08	.031605	.001715	.025971	.051771
.10	.030057	.002104	.032094	.075128
.12	.026826	.002327	.035789	.100124
.14	.022645	.002370	.036760	.125652
.16	.018092	.002241	.035070	.150702
.18	.013610	.001967	.031075	.174367
.20	.009526	.001590	.025359	.195847
.22	.006061	.001158	.018668	.214445
.24	.003348	.000728	.011864	.229578
.26	.001445	.000356	.005867	.240773
.28	.000347	.000096	.001611	.247665
.30	0	0	0	.250000
.32	.000310	.000109	.001876	.247633
.34	.001155	.000457	.007981	.240526
.36	.002393	.001065	.018930	.228740
.38	.003871	.001943	.035190	.212435
.40	.005434	.003087	.057057	.191869
.42	.006931	.004479	.084645	.167385
.44	.008221	.006088	.117874	.139403
.46	.009179	.007868	.156469	.108425
.48	.009698	.009760	.199949	.075014
.50	.009696	.011696	.247641	.039792
.52	.009117	.013594	.298688	0.003427
.54	.007934	.015366	.352054	-0.033377
.56	.006146	.016917	.406548	.069888
.58	.003784	.018149	.460844	.105364
.60	0.000906	.018963	.513509	.139061
.62	-0.002402	.019266	.563032	.170254
.64	.006030	.018969	.607863	.198242
.66	.009847	.017997	.646448	.222368
.68	.013702	.016290	.677280	.242039
.70	.017434	.013811	.698943	.256739
.72	.020875	.010549	.710171	.266047
.74	.023856	.006527	.709902	.269659
.76	.026216	-0.001807	.697346	.267401
.78	.027808	+0.003503	.672048	.259260
.80	.028512	.009242	.633962	.245395
.82	.028239	.015185	.583529	.226165
.84	.026949	.021043	.521758	.202150
.86	.024657	.026445	.450311	.174176
.88	.021448	.030936	.371593	.143336
.90	.017491	.033966	.288849	.111021
.92	.013049	.034878	.206263	.078942
.94	.008500	.032903	.129063	.049160
.96	.004348	.027145	.063626	.024109
0.98	-0.001244	+0.016570	0.017597	-0.006631
1.00	0	0	0	0

m = 0.32

x	$R_0(x)$	$R_1(x)$	$P(x)$	$Q(x)$
0	0	0	0	0
0.02	0.016307	-0.000249	0.003807	0.002958
.04	.026301	.000826	.012730	.011375
.06	.031423	.001526	.023677	.024503
.08	.032899	.002199	.034356	.041532
.10	.031765	.002742	.043167	.061609
.12	.028882	.003096	.049114	.083858
.14	.024951	.003233	.051716	.107400
.16	.020533	.003155	.050917	.131368
.18	.016065	.002887	.047017	.154925
.20	.011869	.002468	.040591	.177278
.22	.008174	.001951	.032424	.197688
.24	.005123	.001395	.023446	.215485
.26	.002789	.000863	.014673	.230076
.28	.001186	.000416	.007161	.240952
.30	.000281	.000111	.001942	.247700
.32	0	0	0	.250004
.34	.000244	.000123	.002218	.247648
.36	.000891	.000511	.009352	.240523
.38	.001812	.001181	.022003	.228628
.40	.002869	.002136	.040591	.212066
.42	.003929	.003364	.065340	.191045
.44	.004864	.004840	.096265	.165877
.46	.005561	.006522	.133163	.136964
.48	.005923	.008356	.175611	.104805
.50	.005873	.010272	.222971	.069980
.52	.005355	.012192	.274393	0.033147
.54	.004340	.014026	.328830	-0.004978
.56	.002824	.015677	.385059	.043621
.58	0.000827	.017044	.441697	.081969
.60	-0.001603	.018025	.497235	.119192
.62	.004397	.018518	.550067	.154440
.64	.007460	.018431	.598533	.186880
.66	.010680	.017681	.640953	.215704
.68	.013929	.016199	.675687	.240153
.70	.017068	.013941	.701179	.259539
.72	.019949	.010888	.716018	.273266
.74	.022426	.007054	.719004	.280859
.76	.024357	-0.002493	.709215	.281985
.78	.025616	+0.002693	.686077	.276486
.80	.026094	.008347	.649449	.264400
.82	.025717	.014251	.599702	.246003
.84	.024448	.020116	.537809	.221833
.86	.022301	.025571	.465440	.192724
.88	.019352	.030158	.385059	.159842
.90	.015751	.033320	.300030	.124724
.92	.011732	.034390	.214725	.089310
.94	.007633	.032581	.134637	.055990
.96	.003901	.026978	.066504	.027635
0.98	-0.001116	+0.016522	0.018426	-0.007647
1.00	0	0	0	0

m = 0.34

x	$R_0(x)$	$R_1(x)$	P(x)	Q(x)
0	0	0	0	0
0.02	0.016410	-0.000304	0.004812	0.001916
.04	.026659	.001019	.016233	.007743
.06	.032120	.001903	.030506	.017426
.08	.033965	.002774	.044793	.030715
.10	.033185	.003509	.057070	.047202
.12	.030608	.004028	.066020	.066342
.14	.026911	.004292	.070928	.087496
.16	.022642	.004295	.071588	.109954
.18	.018226	.004056	.068221	.132965
.20	.013986	.003614	.061376	.155757
.22	.010152	.003023	.051869	.177561
.24	.006875	.002346	.040696	.197637
.26	.004239	.001651	.028979	.215286
.28	.002270	.001007	.017897	.229867
.30	.000950	.000479	.008631	.240816
.32	.000221	.000127	.002317	.247654
.34	0	0	0	.249999
.36	.000184	.000138	.002597	.247581
.38	.000656	.000566	.010863	.240230
.40	.001296	.001295	.025365	.227903
.42	.001982	.002322	.046455	.210672
.44	.002600	.003626	.074265	.188729
.46	.003045	.005172	.108684	.162380
.48	.003228	.006908	.149363	.132047
.50	.003076	.008769	.195713	.098264
.52	.002538	.010676	.246908	.061657
.54	.001583	.012541	.301699	0.022947
.56	0.000203	.014264	.359440	-0.017065
.58	-0.001585	.015743	.418076	.057516
.60	.003741	.016870	.476287	.097493
.62	.006205	.017539	.532312	.136065
.64	.008895	.017652	.584395	.172266
.66	.011713	.017118	.630732	.205158
.68	.014543	.015861	.669537	.233837
.70	.017263	.013828	.699106	.257456
.72	.019741	.010991	.717873	.275261
.74	.021846	.007357	.724485	.286616
.76	.023452	-0.002972	.717873	.291037
.78	.024445	+0.002071	.697331	.288228
.80	.024729	.007620	.662605	.278113
.82	.024238	.013463	.613981	.260875
.84	.022941	.019311	.552380	.237002
.86	.020851	.024797	.479467	.207322
.88	.018041	.029458	.397755	.173050
.90	.014648	.032730	.310714	.135837
.92	.010889	.033939	.222901	.097813
.94	.007072	.032281	.140075	.061644
.96	.003609	.026822	.069334	.030577
0.98	-0.001031	+0.016477	0.019247	-0.008501
1.00	0	0	0	0

m = 0.36

x	$R_0(x)$	$R_1(x)$	P(x)	Q(x)
0	0	0	0	0
0.02	0.016489	-0.000368	0.006022	0.000810
.04	.026937	.001242	.020473	.003865
.06	.032663	.002340	.038818	.009824
.08	.034800	.003447	.057583	.019019
.10	.034306	.004412	.074243	.031498
.12	.031983	.005135	.087091	.047074
.14	.028491	.005564	.095131	.065366
.16	.024362	.005681	.097967	.085839
.18	.020018	.005500	.095707	.107841
.20	.015776	.005058	.088860	.130638
.22	.011869	.004409	.078255	.153450
.24	.008452	.003620	.064958	.175474
.26	.005615	.002765	.050192	.195917
.28	.003394	.001919	.035268	.214015
.30	.001780	.001156	.021526	.229061
.32	.000728	.000544	.010272	.240420
.34	.000165	.000142	.002731	.247546
.36	0	0	0	.250003
.38	.000129	.000152	.003010	.247459
.40	.000441	.000618	.012496	.239720
.42	.000825	.001401	.028966	.226714
.44	.001177	.002491	.052684	.208507
.46	.001399	.003855	.083662	.185304
.48	.001406	.005450	.121640	.157439
.50	.001133	.007211	.166096	.125380
.52	0.000528	.009064	.216248	.089720
.54	-0.000437	.010920	.271064	.051167
.56	.001770	.012679	.329280	0.010531
.58	.003458	.014236	.589425	-0.031290
.60	.005465	.015481	.449851	.073325
.62	.007738	.016303	.508770	.114546
.64	.010201	.016596	.564253	.153901
.66	.012764	.016265	.614485	.190327
.68	.015323	.015225	.657413	.222785
.70	.017764	0.013415	.691226	.250288
.72	.019966	.010799	.714184	.271931
.74	.021811	.007374	.724771	.286926
.76	.023181	-0.003178	.721761	.294643
.78	.023972	+0.001702	.704298	.294651
.80	.024099	.007124	.672000	.286755
.82	.023499	.012879	.625047	.271044
.84	.022148	.018683	.564293	.247942
.86	.020060	.024169	.491379	.218255
.88	.017305	.028873	.408845	.183227
.90	.014016	.032227	.320255	.144590
.92	.010397	.033547	.230333	.104630
.94	.006740	.032017	.145090	.066243
.96	.003434	.026683	.071976	.033000
.98	-0.000980	+0.016436	0.020023	-0.009212
1.00	0	0	0	0

m = 0.38

x	$R_0(x)$	$R_1(x)$	P(x)	Q(x)
0	0	0	0	0
0.02	0.016544	-0.000439	0.007452	-0.000381
.04	.027133	.001494	.025511	-0.000333
.06	.033050	.002835	.048748	+0.001550
.08	.035398	.004214	.072963	.006206
.10	.035115	.005450	.095039	.014174
.12	.032982	.006420	.112817	.025644
.14	.029649	.007053	.124962	.040518
.16	.025638	.007322	.130842	.058458
.18	.021364	.007233	.130417	.078932
.20	.017144	.006818	.124132	.101264
.22	.013208	.006134	.112817	.124672
.24	.009715	.005249	.097591	.148306
.26	.006758	.004241	.079777	.171291
.28	.004380	.003194	.060826	.192747
.30	.002576	.002189	.042240	.211830
.32	.001310	.001303	.025511	.227754
.34	.000517	.000607	.012058	.239806
.36	.000112	.000157	.003[illegible]77	.247386
.38	0	0	0	.249999
.40	.000077	.000164	.003448	.247289
.42	.000240	.000663	.014210	.239038
.44	.000390	.001492	.032711	.225177
.46	.000434	.002628	.059100	.205792
.48	0.000293	.004030	.093237	.181123
.50	-0.000096	.005642	.134692	.151561
.52	.000781	.007391	.182745	.117647
.54	.001789	.009183	.236398	.080056
.56	.003129	.010957	.294392	+0.039597
.58	.004788	.012529	.355234	-0.002811
.60	.006733	.013852	.417223	.046152
.62	.008912	.014791	.478490	.089336
.64	.011255	.015237	.537046	.131224
.66	.013676	.015084	.590829	.170645
.68	.016078	.014245	.637765	.206446
.70	.018352	.012648	.675832	.237500
.72	.020384	.010249	.703135	.262766
.74	.022062	.007036	.717984	.281315
.76	.023276	-0.003040	.718980	.292370
.78	.023927	+0.001661	.705106	.295369
.80	.023936	.006932	.675832	.289990
.82	.023246	.012570	.631216	.276225
.84	.021835	.018296	.572024	.254421
.86	.019720	.023745	.499846	.225342
.88	.016971	.028454	.417223	.190235
.90	.013716	.031851	.327787	.150887
.92	.010156	.033244	.236398	.109699
.94	.006574	.031807	.149291	.069753
.96	.003345	.026570	.074236	.034888
0.98	-0.000953	+0.016402	0.020698	-0.009775
1.00	0	0	0	0

m = 0.40

x	$R_o(x)$	$R_1(x)$	P(x)	Q(x)
0	0	0	0	0
0.02	0.016577	-0.000517	0.009101	-0.001666
.04	.027247	.001770	.031352	.004880
.06	.033277	.003382	.060329	.007464
.08	.035752	.005066	.091006	.007837
.10	.035596	.006609	.119602	-0.004944
.12	.033582	.007865	.143435	+0.001815
.14	.030350	.008744	.160772	.012649
.16	.026418	.009204	.170699	.027440
.18	.022196	.009242	.172993	.045801
.20	.018000	.008889	.168000	.067130
.22	.014060	.008197	.156527	.090663
.24	.010535	.007238	.139734	.115520
.26	.007522	.006094	.119036	.140752
.28	.005062	.004855	.096018	.165375
.30	.003158	.003609	.072352	.188416
.32	.001776	.002445	.049718	.208938
.34	.000854	.001440	.029741	.226069
.36	.000313	.000664	.013934	.239041
.38	.000062	.000170	.003642	.247191
.40	0	0	0	.250001
.42	.000029	.000175	.003895	.247094
.44	0.000052	.000699	.015937	.238259
.46	-0.000019	.001557	.036443	.223451
.48	.000260	.002716	.065415	.202800
.50	.000732	.004123	.102539	.176606
.52	.001483	.005711	.147184	.145342
.54	.002537	.007394	.198412	.109633
.56	.003903	.009076	.254994	.070271
.58	.005568	.010650	.315435	+0.028171
.60	.007500	.012000	.378000	-0.015624
.62	.009648	.013010	.440764	.059977
.64	.011944	.013566	.501646	.103680
.66	.014304	.013558	.558472	.145477
.68	.016633	.012889	.609036	.184104
.70	.018826	.011484	.651164	.218323
.72	.020773	.009289	.682795	.246960
.74	.022363	.006286	.702065	.268951
.76	.023401	-0.002495	.707398	.283385
.78	.024062	+0.002014	.697607	.289561
.80	.024000	.007111	.672000	.287038
.82	.023250	.012603	.630494	.275691
.84	.021791	.018214	.573738	.255780
.86	.019645	.023584	.503244	.228010
.88	.016879	.028251	.421521	.193600
.90	.013623	.031641	.332227	.154358
.92	.010075	.033059	.240311	.112756
.94	.006514	.031671	.152180	.072008
.96	.003312	.026493	.075866	.036160
0.98	-0.000943	+0.016378	0.021202	-0.010169
1.00	0	0	0	0

m = 0.42

x	$R_0(x)$	$R_1(x)$	P(x)	Q(x)
0	0	0	0	0
0.02	0.016587	-0.000599	0.010947	-0.003034
.04	.027282	.002061	.037920	.009749
.06	.033346	.003963	.073416	.017167
.08	.035860	.005977	.111518	.023045
.10	.035744	.007857	.147712	.025784
.12	.033768	.009432	.178732	.024353
.14	.030568	.010591	.202396	.018208
.16	.026663	.011279	.217458	-0.007223
.18	.022460	.011483	.223474	+0.008381
.20	.018275	.011226	.220657	.028099
.22	.014337	.010559	.209761	.051205
.24	.010806	.009555	.191967	.076808
.26	.007779	.008300	.168764	.103899
.28	.005298	.006888	.141862	.131406
.30	.003366	.005416	.113092	.158232
.32	.001951	.003979	.084324	.183303
.34	.000992	.002666	.057392	.205595
.36	.000411	.001554	.034033	.224167
.38	.000117	.000709	.015816	.238195
.40	0.000013	.000180	.004103	.246987
.42	-0.000000	0	0	.250010
.44	.000017	.000181	.004326	.246897
.46	.000126	.000718	.017581	.237462
.48	.000400	.001583	.039941	.221714
.50	.000903	.002732	.071235	.199841
.52	.001678	.004101	.110948	.172227
.54	.002752	.005610	.158234	.139443
.56	.004132	.007164	.211918	.102223
.58	.005805	.008656	.270537	.061473
.60	.007740	.009974	.332353	+0.018229
.62	.009885	.010998	.395404	-0.026344
.64	.012172	.011610	.457553	.070987
.66	.014520	.011697	.516535	.114385
.68	.016832	.011160	.570027	.155185
.70	.019005	.009913	.615724	.192055
.72	.020930	.007899	.651411	.223702
.74	.022496	.005090	.675059	.248940
.76	.023600	-0.001500	.684918	.266732
.78	.024148	+0.002808	.679627	.276246
.80	.024062	.007715	.658323	.276912
.82	.023292	.013032	.620761	.268491
.84	.021816	.018490	.567452	.251134
.86	.019655	.023734	.499793	.225465
.88	.016879	.028306	.420219	.192645
.90	.013617	.031633	.332353	.154461
.92	.010067	.033019	.241172	.113402
.94	.006507	.031625	.153178	.072753
.96	.003307	.026461	.076574	.036687
0.98	-0.000941	+0.016367	0.021454	-0.010357
1.00	0	0	0	0

m = 0.44

x	$R_0(x)$	$R_1(x)$	P(x)	Q(x)
0	0	0	0	0
0.02	0.016576	-0.000681	0.012929	-0.004454
.04	.027242	.002355	.045009	.014829
.06	.033267	.004552	.087627	.027352
.08	.035735	.006905	.133928	.039106
.10	.035571	.009138	.178630	.047942
.12	.033549	.011052	.217841	.052381
.14	.030308	.012517	.248889	.051528
.16	.026368	.013462	.270163	.044994
.18	.022138	.013863	.280951	.032809
.20	.017934	.013737	.281302	-0.015361
.22	.013988	.013132	.271888	+0.006685
.24	.010457	.012118	.253875	.032446
.26	.007439	.010786	.228806	.060876
.28	.004977	.009232	.198487	.090832
.30	.003072	.007560	.164894	.121115
.32	.001689	.005872	.130075	.150524
.34	.000769	.004265	.096062	.177891
.36	0.000232	.002826	.064812	.202118
.38	-0.000014	.001630	.038122	.222220
.40	.000069	.000736	.017582	.237337
.42	.000032	.000185	.004529	.246772
.44	0	0	0	.250004
.46	.000060	.000183	.004708	.246695
.48	.000289	.000715	.019016	.236716
.50	.000750	.001558	.042923	.220133
.52	.001488	.002655	.076064	.197228
.54	.002529	.003930	.117711	.168484
.56	.003882	.005293	.166785	.134566
.58	.005534	.006641	.221881	.096336
.60	.007454	.007859	.281302	.054805
.62	.009591	.008831	.343089	+0.011138
.64	.011876	.009437	.405076	-0.033391
.66	.014228	.009562	.464945	.077429
.68	.016549	.009101	.520297	.119564
.70	.018737	.007968	.568717	.158386
.72	.020680	.006097	.607867	.192517
.74	.022268	.003456	.635570	.220662
.76	.023397	-0.000052	.649920	.241673
.78	.023971	+0.004059	.649386	.254596
.80	.023912	.008766	.632930	.258736
.82	.023169	.013884	.600137	.253731
.84	.021719	.019156	.551353	.239616
.86	.019582	.024229	.487823	.216904
.88	.016827	.028652	.411854	.186666
.90	.013583	.031856	.326973	.150616
.92	.010046	.033146	.238095	.111204
.94	.006496	.031686	.151708	.071705
.96	.003303	.026482	.076064	.036324
0.98	-0.000940	+0.016371	0.021370	-0.010297
1.00	0	0	0	0

m = 0.46

x	$\bar{R}_0(x)$	$R_1(x)$	P(x)	Q(x)
0	0	0	0	0
0.02	0.016547	-0.000758	0.014948	-0.005870
.04	.027141	.002633	.052281	.019925
.06	.033062	.005113	.102297	.037631
.08	.035410	.007796	.157221	.055426
.10	.035119	.010375	.210997	.070620
.12	.032972	.012629	.259103	.081291
.14	.029616	.014408	.298360	.086188
.16	.025575	.015625	.326759	.084645
.18	.021265	.016246	.343300	.076493
.20	.017002	.016279	.347835	.061973
.22	.013020	.015769	.340913	.041672
.24	.009478	.014784	.323655	-0.016440
.26	.006473	.013414	.297616	+0.012670
.28	.004047	.011760	.264675	.044464
.30	.002199	.009929	.226916	.077668
.32	.000894	.008028	.186536	.110974
.34	0.000069	.006162	.145746	.143089
.36	-0.000360	.004425	.106697	.172781
.38	.000486	.002900	.071404	.198909
.40	.000413	.001654	.041679	.220463
.42	.000242	.000739	.019085	.236584
.44	.000074	.000184	.004862	.246592
.46	0	0	0	.250003
.48	.000099	.000177	.005009	.246536
.50	.000435	.000684	.020100	.236130
.52	.001058	.001469	.045079	.218943
.54	.001996	.002464	.079373	.195346
.56	.003258	.003582	.122031	.165924
.58	.004834	.004724	.171753	.131457
.60	.006693	.005781	.226916	.092907
.62	.008765	.006635	.285613	.051404
.64	.011043	.007168	.345698	+0.008203
.66	.013384	.007266	.404847	-0.035329
.68	.015710	.006822	.460628	.077767
.70	.017918	.005747	.510561	.117651
.72	.019895	.003973	.552223	.153543
.74	.021530	-0.001462	.583327	.184073
.76	.022717	+0.001783	.601837	.207997
.78	.023357	.005712	.606068	.224259
.80	.023372	.010219	.594818	.232047
.82	.022705	.015129	.567497	.230875
.84	.021334	.020188	.524267	.220650
.86	.019274	.025054	.466187	.201754
.88	.016592	.029279	.395378	.175131
.90	.013414	.032304	.315193	.142370
.92	.009936	.033440	.230387	.105809
.94	.006433	.031854	.147308	.068626
.96	.003274	.026557	.074095	.034946
0.98	-0.000933	+0.016389	0.020880	-0.009954
1.00	0	0	0	0

$m = 0.48$

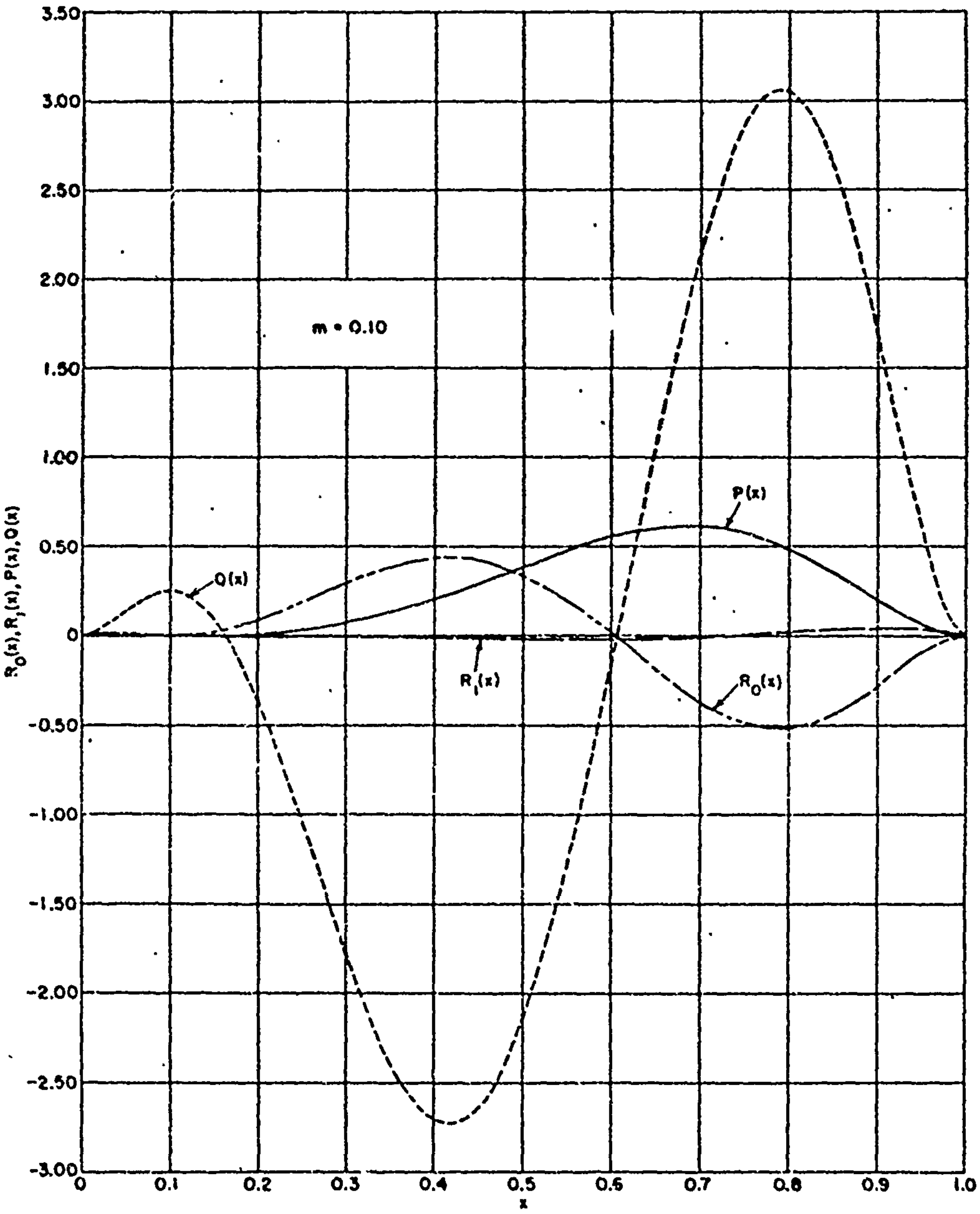
x	$R_0(x)$	$R_1(x)$	$P(x)$	$Q(x)$
0	0	0	0	0
0.02	0.016507	-0.000825	0.016882	-0.007206
.04	.026996	.002876	.059285	.024766
.06	.032768	.005607	.116530	.047466
.08	.034939	.008587	.179993	.071163
.10	.034460	.011483	.242904	.092669
.12	.032124	.014054	.300135	.109644
.14	.028590	.016132	.348012	.120498
.16	.024389	.017617	.384133	.124290
.18	.019944	.018464	.407191	.120640
.20	.015575	.018674	.416810	.109635
.22	.011520	.018285	.413394	.091755
.24	.007940	.017363	.397974	.067793
.26	.004931	.015998	.372081	.038783
.28	.002536	.014291	.337616	-0.005935
.30	0.000752	.012350	.296733	+0.029428
.32	-0.000458	.010289	.251732	.065925
.34	.001160	.008215	.204968	.102171
.36	.001441	.006229	.158748	.136818
.38	.001398	.004419	.115274	.168600
.40	.001139	.002862	.076557	.196367
.42	.000771	.001613	.044366	.219120
.44	.000396	.000711	.020174	.236029
.46	.000111	.000175	.005126	.246465
.48	0	0	0	.250001
.50	.000132	.000163	.005192	.246435
.52	.000558	.000620	.020701	.235791
.54	.001311	.001308	.046131	.218316
.56	.002405	.002146	.080694	.194487
.58	.003832	.003039	.123236	.164980
.60	.005563	.003883	.172253	.130677
.62	.007551	.004563	.225937	.092635
.64	.009729	.004965	.282219	.052061
.66	.012016	.004975	.338822	+0.010288
.68	.014315	.004488	.393333	-0.031275
.70	.016522	.003413	.443268	.071160
.72	.018524	-0.001682	.486167	.107907
.74	.020709	+0.000744	.519684	.140104
.76	.021467	.003867	.541687	.166443
.78	.022200	.007642	.550376	.185785
.80	.022324	.011966	.544405	.197221
.82	.021780	.016670	.523013	.200146
.84	.020540	.021508	.486167	.194332
.86	.018618	.026144	.434714	.180014
.88	.016074	.030139	.370536	.157971
.90	.013029	.032943	.296733	.129619
.92	.009673	.033876	.217792	.097110
.94	.006276	.032114	.139783	.063428
.96	.003200	.026680	.070555	.032501
0.98	-0.000914	+0.016422	0.019945	-0.009309
1.00	0	0	0	0

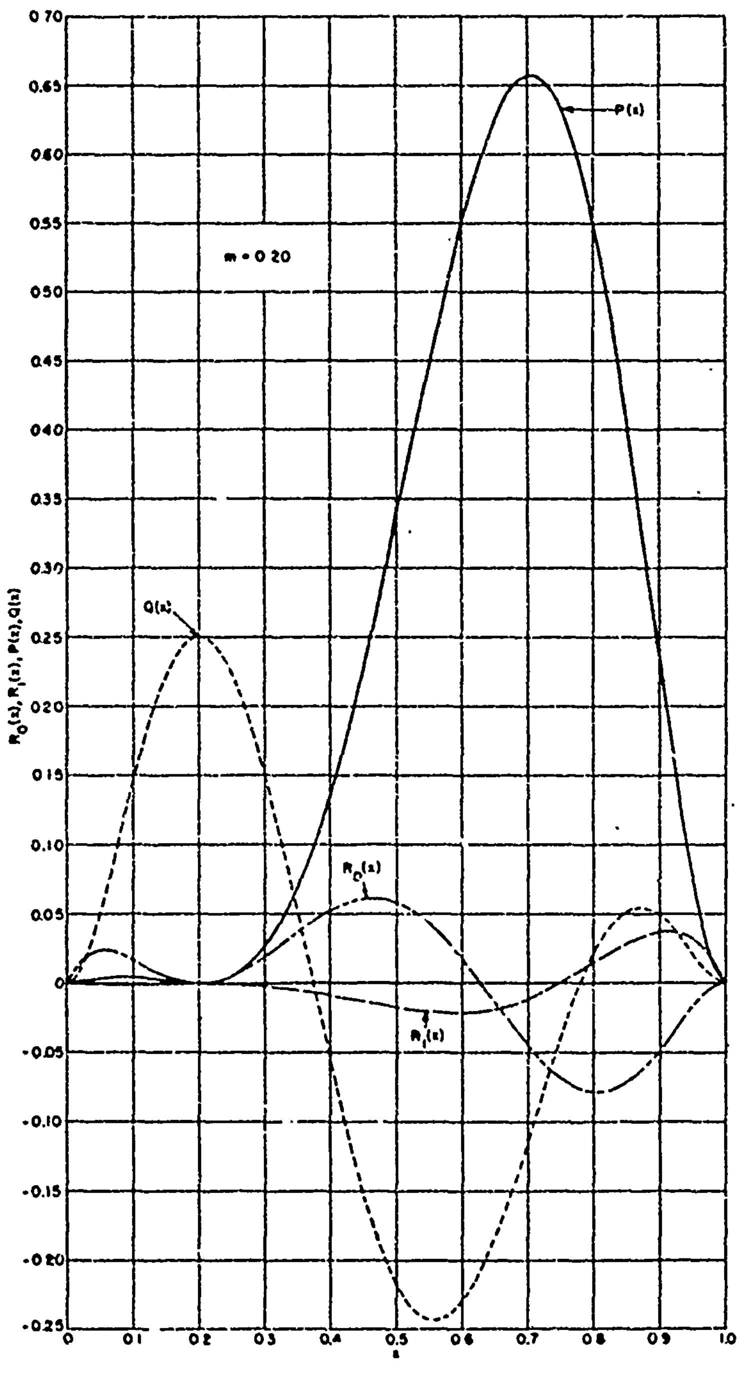
$m = 0.50$

x	$R_0(x)$	$R_1(x)$	$P(x)$	$Q(x)$
0	0	0	0	0
0.02	0.016463	-0.000878	0.018587	-0.008377
.04	.026833	.003068	.065524	.029048
.06	.032434	.006001	.129324	.056250
.08	.034400	.009223	.200666	.085354
.10	.033696	.012384	.272160	.112752
.12	.031132	.015224	.338155	.135743
.14	.027375	.017564	.394527	.152430
.16	.022969	.019290	.438505	.161617
.18	.018343	.020350	.468481	.162714
.20	.013824	.020736	.483840	.155648
.22	.009654	.020482	.484808	.140778
.24	.005997	.019649	.472299	.118813
.26	.002952	.018326	.447766	.090738
.28	0.000562	.016611	.413091	.057745
.30	-0.001176	.014616	.370440	-0.021168
.32	.002301	.012454	.322167	+0.017576
.34	.002882	.010235	.270711	.057043
.36	.003006	.008063	.218495	.095806
.38	.002776	.006033	.167855	.132507
.40	.002304	.004224	.120960	.165888
.42	.001700	.002698	.079754	.194828
.44	.001073	.001499	.045900	.218372
.46	.000524	.000651	.020731	.235753
.48	.000141	.000157	.005233	.246410
.50	0	0	0	.250000
.52	.000157	.000141	.005233	.246410
.54	.000651	.000524	.020731	.235753
.56	.001499	.001073	.045900	.218372
.58	.002698	.001700	.079754	.194828
.60	.004224	.002304	.120960	.165888
.62	.006033	.002777	.167855	.132507
.64	.008063	.003006	.218495	.095806
.66	.010235	.002882	.270711	.057043
.68	.012454	.002301	.322167	+0.017576
.70	.014616	-0.001176	.370440	-0.021168
.72	.016611	+0.000562	.413091	.057745
.74	.018326	.002952	.447766	.090738
.76	.019649	.005997	.472299	.118813
.78	.020482	.009654	.484808	.140778
.80	.020736	.013824	.483840	.155648
.82	.020350	.018343	.468481	.162714
.84	.019290	.022969	.438505	.161617
.86	.017564	.027375	.394527	.152430
.88	.015224	.031132	.338155	.135743
.90	.012384	.033696	.272160	.112752
.92	.009223	.034400	.200666	.085354
.94	.006001	.032434	.129324	.056250
.96	.003068	.026833	.065524	.029048
0.98	-0.000878	+0.016463	0.018587	-0.008377
1.00	0	0	0	0

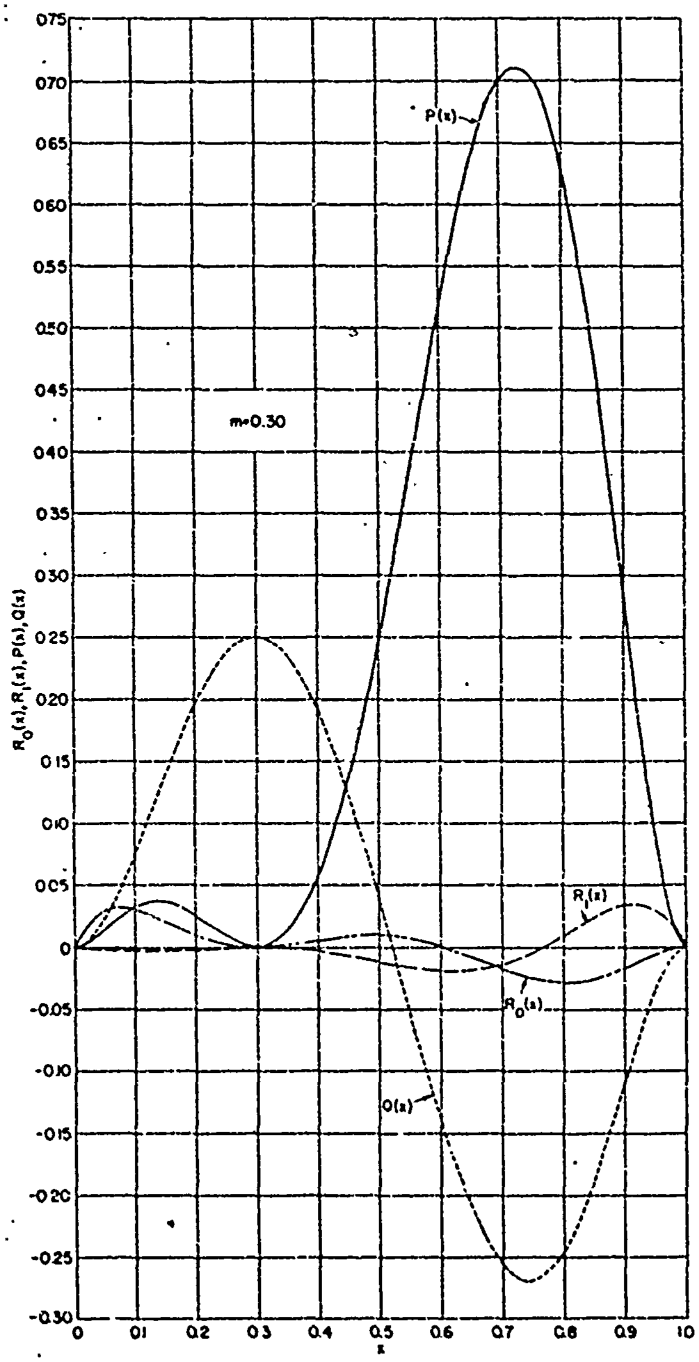
APPENDIX 3

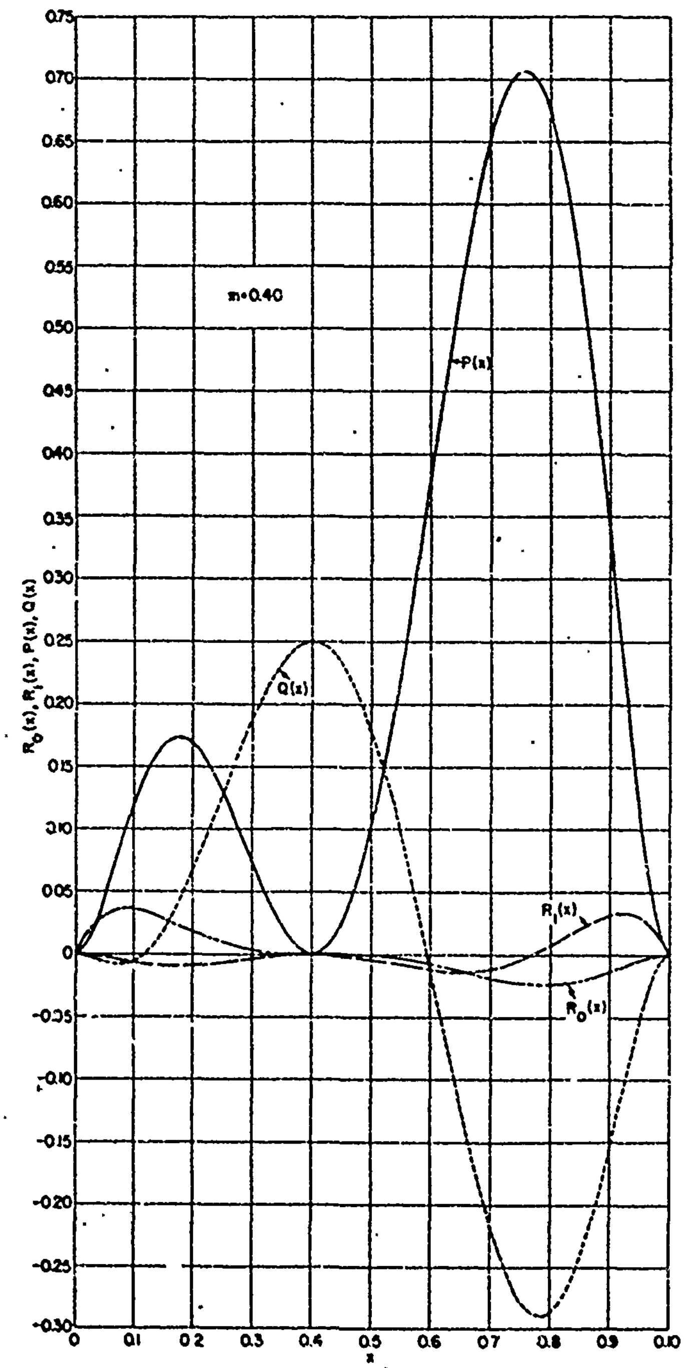
GRAPHS OF BASIC SIXTH-DEGREE POLYNOMIALS FOR SELECTED VALUES OF m

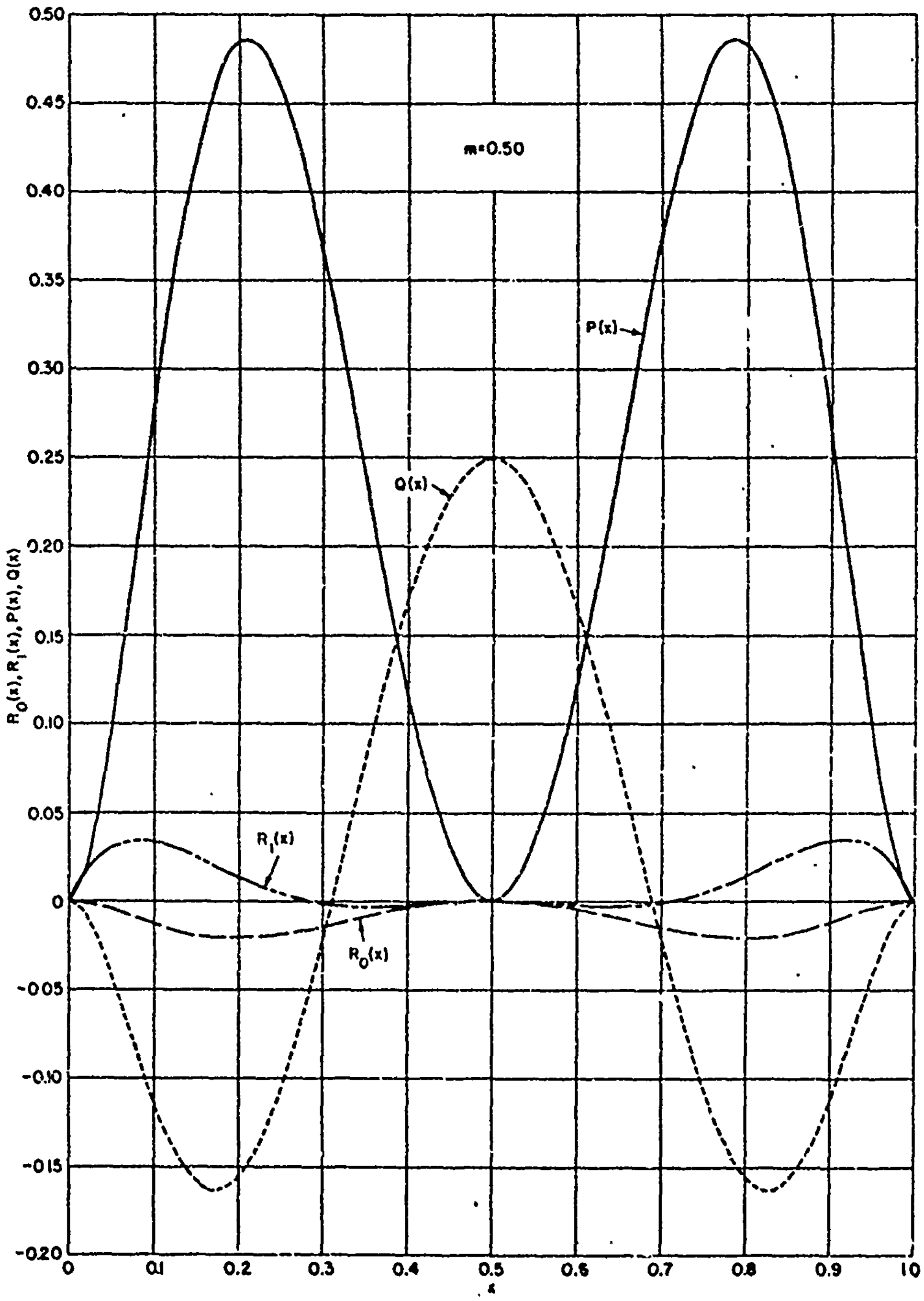

3.50
3.00
2.50
2.00
1.50
1.00
0.50
0
-0.50
-1.00
-1.50
-2.00
-2.50
-3.00
0
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.9
1.0
x
R0(x), R1(x), P(x), Q(x)
m = 0.10
Q(x)
P(x)
R1(x)
R0(x)



P(x)
m = 0.20
Q(x)
$R_0(x)$
$R_1(x)$
$R_0(x), R_1(x), P(x), Q(x)$
x
0.70
0.65
0.60
0.55
0.50
0.45
0.40
0.35
0.30
0.25
0.20
0.15
0.10
0.05
0
-0.05
-0.10
-0.15
-0.20
-0.25
0
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.9
1.0


m=0.30
P(x)
Q(x)
R1(x)
R0(x)
R0(x), R1(x), P(x), Q(x)
x
.075
.070
.065
.060
.055
.050
.045
.040
.035
.030
.025
.020
.015
.010
.005
0
-0.05
-0.10
-0.15
-0.20
-0.25
-0.30
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.9
1.0


m=0.40
P(x)
Q(x)
$R_1(x)$
$R_0(x)$
$R_0(x), R_1(x), P(x), Q(x)$
x


m=0.50
P(x)
Q(x)
$R_1(x)$
$R_0(x)$
$R_0(x), R_1(x), P(x), Q(x)$
0.50
0.45
0.40
0.35
0.30
0.25
0.20
0.15
0.10
0.05
0
-0.05
-0.10
-0.15
-0.20
0
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.9
1.0
x

APPENDIX 4

LEAST SQUARE FIT OF A SEVENTH- BY A SIXTH-DEGREE POLYNOMIAL FORM

The seventh-degree polynomial $y^2 = f(x)$ may be written in the form

$$f(x) = x(1 - x)g(x) \qquad [1]$$

where $g(x)$ is a polynomial of the fifth degree. It will be shown that a least square fit to $g(x)$ by a fourth-degree polynomial $\bar{g}(x)$ is given by

$$\bar{g}(x) = g(x) - C_5\left(x^5 - \frac{5}{2}x^4 + \frac{20}{9}x^3 - \frac{5}{6}x^2 + \frac{5}{42}x - \frac{1}{252}\right) \qquad [2]$$

where C_5 is the coefficient of x^5 in the expansion

$$g(x) = C_0 + C_1x + \ldots + C_5x^5 \qquad [3]$$

Also [2] may be written as

$$\bar{g}(x) = \left(C_0 + \frac{C_5}{252}\right) + \left(C_1 - \frac{5}{42}C_5\right)x + \left(C_2 + \frac{5}{6}C_5\right)x^2 + \left(C_3 - \frac{20}{9}C_5\right)x^3 + \left(C_4 + \frac{5}{2}C_5\right)x^4 \qquad [4]$$

The corresponding sixth-degree polynomial form will then be given by $y^2 = \bar{f}(x)$ where

$$\bar{f}(x) = x(1 - x)\bar{g}(x) \qquad [5]$$

Proof: Since $g(x)$ is a polynomial of the fifth degree, it can be expressed as a linear combination of the first five Legendre polynomials. Furthermore, since here the range of x is from 0 to 1 and the Legendre polynomials are orthogonal over the range -1 to +1, we express $g(x)$ as

$$g(x) = \gamma_0 P_0 + \gamma_1 P_1(\xi) + \ldots + \gamma_5 P_5(\xi) \qquad [6]$$

where γ_0, γ_1, ... γ_5 are coefficients and P_0, $P_1(\xi)$ ... are the Legendre polynomials, and

$$\xi = 2x - 1 \qquad [7]$$

A theorem on orthogonal functions (Reference 6) then states that the best least-square fit of a fourth-degree polynomial to $g(x)$ is

$$\bar{g}(x) = \gamma_0 P_0 + \gamma_1 P_1(\xi) + \ldots + \gamma_4 P_4(\xi)$$

or

$$\bar{g}(x) = g(x) - \gamma_5 P_5(\xi) \qquad [8]$$

Now, from Equation [6],

$$\int_{-1}^{1} g(x) P_5(\xi) d\xi = \gamma_5 \int_{-1}^{1} P_5^2(\xi) d\xi = \frac{2}{11}\gamma_5 \qquad [9]$$

and, from [3] and [7],

$$g(x) = C_5\left(\frac{\xi + 1}{2}\right)^5 + \ldots = \frac{C_5}{32}\xi^5 + \ldots \qquad [10]$$

where only the coefficient of ξ^5 is shown since, by a well-known property of Legendre polynomials, the terms in the powers of ξ less than the fifth give zero when substituted into [9]. Also

$$P_5(\xi) = \frac{1}{8}(63\xi^5 - 70\xi^3 + 15\xi) \qquad [11]$$

Hence, substituting from [10] and [11] into [9], gives

$$\gamma_5 = \frac{a_5}{252} \qquad [12]$$

Also, from [7] and [11],

$$P_5(x) = \frac{1}{8}\left[63(2x - 1)^5 - 70(2x - 1)^3 + 15(2x - 1)\right]$$

$$= 252\left(x^5 - \frac{5}{2}x^4 + \frac{20}{9}x^3 - \frac{5}{6}x^2 + \frac{5}{42}x - \frac{1}{252}\right) \qquad [13]$$

Hence, substituting from [12] and [13] into [8], gives Equation [2] as we wished to prove.

APPENDIX 5

GEOMETRICAL PROPERTIES

The geometrical properties of sixth-degree polynomial forms can be obtained from the following equations when the linear dimension is chosen:

Diameter: $d = \frac{l}{\lambda}$ [1]

Distance of maximum section from nose: $X_m = ml$ [2]

Nose radius: $R_0 = \frac{r_0 l}{\lambda^2}$ [3]

Tail radius: $R_1 = \frac{r_1 l}{\lambda^2}$ [4]

Volume: $\nabla = C_p \frac{\pi l^3}{4\lambda^2}$ [5]

Longitudinal center of buoyancy: $\bar{X} = \frac{4l}{C_p}\int_0^1 xy^2\,dx$ [6]

Volume moment of inertia about the vertical axis through the longitudinal center of buoyancy:

$$I_{yy} = \frac{\pi}{4} l^5 \left[\frac{1}{\lambda^4}\int_0^1 y^4\,dx + \frac{4}{\lambda^2}\int_0^1 y^2x^2\,dx - \frac{1}{\lambda^2} C_p \bar{x}^2\right] \qquad [7]$$

where $\bar{x} = \frac{\bar{X}}{l}$

Surface area: $S = \frac{\pi l^2}{\lambda}\int_0^1 \left[4y^2 + \frac{1}{\lambda^2}\left(\frac{d}{dx}y^2\right)^2\right]^{\frac{1}{2}} dx$ [8]

The integral in Equation [8] may be evaluated numerically using Gauss's formula

$$\int_0^1 \phi(u)\,du = R_1\phi(u_1) + R_2\phi(u_2) + \ldots + R_n\phi(u_n) \qquad [9]$$

The numerical values of the u's and R's for n = 7 are:

$u_1 = 0.02544604383$	$R_1 = R_7 = 0.06474248308$
$u_2 = 0.1292344072$	
$u_3 = 0.2470774243$	$R_2 = R_6 = 0.1398526957$
$u_4 = 0.5$	
$u_5 = 0.7029225757$	$R = R_5 = 0.1909150253$
$u_6 = 0.8707655928$	
$u_7 = 0.9745539562$	$R_4 = 0.2089795918$

The evaluation of Gauss's u's for polynomials of high degree using synthetic division is quite tedious. The calculation may be simplified by linear interpolation between the closely spaced ordinates obtained from the table in Appendix 1. The maximum error introduced by this simplification is only approximately ±0.2 percent.

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