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(6) MAXIMUM LIKELIHOOD ESTIMATION OF THE
DISTRIBUTION OF RADIAL ERRORS. Revision 1.

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PREPARED BY

(10) FRANCIS J. MASON
~~1st LT, USAF~~

ANALYSIS DIVISION

REVISED BY

CHARLES A. BODWELL
(11) 1 OCTOBER 1969
ANALYSIS DIVISION

DIRECTORATE OF GUIDANCE TEST
AIR FORCE MISSILE DEVELOPMENT CENTER
HOLLOMAN AIR FORCE BASE, NEW MEXICO

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MAXIMUM LIKELIHOOD ESTIMATION
OF THE
DISTRIBUTION OF RADIAL ERROR

INTRODUCTION: Unlike most methods for determining the distribution of radial errors, the method presented in this paper makes no assumptions concerning the means, standard deviations, or distribution functions of the individual channels (latitude and longitude) nor of the correlation existing between them. The calculations required to estimate the radial error distributions are trivial. In addition, the method is sufficiently general to cover both the two and three dimensional cases.

GENERAL APPROACH: A complete derivation of the solution is presented in Appendices A, B, and C; however, the general approach is as follows:

(1) Assume that the probability density of (r^2/a^2) can be approximated by the chi-squared (χ^2) distribution where "r" is the radial error and "a" is the normalizing factor, then the probability element for (r^2/a^2) is given by:

$$dp(r^2/a^2) = \frac{1}{2^{n/2}\,\Gamma(n/2)}\, e^{-r^2/2a^2} (r^2/a^2)^{\frac{n-2}{2}}\, d(r^2/a^2)$$

(2) Estimate by the method of maximum likelihood, the two parameters of this distribution ("n" and "a"); that is, the values of these parameters for which the probability of obtaining the given set of observations is a maximum. It should be emphasized that there is no relationship between n (degrees of freedom) and the number of observations for the analysis being discussed. "n" as used in this paper is strictly an arbitrary parameter to be determined in the maximum likelihood sense.

(3) Integrate the assumed distribution for all possible combinations of "a" and "n" to obtain percentiles as functions of these parameters.

The results of these calculations are shown in Figure 1, which gives the best estimate of the percentiles as functions of only the geometric mean,

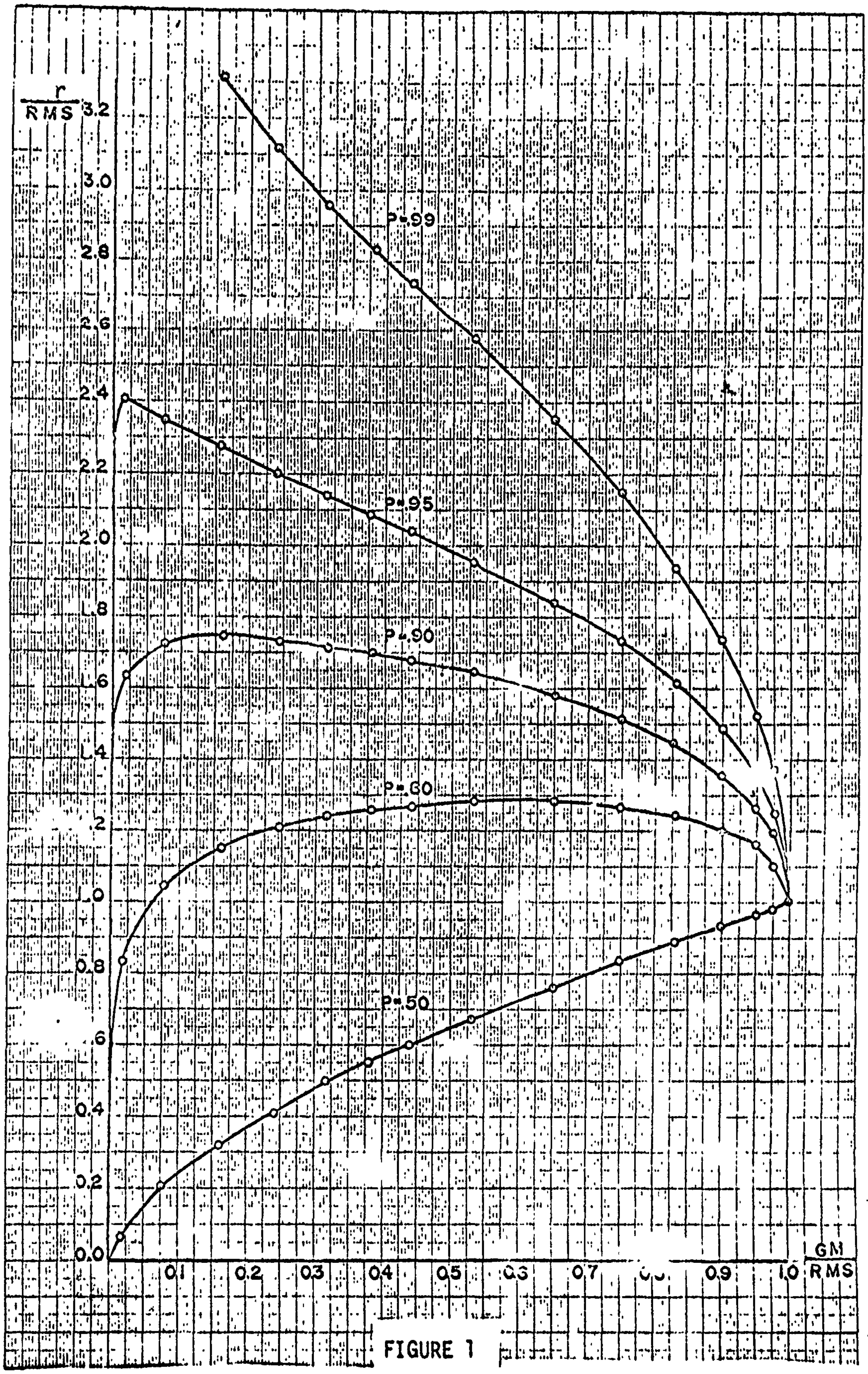


FIGURE 1

PERCENTILES OF THE MAXIMUM LIKELIHOOD ESTIMATE

OF THE DISTRIBUTION OF RADIAL ERROR

GM, and the root-mean-square, RMS, of the observed radial errors, where

$$GM = \sqrt[N]{\prod_{i=1}^{N} r_i}$$

$$RMS = \sqrt{\left(\sum_{i=1}^{N} r_i^2\right)/N}$$

N = Number of observations of radial error

Note the simplicity of the method. To obtain an estimate of any desired percentile of the distribution, only the two quantities: the geometric mean and the root mean square need be calculated.

EXAMPLE APPLICATIONS: The data of Tables IA and IIA were taken from actual flight tests at AFMDC. All units are nautical miles.

TABLE 1-A
SAMPLE DATA SET #1

FLIGHT NUMBER	ERROR AT TIME = 1 HOUR		
	x	y	r
1	-0.81	2.62	2.75
2	1.63	-1.01	1.92
3	1.76	-4.09	4.45
4	-7.27	-3.75	8.18
5	-4.63	-0.84	4.71
6	-4.99	3.03	5.84
7	-3.23	0.29	3.24
8	-2.27	1.21	2.57
9	-2.47	-4.70	5.55
10	-0.90	-2.47	2.62

$\overline{x} = -2.32 \qquad \sigma_x = 2.88$

$\overline{y} = -1.00 \qquad \sigma_y = 2.82$

$RMS(r) = 4.61 \qquad GM(r) = 3.81$

$GM/RMS = .8265$

From Figure 1 for a GM/RMS ratio of 0.826 the values of r/RMS (Column 2 below) associated with the percentage points (Column 1) are obtained. By multiplying the entries of Column 2 by the value of RMS (4.61) the value of the radial error, (Column 3), associated with the corresponding percentage points are obtained.

TABLE 1-B

PERCENTAGE POINTS OF THE THEORETICAL RADIAL ERROR DISTRIBUTION

%	r/RMS	r
50	0.88	4.06
80	1.22	5.62
90	1.42	6.55
95	1.60	7.38

Thus 50% of the time you would expect to have errors less than or equal to 4.06 nautical miles at the end of 1 hour; 80% of the time you would expect to have errors less than or equal to 5.62 nautical miles at the end of 1 hour.

Figure 2 and 3 show the graph of the theoretical radial error distribution as calculated from the sample along with the distribution of the sample for data sets 1 and 2.

TABLE 2-A

SAMPLE DATA SET #2

FLIGHT NUMBER	ERROR AT TIME = 1 HOUR X	Y	r
1	2.28	-13.39	13.60
2	0.18	-24.45	24.45
3	8.17	3.40	8.85
4	3.43	-2.76	4.40
5	4.43	-3.58	5.70
6	0.54	0.49	0.73
7	4.00	-1.43	4.13

$\bar{x} = 3.29 \qquad \sigma_x = 2.71$

$\bar{y} = -5.90 \qquad \sigma_y = 9.73$

$RMS(r) = 11.52 \qquad GM(r) = 5.82$

$GM/RMS = 0.5052$

TABLE 2-B

PERCENTAGE POINTS OF THE THEORETICAL RADIAL ERROR DISTRIBUTION

%	r/RMS	r
50	0.64	7.37
80	1.27	14.63
90	1.65	19.01
95	1.96	22.58

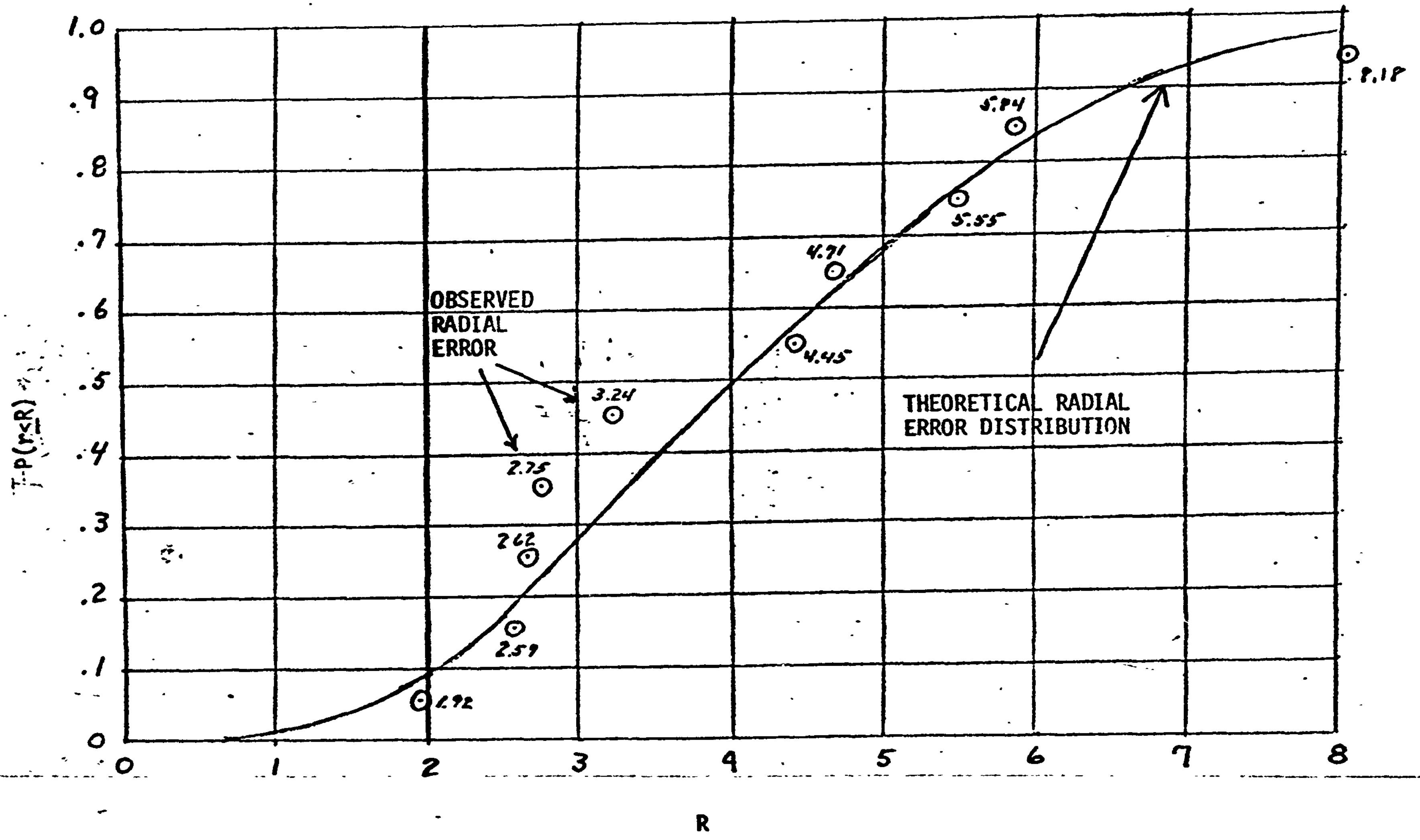


FIGURE 2. RADIAL ERROR DISTRIBUTION
(DATA SET NO. 1)

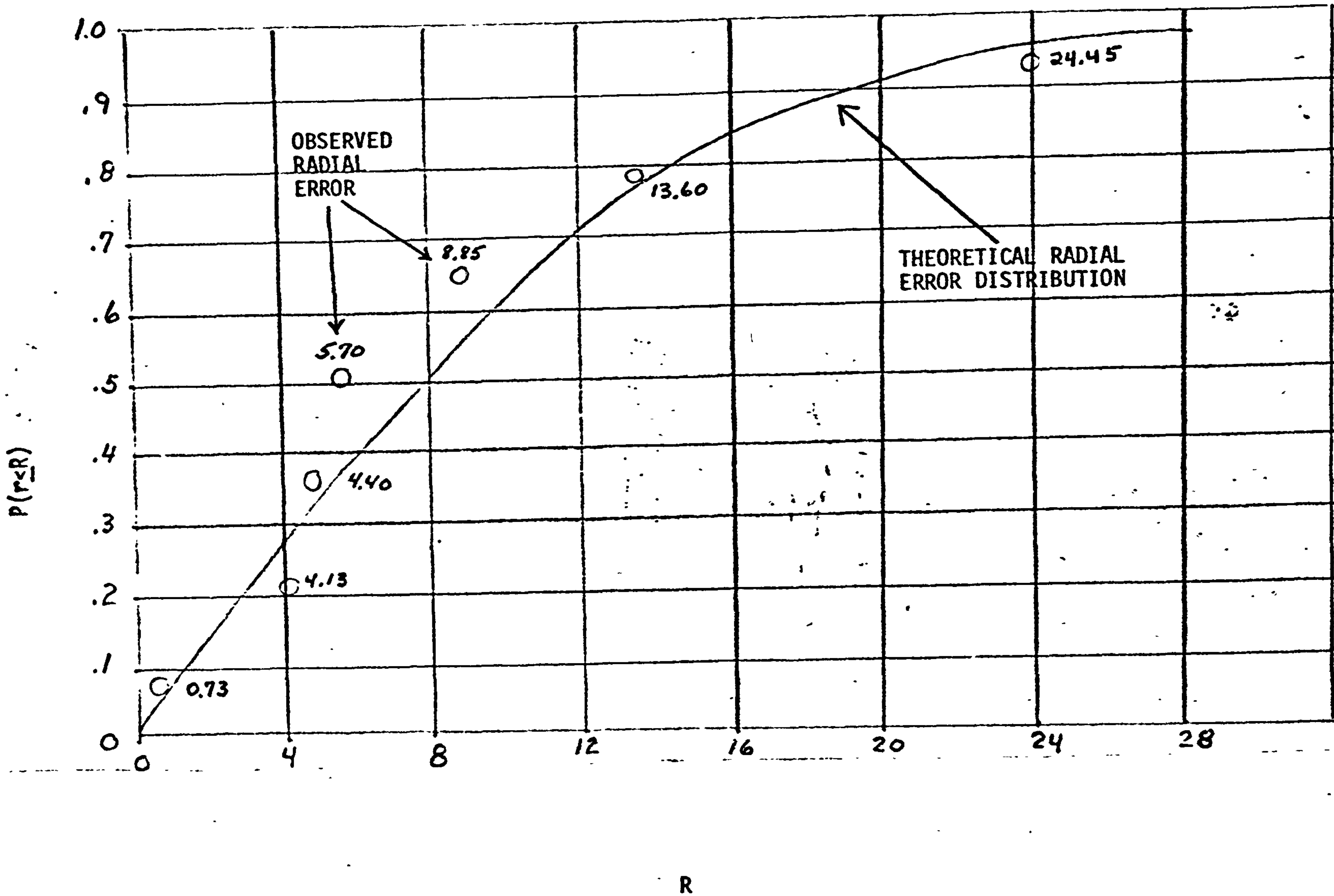


FIGURE 3. RADIAL ERROR DISTRIBUTION
(DATA SET NO. 2)

CONCLUSIONS: A simple method of wide applicability has been presented for estimating the distribution of radial error.

APPENDIX A

DERIVATION OF THE MAXIMUM LIKELIHOOD ESTIMATE

Assume that the probability density of (r^2/a^2) can be approximated by the chi-squared (χ^2) distribution where "r" is the radial error and "a" is the normalizing factor, then the probability element for (r^2/a^2) is given by:

$$dp(r^2/a^2) = \frac{1}{2^{n/2}\,\Gamma(n/2)}\, e^{-r^2/2a^2} \left(\frac{r^2}{a^2}\right)^{\frac{n-2}{2}} d\left(\frac{r^2}{a^2}\right) \tag{A-1}$$

where "n" and "a" are to be determined so as to yield a maximum likelihood estimate. [Note that "n" which is usually associated with degrees of freedom, is in this application considered to be an unknown parameter of the distribution.]

Let $m = n/2$, then the probability of obtaining the given set of N observations, which is simply the product of the individual probabilities, is given by $P(\hat{r})$ where:

$$P(\hat{r}) = \prod_{i=1}^{N} \frac{1}{2^m\,\Gamma(m)}\, e^{-r_i^2/2a^2} (r_i^2/a^2)^{m-1}\, d(r_i^2/a^2)$$

$$= \prod_{i=1}^{N} \frac{1}{(2a^2)^m\,\Gamma(m)}\, e^{-r_i^2/2a^2} (r_i^2)^m\, \frac{d(r_i^2)}{r_i^2}$$

taking logarithms:

$$\text{Ln } P(\hat{r}) = -m\,N\,\text{Ln}\,(2a^2) - N\,\text{Ln}\,\Gamma(m) - \frac{1}{2a^2}\sum r_i^2$$

$$+\, 2m \sum \text{Ln}\, r_i + \sum \text{Ln}\, \frac{d(r_i^2)}{r_i^2}$$

$$\frac{1}{N}\ \mathrm{Ln}\ P(\hat{r}) = -m\ \mathrm{Ln}\ (2a^2) - \mathrm{Ln}\ \Gamma\ (m) - \frac{1}{2a^2} \sum \frac{r_i^2}{N} \qquad \text{(A-2)}$$

$$+ 2m \sum \frac{\mathrm{Ln}\ r_i}{N} + \frac{1}{N} \sum \mathrm{Ln}\ \frac{d(r_i^2)}{r_i^2}$$

Since MS (mean square) $= \sum r_i^2/N$,

GM (geometric mean) $= \sqrt[N]{\prod_{i=1}^{N} r_i}$,

and $\mathrm{Ln\ GM} = \frac{1}{N} \sum \mathrm{Ln}\ r_i$,

it follows that

$$\frac{1}{N}\ \mathrm{Ln}\ P(\hat{r}) = -m\ \mathrm{Ln}(2a^2) - \mathrm{Ln}\ \Gamma(m) - \frac{MS}{2a^2} \qquad \text{(A-3)}$$

$$+ 2m\ \mathrm{Ln\ GM} + \frac{1}{N} \sum \mathrm{Ln}\ \frac{d(r_i^2)}{r_i^2}$$

Setting the partial derivatives of (A-3) with respect to $(2a^2)$ and with respect to (m) each equal to zero: (The condition for obtaining maximum probability.)

$$\frac{\partial}{\partial(2a^2)} \left[\frac{1}{N}\ \mathrm{Ln}\ P(\hat{r}) \right] = -\frac{m}{2a^2} + \frac{MS}{(2a^2)^2} = 0 \qquad \text{(A-4)}$$

$$\frac{\partial}{\partial m} \left[\frac{1}{N}\ \mathrm{Ln}\ P(\hat{r}) \right] = -\mathrm{Ln}\ 2a^2 - \psi(m) + 2\ \mathrm{Ln\ GM} = 0 \qquad \text{(A-5)}$$

where $\psi\ (m) = \frac{d}{dm} \mathrm{Ln}\ [\Gamma(m)]$, the psi function.

From Equation (A-4)

$$m = \frac{MS}{2a^2}$$

$$2a^2 = \frac{MS}{m} = \frac{MS}{n/2} \qquad \text{(A-6)}$$

since $m = n/2.$

From Equation (A-5)

$$\text{LN}\ \frac{GM^2}{2a^2} = \Psi(m)$$

$$\text{LN}\ [\frac{GM^2}{MS} \cdot \frac{n}{2}] = \Psi(n/2)$$

From which:

$$\frac{GM}{RMS} = \frac{e^{[\Psi(n/2)]/2}}{(n/2)^{1/2}} \qquad \text{(A-7)}$$

The solution of this equation is plotted in Figure A-1.

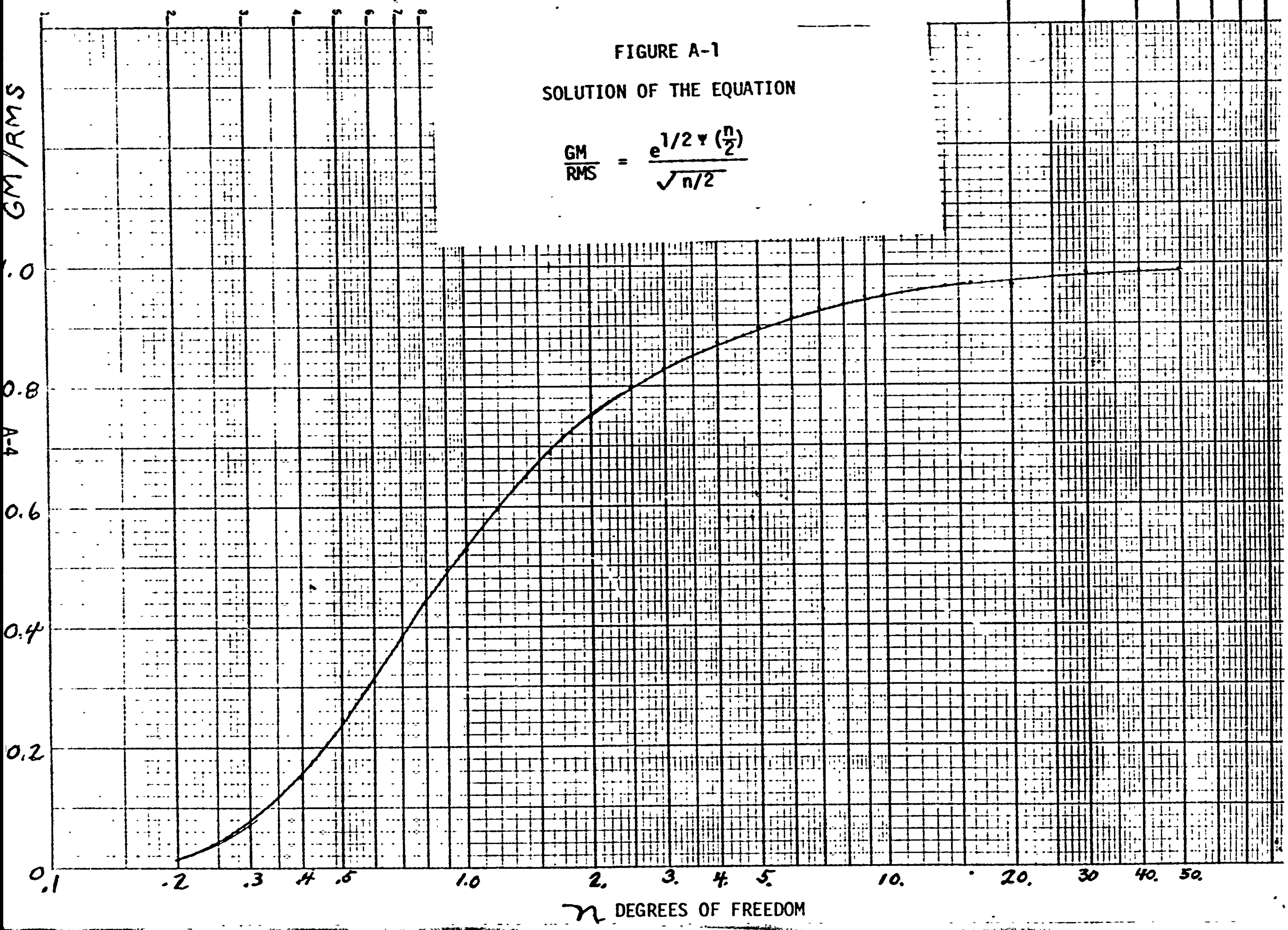
By computing the ratio of GM/RMS and using the graph of Figure A-1, the value of "n" corresponding to the maximum likelihood solution is obtained.

Thus since the value of "a" is given by eq (A-6) and the value of "n" is determined by Figure A-1 the most probable values of the parameters of equation (1) are known. Consequently the distribution function for radial range, equation (A-1) is uniquely determined.

FIGURE A-1

SOLUTION OF THE EQUATION

$$\frac{GM}{RMS} = \frac{e^{1/2\,\psi(\frac{n}{2})}}{\sqrt{n/2}}$$



APPENDIX B

CONSTRUCTION OF FIGURE 1

This appendix gives the procedure for constructing Figure 1. "Percentiles of the Maximum Likelihood Estimate of the Distribution of Radial Error as a Function of the GM/RMS Ratio."

Since by Hypothesis $(\frac{r^2}{a^2})$ follows a χ^2 distribution, then by substitution from Equation (A-6)

$$\frac{r^2}{a^2} = \frac{r^2 n}{MS} = \text{a chi-square distribution}$$

But since the chi-distribution* with n degrees of freedom (χ_n) is defined by

$$\chi_n = \sqrt{\chi^2/n}$$

it follows that

$$\frac{r^2 n}{MS}/n = \frac{r}{rms} = \text{a chi-distribution}$$

Where the percentage points of the chi-distribution for integral values of n $(n \neq 0)$ can be found in tables of the chi-distribution**.

Thus the values of r/RMS corresponding to integral values of "n" were obtained from tables of the percentage points for the chi-distribution, whereas for degrees of freedom less than one, the technique developed in Appendix C was used.

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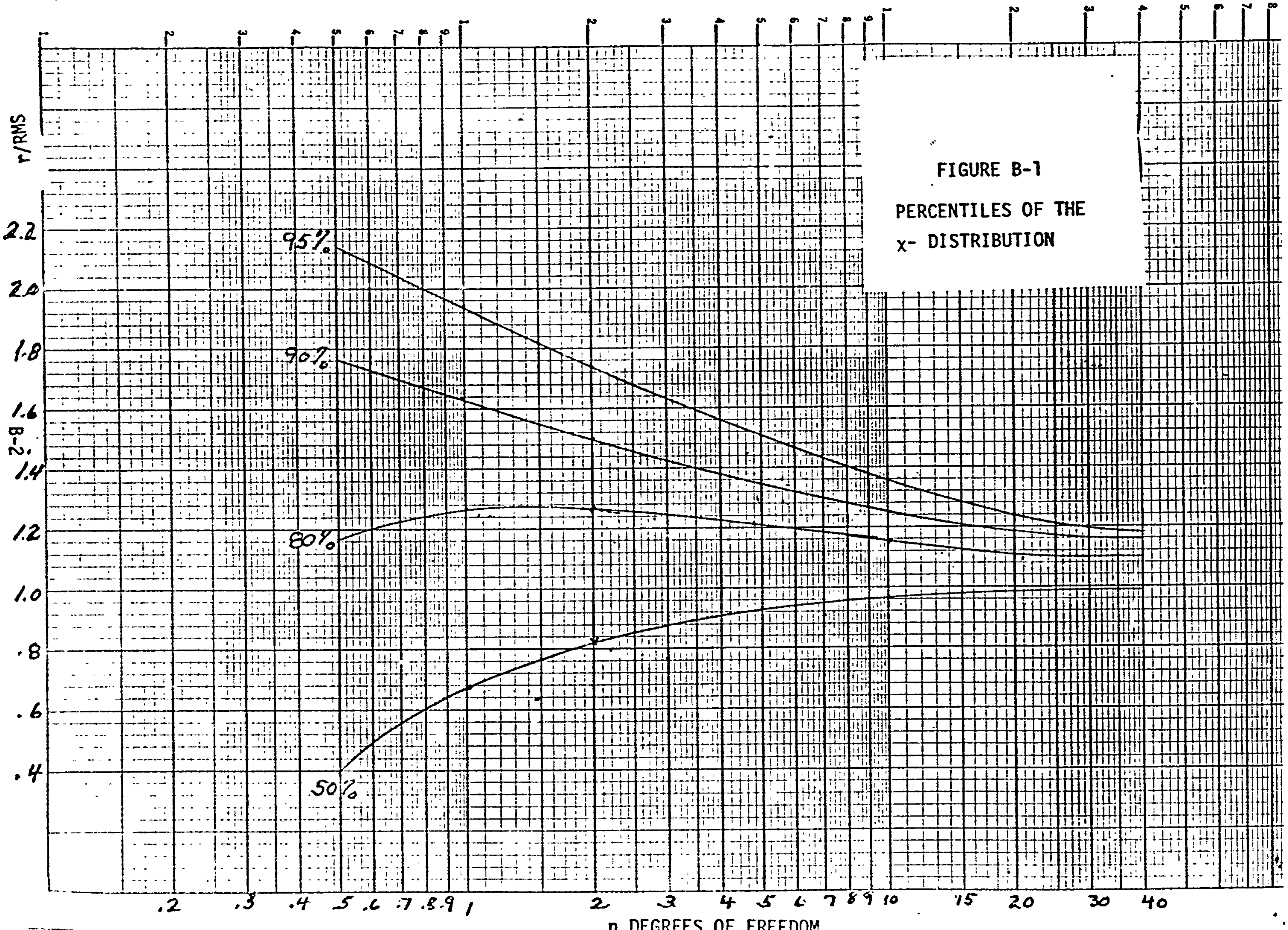
* See Page L-2 of Third Inertial Guidance Test Symposium, 1966.

** See Page M-1 of Third Inertial Guidance Test Symposium, 1966.

FIGURE B-1
PERCENTILES OF THE
χ- DISTRIBUTION



To obtain the percentiles of radial error (normalized by rms), Figure 1, simply cross-plot Figures A-1 and B-1.

APPENDIX C

CALCULATION OF THE χ^2 DISTRIBUTION

FOR DEGREES OF FREEDOM < 1.

The cumulative distribution function for the χ^2 distribution for n degrees of freedom is given by:

$$F(\chi^2, n) = \frac{1}{2^{n/2}\,\Gamma(n/2)} \int_0^{\chi^2} \phi^{(n-2)/2} \; e^{-\phi/2} \, d\phi \tag{C-1}$$

Making the substitution that 2 m=n, this expression becomes

$$F(\chi^2, 2m) = \frac{1}{2^m\,\Gamma(m)} \int_0^{\chi^2} \phi^{m-1} \; e^{-\phi/2} \; d\phi \tag{C-2}$$

Integrating this expression by parts, where

$$u = \phi^{m-1} \; e^{-\phi/2} \qquad du = \left(\frac{m-1}{\phi} - 1/2\right) u d\phi$$

$$v = \phi \qquad dv = d\phi$$

Equation (C-1) becomes:

$$F(\chi^2, 2m) = \frac{1}{2^m\,\Gamma(m)} \left[\phi^m \; e^{-\phi/2}\right]_0^{\chi^2} - \frac{1}{2^m\,\Gamma(m)} \int_0^{\chi^2} \left(\frac{m-1}{\phi} - 1/2\right)\phi^m \; e^{-\phi/2} d\phi$$

(C-3)

$$= \frac{\chi^{2m} \; e^{-\chi^2/2}}{2^m\,\Gamma(m)} - \frac{(m-1)}{2^m\,\Gamma(m)} \int_0^{\chi^2} \phi^{m-1} \; e^{-\phi/2} \; d\phi$$

$$+ \frac{1}{2^{m+1}\,\Gamma(m)} \int_0^{\chi^2} \phi^m \; e^{-\phi/2} \; d\phi$$

But from Equation (C-2) the middle term of the expression on the right is $-(m - 1)\, F(\chi^2, 2m)$. Transposing this term to the left side, Equation (C-3) becomes:

$$mF(\chi^2, 2m) = \frac{\chi^{2m}\, e^{-\chi^2/2}}{2^m\, \Gamma(m)}$$

$$+ \frac{\Gamma(m + 1)}{\Gamma(m)} \left[\frac{1}{2^{m+1}\,\Gamma(m+1)} \int_0^{\chi^2} \phi^m\, e^{-\phi/2}\, d\phi\right]$$

where the expression in brackets is the cumulative distribution function of the χ^2 distribution with (2m+2) degrees of freedom.

$$\therefore \quad F(\chi^2, 2m) = \left(\frac{\chi^2}{2}\right)^m \frac{e^{-\chi^2/2}}{\Gamma(m + 1)} + F(\chi^2, 2m + 2)$$

or since $2m = n$

$$F(\chi^2, n) = \left(\frac{\chi^2}{2}\right)^{n/2} \frac{e^{-\chi^2/2}}{\Gamma\left(\frac{n + 2}{2}\right)} + F(\chi^2, n + 2) \qquad \text{(C-4)}$$

By interpolating between 2 and 3 degrees of freedom in a table of the χ^2 distribution, and then applying the recursion relation above, the distribution can be calculated for DF < 1. Then since the chi-distribution in terms of the χ^2 distribution is defined by

$$\chi = \sqrt{\chi^2/n}\ ,$$

the corresponding percentage points of the χ-distribution can be readily obtained.

Numerical example:

Given that $F(\chi^2, 2.5) = 0.5$ for $\chi^2 = 1.88$ (obtained by interpolation from tables of the χ^2 distribution) find the probability associated with $F(\chi^2, 0.5)$ for $\chi^2 = 1.88$.

From Equation (C-4)

$$F(\chi^2, n) = \left(\frac{\chi^2}{2}\right)^{n/2} \frac{e^{-\chi^2/2}}{\Gamma\left(\frac{n+2}{2}\right)} + F(\chi^2, n+2)$$

Since for the example given

$$n = 0.5$$

$$\chi^2 = 1.88$$

and from tables

$$\Gamma\left(\frac{n+2}{2}\right) = \Gamma(1.25) = 0.90640$$

$$F(\chi^2, n+2) = F(1.88, 2.5) = 0.50$$

it follows that

$$F(1.88, 0.5) = (.94)^{.25} \frac{e^{-.94}}{0.90640} + 0.5$$

$$= (.984650) \left(\frac{.390628}{.90640}\right) + 0.5$$

$$= .424351 + 0.5$$

$$= 0.924351$$

In order to compare this with the value obtained from Figure A-1, it is necessary to transform from the χ^2 distribution to the χ distribution. Since by definition

$$\chi = \sqrt{\chi^2/n}$$

$$\chi = \sqrt{\frac{1.88}{0.5}} = 1.939$$

From Figure A-1 for 0.5 degrees of freedom, the probability of obtaining a $\chi \leq 1.939$ falls about halfway between the 90% and 95% probability contours. This is consistent with the 0.9247. probability as calculated above.

Note that the recursion formulas as given by the National Bureau of Standards, Equations 26.4.8 Pg 941, Handbook of Mathematical Functions, AMS-55, 1964 would appear to be inconsistent with Equation C-4 as given above. However, the values of the χ^2 distribution as given by NBS are for the areas under the curve integrated from χ^2 to infinity. Whereas the areas considered in this paper are from 0 to χ^2. This accounts for the discrepancy of the sign of the constant term.