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BENDING OF RECTANGULAR PLATES WITH LARGE DEFLECTIONS

By Chi-Teh Wang

Brown University

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SUMMARY

Von Kármán's equations for thin plates with large deflections are solved for the special cases of rectangular plates having ratios of length to width of 1.5 and 2 and loaded by uniform normal pressure. The boundary conditions are such as to approximate panels with riveted edges under normal pressure greater than that of the surrounding panels. Center deflections, membrane stresses, and extreme-fiber bending stresses are given as functions of the pressure for center deflections up to twice the thickness of the plate. For small deflections the results are consistent with those given by Timoshenko.

INTRODUCTION

A general numerical method for solving Von Kármán's equations for thin plates with large deflections was developed in reference 1. A square plate loaded by uniform normal pressure with simply supported edges was studied. The boundary conditions approximate the case when a riveted sheet-stringer panel is under normal pressure greater than that of the surrounding ones.

It was subsequently decided to extend the investigation to rectangular plates of various ratios of length to width. Two special cases are studied in this report; namely, rectangular plates for which the ratio of length to width has the value of 1.5 or 2. (For rectangular plates having a length equal to or greater than twice their width, experimental evidence (reference 2) indicates that they can practically be regarded as infinitely long.) Center deflections, membrane stresses, and extreme-fiber bending stresses are given as functions of the pressure for center deflections up to twice the thickness of the plate. For small deflections, the results are consistent with those obtained by Timoshenko (reference 3).

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SYMBOLS

\(a, b\) length and width of plate; \(a\), shorter side of rectangular plate

\(h\) thickness of plate

\(x, y, z\) coordinates of a point in plate

\(u, v\) horizontal displacements in \(x\)- and \(y\)-directions of points in middle surface; nondimensional forms are \(ua/h^2\) and \(va/h^2\), respectively

\(w\) deflection of middle surface out of its initial plane; nondimensional form is \(w/h\)

\(p\) normal load on plate per unit area; nondimensional form is \(pa^4/Eh^4\)

\(E, \mu\) Young's modulus and Poisson's ratio, respectively

\(D\) flexural rigidity of plate \(\left(\frac{Eh^3}{12(1-\mu^2)}\right)\)

\(\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\)

\(\nabla^4 = \frac{\partial^4}{\partial x^4} + 2 \frac{\partial^4}{\partial x^2 \partial y^2} + \frac{\partial^4}{\partial y^4}\)

\(\sigma_x', \sigma_y', \tau_{xy}'\) membrane stresses in middle surface; nondimensional forms are \(\sigma_x'a^2/Eh^2\), \(\sigma_y'a^2/Eh^2\), and \(\tau_{xy}'a^2/Eh^2\), respectively

\(\sigma_x'', \sigma_y'', \tau_{xy}''\) extreme-fiber bending and shearing stresses; nondimensional forms are \(\sigma_x''a^2/Eh^2\), \(\sigma_y''a^2/Eh^2\), and \(\tau_{xy}''a^2/Eh^2\), respectively

\(\varepsilon_x', \varepsilon_y', \gamma_{xy}'\) membrane strains in middle surface; nondimensional forms are \(\varepsilon_x'a^2/h^2\), \(\varepsilon_y'a^2/h^2\), and \(\gamma_{xy}'a^2/h^2\), respectively

\(\varepsilon_x'', \varepsilon_y'', \gamma_{xy}''\) extreme-fiber bending and shearing strains; nondimensional forms are \(\varepsilon_x''a^2/h^2\), \(\varepsilon_y''a^2/h^2\), \(\gamma_{xy}''a^2/h^2\), respectively

\(F\) stress function; nondimensional form is \(F/Eh^2\)
Δ, Δ², . . . Δⁿ  
first-, second-, . . . and nth-order differences, respectively

Δₓ, Δᵧ  
first-order differences in x- and y-directions, respectively

FUNDAMENTAL EQUATIONS

The deformation of a thin plate, the deflections of which are large in comparison with its thickness but are still small as compared with the other dimensions, is governed by von Kármán's equations:

\[
\begin{align*}
\frac{\partial^4 F}{\partial x^4} + 2\frac{\partial^4 F}{\partial x^2 \partial y^2} + \frac{\partial^4 F}{\partial y^4} &= E \left[ \left( \frac{\partial^2 w}{\partial x \partial y} \right)^2 - \frac{\partial^2 w}{\partial x^2} \frac{\partial^2 w}{\partial y^2} \right] \\
\frac{\partial^4 w}{\partial x^4} + 2\frac{\partial^4 w}{\partial x^2 \partial y^2} + \frac{\partial^4 w}{\partial y^4} &= \frac{p}{D} + \frac{h}{D} \left( \frac{\partial^2 F}{\partial x^2 \partial y^2} + \frac{\partial^2 F}{\partial y^2 \partial x^2} - 2 \frac{\partial^2 F}{\partial x \partial y \partial x \partial y} \right)
\end{align*}
\]

where \( D = \frac{Eh^3}{12(1 - \mu^2)} \). The median-fiber stresses are

\[
\begin{align*}
\sigma_x' &= \frac{\partial^2 F}{\partial y^2} \\
\sigma_y' &= \frac{\partial^2 F}{\partial x^2} \\
\tau_{xy}' &= -\frac{\partial^2 F}{\partial x \partial y}
\end{align*}
\]

and the median-fiber strains are
The extreme-fiber bending and shearing stresses are

\[
\begin{align*}
\epsilon_x' &= \frac{1}{E} \left( \frac{\partial^2 F}{\partial y^2} - \mu \frac{\partial^2 F}{\partial x^2} \right) \\
\epsilon_y' &= \frac{1}{E} \left( \frac{\partial^2 F}{\partial x^2} - \mu \frac{\partial^2 F}{\partial y^2} \right) \\
\gamma_{xy}' &= -\frac{2(1 + \mu)}{E} \frac{\partial^2 F}{\partial x \partial y} \tag{3}
\end{align*}
\]

For a riveted panel under normal pressure greater than that of the surrounding panels, the boundary conditions are formulated in reference 1 and are as follows:

\[
\begin{align*}
\sigma_x'' &= -\frac{Eh}{2(1 - \mu^2)} \left( \frac{\partial^2 w}{\partial x^2} + \mu \frac{\partial^2 w}{\partial y^2} \right) \\
\sigma_y'' &= -\frac{Eh}{2(1 - \mu^2)} \left( \frac{\partial^2 w}{\partial y^2} + \mu \frac{\partial^2 w}{\partial x^2} \right) \\
\tau_{xy}'' &= -\frac{Eh}{2(1 + \mu)} \frac{\partial^2 w}{\partial x \partial y} \tag{4}
\end{align*}
\]

\[
\begin{align*}
w &= 0 \\
\frac{\partial^2 w}{\partial y^2} &= 0 \\
\frac{\partial^2 F}{\partial y^2} - \mu \frac{\partial^2 F}{\partial x^2} &= 0 \tag{5}
\end{align*}
\]

along \( y = \pm \frac{b}{2} \) and

\[
\int_{y=0}^{y=b/2} \left[ \frac{\partial^2 F}{\partial x^2} - \mu \frac{\partial^2 F}{\partial y^2} - \frac{E}{2} \left( \frac{\partial w}{\partial y} \right)^2 \right] \, dy = 0
\]
along any line \( x = \text{Constant} \).

And,

\[
\begin{align*}
  w &= 0 \\
  \frac{\partial^2 w}{\partial x^2} &= 0 \\
  \frac{\partial^2 F}{\partial x^2} - \mu \frac{\partial^2 F}{\partial y^2} &= 0
\end{align*}
\]

along \( x = \pm \frac{a}{2} \) and

\[
\int_{x=0}^{x=\frac{a}{2}} \left[ \frac{\partial^2 F}{\partial y^2} - \mu \frac{\partial^2 F}{\partial x^2} - \frac{E}{2} \left( \frac{\partial w}{\partial x} \right)^2 \right] \, dx = 0
\]

along any line \( y = \text{Constant} \).

These expressions can be made nondimensional by writing \( F, w, p, \sigma, x, y, \) and \( \epsilon \), respectively, where \( a \) is the smaller side of the rectangular plate. (These latter symbols are used to effect a reduction in the amount of writing involved in the equations.) The resulting differential equations can then be transformed into finite-difference equations. In terms of finite differences, the differential equations (1) are replaced by the following equations:

\[
\begin{align*}
  \Delta_x^h F + 2 \Delta_{xy}^h F + \Delta_y^h F &= \left[ (\Delta_{xy} w)^2 - \Delta_x^2 w \Delta_y^2 w \right] \\
  \Delta_x^h w + 2 \Delta_{xy}^h w + \Delta_y^h w &= 10.8(\Delta)^4 p + 10.8 \left( \frac{\partial^2 F}{\partial x^2} \Delta^2 w + \frac{\partial^2 F}{\partial y^2} \Delta^2 w \right) - 2 \Delta_{xy}^2 w \Delta_{xy}^2 w
\end{align*}
\]
where $\Delta l = \Delta x = \Delta y$ and $\mu^2$ is taken to be 0.1, which value is characteristic for aluminum alloys.

In terms of finite differences, the boundary conditions are

\[
\begin{align*}
    w_{m,k} &= 0 \\
    (\Delta_y^2 w)_{m,k} &= 0 \\
    (\Delta_x^2 F - \mu \Delta_y^2 F)_{m,k} &= 0
\end{align*}
\]

along $y = \pm \frac{b}{2}$, and

\[
\sum_{n=0}^{k-1} \left[ \Delta_x^2 F - \mu \Delta_y^2 F - \frac{1}{2} (\Delta_y w)^2 \right]_{1,n} = 0
\]

where $n = k$ denotes points along the edges $y = \pm \frac{b}{2}$, $n = 0$ denotes points along the center lines $y = 0$, and $m = 1$ denotes any point along the line $x = \text{Constant}$ in the plate. Similarly, along $x = \pm \frac{a}{2}$, the boundary conditions are

\[
\begin{align*}
    w_{k,n} &= 0 \\
    (\Delta_x^2 w)_{k,n} &= 0 \\
    (\Delta_x^2 F - \mu \Delta_y^2 F)_{k,n} &= 0
\end{align*}
\]

and

\[
\sum_{m=0}^{k-1} \left[ \Delta_y^2 F - \mu \Delta_x^2 F - \frac{1}{2} (\Delta_x w)^2 \right]_{m,1} = 0
\]

where $m = k$ denotes points along the edges $x = \pm \frac{a}{2}$, $m = 0$ denotes points along the center line $x = 0$ and $n = 1$ denotes any point along the line $y = \text{Constant}$ in the plate.
The normal median-fiber stresses are

\[
\begin{align*}
\sigma'_x &= \frac{\Delta y^2 F}{(\Delta l)^2} \\
\sigma'_y &= \frac{\Delta x^2 F}{(\Delta l)^2}
\end{align*}
\]

and the extreme-fiber bending stresses are

\[
\begin{align*}
\sigma''_x &= -\frac{1}{2(1-\mu^2)(\Delta l)^2} \left( \Delta x^2 w + \mu \Delta y^2 w \right) \\
\sigma''_y &= -\frac{1}{2(1-\mu^2)(\Delta l)^2} \left( \Delta y^2 w + \mu \Delta x^2 w \right)
\end{align*}
\]

RECTANGULAR PLATE WITH LENGTH-WIDTH RATIO OF 1.5 UNDER UNIFORM NORMAL PRESSURE

The plate is divided into 24 square meshes. (See fig. 1.) The points 2', 5', 8', 9', 10', 11', and 11" are fictitious points outside the plate in order to make possible a better approximation to the boundary conditions. Since the plate is symmetrical with respect to the center lines, only one-quarter of the plate needs to be considered. In order to get satisfactory results in the subsequent computations, it is convenient to use a number of figures beyond those normally considered justifiable in view of the precision of the basic data.
With $\mu^2 = 0.1$ or $\mu = 0.316228$, the compatibility equations become

\[
\begin{align*}
20F_0 - 16F_1 + 2F_2 - 16F_3 + 8F_4 + 2F_6 &= K_0 \\
-8F_0 + 21F_1 - 8F_2 + 4F_3 - 16F_4 + 4F_5 + 2F_7 + F_2 &= K_1 \\
-8F_0 + 4F_1 + 21F_3 - 16F_4 + 2F_5 - 8F_6 + 4F_7 + F_9 &= K_3 \\
2F_0 - 8F_1 + 2F_2 - 8F_3 + 22F_4 - 8F_5 + 2F_6 - 8F_7 + 2F_8 + F_{10} + F_5 &= K_4 \\
F_0 - 8F_3 + 4F_4 + 20F_6 - 16F_7 + 2F_8 - 8F_9 + 4F_{10} + F_9 &= K_6 \\
F_1 + 2F_3 - 8F_4 + 2F_5 - 8F_6 + 21F_7 - 8F_8 + 2F_9 - 8F_{10} + 2F_{11} + F_{10} &= K_7
\end{align*}
\]

where $K_0$, $K_1$, $K_3$, $K_4$, $K_6$, and $K_7$ are equal to $\left[(\Delta_{xy})^2 - \Delta_x^2 \Delta_y^2 \right]$ at points 0, 1, 3, 4, 6, and 7, respectively.

The equilibrium equations are

\[
\begin{align*}
20w_0 - 16w_1 - 16w_3 + 2w_2 + 8w_4 + 2w_6 &= \gamma + 10.8 \left[\alpha' \left(2w_3 - 2w_0\right)\right] \\
+ \beta' \left(2w_1 - 2w_0\right) - 2\gamma' \left(w_2 + w_0 - w_1 - w_3\right) \\
21w_1 - 8w_0 - 8w_2 - 16w_4 + 4w_3 + 4w_5 + 2w_7 + w_2 &= \gamma + 10.8 \left[\alpha' \left(2w_4 - 2w_1\right)\right] \\
+ \beta' \left(w_2 - 2w_1 + w_0\right) - 2\gamma' \left(w_5 + w_1 - w_2 - w_4\right) \\
21w_3 - 8w_0 - 16w_4 - 8w_6 + 8w_1 + 2w_5 + 4w_7 + w_9 &= \gamma + 10.8 \left[\alpha' \left(8w_6 - 2w_3 + w_0\right)\right] \\
+ \beta' \left(2w_4 - 2w_3\right) - 2\gamma' \left(w_7 + w_3 - w_4 - w_6\right) \\
22w_4 + 2w_0 - 8w_1 + 2w_2 - 8w_3 - 8w_5 - 8w_7 - 2w_6 + 2w_8 + w_{10} + w_5 &= \gamma + 10.8 \left[\alpha' \left(7w_7 - 2w_4 + w_1\right)\right] \\
+ \beta' \left(w_5 - 2w_4 + w_3\right) - 2\gamma' \left(w_8 + w_4 - w_5 - w_7\right) \\
20w_6 + w_0 - 8w_3 + 4w_4 - 16w_7 - 8w_9 + 2w_8 + 4w_{10} + w_9 &= \gamma + 10.8 \left[\alpha' \left(w_3 - 2w_6 + w_9\right)\right] \\
+ \beta' \left(2w_7 - 2w_6\right) - 2\gamma' \left(w_{10} + w_6 - w_7 - w_9\right) \\
21w_7 + w_1 + 2w_3 - 8w_4 + 2w_5 - 8w_6 - 8w_8 + 2w_9 - 8w_{10} + 2w_{11} + w_8 + w_{10} &= \gamma + 10.8 \left[\alpha' \left(2w_7 + w_4\right)\right] \\
+ \beta' \left(w_8 - 2w_7 + w_6\right) - 2\gamma' \left(w_{11} + w_7 - w_8 - w_{10}\right)
\end{align*}
\]

(13)
where \( p' = 12(1 - \mu^2)(\Delta l)\frac{2}{4} = 0.0421875p \) since \( \Delta l = \frac{1}{4} \), and \( \alpha', \beta', \) and \( \gamma' \) are \( \Delta x^2F, \Delta y^2F, \) and \( \Delta xyF \) at the points indicated by the subscripts, respectively.

The conditions for zero edge displacements are

\[
\begin{align*}
(S_1) & \quad (F_3 - F_0) - \mu (F_1 - F_0) + (4F_4 - 4F_1) - \mu (2F_2 - 4F_1 + 2F_0) + (2F_5 - 2F_2) \\
& \quad - \mu (F_1 - 2F_2 + F_2') = S_1 \\
(S_2) & \quad (F_0 - 2F_3 + F_6) - \mu (2F_4 - 2F_3) + (2F_7 - 4F_4 + 2F_1) - \mu (2F_5 - 4F_4 + 2F_3) \\
& \quad + (F_2 - 2F_5 + F_8) - \mu (F_4 - 2F_5 + F_5') = S_2 \\
(S_3) & \quad (F_3 - 2F_6 + F_9) - \mu (2F_7 - 2F_6) + (2F_4 - 4F_7 + 2F_10) - \mu (2F_6 - 4F_7 + 2F_8) \\
& \quad + (F_5 - 2F_8 + F_{11}) - \mu (F_8 - 2F_8 + F_7') = S_3 \\
(S_4) & \quad (F_1 - 2F_0) - \mu (2F_3 - 2F_0) + (4F_4 - 4F_3) - \mu (2F_0 - 4F_3 + 2F_6) + (4F_7 - 4F_6) \\
& \quad - \mu (2F_3 - 4F_6 + 2F_9) + (2F_{10} - 2F_9) - \mu (F_6 - 2F_9 + F_{9'}) = S_4 \\
(S_5) & \quad (F_2 - 2F_1 + F_0) - \mu (2F_4 - 2F_1) + (2F_5 - 4F_4 + 2F_3) - \mu (2F_1 - 4F_4 + 2F_7) \\
& \quad + (2F_8 - 4F_7 + 2F_6) - \mu (2F_4 - 4F_7 + 2F_10) + (F_{11} - 2F_{10} + F_9) \\
& \quad - \mu (F_7 - 2F_{10} + F_{10'}) = S_5
\end{align*}
\]

where

\[
\begin{align*}
S_1 & = (w_2 - w_1)^2 + (w_1 - w_0)^2 \\
S_2 & = (w_5 - w_4)^2 + (w_4 - w_3)^2 \\
S_3 & = (w_8 - w_7)^2 + (w_7 - w_6)^2 \\
S_4 & = (w_9 - w_6)^2 + (w_6 - w_3)^2 + (w_3 - w_0)^2 \\
S_5 & = (w_{10} - w_7)^2 + (w_7 - w_4)^2 + (w_4 - w_1)^2
\end{align*}
\]
The boundary conditions are

\[ w_2 = 0, \ w_5 = 0, \ w_8 = 0, \ w_9 = 0, \ w_{10} = 0, \ w_{11} = 0 \]

\[ w_2' - 2w_2 + w_1 = 0 \]
\[ w_5' - 2w_5 + w_4 = 0 \]
\[ w_8' - 2w_8 + w_7 = 0 \]
\[ w_9' - 2w_9 + w_6 = 0 \]
\[ w_{10}' - 2w_{10} + w_7 = 0 \]

\[ F_2' - 2F_2 + F_1 - \mu (2F_5 - 2F_2) = 0 \]
\[ F_5' - 2F_5 + F_4 - \mu (F_8 - 2F_5 + F_2) = 0 \]
\[ F_8' - 2F_8 + F_7 - \mu (F_{11} - 2F_8 + F_5) = 0 \]
\[ F_9' - 2F_9 + F_6 - \mu (2F_{10} - 2F_9) = 0 \]
\[ F_{10}' - 2F_{10} + F_7 - \mu (F_{11} - 2F_{10} + F_9) = 0 \]

Now the boundary-value problem determines the values of \( w \) uniquely and the values of \( F \) to within an unknown constant. Since the actual value of the constant is irrelevant, it may be defined by letting \( F_{11} = 0 \).

On combining the boundary conditions with equations (12), (13), and (14), the final equations are
\[
\begin{align*}
20F_0 - 16F_1 + 2F_2 - 16F_3 + 8F_4 + 2F_6 &= K_0 \\
- 8F_0 + 20F_1 - 6.632456F_2 + 4F_3 - 16F_4 + 4.632456F_5 + 2F_7 &= K_1 \\
- 8F_0 + 4F_1 + 21F_3 - 16F_4 + 2F_5 - 8F_6 + 4F_7 + F_9 &= K_3 \\
2F_0 - 8F_1 + 2.316228F_2 - 8F_3 + 21F_4 - 6.632456F_5 + 2F_6 - 8F_7 + 2.316228F_8 + F_{10} &= K_4 \\
F_0 - 8F_3 + 4F_4 + 19F_6 - 16F_7 + 2F_8 - 6.632456F_9 + 4.632456F_{10} &= K_6 \\
F_1 + 2F_3 - 8F_4 + 2.316228F_5 - 8F_6 + 19F_7 - 6.632456F_8 + 2.316228F_9 - 6.632456F_{10} &= K_7 \\
- 2F_0 - 3.367544F_1 - 2.432456F_2 + 2F_3 + 4F_4 + 1.8F_5 &= S_1 \\
F_0 + 2F_1 + 0.9F_2 - 2F_3 - 3.367544F_4 - 2.432456F_5 + F_6 + 2F_7 + 0.9F_8 &= S_2 \\
F_3 + 2F_4 + 0.9F_5 - 2F_6 - 3.367544F_7 - 2.432456F_8 + F_9 + 2F_{10} + F_{11} &= S_3 \\
- 2F_0 + 2F_1 - 4F_3 + 4F_4 - 3.367544F_6 + 4F_7 - 2.432456F_9 + 1.8F_{10} &= S_4 \\
F_0 - 2F_1 + F_2 + 2F_3 - 4F_4 + 2F_5 + 2F_6 - 3.367544F_7 + 2F_8 - 0.9F_9 - 2.432456F_{10} &= S_5 \\
\end{align*}
\]
By following the procedures outlined in reference 1, equations (15) and (16) can be solved simultaneously for $w$ and $F$ by the method of successive approximations.

RECTANGULAR PLATE WITH LENGTH–WIDTH RATIO OF 2

UNDER UNIFORM NORMAL PRESSURE

The plate is first divided into 8 square meshes and then into 32 square meshes. (See figs. 2 and 3, respectively.) On referring first to figure 2, points 1', 3', 4', 5', and 5'' are fictitious points outside the plate in order to give a better approximation to the boundary conditions. Consider one-quarter of the plate; the compatibility equations are

\[
\begin{align*}
20F_0 & - 16F_1 - 16F_2 + 8F_3 + 2F_4 + 2F_1' = K_0 \\
20F_2 & - 8F_0 - 16F_3 - 8F_4 + 4F_1 + 4F_5 + F_2 + 2F_3 + F_4' = K_1
\end{align*}
\]
where $K_0$ and $K_1$ are equal to $[(\Delta_{xy}w)^2 - \Delta_x^2w\Delta_y^2w]$ at points 0 and 1, respectively.

The equilibrium equations are

$$20w_0 - 16w_1 + 16w_2 + 8w_3 + 2w_4 + 2w_5 + 2w_6 = p' + 10.8 \left[ \alpha_0'(2w_2 - 2w_0) + \beta_0'(2w_1 - 2w_0) - 2\gamma_0'(w_3 + w_0 - w_1 - w_2) \right]$$

$$20w_2 - 8w_0 - 16w_3 - 8w_4 + 4w_5 + w_2 + 2w_3 + w_4$$

$$= p' + 10.8 \left[ \alpha_2'(w_0 - 2w_2 + w_4) + \beta_2'(2w_3 - 2w_2) - 2\gamma_2'(w_5 + w_2 - w_3 - w_4) \right]$$

(18)

where $p' = 12(1 - \mu^2)(\Delta l)^4 = 0.675p$ since $\Delta l = \frac{1}{2}$, and $\alpha'$, $\beta'$, and $\gamma'$ are $\Delta_x^2F$, $\Delta_y^2F$, and $\Delta_{xy}^2F$ at the points indicated by the subscripts, respectively.

The conditions for zero edge displacements are

$$\begin{align*}
(2F_2 - 2F_0) - \mu(2F_1 - 2F_0) + (2F_3 - 2F_1) - \mu(F_0 - 2F_1 + F_1') &= w_0^2 \\
(F_0 - 2F_2 + F_4) - \mu(2F_3 - 2F_2) + (F_1 - 2F_3 + F_5) - \mu(F_2 - 2F_3 + F_3') &= w_2^2 \\
(2F_1 - 2F_0) - \mu(2F_2 - 2F_0) - 2\mu(F_0 - 2F_2 + F_4) + 2(2F_3 - 2F_2) - \mu(F_2 - 2F_1 + F_1') + (2F_5 - 2F_4) &= (w_2 - w_0)^2 + w_2^2
\end{align*}$$

(19)
The boundary conditions are

\[ w_1 = 0, \; w_3 = 0, \; w_4 = 0, \; w_5 = 0 \]
\[ w_1' + w_0 = 0 \]
\[ w_3' + w_2 = 0 \]
\[ w_4' + w_2 = 0 \]

\[ F_1' - 2F_1 + F_0 - \mu \left( 2F_3 - 2F_1 \right) = 0 \]
\[ F_3' - 2F_3 + F_2 - \mu \left( F_1 - 2F_3 + F_5 \right) = 0 \]
\[ F_4' - 2F_4 + F_2 - \mu \left( 2F_5 - 2F_4 \right) = 0 \]

The problem can now be solved uniquely for the values of \( w \) and the values of \( F \), to within an unknown constant. As the actual value of the constant is irrelevant, it may be defined by letting \( F_5 = 0 \).

Combined with the boundary conditions, equations (17), (18), and (19) are then

\[
\begin{align*}
18F_0 - 13.264912F_1 - 16F_2 + 9.264912F_3 + 2F_4 &= K_0 \\
-8F_0 + 4.632456F_1 + 18F_2 - 13.264912F_3 - 6.632456F_4 &= K_1 \\
-1.367544F_0 - 2.532456F_1 + 2F_2 + 1.8F_3 &= w_0^2 \\
F_0 + 0.9F_1 - 1.367544F_2 - 2.532456F_3 + F_4 &= w_2^2 \\
-2F_0 + 2F_1 - 3.367544F_2 + 4F_3 - 2.532456F_4 &= (w_2 - w_0)^2 + w_2^2
\end{align*}
\]

\[
\begin{align*}
[18 + 21.6(\alpha' + \beta' + \gamma')]w_0 - [16 + 21.6(\alpha' + \gamma')]w_2 &= p' \\
-(8 + 10.8\alpha')w_0 + [18 + 21.6(\alpha' + \beta' + \gamma')]w_2 &= p'
\end{align*}
\]
Equations (20) and (21) can now be solved simultaneously for \( w \) and \( F \) by the method of successive approximations.

With the values of \( w \) thus computed as a first approximation, the case wherein the plate is divided into 32 square meshes is now to be studied. On referring to figure 3, points 2', 5', 8', 11', 12', 13', 14', and 14'' are again fictitious points outside the plate in order to give a better approximation to the boundary conditions. Consider one-quarter of the plate; the compatibility equations are

\[
\begin{align*}
20F_0 - 16F_1 + 2F_2 - 16F_3 + 8F_4 + 2F_6 &= K_0 \\
-8F_0 + 21F_1 - 8F_2 + 4F_3 - 16F_4 + 4F_5 + 2F_7 + F_2 &= K_1 \\
-8F_0 + 4F_1 + 21F_3 - 16F_4 + 2F_5 - 8F_6 + 4F_7 + F_9 &= K_3 \\
2F_0 - 8F_1 + 2F_2 - 8F_3 + 22F_4 - 8F_5 + 2F_6 - 8F_7 + 2F_8 + F_{10} + F_{5'} &= K_4 \\
F_0 - 8F_3 + 4F_4 + 20F_6 - 16F_7 + 2F_8 - 8F_9 + 4F_{10} + F_{12} &= K_6 \\
F_1 + 2F_3 - 8F_4 + 2F_5 - 8F_6 + 21F_7 - 8F_8 + 2F_9 - 8F_{10} + 2F_{11} + F_{13} + F_{8'} &= K_7 \\
F_3 - 8F_6 + 4F_7 + 20F_9 - 16F_{10} + 2F_{11} - 8F_{12} + 4F_{13} + F_{12'} &= K_9 \\
F_4 + 2F_6 - 8F_7 + 2F_8 - 8F_9 + 21F_{10} - 8F_{11} + 2F_{12} - 8F_{13} + 2F_{14} + F_{11'} + F_{13'} &= K_{10}
\end{align*}
\]

where \( K = \left[ (\Delta_{xy}^2 - \Delta_x^2 \Delta_y^2)^2 \right] \) at the points indicated by the subscripts.
The equilibrium equations are

\[ 20w_0 - 16w_1 - 16w_3 + 8w_4 + 2w_2 + 2w_6 = p' + 10.8\left[ a'_0 (2w_3 - 2w_0) + \beta'_{0}(2w_1 - 2w_0) - 2\gamma'_0 (w_4 - w_1 - w_3 + w_0) \right] \]

\[-8w_0 + 21w_1 - 8w_2 + 4w_3 - 16w_4 + 4w_5 + 2w_7 + w_9 = p' + 10.8\left[ a'_1 (2w_4 - 2w_1) + \beta'_{1}(w_2 - 2w_1 + w_0) - 2\gamma'_1 (w_5 - w_2 - w_4 + w_1) \right] \]

\[-8w_0 + 4w_1 + 21w_3 - 16w_4 + 2w_5 - 8w_6 + 4w_7 + w_9 = p' + 10.8\left[ a'_3 (w_0 - 2w_3 + w_6) + \beta'_{3}(2w_4 - 2w_3) - 2\gamma'_3 (w_7 - w_4 - w_6 + w_3) \right] \]

\[2w_0 - 8w_1 + 2w_2 - 8w_3 + 22w_4 - 8w_5 + 2w_6 - 8w_7 + 2w_8 + w_{10} + w_{5,1} = p' + 10.8\left[ a'_{4} (w_7 - 2w_4 + w_1) + \beta'_{4}(w_5 - 2w_4 + w_3) - 2\gamma'_{4}(w_8 - w_5 - w_7 + w_4) \right] \]

\[w_0 - 3w_1 + 4w_4 + 20w_6 - 16w_7 + 2w_8 - 8w_9 + 4w_{10} + w_{12} = p' + 10.8\left[ a'_{6} (w_3 - 2w_6 + w_9) + \beta'_{6}(2w_7 - 2w_6) - 2\gamma'_{6}(w_{10} - w_7 - w_9 + w_6) \right] \]

\[w_1 + 2w_3 - 8w_4 + 2w_5 - 8w_6 + 21w_7 - 8w_8 + 2w_9 - 8w_{10} + 2w_{11} + w_{13} + w_{8,1} = p' + 10.8\left[ a'_{7} (w_4 - 2w_7 + w_{10}) + \beta'_{7}(w_8 - 2w_7 + w_6) - 2\gamma'_{7}(w_{11} - w_8 - w_{10} + w_7) \right] \]

\[w_3 - 8w_6 + 4w_7 + 20w_9 - 16w_{10} + 2w_{11} - 8w_12 + 4w_{13} + w_{12,1} = p' + 10.8\left[ a'_{9} (w_6 - 2w_9 + w_{12}) - 2\gamma'_{9}(w_{13} - w_{10} - w_{12} + w_9) \right] \]

\[w_4 + 2w_6 - 8w_7 + 2w_8 - 8w_9 + 21w_{10} - 8w_{11} + 2w_{12} - 8w_{13} + 2w_{14} + w_{11,1} + w_{13,1} = p' + 10.8\left[ a'_{10} (w_7 - 2w_{10} + w_{13}) + \beta'_{10}(w_{11} - 2w_{10} + w_9) - 2\gamma'_{10}(w_{14} - w_{11} - w_{13} + w_{10}) \right] \]
where \( p' = 12(1 - \mu^2)(\Delta l)^4 \) since \( \Delta l = \frac{1}{4} \), and \( \alpha', \beta', \gamma' \) are \( \Delta x 2F, \Delta y 2F, \) and \( \Delta z F \) at the points indicated by the subscripts, respectively.

The conditions for zero edge displacements are

\[
2F_3 - 2F_0 - \mu(2F_4 - 2F_0) + 4F_4 - 4F_1 - 2\mu(F_2 - 2F_1 + F_0) + 2F_5 - 2F_2
- \mu(F_2 - 2F_1 + F_0) = S_1
\]

\[
F_0 - 2F_3 + 2F_6 - \mu(2F_4 - 2F_3) + 2F_6 + 4F_4 + 4F_7 - 2\mu(F_3 - 2F_4 + F_5)
+ (F_2 - 2F_5 + F_8) - \mu(F_5 - 2F_5 + F_4) = S_2
\]

\[
F_3 - 2F_4 + F_9 - \mu(2F_7 - 2F_6) + 2F_4 + 4F_7 + 2F_7 - 2\mu(F_6 - 2F_7 + F_8)
+ (F_5 - 2F_8 + F_9) - \mu(F_7 - 2F_7 + F_8) = S_3
\]

\[
F_4 - 2F_6 + F_9 - \mu(2F_7 - 2F_6) + 2F_4 + 4F_7 + 2F_7 - 2\mu(F_6 - 2F_7 + F_8)
+ (F_5 - 2F_8 + F_9) - \mu(F_7 - 2F_7 + F_8) = S_3
\]

\[
F_6 - 2F_8 + F_9 - \mu(2F_10 - 2F_9) + 2F_7 - 2F_10 + 2F_13 - 2\mu(F_9 - 2F_10 + F_11)
+ F_8 - 2F_11 + F_14 - \mu(F_10 - 2F_11 + F_11) = S_4
\]

\[
2F_1 - 2F_0 - \mu(2F_3 - 2F_0) + 4F_4 - 4F_1 - 2\mu(F_0 - 2F_3 + F_6) + 4F_7 - 4F_6
- 2\mu(F_3 - 2F_6 + F_9) + 4F_10 - 4F_9 - 2\mu(F_6 - 2F_9 + F_12) + 2F_13 - 2F_12
- \mu(F_9 - 2F_12 + F_12) = S_5
\]

\[
F_2 - 2F_1 + F_0 - \mu(2F_4 - 2F_1) + 2F_5 - 4F_4 + 2F_3 - 2\mu(F_1 - 2F_4 + F_7) + 2F_8 - 4F_7
+ 2F_6 - 2\mu(F_4 - 2F_7 + F_10) + 2F_7 - 4F_10 + 2F_9 - 2\mu(F_7 - 2F_10 + F_13)
+ F_14 - 2F_13 + F_12 - \mu(F_10 - 2F_13 + F_13) = S_6
\]

where \( S_i = \sum_{m=0}^{k-l} (w_{m+1,n} - w_{m,n})^2 \) for \( i = 1, 2, 3, 4 \)

and \( S_j = \sum_{n=0}^{k-l} (w_{m,n+1} - w_{m,n})^2 \) for \( j = 5, 6 \)
The boundary conditions are

\[ w_2 = 0, \quad w_5 = 0, \quad w_8 = 0, \quad w_{11} = 0, \quad w_{12} = 0, \quad w_{13} = 0, \quad w_{14} = 0 \]

\[ w_{2t} - 2w_2 + w_1 = 0 \]

\[ w_{5t} - 2w_5 + w_4 = 0 \]

\[ w_{8t} - 2w_8 + w_7 = 0 \]

\[ w_{11t} - 2w_{11} + w_{10} = 0 \]

\[ w_{12t} - 2w_{12} + w_9 = 0 \]

\[ w_{13t} - 2w_{13} + w_{10} = 0 \]

\[ F_1 - 2F_2 + F_2, - \mu (2F_5 - 2F_2) = 0 \]

\[ F_4 - 2F_5 + F_5, - \mu (F_2 - 2F_5 + F_8) = 0 \]

\[ F_7 - 2F_8 + F_8, - \mu (F_5 - 2F_8 + F_{11}) = 0 \]

\[ F_{10} - 2F_{11} + F_{11}, - \mu (F_8 - 2F_{11} + F_{14}) = 0 \]

\[ F_9 - 2F_{12} + F_{12}, - \mu (2F_{13} - 2F_{12}) = 0 \]

\[ F_{10} - 2F_{13} + F_{13}, - \mu (F_{14} - 2F_{13} + F_{12}) = 0 \]

The problem can now be solved uniquely again for the values for w and the values of f to within an unknown constant. As the actual value of the constant is irrelevant, it may be defined by letting \( F_{14} = 0 \).

On combining with the boundary conditions, equations (22), (23), and (24) are then
\[2F_0 - 16F_1 + 2F_2 - 16F_3 + 8F_4 + 2F_6 = K_0\]
\[-8F_0 + 20F_1 - 6.632456F_2 + 4F_3 - 16F_4 + 4.632456F_5 + 2F_7 = K_1\]
\[-8F_0 + 4F_1 + 21F_3 - 16F_4 + 2F_5 - 8F_6 + 4F_7 + F_9 = K_3\]
\[2F_0 - 8F_1 + 2.316228F_2 - 8F_3 + 21F_4 - 6.632456F_5 + 2F_6 - 8F_7 + 2.316228F_8 + F_{10} = K_4\]
\[F_0 - 8F_3 + 4F_4 + 20F_6 - 16F_7 + 2F_8 - 8F_9 + 4F_{10} + F_{12} = K_6\]
\[F_1 + 2F_3 - 8F_4 + 2.316228F_5 - 8F_6 + 20F_7 - 6.632456F_8 + 2F_9 - 8F_{10} + 2.316228F_{11} + F_{13} = K_7\]
\[F_3 - 8F_6 + 4F_7 + 16F_9 - 16F_{10} + 2F_{11} - 6.632456F_{12} + 4.632456F_{13} = K_9\]
\[F_4 + 2F_6 - 8F_7 + 2.316228F_8 - 8F_9 + 16F_{10} - 6.632456F_{11} + 2.316228F_{12} - 6.632456F_{13} = K_{10}\]
\[-2F_0 - 3.367544F_1 - 2.432456F_2 + 2F_3 + 4F_4 + 1.8F_5 = S_1\]
\[F_0 + 2F_1 + 0.9F_2 - 2F_3 - 3.367544F_4 - 2.432456F_5 + F_6 + 2F_7 + 0.9F_8 = S_2\]
\[F_3 + 2F_4 + 0.9F_5 - 2F_6 - 3.367544F_7 - 2.432456F_8 + F_9 + 2F_{10} + 0.9F_{11} = S_3\]
\[F_6 + 2F_7 + 0.9F_8 - 2F_9 - 3.367544F_{10} - 2.432456F_{11} + F_{12} + 2F_{13} = S_4\]
\[-2F_0 + 2F_1 - 4F_3 + 4F_4 - 4F_6 + 4F_7 - 3.367544F_9 + 4F_{10} - 2.432456F_{12} + 1.8F_{13} = S_5\]
\[F_0 - 2F_1 + F_2 + 2F_3 - 4F_4 + 2F_5 + 2F_6 - 4F_7 + 2F_8 + 2F_9 - 3.367544F_{10} + 2F_{11} + 0.9F_{12} - 2.432544F_{13} = S_6\]
and

\[
\begin{align*}
20 + 21.6 (\alpha'_{i0} + \beta'_{i0} + \gamma'_{i0}) w_0 - & \left[ 16 + 21.6 (\beta'_{i0} - \gamma'_{i0}) \right] w_1 \\
& - \left[ 16 + 21.6 (\alpha'_{i0} + \gamma'_{i0}) \right] w_3 + \left( 8 + 10.8 \gamma'_{i0} \right) w_4 + 2w_6 = p' \\
& - \left( 8 + 10.8 \beta'_{i1} \right) w_0 + \left[ 20 + 21.6 (\alpha'_{i1} + \beta'_{i1} + \gamma'_{i1}) \right] w_1 + 4w_3 \\
& - \left[ 16 + 21.6 (\alpha'_{i1} + \gamma'_{i1}) \right] w_4 + \left[ 8 + 10.8 (\alpha'_{i3} + \beta'_{i3}) \right] w_6 \\
& + \left( 4 + 21.6 \gamma'_{i3} \right) w_7 + w_9 = p' \\
2w_0 - & \left( 8 + 10.8 \alpha'_{i4} \right) w_1 - \left( 8 + 10.8 \beta'_{i4} \right) w_3 + \left[ 20 + 21.6 (\alpha'_{i4} + \beta'_{i4} + \gamma'_{i4}) \right] w_4 \\
& + 2w_6 - \left[ 8 + 10.8 (\alpha'_{i4} + 2\gamma'_{i4}) \right] w_7 + w_{10} = p' \\
v_0 - & \left( 8 + 10.8 \alpha'_{i6} \right) w_3 + 4w_4 + \left[ 20 + 21.6 (\alpha'_{i4} + \beta'_{i4} + \gamma'_{i4}) \right] w_6 \\
& - \left[ 16 + 21.6 (\beta'_{i6} + \gamma'_{i6}) \right] w_7 - \left[ 8 + 10.8 (\alpha'_{i6} + 2\gamma'_{i6}) \right] w_9 \\
& + \left( 4 + 21.6 \gamma'_{i6} \right) w_{10} = p' \\
w_1 + 2w_3 - & \left( 8 + 10.8 \alpha'_{i7} \right) w_4 - \left( 8 + 10.8 \beta'_{i7} \right) w_6 + \left[ 20 + 21.6 (\alpha'_{i7} + \beta'_{i7} + \gamma'_{i7}) \right] w_7 \\
& + 2w_9 - \left[ 8 + 10.8 (\alpha'_{i7} + 2\gamma'_{i7}) \right] w_{10} = p' \\
w_3 - & \left( 8 + 10.8 \alpha'_{i9} \right) w_6 + 4w_7 + \left[ 19 + 21.6 (\alpha'_{i9} + \beta'_{i9} + \gamma'_{i9}) \right] w_9 \\
& - \left[ 16 + 21.6 (\beta'_{i9} + \gamma'_{i9}) \right] w_{10} = p' \\
w_4 + 2w_6 - & \left( 8 + 10.8 \alpha'_{i10} \right) w_7 - \left( 8 + 10.8 \beta'_{i10} \right) w_9 \\
& + \left[ 19 + 21.6 (\alpha'_{i10} + \beta'_{i10} + \gamma'_{i10}) \right] w_{10} = p'
\end{align*}
\]

By the method of successive approximations, equations (25) and (26) can be solved simultaneously for values of \( w \) and \( F \).
RESULTS AND DISCUSSION

The bending problem of rectangular plates under uniform normal pressure with simply supported edges is studied by finite-difference approximations. The difference equations are solved by the method of successive approximation. The case of a square plate is studied in reference 1. In the present report, the computations are extended to rectangular plates with ratios of length to width of 1.5 or 2. Experimental work (reference 2) indicates that a rectangular plate having a length equal to twice its width may be regarded practically as an infinitely long plate. The maximum normal pressure calculated is \( \frac{p a}{E h^2} = 250 \), which gives the center deflections about twice the thickness of the plate.

The deflections at various points in the plate under different pressure ratios are tabulated in tables 1, 2, and 3. The center deflections are plotted against the normal pressure ratio in figures 4 and 5. The membrane stresses in the centers of the plates and at the centers of the edges are tabulated in tables 4, 5, and 6 and are plotted in figures 6 and 7. The bending and total stresses at the centers of the plates are tabulated in tables 7 and 8 and are plotted in figures 8 and 9.

As indicated in reference 1, the plate should be designed for strength on the basis of clamped edges and for maximum deflection or washboarding according to the boundary conditions such as formulated in the present report. The finite-difference equations give the values of deflections with good approximations, whereas the stresses, which are obtained by a second-order numerical differentiation, are always less accurate.

Brown University
Providence, R. I.
November 4, 1946
REFERENCES


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### TABLE 2

**DEFLECTION AT VARIOUS POINTS**

\[
\frac{b}{a} = 1.5; \quad \Delta l = \frac{a}{4}
\]

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### TABLE 3

**DEFLECTIONS AT VARIOUS POINTS**

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\left[ \frac{b}{a} = 2; \quad \Delta l = \frac{a}{4} \right]
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TABLE 4a

MEMBRANE STRESSES

\[ \frac{b}{a} = 1.5; \Delta = \frac{a}{b} \]

<table>
<thead>
<tr>
<th>( \frac{p}{Eh^2} )</th>
<th>( \sigma_{x0} )</th>
<th>( \sigma_{y0} )</th>
<th>( \sigma_{x1} )</th>
<th>( \sigma_{y1} )</th>
<th>( \sigma_{x2} )</th>
<th>( \sigma_{y2} )</th>
</tr>
</thead>
<tbody>
<tr>
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<td>0</td>
<td>0</td>
<td>0</td>
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<td>0</td>
</tr>
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<td>12.5</td>
<td>1.064</td>
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<td>1.072</td>
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<td>0.2316</td>
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</tr>
<tr>
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<td>1.213</td>
<td>2.150</td>
<td>0.6799</td>
<td>0.4775</td>
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<td>2.202</td>
<td>3.903</td>
<td>1.234</td>
<td>0.8930</td>
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</tr>
<tr>
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<td>5.176</td>
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<td>5.381</td>
<td>1.702</td>
<td>1.251</td>
<td>3.955</td>
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<tr>
<td>100</td>
<td>6.410</td>
<td>3.784</td>
<td>6.694</td>
<td>2.117</td>
<td>1.574</td>
<td>4.977</td>
</tr>
<tr>
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<td>8.605</td>
<td>5.110</td>
<td>9.037</td>
<td>2.858</td>
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<td>6.292</td>
<td>11.13</td>
<td>3.518</td>
<td>2.673</td>
<td>8.452</td>
</tr>
</tbody>
</table>

aSubscript 0 denotes the center of the plate; subscript 1 denotes the center of the sides \( x = \pm \frac{a}{2} \); and subscript 2 denotes the center of the sides \( y = \pm \frac{b}{2} \).
### TABLE 5a

MEMBRANE STRESSES

\[
\begin{bmatrix}
\frac{b}{a} = 2; \quad \Delta l = \frac{a}{2}
\end{bmatrix}
\]

<table>
<thead>
<tr>
<th>( \frac{P a^4}{Eh^4} )</th>
<th>( \frac{\sigma_{x0}a^2}{Eh^2} )</th>
<th>( \frac{\sigma_{y0}a^2}{Eh^2} )</th>
<th>( \frac{\sigma_{xl}a^2}{Eh^2} )</th>
<th>( \frac{\sigma_{yl}a^2}{Eh^2} )</th>
<th>( \frac{\sigma_{x2}a^2}{Eh^2} )</th>
<th>( \frac{\sigma_{y2}a^2}{Eh^2} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>12.5</td>
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<td>1.115</td>
<td>.3525</td>
<td>.1897</td>
<td>.5998</td>
</tr>
<tr>
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<td>.3751</td>
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<td>1.589</td>
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<td>1.597</td>
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<td>3.233</td>
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<td>11.75</td>
<td>6.928</td>
<td>11.96</td>
<td>3.782</td>
<td>2.316</td>
<td>7.324</td>
</tr>
</tbody>
</table>

*aSubscript 0 denotes the center of the plate; subscript 1 denotes the center of the sides \( x = \pm \frac{a}{2} \); and subscript 2 denotes the center of the sides \( y = \pm \frac{b}{2} \).*
TABLE 6<sup>a</sup>

MEMBRANE STRESSES

\[
\begin{bmatrix}
\frac{b}{a} = 2; \quad \Delta l = \frac{a}{4}
\end{bmatrix}
\]

<table>
<thead>
<tr>
<th>( \frac{p a_{4}}{E h_{4}} )</th>
<th>( \sigma_{x0}^{a} )</th>
<th>( \sigma_{y0}^{a} )</th>
<th>( \sigma_{x1}^{a} )</th>
<th>( \sigma_{y1}^{a} )</th>
<th>( \sigma_{x2}^{a} )</th>
<th>( \sigma_{y2}^{a} )</th>
</tr>
</thead>
<tbody>
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<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
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<td>1.293</td>
<td>.5421</td>
<td>1.298</td>
<td>.4106</td>
<td>.2372</td>
<td>.7500</td>
</tr>
<tr>
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<td>2.403</td>
<td>1.035</td>
<td>2.441</td>
<td>.7718</td>
<td>.4718</td>
<td>1.492</td>
</tr>
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<td>1.827</td>
<td>4.264</td>
<td>1.349</td>
<td>.8730</td>
<td>2.761</td>
</tr>
<tr>
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<td>5.788</td>
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<td>1.220</td>
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</tr>
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<td>6.911</td>
<td>3.094</td>
<td>7.149</td>
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<td>1.534</td>
<td>4.852</td>
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<td>5.111</td>
<td>11.72</td>
<td>3.707</td>
<td>2.606</td>
<td>8.240</td>
</tr>
</tbody>
</table>

<sup>a</sup> Subscript 0 denotes the center of the plate; subscript 1 denotes the center of the sides \( x = \pm \frac{a}{2}; \) and subscript 2 denotes the center of the sides \( y = \pm \frac{b}{2}. \)
**TABLE 7**

**EXTREME-FIBER BENDING AND TOTAL STRESSES AT CENTER OF PLATE**

\[
\frac{b}{a} = 1.5; \quad \Delta l = \frac{a}{h}
\]

<table>
<thead>
<tr>
<th>( \frac{p a}{E h^4} )</th>
<th>( \sigma''_{x0} ) ( \frac{a}{E h^2} )</th>
<th>( \sigma''_{y0a} ) ( \frac{E h^2}{a} )</th>
<th>( \sigma''_{x0a} ) ( \frac{E h^2}{a} )</th>
<th>( \sigma''_{y0a} ) ( \frac{E h^2}{a} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
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<td>3.413</td>
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<td>4.477</td>
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<tr>
<td>25</td>
<td>4.700</td>
<td>2.761</td>
<td>6.809</td>
<td>3.974</td>
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<tr>
<td>50</td>
<td>6.137</td>
<td>3.476</td>
<td>9.920</td>
<td>5.679</td>
</tr>
<tr>
<td>75</td>
<td>7.072</td>
<td>3.914</td>
<td>12.25</td>
<td>6.953</td>
</tr>
<tr>
<td>100</td>
<td>7.813</td>
<td>4.260</td>
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<td>8.044</td>
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<tr>
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<td>4.777</td>
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<td>9.770</td>
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<td>20.32</td>
<td>11.47</td>
</tr>
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<td>5.519</td>
<td>22.83</td>
<td>12.89</td>
</tr>
</tbody>
</table>

\(^a\)Subscript 0 denotes the center of the plate.
### TABLE 8a

**EXTREME-FIBER BENDING AND TOTAL STRESSES AT CENTER OF PLATE**

\[
\begin{bmatrix}
\frac{b}{a} = 2
\end{bmatrix}
\]

<table>
<thead>
<tr>
<th>( \frac{pa^4}{Eh^4} )</th>
<th>( \frac{\sigma''_{xoa^2}}{Eh^2} )</th>
<th>( \frac{\sigma''_{yoa^2}}{Eh^2} )</th>
<th>( \frac{\sigma_{xoa^2}}{Eh^2} )</th>
<th>( \frac{\sigma_{yoa^2}}{Eh^2} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \Delta l = \frac{a}{2} )</td>
<td>( \Delta l = \frac{a}{4} )</td>
<td>( \Delta l = \frac{a}{2} )</td>
<td>( \Delta l = \frac{a}{4} )</td>
<td>( \Delta l = \frac{a}{2} )</td>
</tr>
<tr>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>12.5</td>
<td>3.431</td>
<td>3.577</td>
<td>1.588</td>
<td>1.532</td>
</tr>
<tr>
<td>75</td>
<td>7.116</td>
<td>7.005</td>
<td>3.003</td>
<td>2.637</td>
</tr>
<tr>
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<td>7.895</td>
<td>7.703</td>
<td>3.300</td>
<td>2.864</td>
</tr>
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<td>9.118</td>
<td>8.792</td>
<td>3.767</td>
<td>3.224</td>
</tr>
<tr>
<td>200</td>
<td>10.08</td>
<td>9.651</td>
<td>4.138</td>
<td>3.519</td>
</tr>
</tbody>
</table>

\*Subscript 0 denotes the center of the plate.*
Figure 1.- Plate divided into 36 square meshes.

Figure 2.- Plate divided into 8 square meshes.

Figure 3.- Plate divided into 32 square meshes.
Figure 4. - Center deflections for a rectangular plate under normal pressure. \( \frac{b}{a} = 1.5 \).

Figure 5. - Center deflections for a rectangular plate under normal pressure. \( \frac{b}{a} = 2 \).
Figure 6. Membrane stresses. \( \frac{b}{a} = 1.5; \frac{A_1}{a} = \frac{8}{4} \).

Figure 7. Membrane stresses. \( \frac{b}{a} = 2; \frac{A_1}{a} = \frac{8}{4} \).
Figure 8. - Extreme-fiber total and bending stresses at center of plate. $\alpha / a = 1.5; \Delta t = 0.4$.

Figure 9. - Extreme-fiber total and bending stresses at center of plate. $b / a = 2; \Delta t = 0.4$. 
Von Karman's equations for thin plates with large deflections are solved for the special cases of rectangular plates having ratios of length to width of 1.5 and 2 and loaded by uniform normal pressure. The boundary conditions are such as to approximate panels with riveted edges under normal pressure greater than that of the surrounding panels. Center deflections, membrane stresses, and extreme-fiber bending stresses are given as functions of the pressure for center deflections up to twice the thickness of the plate. For small deflections the results are consistent with those given by Timoshenko.
ABSTRACT:

Von Karman’s equations for thin plates with large deflections are solved for the special cases of rectangular plates having ratios of length to width of 1.5 and 2 and loaded by uniform normal pressure. The boundary conditions are such as to approximate panels with riveted edges under normal pressure greater than that of the surrounding panels. Center deflections, membrane stresses, and extreme-fiber bending stresses are given as functions of the pressure for center deflections up to twice the thickness of the plate. For small deflections the results are consistent with those given by Timoshenko.