REEL-C 6886

A.T.I 147 291

The Radio Direction Finding
Research Laboratory
Department of Electrical Engineering
University of Illinois

DOPPLER-TYPE DIRECTION FINDING B

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October 1, 1948

Technical Report No. 8

Contract N6-ori-71
Task Order XV

Office of Naval Research
Project No. 076 161

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Copy No. 61

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Approved by E. C. Jordan
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I FOREWORD

As part of its general research program, the Radio Direction Finding Research Laboratory at the University of Illinois has undertaken to study new systems for radio direction finding. One of the new systems proposed was a "Doppler-Effect" system. The initial theoretical work by this group on this system was carried out by Mr. J. Lionel Boulet and was written up in the Masters' Degree thesis (4) referred to in this report. Unfortunately, due to a prolonged illness, it became necessary for Mr. Boulet to withdraw from school and further work on the problem. The work was continued by Mr. John Anderson and Mr. Thomas O'Meara, and this present report has been written by them.

Soon after the problem was undertaken, it was found that some experimental work on this idea had already been carried out at Camp Cole Signal Laboratory, the work being summarized in a series of patent applications (2) by Paul Hansell. Considerably later, there came to the attention of this group the British report (3) by C. W. Earp and R. M. Godfrey on "Radio Direction Finding by the Cyclical Differential Measurement of Phase." This excellent paper discussed in some detail certain aspects of the theory and described a working system which had been built.

The theoretical approach in this report is aimed chiefly at determining the ability of the system to discriminate against interfering waves such as those produced by site reflections. Several definite conclusions in this respect have been drawn and are stated herein. A 'model' Doppler Effect system has been built and is in operation. At present, it is being tried out and compared with a simple Adcock system on the laboratory's R-D-F System Analyzer. The results of these tests are to be given in a later report.

E. C. Jordan

II ABSTRACT

This report develops and discusses the basic theory of the Doppler Type of Direction Finder. The most practical antenna system for this purpose appears to be a circular array of fixed antennas with gradual or abrupt switching from antenna to antenna around the circle.

A mathematical discussion of site errors is included for the purpose of showing the effect of antenna aperture upon reduction of such errors. It is shown that the Doppler Direction Finder offers no advantage over the Adcock System when the antenna aperture is a small fraction of a wavelength, but that an increase in aperture of the Doppler soon reduces site errors to the order of instrumentation errors. This comparison with Adcock Systems is made on a figure of merit basis.

Some discussion of antenna commutation difficulties and number of antennas necessary to place the Doppler in the class of wide aperture systems is included.

To correlate this with the previous discussion it must be recalled that the rate of change of phase with time is proportional to the instantaneous frequency. That is,

$$f_{inst} = \frac{1}{2\pi} \frac{d\varphi}{dt} \tag{8}$$

where ϕ for this case $= \omega_c t + \frac{\pi d}{\lambda} \cos(\theta - \omega_r t)$

Then

$$2\pi f_{inst} = \omega_c + \frac{\pi d \omega_r}{\lambda} \sin(\theta - \omega_r t) \tag{9}$$

$$f_{inst} = f_c + \frac{\pi d f_r}{\lambda} \sin(\theta - \omega_r t) \tag{10}$$

The instantaneous frequency may be expressed as a function of the linear velocity, $v_r = \omega_r r$, of the antenna and as a function of the free space propagation velocity, $c = \lambda f_c$, of the incoming wave.

$$f_{inst} = f_c[1 + \frac{v_r}{c} \sin(\theta - \omega_r t)] \tag{11}$$

Notice the similarity between (11) and (5), the only difference being that the relative velocity is varying as a sinusoidal function.

The expression (7), if phase detected, will yield a term

$$\frac{\pi d}{\lambda} \cos(\theta - \omega_r t) \tag{12}$$

which is sinusoidal in nature at the frequency of rotation of the antenna, and has a phase dependent upon θ, the angle of arrival of the incoming wave. It is possible to obtain a sinusoidal voltage from the antenna motion directly

$$\frac{\pi d}{\lambda} \cos(\omega_r t) \tag{13}$$

which when compared with (12) will give a means of determining θ. Thus the basic requirements for a direction finder are fulfilled.

(b) Non-Cooperative Systems

In order for the direction finder to be classified as a Doppler type it must possess the equivalent of a moving antenna element of some form. However, this does not stipulate the path the antenna must follow. In the thesis, previously mentioned, Mr. Boulet analyzed the case of an antenna moving at a uniform speed on a linear path and retracing its course periodically. This produces a frequency change depending upon speed of motion and upon the angle between the path of the antenna and the incoming wave. If the speed is constant the frequency change is a measure of the angle. One difficulty arises when the signal arrives from the same angle on the other side of the axis. To remove this ambiguity a second antenna is made to traverse a path perpendicular to the first.

What appears to be the most practical case to consider is the revolution of the antenna about a vertical axis as was discussed in the introduction. Greater linear circumferential speeds are obtained in this case and the problem of getting the antennae back to their starting points in infinitesimal time does not enter, as it does in the linear path. Certain transients present in the linear case are not present in the circular case, making for smoother operation.

In the final analysis the rotating antenna will be replaced by a circular array of antennas, with electronic switching from antenna to antenna.

(c) Cooperative Systems

These two types of Doppler Direction Finders are classified as non-cooperative, since no knowledge is known of the nature of the transmitting source. It should be mentioned that this type of direction finder can be converted to the so-called beacon system by reversing the role of the rotating antenna from receiving to transmitting. It becomes necessary to transmit a synchronizing pulse, which will be received along with the phase modulated wave. Since the phase of the phase envelope will vary according to the azimuthal angle from the station, a comparison with the synchronizing pulse will give the radial line from the station.

IV CIRCULAR ANTENNA ARRAY FOR DOPPLER

(a) Advantages

Consider the aforementioned rotating antenna and pick a set of representative values for the variables involved. Assuming a rotating arm, one meter in length, an angular velocity of 60π radians/sec. (1800 R.P.M.) the maximum instantaneous frequency difference will be given by (9)

$$\Delta f_{max} = .628 \times 10^{-6} f_c \text{ cycles/sec.} \qquad (13)$$

The immediate important observation is the small value of Δf unless the frequency involved is very large, or the practical length of the arm is exceeded.

The possibility of producing an effect similar to that described for the single rotating antenna, by using fixed antennae connected successively to a receiver was investigated. This system should be strictly referred to as indirect Doppler since the motion is simulated by rapid switching. One of the advantages of this indirect approach is the speed at which electronic switching can be accomplished, thus eliminating the problem of small Δf. In addition, the use of fixed antennas makes it possible to build arrays of large aperture—a very necessary requirement for accurate bearings in the presence of site reflections. Both British and American systems to date utilize a circular array of antennae with electronic switching. To appreciate the shape of phase variation involved, plate (3) shows an array with a signal arriving at an angle of 45° azimuthal, and also the phase of the r.f. voltage picked up as commutation progresses around the circle. If the number of antennae used is sufficient, the step function of phase, when detected, will yield a fundamental sinusoidal variation similar to the rotating antenna. This variation will change phase with the angle of incoming wave in a linear manner, again giving the basic requirement for a Doppler System.

(b) Law of Coupling

The abrupt switching from one antenna to another presents problems in detection inasmuch as discrete jumps in phase cause accompanying infinite frequency change, Eq. (8). To alleviate this a law of coupling to the antennae was studied. Consider two elements in a circular array of fixed antennae. Call one element the "nth" element and the next, the "(n + 1th)" element. It is desired to find a law of coupling as switching is completed between the two antennae such that the voltage output will have a sinusoidal variation of phase. Let the voltage received by the "nth" antenna be

$$e_n = \cos \left\{ \omega t + \frac{2\pi r}{\lambda} \cos \left[\theta - \frac{2\pi (n-1)}{k} \right] \right\} \text{ and} \tag{14}$$

by the "$(n+1)^{th}$" antenna be

$$e_{n+1} = \cos \left[\omega t + \frac{2\pi r}{\lambda} \cos \left(\theta - \frac{2\pi n}{k} \right) \right] \tag{15}$$

where the notation is the same as plate (3).

The addition of these two voltages with the appropriate law of coupling would be required to take the form:

$$A \cos \left[\omega t + \frac{2\pi r}{\lambda} \cos (\theta - 2\pi a t) \right] \tag{16}$$

That is:

$$G(t) \cos\left\{\omega t + \frac{2\pi r}{\lambda} \cos\left[\theta - \frac{2\pi (n-1)}{k} \right] \right\} \tag{17}$$

$$+ H(t) \cos\left[\omega t + \frac{2\pi r}{\lambda} \cos\left(\theta - \frac{2\pi n}{k}\right)\right] = A \cos\left[\omega t + \frac{2\pi r}{\lambda} \cos(\theta - 2\pi a t)\right]$$

Consider plate (3). The difference in phase between e_n and e_{n+1} is

$$\phi = \alpha\left\{ \cos\left(\theta - \frac{2\pi n}{k}\right) - \cos\left[\theta - \frac{2\pi (n-1)}{k}\right] \right\} \tag{18}$$

and the phase angle (β) of the resultant R is given by

$$\sin \beta = \frac{H(t)\ e_{n-1} \sin \phi}{R} \tag{19}$$

$$\tan\beta = \frac{H(t)\ e_{n+1}\ \sin\varphi}{G(t)\ e_n \ddagger H(t)\ \ddot{e}_{n+1}\ \cos\varphi} \tag{20}$$

If the arbitrary phase of the e_n voltage is called γ the phase of R at any instant t, where $\frac{n-1}{ka} < t < \frac{n}{ka}$ is giv en by

$$\delta(t) = \gamma + \text{arc}\tan\left[\frac{H(t)\ \sin\varphi}{G(t)\frac{e_n}{e_{n+1}} + H(t)\ \cos\varphi}\right] \tag{21}$$

$$\frac{e_n}{e_{n+1}} = \text{constant} = c$$

This variation must be equal or close to the variation $\alpha\cos(\theta - 2\pi at)$.The integral to be minimized is then:

$$\int_{\frac{n-1}{ka}}^{\frac{n}{ka}} [\alpha\cos(\theta - 2\pi at) - \delta(t)]\ dt \tag{22}$$

The necessary condition for a minimum must satisfy Euler equations, that is if $\int_a^b G(x,y,y')\ dx$ is to be minimized

$$\frac{d}{dx}\frac{\partial G}{\partial y'} = \frac{\partial G}{\partial y} \quad \text{or} \quad \frac{d}{dx}\left(G - y'\frac{\partial G}{\partial y'}\right) = \frac{\partial G}{\partial x} \tag{23}$$

must be satisfied.

In our case $t = x,$

$$\frac{G(t)}{H(t)} = F(t) = y \tag{24}$$

and

$$G = \alpha\cos(\theta - 2\pi at) - \gamma - \text{arc}\tan\left[\frac{\sin\varphi}{cF(t) + \cos\varphi}\right] \tag{25}$$

Then

$$\frac{d}{dt}\frac{\partial G}{\partial F'(t)} = \frac{\partial G}{\partial F(t)} \tag{26}$$

Since F' (t) is not written explicitely in G

$$\frac{\partial G}{\partial F'(t)} = 0 \tag{27}$$

Then

$$\frac{\partial G}{\partial F(t)} = 0 \tag{28}$$

and

$$\frac{c\sin\varphi}{[cF(t) + \cos\varphi]^2 + \sin^2\varphi} = 0 \tag{29}$$

Because $c \neq 0, \sin\varphi = 0$ which leaves F (t) undetermined. Hence, there appears to be no general law of coupling which will give the desired result. However, Boulet has shown (4) that linear variations of G (t) and H (t) will give a phase curve which closely approximates the ideal curve.

V DISCUSSION OF SITE ERRORS

(a) Constant Phase Front Surface

The errors encountered in Direction Finders can be classified in general as follows:

(1) Propagation Errors

The greatest source of this type of error is the nonuniform nature of the ionosphere which gives rise to a lateral deviation of the main or desired ray. For the most part these errors are random and may vary rapidly (within the minute) or slowly (within the hour). The approach to the elimination of errors due to these variations seems to be the application of mathematics of probability. Spot readings are taken over a period of time and the reading of greatest probability calculated. The ionosphere can also be responsible for the arrival of several rays of slightly different azimuth. Minimizing of these errors also follows statistical lines.

(2) Instrumentation Errors

These are errors that occur in the equipment and as a result of the type of equipment used. Polarization errors, that is errors due to random polarization of the incoming signal, are in general classified under this heading since their effect can be virtually eliminated by proper antenna arrays. Errors due to observation by the operator can also fall under this heading.

(3) Site Errors

These are errors due to reradiation and reflection from objects in the localeof the direction finder. Scattering and reradiation by ionic clouds are classified here since their effect is the same as that for near-by objects. This section is mainly interested in the subject of site errors since their effect can be materially reduced by the choice of the proper type of direction finder.

When examining the field due to a single incoming wave, a considerable distance from its source, it will be found that the surface defining a constant phase is a plane perpendicular to the direction of travel. Any direction finder that operates on the phase front principle, (such as the Doppler) will indicate an errorless bearing in the direction from which the wave is arriving. However, if there is present another signal of appreciable amplitude and of arbitrary phase and direction of arrival, the phase front indicated above will no longer be plane. The effect of the interfering signal is to corrugate the surface in a manner depending upon the relative characteristics of the two signals. When the resultant voltage due to the space addition of the two signals is examined, it is found to describe a standing wave pattern with points of minimum and maximum amplitude spaced at uniform intervals. The phase of the resultant voltage also follows definite space patterns and, to get a line of constant phase, it is necessary to set the expression for phase at any point equal to a constant and plot the resultant curve. This procedure has been outlined in a previous technical report [6] issued by this laboratory and only the resultant expression is given here:

$$y = -\frac{\lambda}{2\pi \sin\alpha} \text{ arc tan } \left[\frac{1+k}{1-k} \tan\left(\frac{2\pi x}{\lambda} \cos\alpha\right)\right] - \frac{n\lambda}{\sin\alpha} \qquad (30)$$

where $n = (0, 1, 2, 3 \ldots\ldots\ldots -1, -2, -3 \ldots\ldots)$

k = ratio of amplitude of signal #2 to signal #1

and 2α = azimuthal separation of the two rays.

A plot of constant phase surfaces is shown on plate (5), this plate being a reproduction of plate #1 in the above mentioned Summary Technical Report.

(b) Similarity of all Small Aperture Systems

If a direction finder with small aperture, that is small dimensions of antenna array, were operating at different points along the constant phase front surface, it is clear that the direction of arrival indicated would be a perpendicular to the tangent of the phase front curve at the specified point. Since this phase front curve is not a straight line, different points would indicate different directions of arrival. It will be seen in a later section that maximum error in bearing will occur at points of maximum and minimum amplitude of standing wave pattern and that at one point between a maximum and minimum there will be indicated the true bearing. From phase front considerations, it can be concluded that any direction finder operating on phase front principle and with aperture small in wave length will be subject to the same site errors. It might seem that a direction finder using an antenna system with a very narrow beam might be an exception to this statement if the antenna could be made to have small aperture. However, it has been shown* that such antenna systems are not practical, because of extremely low sensitivity. Thus to get the desired sensitivity, it must be made large in wavelengths and falls in the group of wide aperture systems.

* "Small High Gain Arrays for Direction Finding", N. Yaru, Technical Report No.6, Univ. of Ill. Direction Finding Research Laboratory, September 1, 1948.

Now if the aperture of the system is made larger such that the antenna operates over a greater distance on the phase front curve, the maximum error variations would be less. This is somewhat analogous to the length of wheelbase on an automobile. On a road of given roughness, the longer the wheelbase, the smoother the ride. The Adcock System does not lend itself to wide aperture operation due to ambiguities which arise. However, the Doppler System is theoretically not restricted on this account and can be increased in aperture to the limit of physical structure and electronic equipment techniques. This tends to put it in the class of wide aperture systems of which there are several other examples. The site errors of the Doppler System under different apertures will be given in a later section.

When increasing the aperture of the Doppler, from a practical viewpoint, the two problems encountered when fixed antennas are used are: where to sample the field, and how to detect the resultant phase variation. Consider the phase front diagram (Plate 5) and notice that with a Doppler System of only four antennas, properly spaced, it is possible to get no phase variation whatsoever as the receiver is switched from antenna to antenna. That is, locate the antennas at points of equal phase. This is an extreme example, but it is evident that to get a phase variation that is at all close to the desired, the number of antennas must be sufficient to prevent ambiguities and give a phase pattern that is geometrically consistent. Because of limitations on the phase discriminator employed, the steps between antennas must not be greater than approximately 90^{o}. This requires that the number of antenna be proportional to the aperture when the aperture is very large. When using small apertures, the number of antennas must be enough to cause the fundamental of the step phase variation to follow the angle of arrival of incoming signal. The least number of antennas that will give a shifting phase variation is three but aperture requirements set the lower limit at around 6 or 8 antennas.

VI ANALYSIS OF EFFECT OF AN INTERFERING WAVE

(a) General Mathematical Discussion

When one or more interfering signals arrive with random direction and time phase with respect to the desired signal, the effect upon the Doppler Direction Finder indication is to change the effective position of the desired phase variation and thus bring about a bearing error. For wide aperture Doppler Systems, the larger in amplitude of the two signals will make by far the largest contribution to the resultant phase variation. The stronger signal will "take over" the bearing in a manner analogous to the "capture effect" shown by the stronger signal in a wide-band frequency modulation system.

The case which considers a single interfering signal (of the same frequency) is treated mathematically in this report. A general formula has been developed to give the bearing error ($\Delta\rho$). Fig. (1a) Plate 6 gives the space diagram illustrating the arrival of the primary and interfering waves. The ratio of the interfering signal magnitude to that of the primary signal is designated by α , θ is the arbitrary angle of arrival of the interfering signal with respect to the other, and γ is the time phase difference between the interfering and primary signals.

For the primary wave

$$e_1 = \cos(\omega t + \beta \cos\omega_1 t) \tag{31}$$

and for the interfering wave

$$e_2 = \alpha \cos[\omega t + \beta \cos(\omega_1 t - \theta) + \gamma] \tag{32}$$

where ω = frequency of the carrier

$\beta = \frac{2\pi a}{\lambda}$ = electrical aperture of the system

ω_1 = rotation or switching rate

a = radius of circle.

If e_1 and e_2 are represented as vectors they can be combined as shown in Fig. (16) of Plate 6. The magnitude of the vector, R, is of little importance as the Doppler System operates purely on a phase comparison basis. In order to determine the change in phase, the angle Δe must be determined:

$$\Delta e = \arc \tan \left(\frac{\alpha \sin\varphi}{1 + \alpha \cos\varphi} \right) \tag{33}$$

where

$$\varphi = [\omega t + \beta \cos(\omega_1 t - \theta) + \gamma] - [\omega t + \beta \cos\omega_1 t]$$

$$= \beta[\cos(\omega_1 t - \theta) - \cos \omega_1 t] + \gamma \tag{34}$$

The complete expression for the vector resultant of e_1 and e_2 is :

$$R \cos[\omega t + \beta \cos\omega_1 t + \arc \tan \left(\frac{\alpha \sin\varphi}{1 + \alpha \cos\varphi} \right)] \tag{35}$$

Let ψ be the phase angle of this vector:

$$\psi = \omega t + \beta \cos\omega_1 t + \arc \tan \left(\frac{\alpha \sin\varphi}{1 + \alpha \cos\varphi} \right) \tag{36}$$

Without any interfering signal, the phase would be:

$$\psi = \omega t + \beta \cos \omega_1 t \tag{37}$$

Hence, the extra arc tan term must be an error term:

$$\Delta e = \text{error term} = \arc \tan \left(\frac{\alpha \sin \varphi}{1 + \alpha \cos\varphi} \right) \tag{38}$$

This Δe term will contain some constant terms independent of $\omega_1 t$ (which will cause no phase shift of the β cos $\omega_1 t$ term), and some cos $\omega_1 t$ terms as well as harmonics of these terms. The content of Δe can be resolved by expanding it in an infinite series as follows (see Appendix I for mathematical derivation).

$$\Delta e = \text{arc tan}\left(\frac{\alpha \sin\varphi}{1 + \alpha \cos\varphi}\right) \tag{39}$$

$$= \alpha \sin\varphi - \frac{\alpha^2}{2} \sin 2\varphi + \frac{\alpha^3}{3} \sin 3\varphi - \dots + \dots \tag{40}$$

The expression for ϕ can be simplified to some extent..

Refering to Fig. (1c)

$$\varphi = \beta[\cos(\omega_1 t - \theta) - \cos\omega_1 t] + \gamma \tag{34}$$

Combining $\beta \cos(\omega_1 t - \theta)$ and $-\beta \cos\omega_1 t$ as vectors gives a vector X

$$X = \sqrt{2 - 2\cos\theta}\,\cos\left(\omega_1 t - \frac{\theta}{2} - \frac{\pi}{2}\right) \tag{41}$$

$$= \sqrt{2 - 2\cos\theta}\,\sin\left(\omega_1 t - \frac{\theta}{2}\right) \tag{42}$$

Let $\beta\sqrt{2 - 2\cos\theta} = g$ and $\left(\omega_1 t - \frac{\theta}{2}\right) = r$

then $$\varphi = g \sin r + \gamma \tag{43}$$

$$\Delta e = \text{arc tan}\left[\frac{\alpha \sin(g \sin r + \gamma)}{1 + \alpha\cos(g \sin r + \gamma)}\right] \tag{44}$$

$$= \alpha \sin(g \sin r + \gamma) - \frac{\alpha^2}{2} \sin(2g \sin r + 2\gamma) + \dots \tag{45}$$

By using the trignometric expansion

$$\sin(g \sin r + \gamma) = \sin(g \sin r)\cos\gamma + \cos(g \sin r)\sin\gamma \tag{46}$$

Δe becomes $\alpha \sin(g \sin r)\cos\gamma + \alpha\cos(g \sin r)\sin\gamma$

$$- \frac{\alpha^2}{2}\sin(2g \sin r)\cos 2\gamma - \frac{\alpha^2}{2}\cos(2g \sin r)\sin 2\gamma + \dots \tag{47}$$

It is known that

$$\cos(g \sin r) = J_0(g) + 2\sum_{n=1}^{\infty} J_{2n}(g)\cos(2nr) \tag{48}$$

$$\sin(g \sin r) = 2\sum_{n=0}^{\infty} J_{2n+1}(g)\sin[(2n+1)r]$$

There is need to consider only the terms involving $(\omega_1 t - \frac{\theta}{2})$ as these are the only terms which can contribute to a phase shift of the original $\beta \cos \omega_1 t$ term. The $[2(\omega_1 t - \frac{\theta}{2})]$ and higher frequency terms are generally removed after detection by a band-pass filter, to facilitate phase comparison. Let the portion of Δe involving only $(\omega_1 t - \frac{\theta}{2})$ terms be called $\Delta e'$

The expansions of cos $(g \sin r)$ and sin $(g \sin r)$ give $(\omega_1 t - \frac{\theta}{2})$ terms only for $n = 0$

$$\begin{aligned}\Delta e' &= \alpha[2J_1(g) \sin r] \cos\gamma + 0 \\ &- \frac{\alpha^2}{2}[2J_1(g) \sin r] \cos 2\gamma + 0 \\ &+ \frac{\alpha^3}{3}[\ldots\ldots\ldots] + 0\end{aligned} \tag{49}$$

$$\begin{aligned}\Delta e' &= 2\alpha \cos\gamma \, J_1(\beta\sqrt{2 - 2\cos\theta}) \sin(\omega_1 t - \frac{\theta}{2}) \\ &- \alpha^2 \cos 2\gamma \, J_1(2\beta\sqrt{2 - 2\cos\theta}) \sin(\omega_1 t - \frac{\theta}{2}) \\ &+ \ldots\ldots\ldots\ldots\ldots\ldots\end{aligned} \tag{50}$$

$$\Delta e' = [\sum_{n=1}^{\infty} \frac{2\alpha^n}{n} (-1)^{n+1} \cos n\gamma \, J_1(n\beta\sqrt{2 - 2\cos\theta})] \sin(\omega_1 t - \frac{\theta}{2}) \tag{51}$$

Let the infinite series which is the coefficient of $\sin(\omega_1 t - \frac{\theta}{2})$ be called D. Then

$$\Delta e = D \sin(\omega_1 t - \frac{\theta}{2}) \tag{52}$$

D converges sufficiently rapid . that only four terms ($\alpha, \alpha^2, \alpha^3, \alpha^4,$) usually need be calculated to give three or four place accuracy for D.

As bearing is given by a shift in the phase of the original $\beta \cos\omega_1 t$ term, then bearing error ($\Delta\rho$) will result from any undesired shift in the phase caused by adding the $D \sin(\omega_1 t - \frac{\theta}{2})$ term. The magnitude of this additional shift can be determined by refering to Fig. (1d). Let y be the resultant voltage of combining $\beta \cos \omega_1 t$ and $D \sin(\omega_1 t - \frac{\theta}{2})$

$$y = \beta \cos \omega_1 t + D \sin(\omega_1 t - \frac{\theta}{2}) \tag{53}$$

$$= E \cos(\omega_1 t - \Delta\rho) \tag{54}$$

$$\text{Bearing error} = \Delta\rho = \text{arc tan} \left[\frac{D \cos(\frac{\theta}{2})}{\beta - D \sin(\frac{\theta}{2})} \right] \tag{55}$$

It should be emphasized that this bearing error assumes a linear phase detector and no instrumentation error. In practice a phase detector is linear for small phase deviations, but approaches a sinusoidal response for larger deviations.

(b) Figure of Merit for Doppler Systems

To give some idea of the amount of error encountered in the Doppler System under different conditions of the interfering wave, the expression (55) was plotted. Three different apertures were chosen with the express purpose of showing the effect of increasing the aperture and consequent reduction of site errors. In order to compare the relative error of the Doppler and Adcock Systems, the expression (23) of Appendix II was plotted under similar circumstances. (23) is the bearing

(5)

error for an Adcock System whose aperture is very small in wavelengths."

The curves are divided into the following groups:

1. Plates 7, 8, 9,——Adcock——Small aperture
2. Plates 10, 11, 12——Doppler—— $\lambda/8$ Aperture
3. Plates 13, 14, 15——Doppler—— λ Aperture
4. Plates 16, 17, 18——Doppler——5λ Aperture

The word Aperture above, as throughout the report, refers to the electrical radius of the system.

These groups are further divided into different ratios of signal magnitudes, namely α = .2, .5, and .8. Different values of θ, space separation of the waves are plotted against γ, the time phase difference.

A general observation of the curves will reveal the following facts:

1. Maximum error occurs at $\gamma = 0, 180°$
2. At two points the bearing error is zero as was pointed out earlier in the report.
3. As the aperture of the Doppler increases the maximum error decreases.

Comparing the curves for Adcock and Doppler ($\beta = 45°$) it will be noted that there is a marked similarity. Since an aperture of 45° is reasonably small, it is just what is expected when the phase front curve is considered. It is interesting to let the Doppler Aperture approach zero as a limit and then compare the resultant mathematical expression with that for the error of the Adcock. From Appendix II we extract (31) which is the bearing error of the Doppler as $\beta \rightarrow 0$.

$$\Delta\rho = \tfrac{1}{2} \arctan \left[\frac{\alpha \sin\theta (\cos\gamma + \alpha)(1 + \alpha\cos\gamma + \alpha^2\cos\theta + \alpha\cos\theta\cos\gamma)}{1 + 2\alpha\cos\gamma(1 + \cos\theta) + 2\alpha^2\cos\theta(2 + \cos^2\gamma\cos\theta) + 2\alpha^3\cos\gamma(2\cos^2\theta + \cos\theta - 1) + \alpha^4(2\cos^2\theta - 1)} \right] \qquad (56)$$

And from the same appendix the expression for the Adcock System (23),

$$\theta_{ne} = \tfrac{1}{2} \text{ arc tan } \left[\frac{2\alpha \sin\theta (\cos\gamma + \alpha \cos\theta)}{1 + 2\alpha \cos\theta \cos\gamma + \alpha^2 (\cos^2\theta - \sin^2\theta)} \right] \tag{57}$$

When $\gamma = 0^\circ$ or 360° which is the case for the origin of the phase front curve of Plate (5) the two expressions (56) and (57) become equal, namely,

$$\theta_{ne} = \Delta\rho = \tfrac{1}{2} \text{ arc tan } \left(\frac{\alpha \sin\theta + \alpha^2 \sin\theta \cos\theta}{1 + 2\alpha \cos\theta + \alpha^2 \cos 2\theta} \right) \tag{58}$$

If γ is placed equal to 180° there results another set of equivalent equal formulas.

However, if γ is set equal to any other angle, the two bearing error formulas are not equal. In the derivation of the Adcock formula attention was paid to the magnitude of the interference pattern as well as the phase. At the 0°, 180°, and 360° the magnitude is symetrically disposed and causes no blur in Adcock reading, while at other points the blur causes the minimum reading to be shifted. The difference in actual bearing between Adcock and Doppler is very slight and for all practical cases it would seem that the result could be extended generally to include all small aperture systems, the conclusion being that <u>any small aperture system suffers from the same site errors</u>.

When comparing Direction Finding Systems it becomes necessary to formulate some standard by which a quick comparison of site error reduction can be made. A figure of merit (or demerit) used by H. G. Hopkins of the National Physical Laboratory (England) is the following: The maximum error is found for each angle of arrival of the interfering signal as the time phase varies from 0° to 360°. The r.m.s. value of these maximum errors is used as the figure of demerit of the system.

Following this method, (with the aid of Plates 10 through 18), the r.m.s. error angle for the Doppler under different aperture has been calculated and appears in the following concise table:.

APERTURE / RATIO OF SIGNALS	$\beta = 45^\circ$	$\beta = 360^\circ$	$\beta = 1800^\circ$
$\alpha = .2$	7.68°	$.59^\circ$	$.0321^\circ$
$\alpha = .5$	23.8°	1.66°	$.0892^\circ$
$\alpha = .8$	39.7°	3.1°	$.1712^\circ$

TABLE I

The important observation of the above table is the large reduction of error which is obtainable in an aperture of only one wavelength. As was pointed out earlier, the errors for Doppler ($\beta = 45^\circ$) and Adcock were almost identical, so the above table indicates the superiority of wide aperture Doppler over the Adcock in this respect. Theoretically, when the aperture is increased to $\beta = 1800^\circ$ there is a further reduction of error. However, there is an upper limit on the aperture (due to the constant instrumentation error) above which it becomes unprofitable to further increase β. From the table it would appear that this limit lies somewhere between $\beta = 180^\circ$ ($a = \lambda/2$) and $\beta = 360^\circ$ ($a = \lambda$). Most of the site reradiation signals have amplitudes which are much less than 1 ($\alpha << 1$) and this further helps the situation. . .

It is interesting to compare Doppler and Adcock systems under conditions where the interfering wave is arriving from almost the same direction as the desired signal and has comparable magnitude (this is the usual case of sky-wave multipath propagation). Apply the following conditions to expression (57), which is the error for the Adcock with small aperture. Let $\gamma = 180^\circ$ and $\theta \simeq \Delta\theta$. $\gamma = 180^\circ$ is a point of maximum error as can be seen from plates (7 through 18). The result is:

$$\theta_{re} \approx -\tfrac{1}{2} \text{ arc tan } \left(\frac{2\Delta\theta\alpha}{1-\alpha} \right) \tag{59}$$

Apply the conditions to expression (55), the error for the Doppler. Make the assumption that the aperture of the Doppler is very small and the result is:

$$\Delta\rho \approx -\text{ arc tan } \left(\frac{\Delta\theta\alpha}{1-\alpha} \right) \tag{60}$$

Comparison of (59) and (60) shows that for small apertures the errors of the two systems have the same order of magnitude. In particular for nearly equal to unity the errors approach 90° in both cases. It is concluded that when the apertures of the two systems are small, the bearing error for the case of multiple path transmission is the same. However, when the aperture of the Doppler is increased the expression for error must be approached along different lines. Consider expression (55),

$$\Delta\rho = \text{ arc tan } \left[\frac{D \cos\left(\frac{\theta}{2}\right)}{\beta - D \sin\left(\frac{\theta}{2}\right)} \right] \tag{55}$$

and let $\theta \simeq \Delta\theta$

$$\Delta\rho \approx \text{ arc tan } \left(\frac{D}{\beta} \right) \tag{61}$$

where $D = \sum_{n=1}^{\infty} \frac{2\alpha^n}{n} (-1)^{n+1} \cos n\gamma \, J_1 (n\beta \sqrt{2 - 2 \cos\Delta\theta})$

when $\gamma = 180^\circ$

$$D = -\sum_{n=1}^{\infty} \frac{2\alpha^n}{n} J_1 (n\beta \sqrt{2 - 2 \cos\Delta\theta});$$

If $\alpha < 1$ the summation will converge to a finite value that is not directly dependent on β. Thus $\Delta\rho$ from (61) can be made to decrease by increasing β. The conclusion is that discrimination against the weakest of two signals arriving from almost the same angle can be made arbitrarly great by increasing the aperture.

VII: CONCLUSION

It is concluded that a Direction Finder operating on the Doppler Principle can offer a very practical and accurate solution to the problem of determining the direction of arrival of wave energy. When the dimensions of the antenna system are made comparable with that of the Adcock, it seems to offer no improvement over the later in the matter of site error reduction. However, the Doppler System lends itself admirably to wide-aperture operation under which conditions it can be made to give marked reduction of bearing error caused by site re-radiation. Of course, as a wide aperture system, its use would be limited to ground installations and shipboard for the medium and high frequencies, with the possibility of aircraft application only at very and ultra-high frequencies.

VIII. APPENDIX I

COMBINATION OF PHASE MODULATED WAVES

A more rigorous analysis of the combination of the vectors

$$e_1 = \cos(\omega t + \beta \cos \omega_1 t)$$
$$e_2 = \alpha \cos[\omega t + \beta \cos(\omega_1 t - \theta) + \gamma]$$

is presented.

$$e_1 = \cos\omega t \cos(\beta \cos \omega_1 t) - \sin \omega t \sin(\beta \cos \omega_1 t)$$
$$e_2 = \alpha \cos\omega t \cos[\beta \cos(\omega_1 t - \theta) + \gamma] - \alpha \sin\omega t[\sin\beta \cos(\omega_1 t-\theta) + \gamma]$$

$$e_1 + e_2 = \cos\omega t \{ \cos(\beta \cos\omega_1 t) + \alpha \cos[\beta \cos(\omega_1 t - \theta) + \gamma]\}$$
$$- \sin \omega t\{ \sin(\beta \cos\omega_1 t) + \alpha \sin[\beta \cos(\omega_1 t - \theta) + \gamma]\}$$

In combining two expressions

$$E = A \cos \omega t + B \cos \omega t$$
$$= \sqrt{A^2 + B^2} \cos(\omega t + \psi)$$

where

$$\psi = \arctan\left(\frac{B}{A}\right)$$

We are mainly interested in the phase of the resultant expression (ψ).

$$\psi = \omega t + \arctan\left\{\frac{\sin(\beta \cos\omega_1 t) + \alpha \sin[\beta \cos(\omega_1 t - \theta) + \gamma]}{-\cos(\beta \cos\omega_1 t) - \alpha \cos[\beta \cos(\omega_1 t - \theta) + \gamma]}\right\}$$

which can be written

$$\psi = \omega t + \arctan\left\{\frac{\begin{aligned}&\sin(\beta \cos\omega_1 t) + \alpha \sin(\beta \cos\omega_1 t) \cos(\beta \cos\omega_1 t) \cos[\beta \cos(\omega_1 t-\theta) + \gamma] \\ &\quad - \alpha \sin(\beta \cos\omega_1 t) \cos(\beta \cos\omega_1 t) \cos[\beta \cos(\omega_1 t-\theta) + \gamma] \\ &\quad + \alpha \sin^2(\beta \cos\omega_1 t) \sin[\beta \cos(\omega_1 t - \theta) + \gamma] \\ &\quad + \alpha \cos^2(\beta \cos\omega_1 t) \sin[\beta \cos(\omega_1 t - \theta) + \gamma]\end{aligned}}{\begin{aligned}&\cos(\beta \cos\omega_1 t) + \alpha \sin(\beta \cos \omega_1 t) \cos(\beta \cos\omega_1 t) \sin[\beta \cos(\omega_1 t-\theta) + \gamma] \\ &\quad - \alpha \sin(\beta \cos\omega_1 t) \cos(\beta \cos\omega_1 t) \sin[\beta \cos(\omega_1 t-\theta) + \gamma] \\ &\quad + \alpha \cos^2(\beta \cos\omega_1 t) \cos[\beta \cos(\omega_1 t - \theta) + \gamma] \\ &\quad + \alpha \sin^2(\beta \cos\omega_1 t) \cos[\beta \cos(\omega_1 t - \theta) + \gamma]\end{aligned}}\right\}$$

Divide the numerator and denominator by:

$$\cos(\beta \cos\omega_1 t)\{1 + \alpha \cos(\beta \cos\omega_1 t) \cos[\beta \cos(\omega_1 t - \theta) + \gamma]$$

$$+ \alpha \sin(\beta \cos\omega_1 t) \sin[\beta \cos(\omega_1 t - \theta) + \gamma]\}$$

Then

$$\psi = \omega t + \text{arc tan} \left\{ \frac{\tan(\beta \cos\omega_1 t) + \dfrac{\alpha \sin[\beta \cos(\omega_1 t - \theta) + \gamma - \beta \cos\omega_1 t]}{1 + \alpha \cos[\beta \cos(\omega_1 t - \theta) + \gamma - \beta \cos\omega_1 t]}}{1 - \dfrac{\tan(\beta \cos\omega_1 t)\, \alpha \sin[\beta \cos(\omega_1 t - \theta) + \gamma - \beta \cos\omega_1 t]}{1 + \alpha \cos[\beta \cos(\omega_1 t - \theta) + \gamma - \beta \cos\omega_1 t]}} \right\}$$

But

$$\text{arc tan} \left(\frac{x + y}{1 - xy}\right) = \text{arc tan}\, x + \text{arc tan}\, y$$

Let $y = \dfrac{\alpha \sin[\beta \cos(\omega_1 t - \theta) + \gamma - \beta \cos\omega_1 t]}{1 + \alpha \cos[\beta \cos(\omega_1 t - \theta) + \gamma - \beta \cos\omega_1 t]}$,

$x = \tan(\beta \cos\omega_1 t)$,

and $\varphi = \beta \cos(\omega_1 t - \theta) + \gamma - \beta \cos\omega_1 t$

Then $\psi = \omega t + \text{arc tan}(\tan \beta \cos\omega_1 t) + \text{arc tan}\, y$

$$\psi = \omega t + \beta \cos\omega_1 t + \text{arc tan}\left(\frac{\alpha \sin \varphi}{1 + \alpha \cos\varphi}\right)$$

From identities it is known that

$$\arctan x = \frac{1}{2i} \ln\left(\frac{1 + ix}{1 - ix}\right)$$

Applying this:

$$\psi = \omega t + \beta \cos\omega_1 t + \frac{1}{2i} \ln\left(\frac{1 + \dfrac{i\,\alpha \sin\varphi}{1 + \alpha\cos\varphi}}{1 - \dfrac{i\alpha \sin\varphi}{1 + \alpha\cos\varphi}}\right)$$

$$\psi = \omega t + \beta \cos\omega_1 t + \frac{1}{2i} \ln\left(\frac{1 + \alpha\cos\varphi + i\alpha\sin\varphi}{1 + \alpha\cos\varphi - i\alpha\sin\varphi}\right)$$

$$\psi = \omega t + \beta \cos\omega_1 t + \frac{1}{2i} \left[\ln(1 + \alpha e^{i\varphi}) - \ln(1 + \alpha e^{-i\varphi})\right]$$

No value of ϕ can make $e^{i\varphi}$ greater than ± 1.

Also α was chosen $-1 \leq \alpha \leq 1$. Under these conditions $\alpha e^{i\varphi}$ cannot be greater than 1 or less than -1.

$$\psi = \omega t + \beta \cos\omega_1 t + \frac{1}{2i} \Big[\big(\alpha e^{i\varphi} - \tfrac{1}{2}\alpha^2 e^{2i\varphi} + \tfrac{1}{3}\alpha^3 e^{3i\varphi} - + \ldots\big)$$

$$- \big(\alpha e^{-\varphi} - \tfrac{1}{2}\alpha^2 e^{-2i\varphi} + \tfrac{1}{3}\alpha^3 e^{-3i\varphi} - + \ldots\big)\Big]$$

$$\psi = \omega t + \beta\cos\omega_1 t + \alpha\sin\varphi - \frac{\alpha^2}{2}\sin 2\varphi + \frac{\alpha^3}{3}\sin 3\varphi - \frac{\alpha^4}{4}\sin 4\varphi + \ldots$$

It will be noticed that if $\alpha = 0$ (no interfering wave), the first two turns will give the bearing. However, any interfering signal will cause the infinite series to have a value and thus produce error. This error is the deviation of phase from the desired as the single element progresses around the circle. Analyzing any one of the infinite series terms

$$\alpha \sin \varphi$$

$$\underset{\alpha}{\underset{\downarrow}{\alpha}} \sin\overbrace{[(\underset{\beta}{\underset{\downarrow}{\beta}} \cos(\underset{\omega_1}{\underset{\downarrow}{\omega_1}} t - \underset{\theta}{\underset{\downarrow}{\theta}}) + \underset{\gamma}{\underset{\downarrow}{\gamma}} - \beta\cos\omega_1 t]}$$

it is found that the error is due to (α, β, ω, Θ, and γ). The factors which carry the greatest weight are α, β, and γ. The final bearing error will be found by extracting all terms of the expansion containing $\omega_1 t$ and it will be found (Appendix II) that the reduction of error is mainly dependent on β (the aperture of the system).

APPENDIX II

BEARING ERROR (DUE TO SITE RERADIATION) OF THE DOPPLER SYSTEM AS THE APERTURE APPROACHES ZERO ($\beta \to 0$), AND COMPARISON WITH ADCOCK ERROR.

As previously shown, the general formula for the bearing error of a Doppler system is given by:

$$\Delta\rho = \arc\tan\left[\frac{D\cos\left(\frac{\theta}{2}\right)}{\beta - D\sin\left(\frac{\theta}{2}\right)}\right] \tag{1}$$

where

$$D = 2\sum_{n=1}^{\infty}\frac{\alpha^n\cos n\gamma\,(-1)^{n+1}}{n} J_1(ng) \tag{2}$$

and

$$J_1(ng) = J_1(n\beta\sqrt{2-2\cos\theta}) \tag{3}$$

For any value of n which can be chosen β can still be made small enough to make valid the following approximation

$$J_1(ng) \approx \frac{ng}{2} \tag{4}$$

This causes D to now become:

$$D_1 = 2\sum_{n=1}^{\infty}\frac{\alpha^n\cos n\gamma(-1)^{n+1}}{n}\left(\frac{ng}{2}\right) \tag{5}$$

$$D_1 = \sum_{n=1}^{\infty}\alpha^n\cos n\gamma\, g\,(-1)^{n+1} \tag{6}$$

$$D_1 = g\,(\alpha\cos\gamma - \alpha^2\cos 2\gamma + \alpha^3\cos 3\gamma + \dots\dots\dots) \tag{7}$$

The series can be synthesised as follows:

It is true that:*

$$\frac{1-\alpha^4}{1-2\alpha^2\cos 2\gamma+\alpha^4} - 1 = 2(\alpha^2\cos 2\gamma + \alpha^4\cos 4\gamma + \alpha^6\cos 6\gamma + \dots\dots)$$

where $\alpha^2 < 1$ (8)

* (Dwights Table of Integrals).

$$1 - \frac{(1-\alpha^4)}{1 - 2\alpha^2 \cos 2\gamma + \alpha^4} = -2(\alpha^2 \cos 2\gamma - \alpha^4 \cos 4\gamma - \alpha^6 \cos 6\gamma \; . \; . \; . \; . \; . \;)$$

$$\text{where } \alpha^2 < 1 \tag{9}$$

$$\frac{1 - \alpha \cos\gamma}{1 - 2\alpha \cos\gamma + \alpha^2} - 1 = \alpha \cos\gamma + \alpha^2 \cos 2\gamma + \alpha^3 \cos 3\gamma + \; . \; . \; . \; . \qquad \text{where } \alpha^2 < 1 \tag{10}$$

Adding the last two identities

$$\left[\frac{-(1-\alpha^4)}{1 + 2\alpha^2 - 4\alpha^2 \cos^2\gamma + \alpha^4} + \frac{1 - \alpha \cos\gamma}{1 + \alpha^2 - 2\alpha \cos\gamma}\right] \tag{11}$$

$$= (\alpha \cos\gamma - \alpha^2 \cos 2\gamma + \alpha^3 \cos 3\gamma \; . \; . \; . \; . \; . \;)$$

Thus:

$$D_1 = g\left(\frac{\alpha^4 - 1}{1 + 2\alpha^2 + \alpha^4 - 4\alpha^2 \cos^2\gamma} + \frac{1 - \alpha \cos\gamma}{1 + \alpha^2 - 2\alpha \cos\gamma}\right) \tag{12}$$

The importance of this equation is that it reduces the previously given bearing error ($\Delta\rho$) expression to an analytic function.

$$\Delta\rho = \text{arc tan} \left[\frac{D_1 \cos\left(\frac{\theta}{2}\right)}{\beta - D_1 \sin\left(\frac{\theta}{2}\right)}\right] \tag{13}$$

To simplify D_1 rewrite it:

$$D_1 = g\left[\frac{\alpha^4 - 1}{(1 + \alpha^2)^2 - 4\alpha^2 \cos^2\gamma} + \frac{(1 - \alpha \cos\gamma)(1 + \alpha^2 + 2\alpha \cos\gamma)}{(1 + \alpha^2 - 2\alpha \cos\gamma)(1 + \alpha^2 + 2\alpha \cos\gamma)}\right] \tag{14}$$

Combining

$$D_1 = g\left[\frac{\alpha^4 + \alpha^2 + \alpha \cos\gamma - 2\alpha^2 \cos^2\gamma - \alpha^3 \cos\gamma}{(1 + \alpha)^2 - 4\alpha^2 \cos^2\gamma}\right] \tag{15}$$

$$D_1 = \alpha g\left[\frac{(1 + \alpha^2 - 2\alpha \cos\gamma)(\alpha + \cos\gamma)}{(1 + \alpha^2 - 2\alpha \cos\gamma)(1 + \alpha^2 + 2\alpha \cos\gamma)}\right] \tag{16}$$

$$= \alpha g\left(\frac{\alpha + \cos\gamma}{1 + \alpha^2 + 2\alpha \cos\gamma}\right) \tag{17}$$

$$= \alpha\beta \sqrt{2 - 2\cos\theta} \left(\frac{\alpha + \cos\gamma}{1 + \alpha^2 + 2\alpha \cos\gamma}\right) \tag{18}$$

$$D_1 = 2\alpha\beta \sin(\frac{\theta}{2}) (\frac{\alpha + \cos\gamma}{1 + \alpha^2 + 2\alpha \cos\gamma}) \tag{19}$$

From the Thesis by R. W. Annis,(5) mentioned in the text, the bearing error for the Adcock System is

$$\theta_{ne} = \frac{\varphi_{n+} + \varphi_{n-}}{2n} \tag{20}$$

If placed in our notation, it would appear

$$\theta_{ne} = \tfrac{1}{2} \arctan [\frac{\alpha \sin(\theta - \gamma)}{1 + \alpha \cos(\theta - \gamma)}] + \tfrac{1}{2} \arctan [\frac{\alpha \sin(\theta + \gamma)}{1 + \alpha \cos(\theta + \gamma)}] \tag{21}$$

which is

$$2\,\theta_{ne} = \arctan\{ \frac{\dfrac{\alpha \sin(\theta - \gamma)}{1 + \alpha \cos(\theta - \gamma)} + \dfrac{\alpha \sin(\theta + \gamma)}{1 + \alpha \cos(\theta + \gamma)}}{1 - \dfrac{\alpha^2 \sin(\theta - \gamma) \sin(\theta + \gamma)}{[1 + \alpha \cos(\theta - \gamma)][1 + \alpha \cos(\theta + \gamma)]}} \} \tag{22}$$

Expanding and reducing:

$$\theta_{ne} = \tfrac{1}{2} \arctan [\frac{2\alpha \sin\theta(\cos\gamma + \alpha \cos\theta)}{1 + 2\alpha \cos\theta \cos\gamma + \alpha^2 (\cos^2\theta - \sin^2\theta)}] \tag{23}$$

Putting the Doppler error expression in this form, the result is:

$$\Delta\rho = \tfrac{1}{2} \arctan [\frac{\beta D_1 \cos(\frac{\theta}{2}) - D_1^2 \tfrac{1}{2} \sin\theta}{\beta^2 - 2\beta D_1 \sin(\frac{\theta}{2}) - D_1^2 \cos\theta}] \tag{24}$$

Now it is necessary to place the expression for D_1 (19) in (24) and compare (24) with (23).

Expanding the numerator of (24)

$$\begin{aligned} D_1\beta \cos(\frac{\theta}{2}) &= \alpha \beta^2 2 \cos(\frac{\theta}{2}) \sin(\frac{\theta}{2}) (\frac{\alpha + \cos\gamma}{1 + \alpha^2 + 2\alpha \cos\gamma}) \\ &= \alpha\beta^2 (\frac{\alpha + \cos\gamma}{1 + \alpha^2 + 2\alpha \cos\gamma}) \sin\theta \end{aligned} \tag{25}$$

$$-D_1^2\ \tfrac{1}{2}\sin\theta = -4\,\alpha^2\beta^2\sin^2\left(\frac{\theta}{2}\right)(\tfrac{1}{2})\sin\theta\left[\frac{\alpha^2+2\alpha\cos\gamma+\cos^2\gamma}{(1+\alpha^2+2\alpha\cos\gamma)^2}\right]$$

$$= -4\alpha^2\beta^2(\tfrac{1}{4})(1-\cos\theta)(\sin\theta)\left[\frac{\alpha^2+2\alpha\cos\gamma+\cos^2\gamma}{(1+\alpha^2+2\alpha\cos\gamma)^2}\right]$$

$$= -\alpha^2\beta^2(\sin\theta-\cos\theta\sin\theta)\left[\frac{\alpha^2+2\alpha\cos\gamma+\cos^2\gamma}{(1+\alpha^2+2\alpha\cos\gamma)^2}\right] \tag{26}$$

Adding $D_1\beta\cos\left(\frac{\theta}{2}\right)$ and $-D_1^2\ \tfrac{1}{2}\sin\theta$

$$\text{Numerator} = \left[\frac{\alpha^2\beta^2}{(1+\alpha^2+2\alpha\cos\gamma)^2}\right]\Big[(\alpha+\cos\gamma)(1+\alpha^2+2\alpha\cos\gamma)\sin\theta - \alpha(\sin\theta-\cos\theta\sin\theta)(\alpha^2+2\alpha\cos\gamma+\cos^2\gamma)\Big]$$

$$=\left[\frac{\alpha^2\beta^2}{(1+\alpha^2+2\alpha\cos\gamma)^2}\right]\left[\sin\theta(\cos\gamma+\alpha)(1+\alpha\cos\gamma)+\alpha\cos\theta\sin\theta(\alpha+\cos\gamma)^2\right]$$

$$=\left[\frac{\alpha\beta^2\sin\theta(\cos\gamma+\alpha)}{(1+\alpha^2+2\alpha\cos\gamma)^2}\right]\left[(1+\alpha\cos\gamma+\alpha^2\cos\theta+\alpha\cos\theta\cos\gamma)\right] \tag{27}$$

Expanding the denominator of (24)

$$-2\beta D_1\sin\left(\frac{\theta}{2}\right) = -2\beta^2\,2\alpha\sin^2\left(\frac{\theta}{2}\right)\left(\frac{\alpha+\cos\gamma}{1+\alpha^2+2\alpha\cos\gamma}\right)$$

$$= -2\alpha\beta^2(1-\cos\theta)\left(\frac{\alpha+\cos\gamma}{1+\alpha^2+2\alpha\cos\gamma}\right) \tag{28}$$

$$-D^2\cos\theta = -4\alpha^2\beta^2\sin^2\left(\frac{\theta}{2}\right)\cos\theta\left(\frac{\alpha+\cos\gamma}{1+\alpha^2+2\alpha\cos\gamma}\right)^2$$

$$= -2\alpha\beta^2(1-\cos\theta)\cos\theta\left(\frac{\alpha+\cos\gamma}{1+\alpha^2+2\alpha\cos\gamma}\right)^2 \tag{29}$$

Combining these two terms

$$\beta^2-2\beta D\sin\left(\frac{\theta}{2}\right)-D^2\cos\theta = \left[\frac{\beta^2}{(1+\alpha^2+2\alpha\cos\gamma)^2}\right]\Big[1+2\alpha\cos\gamma(1+\cos\theta) + 2\alpha^2\cos\theta(2+\cos^2\gamma\cos\theta) + 2\alpha^3\cos\gamma(2\cos^2\theta+\cos\theta-1) + \alpha^4(2\cos^2\theta-1)\Big] \tag{30}$$

This gives as the resultant expression for $\Delta\rho$.

$$\Delta\rho = \tfrac{1}{2}\,\text{arc tan}\left[\frac{\alpha\sin\theta(\cos\gamma+\alpha)(1+\alpha\cos\gamma+\alpha^2\cos\theta+\alpha\cos\theta\cos\gamma)}{1+2\alpha\cos\gamma(1+\cos\theta)+2\alpha^2\cos\theta(2+\cos^2\gamma\cos\theta) + 2\alpha^3\cos\gamma\,(2\cos^2\theta+\cos\theta-1) + \alpha^4\,(2\cos^2\theta-1)}\right] \tag{31}$$

$\Delta\rho$ was placed in the form (31) in order to compare it with equation (23) which comes from the Adcock system. It is possible to express $\Delta\rho$ in a somewhat simpler form by placing the expression for D_1(19) directly in (1).

$$\Delta\rho = \text{arc tan}\left[\frac{2\alpha\beta\sin\left(\frac{\theta}{2}\right)\cos\left(\frac{\theta}{2}\right)\left(\frac{\alpha+\cos\gamma}{1+\alpha^2+2\alpha\cos\gamma}\right)}{\beta-2\alpha\beta\sin^2\left(\frac{\theta}{2}\right)\left(\frac{\alpha+\cos\gamma}{1+\alpha^2+2\alpha\cos\gamma}\right)}\right] \tag{32}$$

$$= \text{arc tan}\left[\frac{\alpha\sin\theta(\alpha+\cos\gamma)}{1+\alpha^2+2\alpha\cos\gamma-2\sin^2\left(\frac{\theta}{2}\right)(\alpha^2+\alpha\cos\theta)}\right]$$

$$\Delta\rho = \text{arc tan}\left[\frac{\alpha\sin\theta(\alpha+\cos\gamma)}{1+\alpha\cos\gamma+\alpha(\alpha+\cos\gamma)\cos\theta}\right] \tag{33}$$

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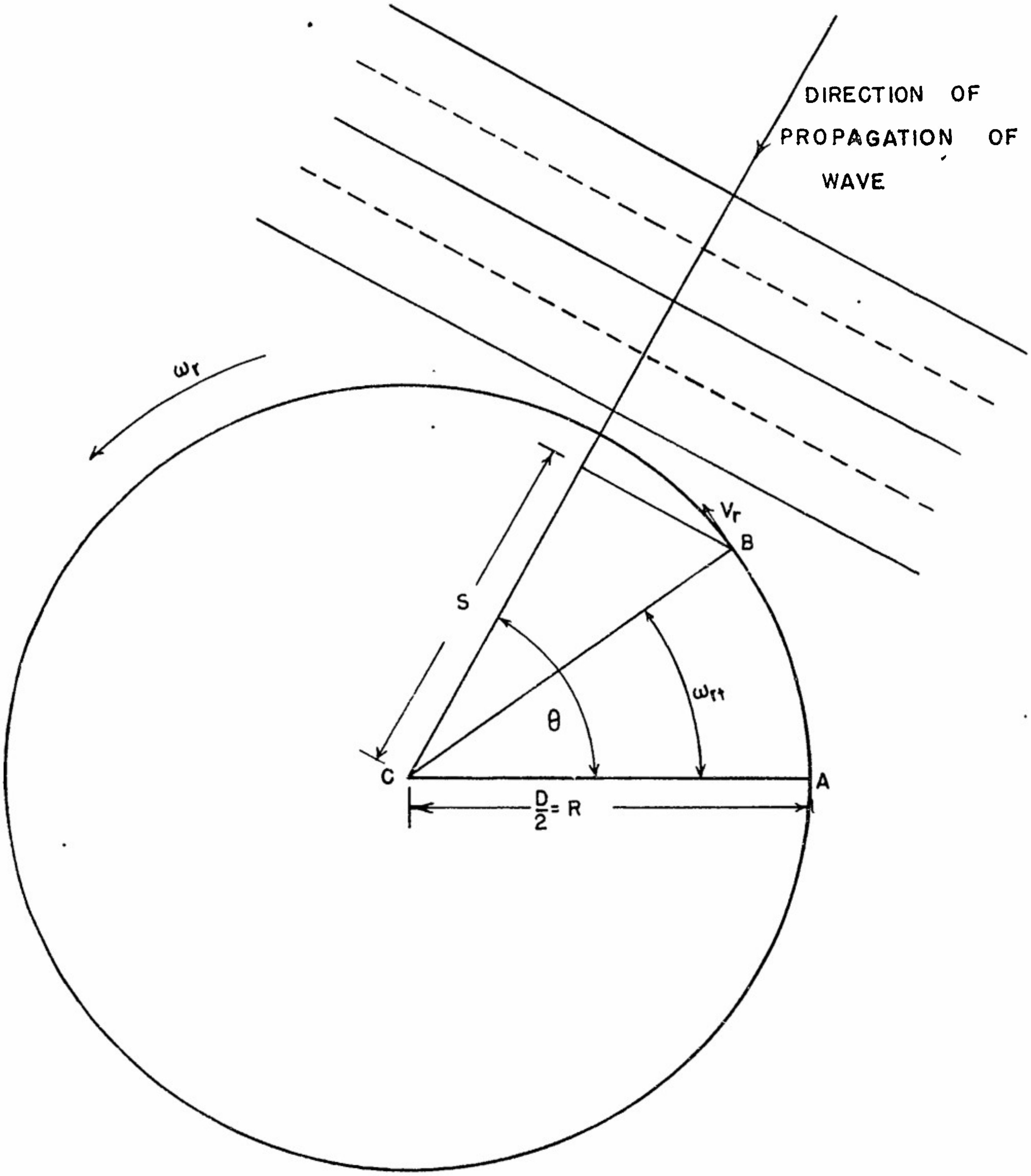


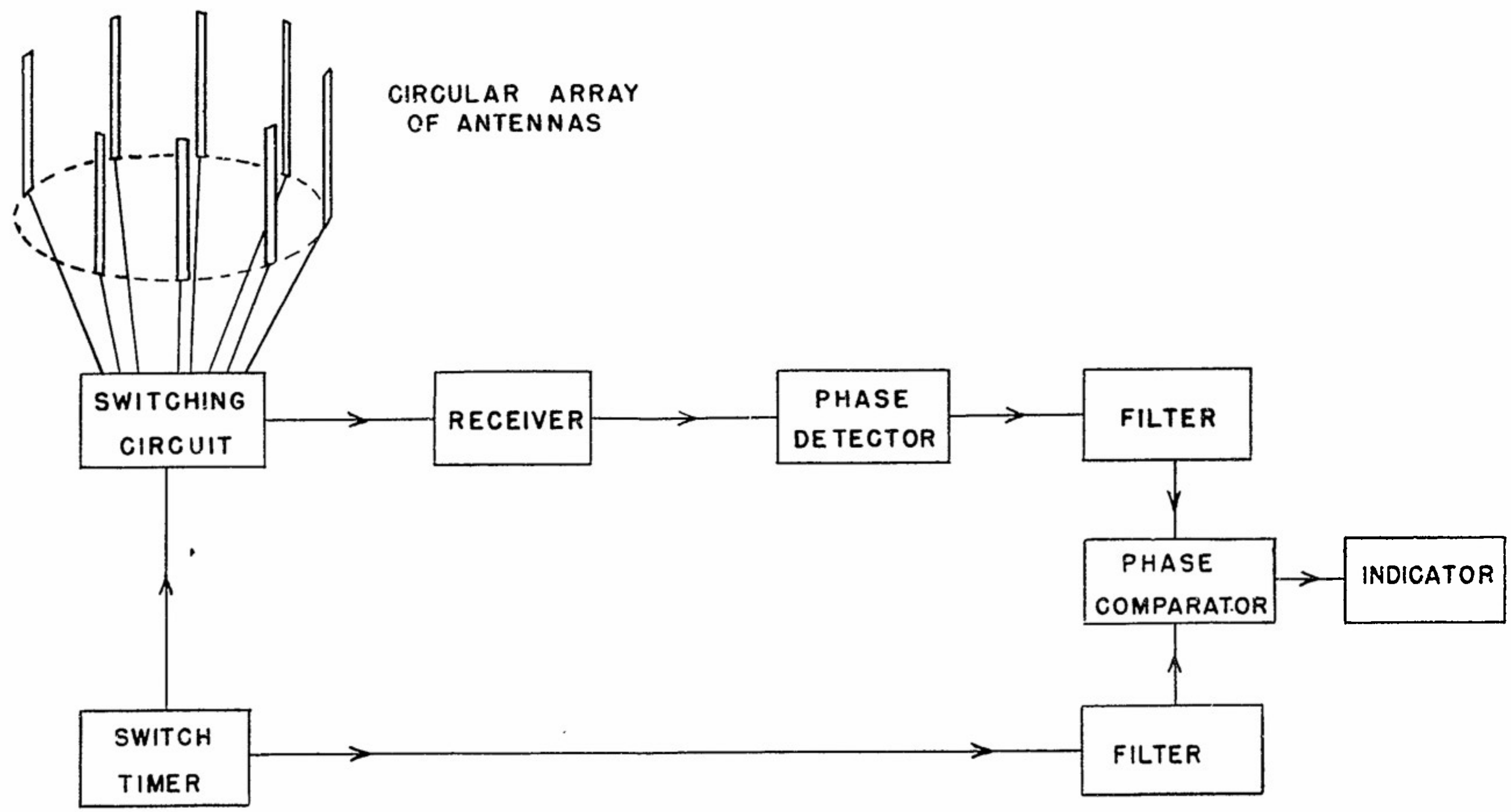
C.A. = Reference Axis

B. = Position of Antenna at time $t = t_1$

THE CASE OF THE SINGLE ROTATING ANTENNA

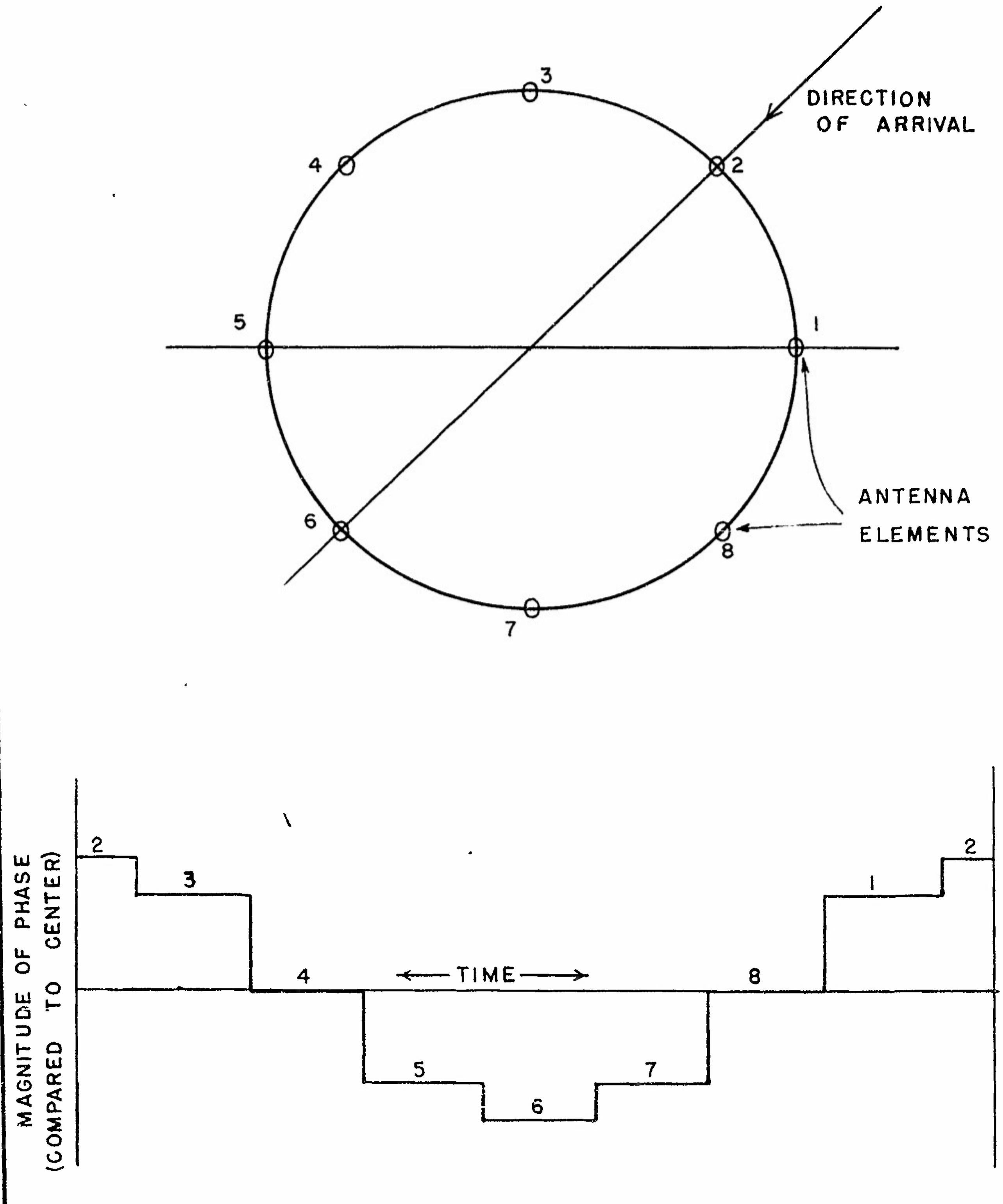
PLATE I

TECH. RPT. 8



BLOCK DIAGRAM OF TYPICAL DOPPLER DIRECTION FINDER

PLATE 2
TECH. RPT. 8



PHASE RELATIONS OF COMMUTATED ANTENNAS

PLATE 3
TECH. RPT. 8

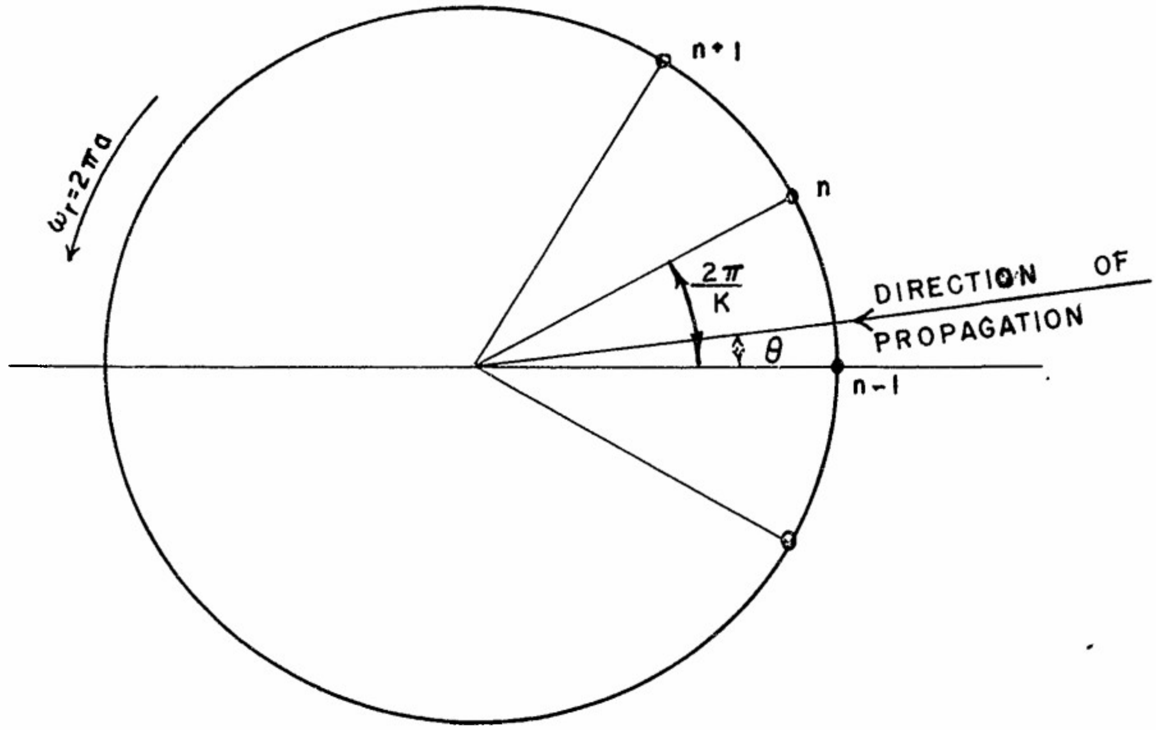


FIG. 1

CIRCULAR ANTENNA ARRAY OF N ELEMENTS

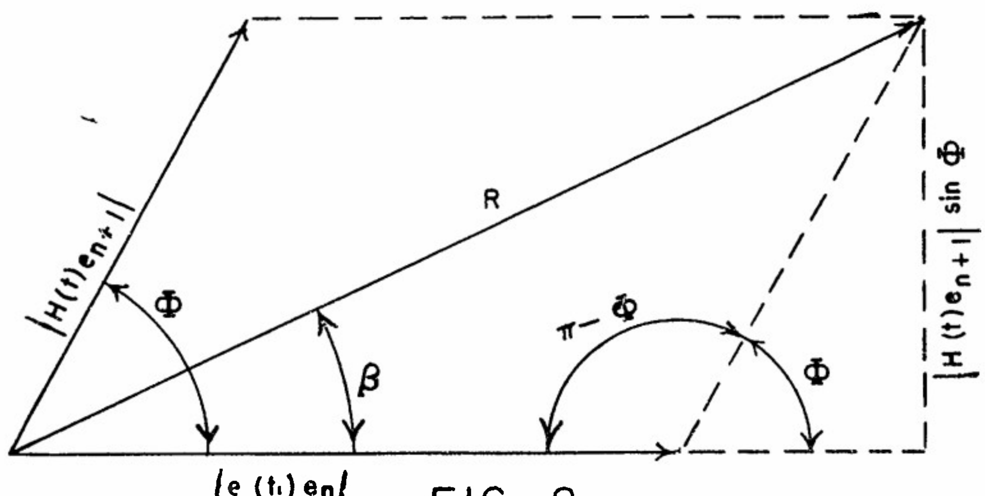


FIG. 2

LAW OF COUPLING—VECTOR DIAGRAM

PLATE 4

TECH. RPT. 8

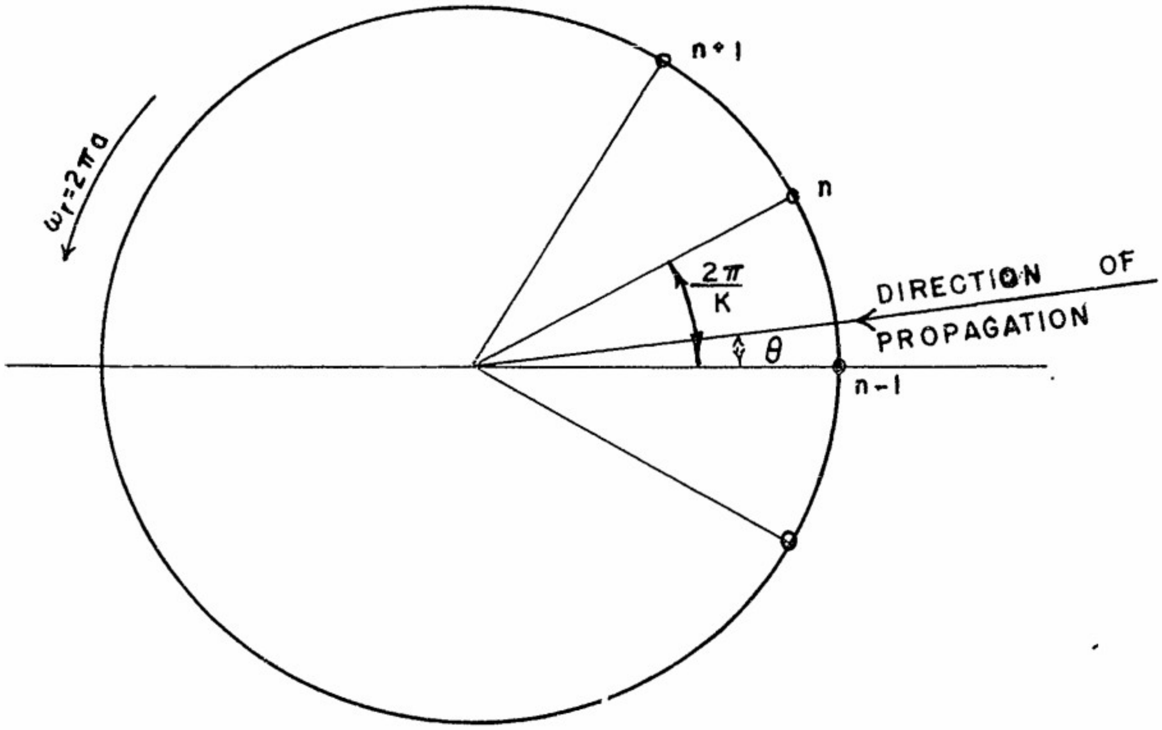


FIG. 1

CIRCULAR ANTENNA ARRAY OF N ELEMENTS

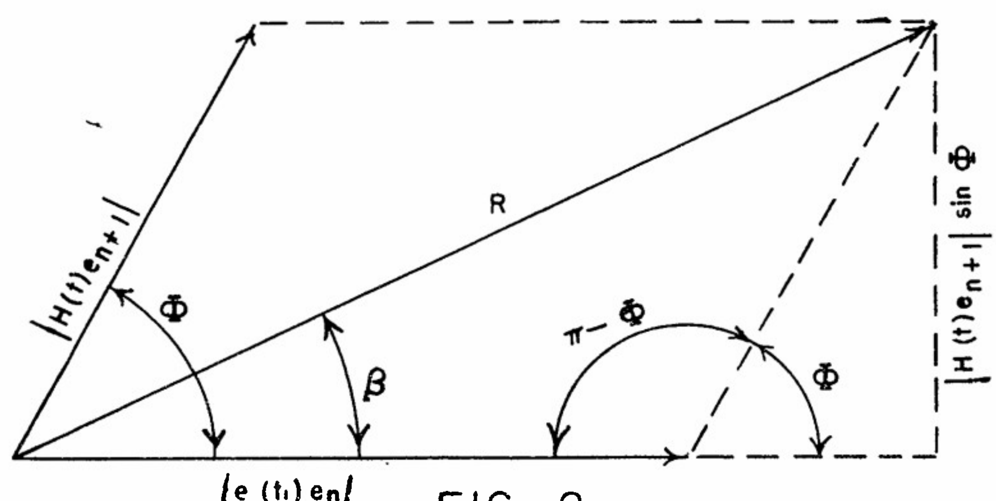


FIG. 2

LAW OF COUPLING — VECTOR DIAGRAM

PLATE 4
TECH. RPT. 8

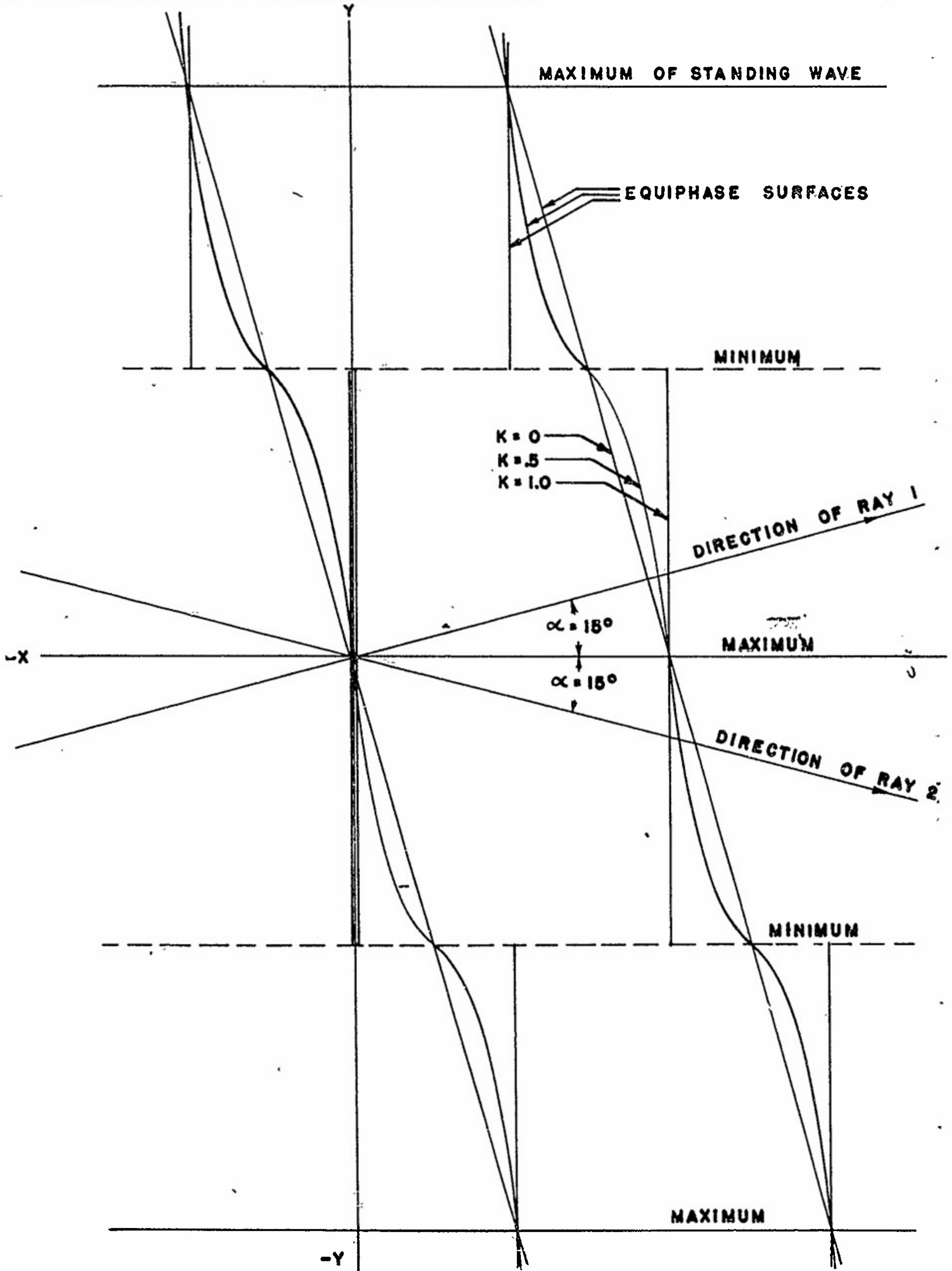


FIG. 1

NOTE: a) $K = \frac{\text{STRENGTH OF RAY 2}}{\text{STRENGTH OF RAY 1}}$

b) ELECTRIC VECTOR ASSUMED PERPENDICULAR TO PAGE

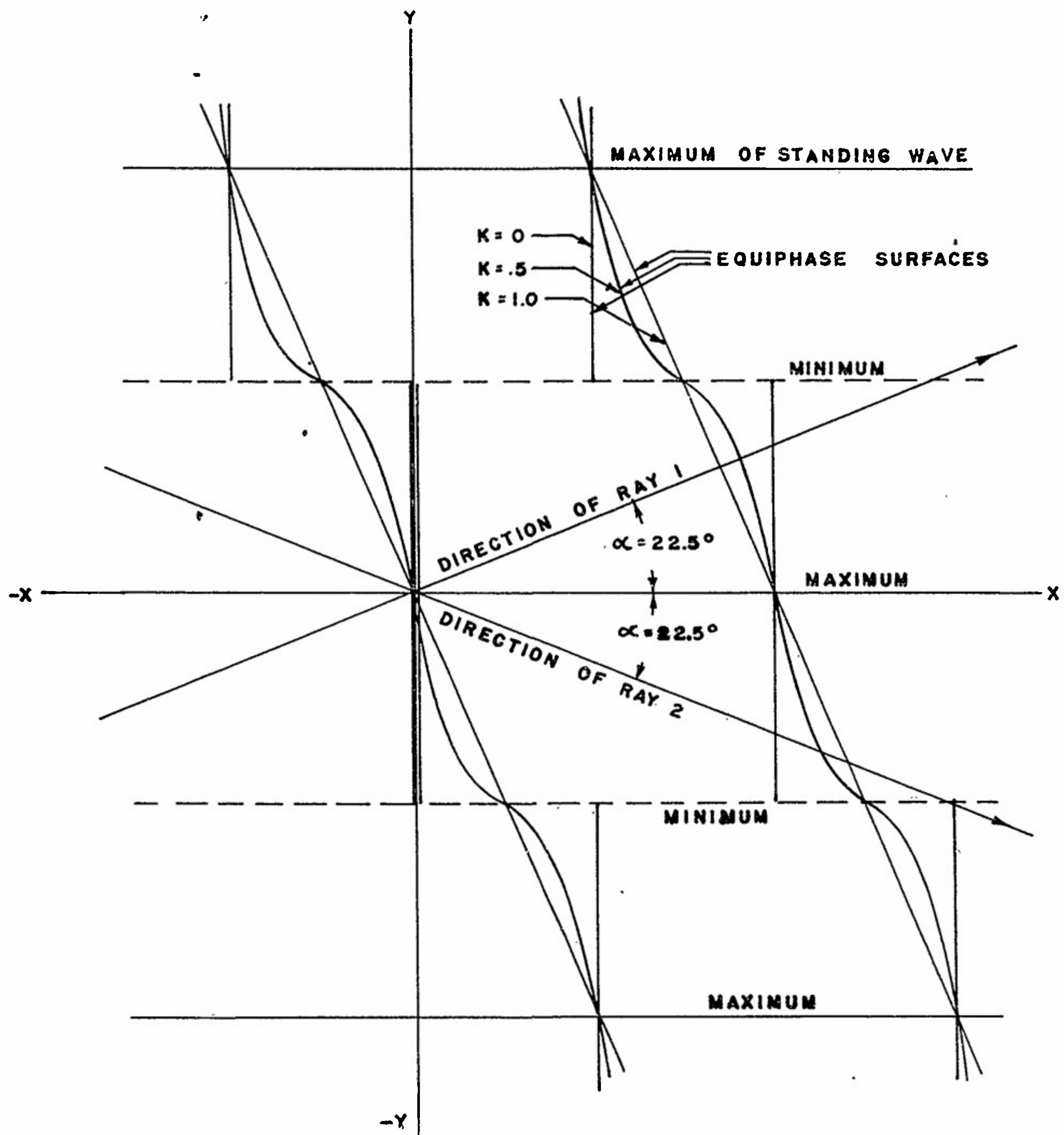


FIG. 2

STANDING WAVE PATTERN IN THE FIELD OF TWO INTERSECTING RAYS

PLATE 5

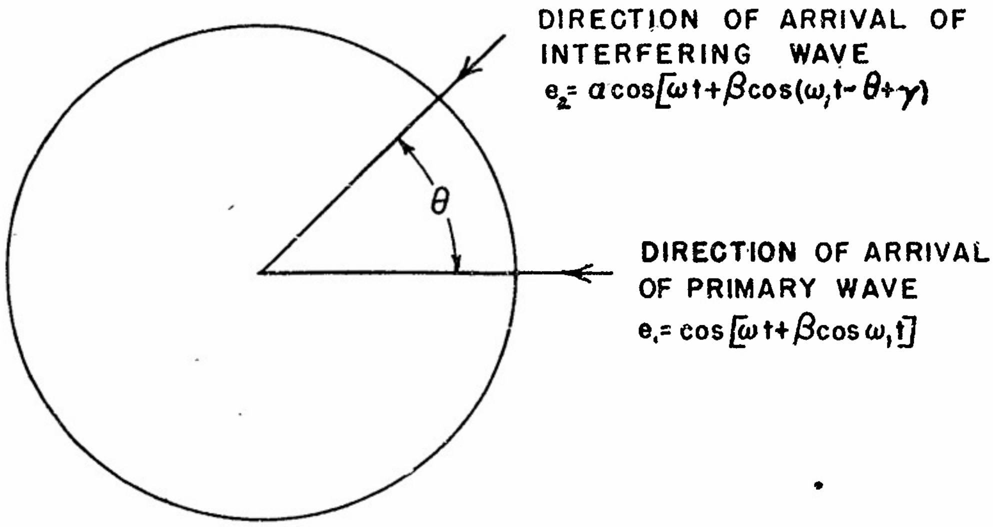
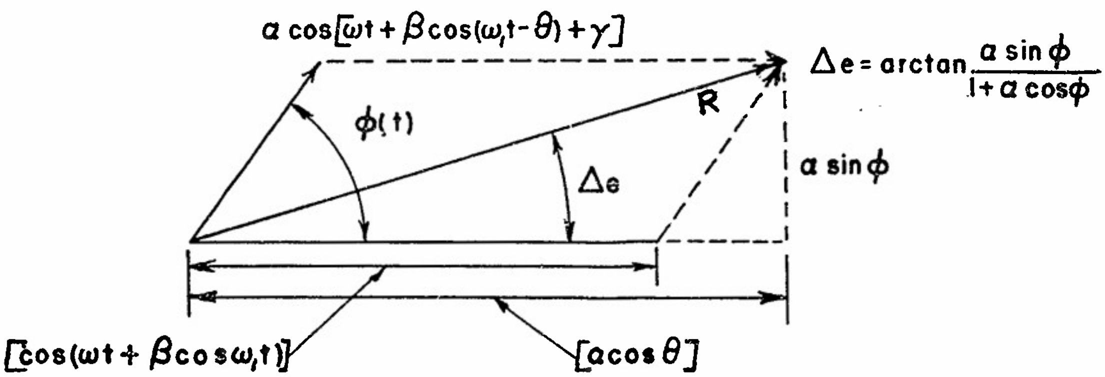


FIG. 1

DIRECTION OF ARRIVAL OF PRIMARY AND INTERFERING SIGNAL



$$\phi(t) = [\omega t + \beta\cos(\omega t - \theta + \gamma)] - [\omega t + \beta\cos\omega_1 t]$$
$$= \beta[\cos(\omega_1 t - \theta) - \cos\omega_1 t] + \gamma$$

FIG. 2

VECTOR DIAGRAM OF SIGNAL PHASE SHIFT

PLATE 6a

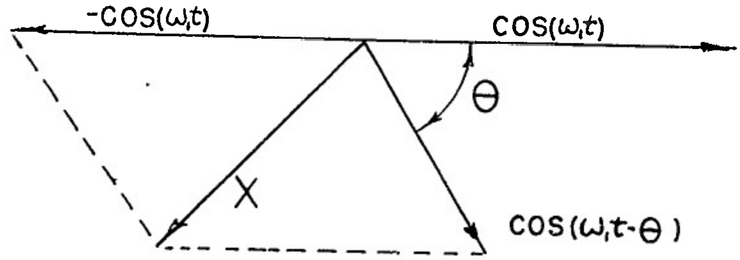


FIG. 3

COMBINATION OF TWO VECTORS

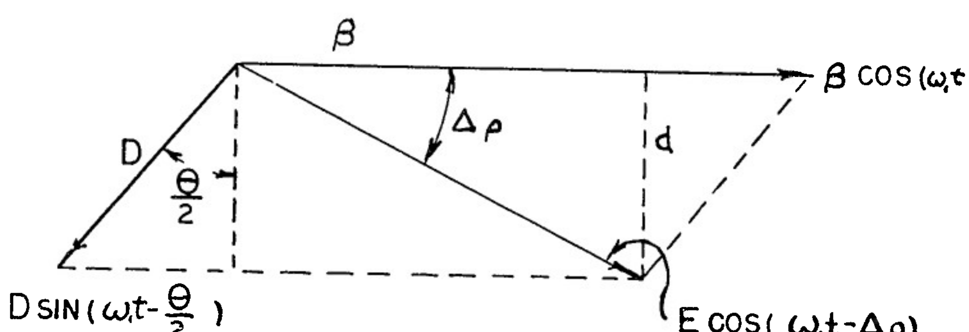
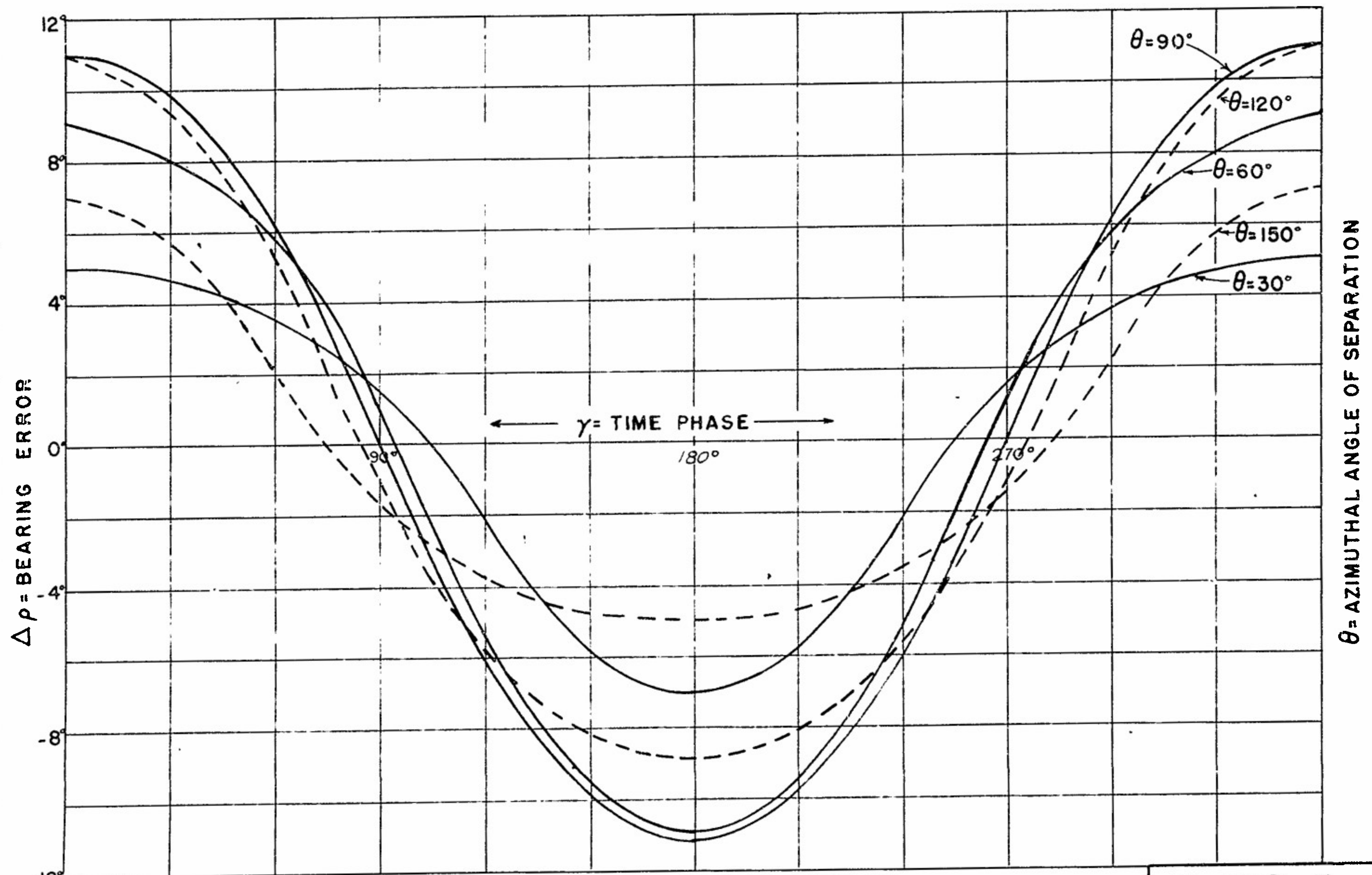


FIG. 4

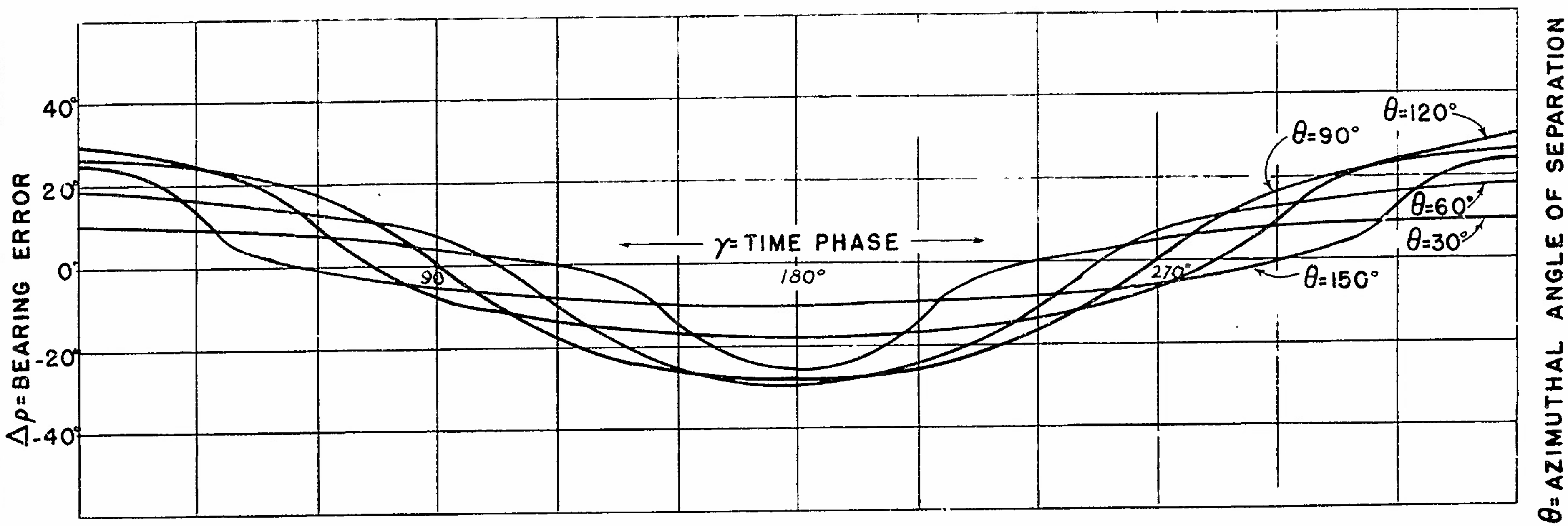
VECTOR DIAGRAM OF BEARING SHIFT

PLATE 6b
TECH. RPT. 8



ADCOCK BEARING ERROR
$\alpha = .2$

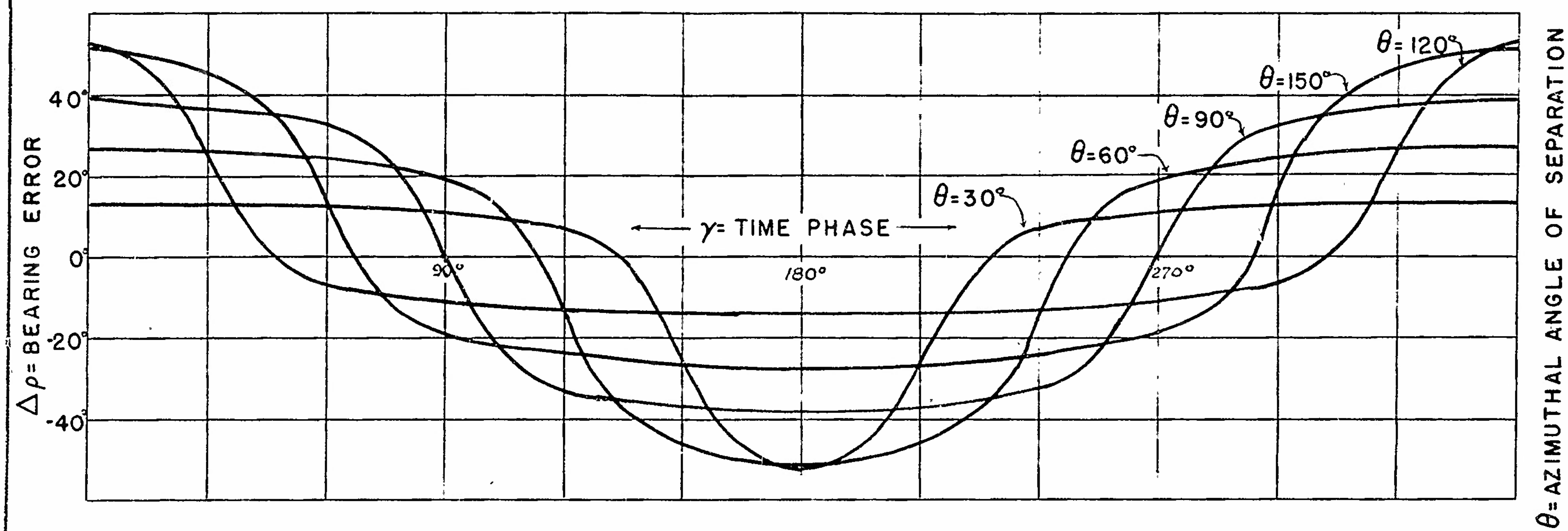
PLATE 7
TECH. RPT. 8



ADCOCK BEARING ERROR

$\alpha = .5$

PLATE 8
TECH. RPT. 8

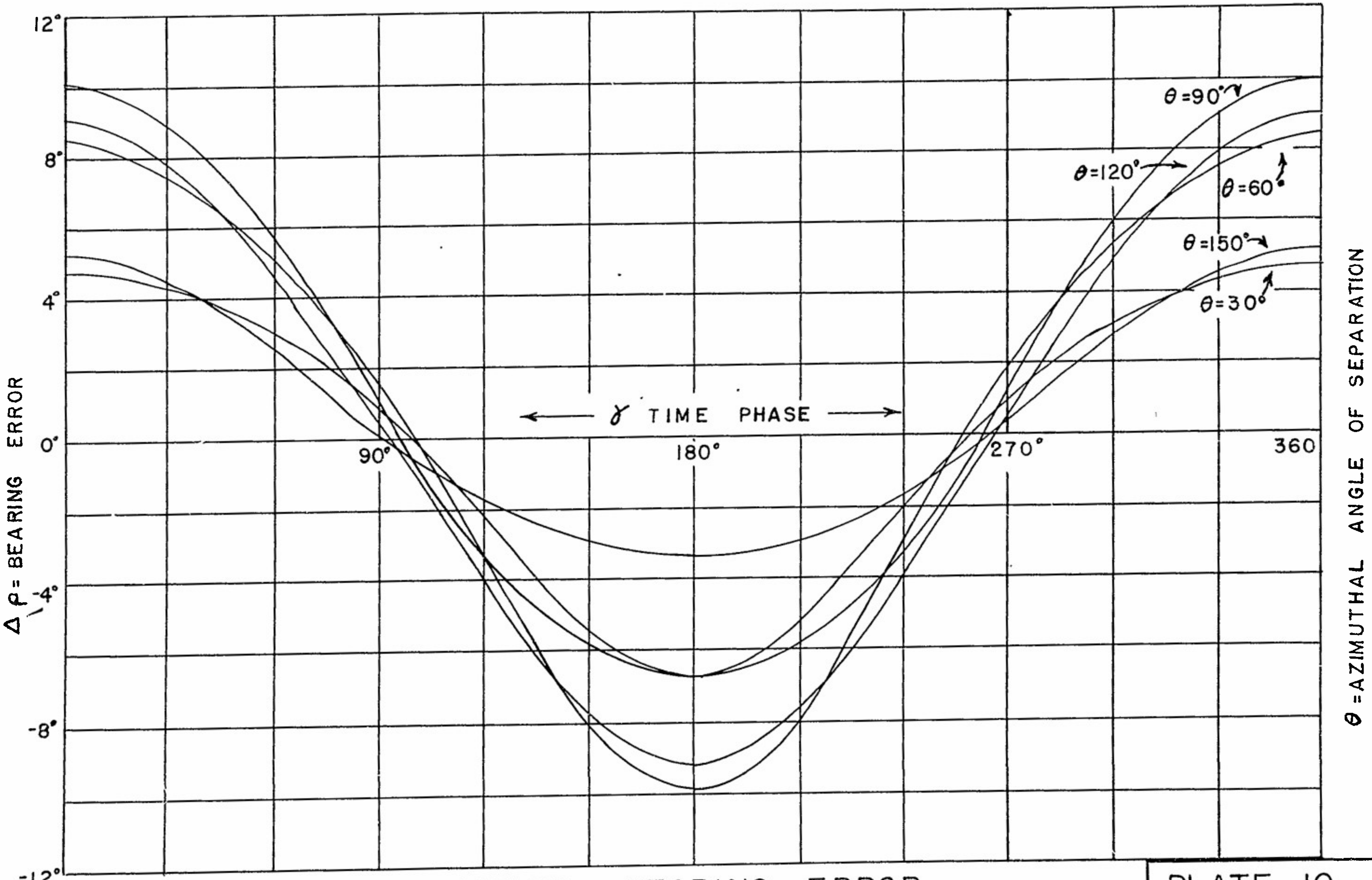


ADCOCK BEARING ERROR

$a=.8$

PLATE 9

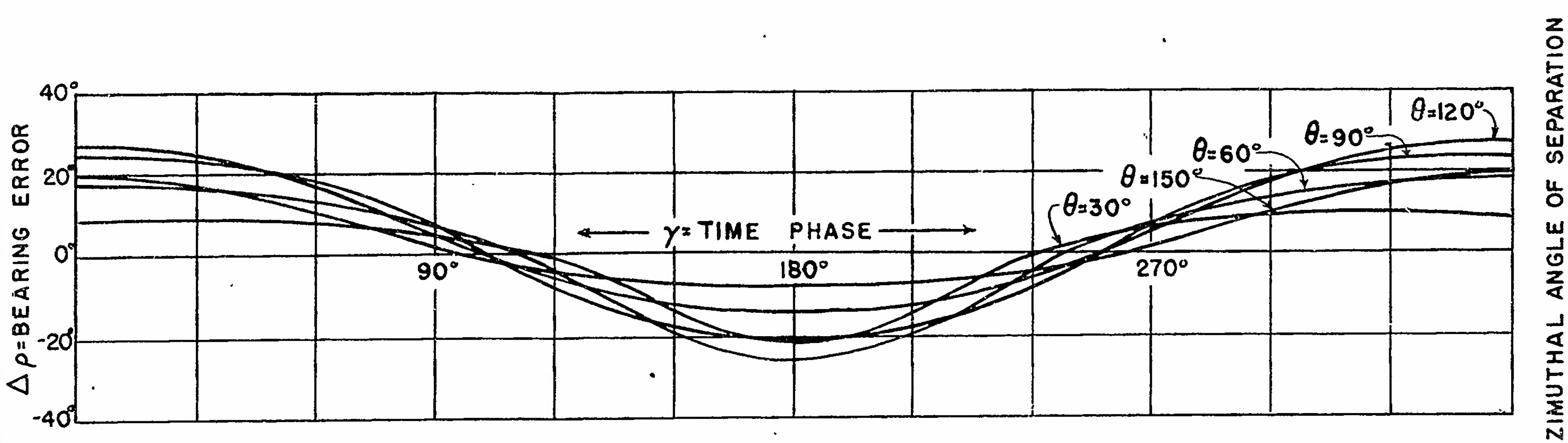
TECH. RPT. 8



DOPPLER BEARING ERROR

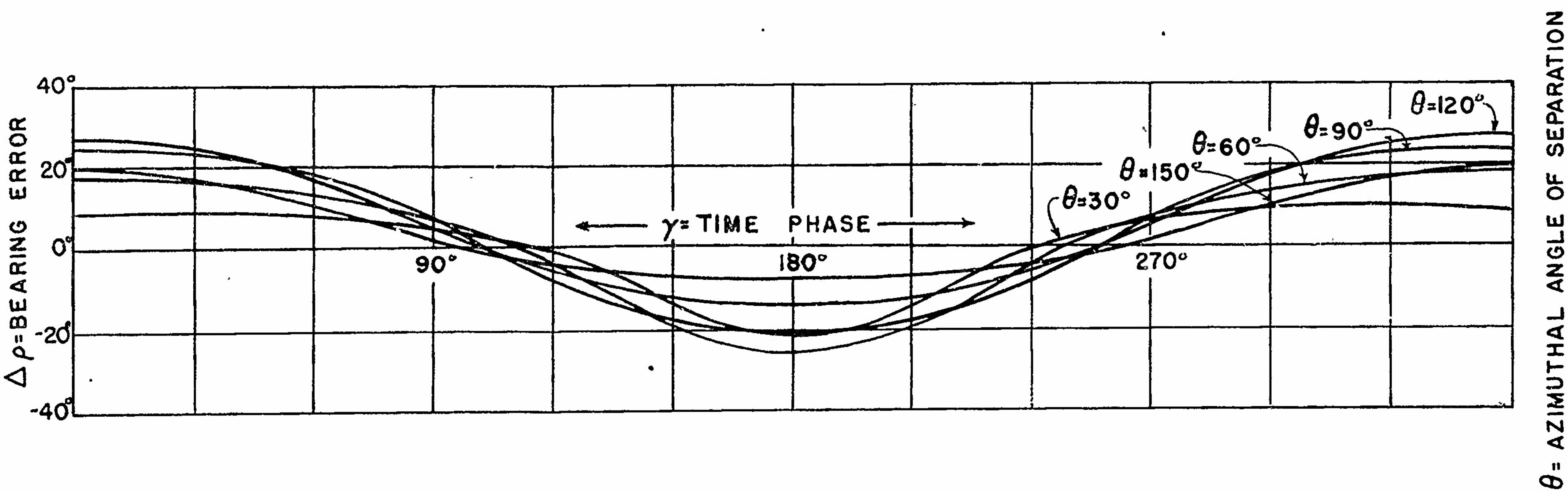
$\alpha = .2 \quad \beta = 45°$

PLATE 10
TECH. RPT. 8



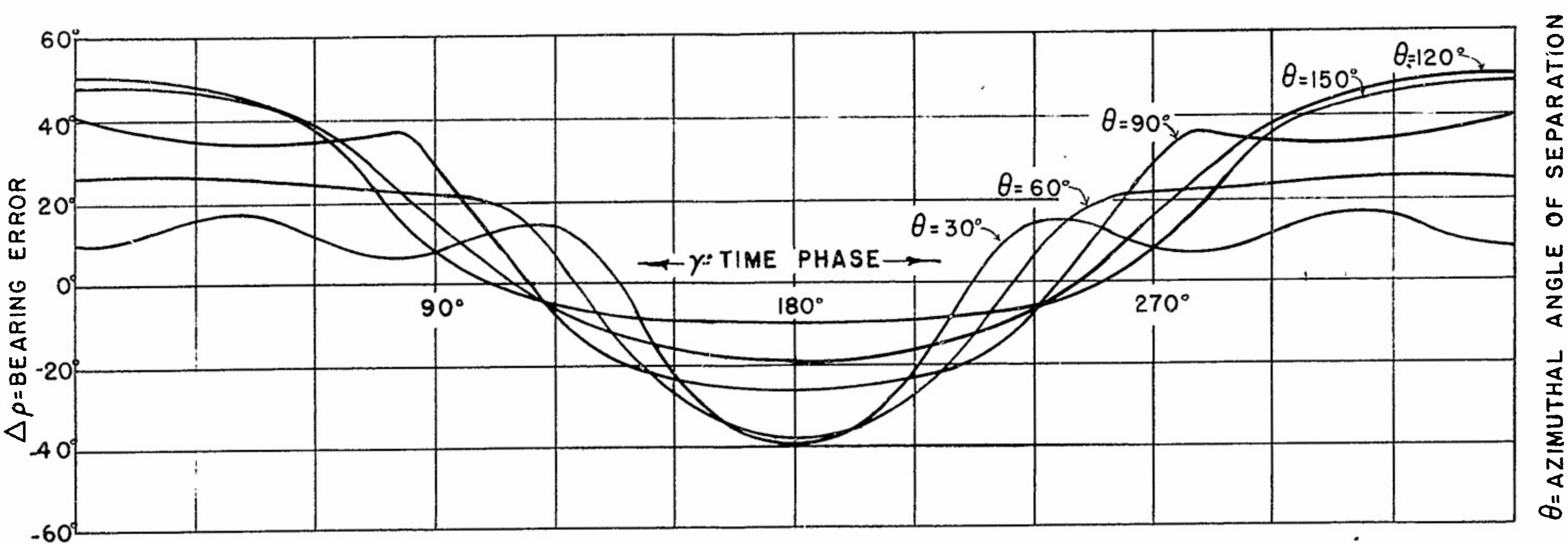
DOPPLER BEARING ERROR
$\beta = 45°$ $\alpha = .5$

PLATE II
TECH. RPT. 8



DOPPLER BEARING ERROR
$\beta = 45°\quad a = .5$

PLATE II
TECH. RPT. 8

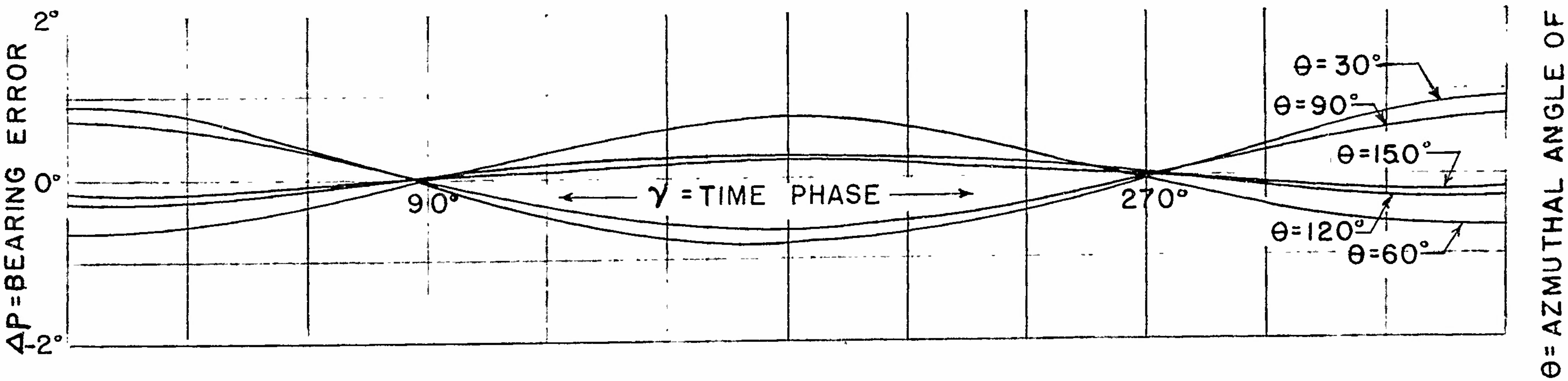


DOPPLER BEARING ERROR

$\beta = 45°$ $\alpha = .8$

PLATE 12

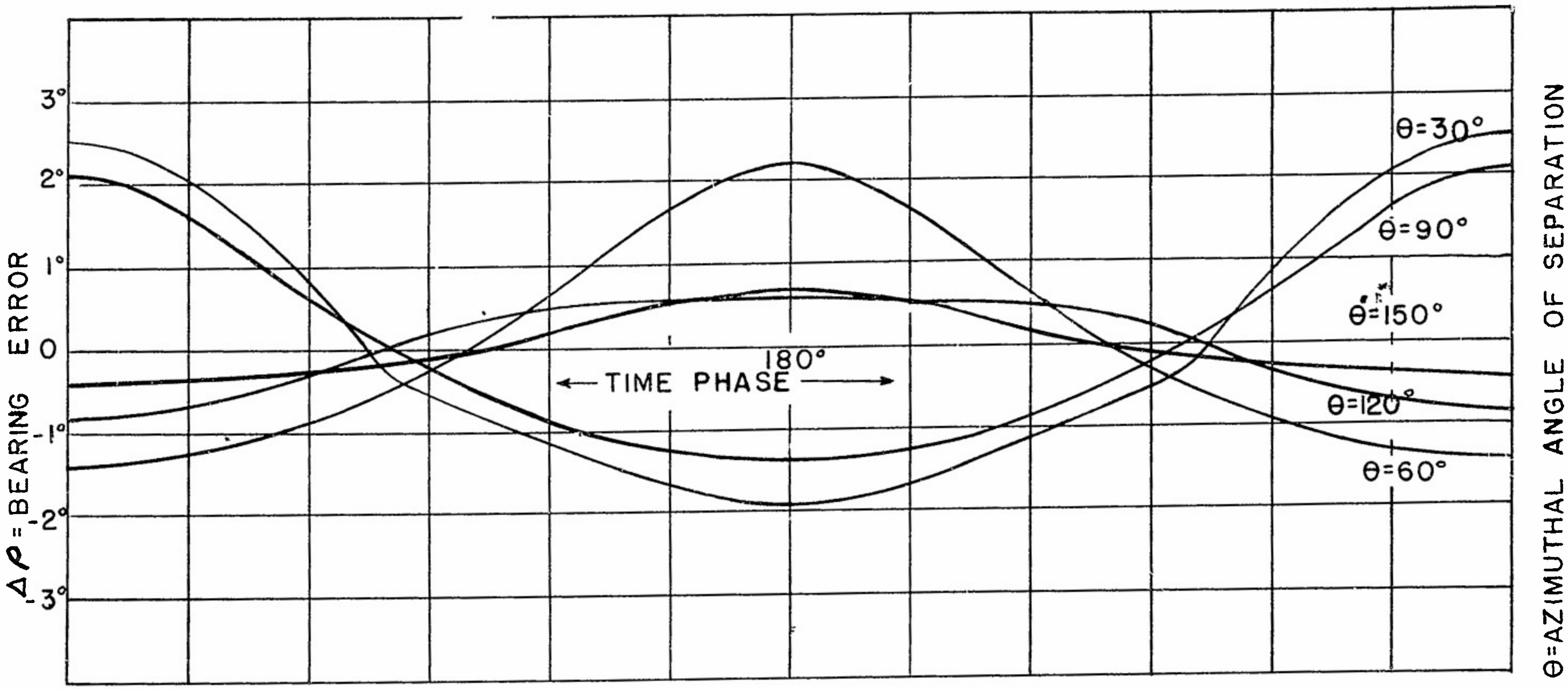
TECH. RPT. 8



DOPPLER BEARING ERROR

$\beta = 360°$ $\alpha = .2$

PLATE 13
TECH. RPT. 8

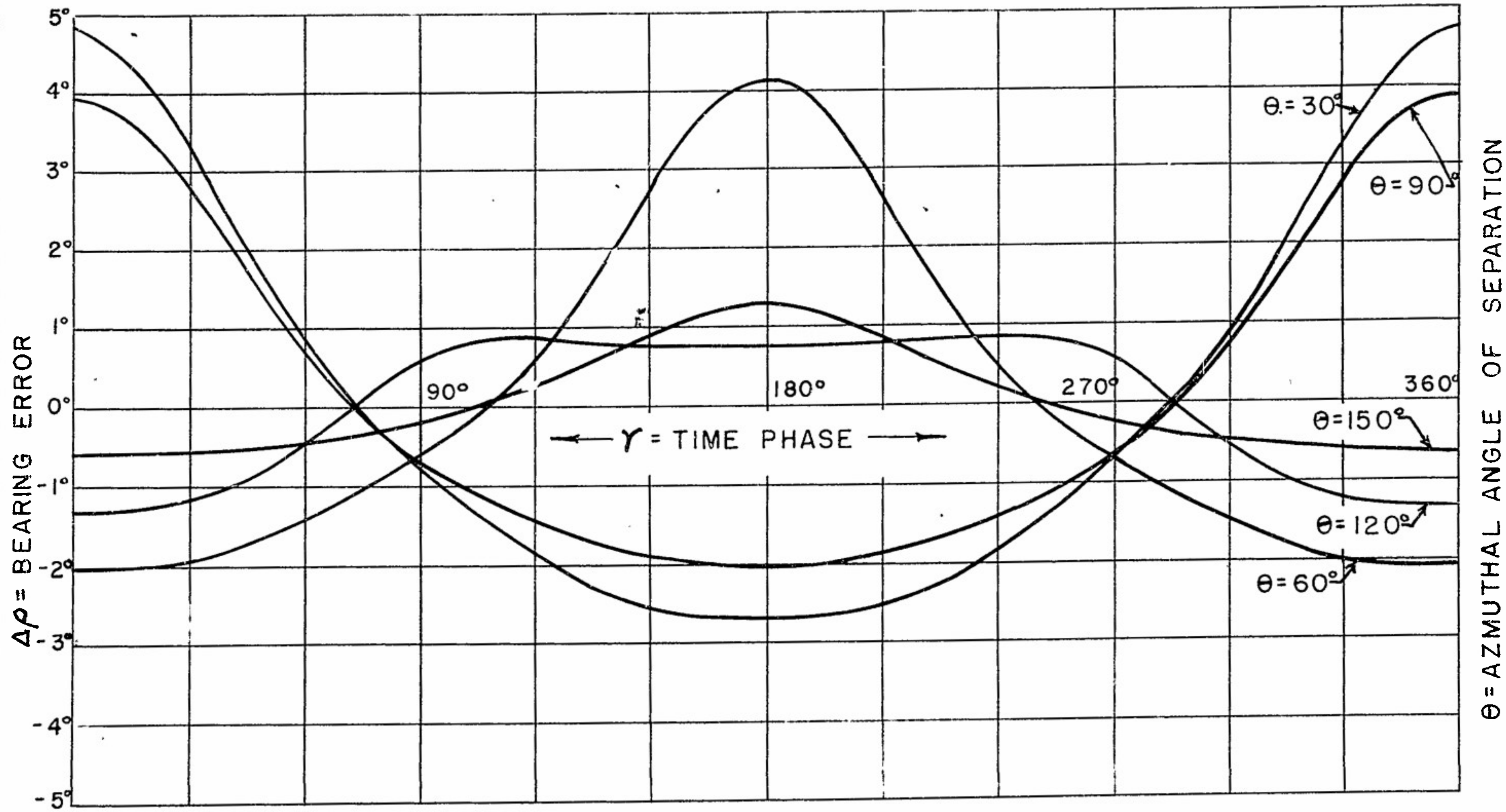


DOPPLER BEARING ERROR

$\alpha = .5 \quad \beta = 360°$

PLATE 14

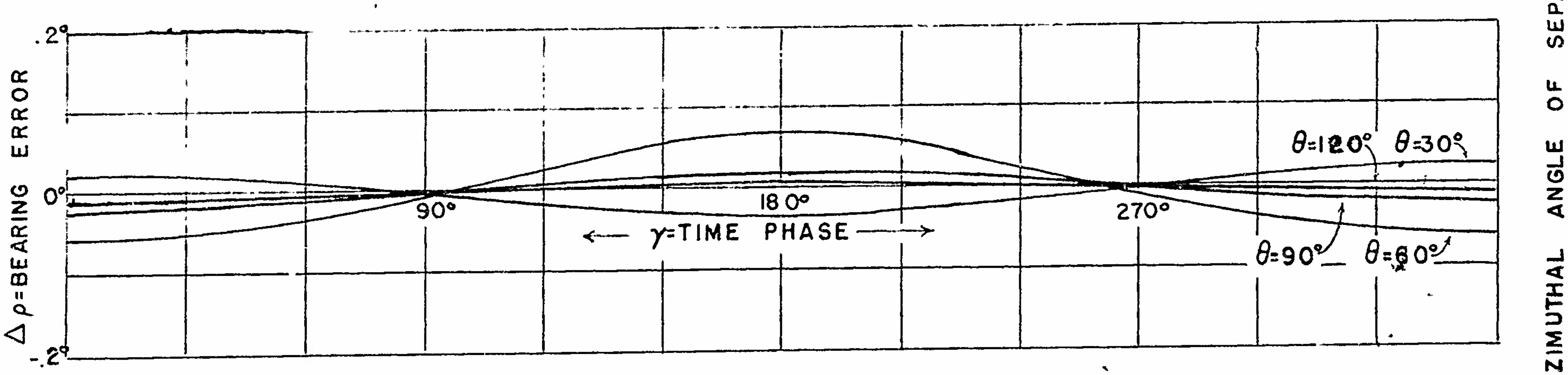
TECH. RPT. 8



DOPPLER BEARING ERROR

$\beta = 360° \quad \alpha = .8$

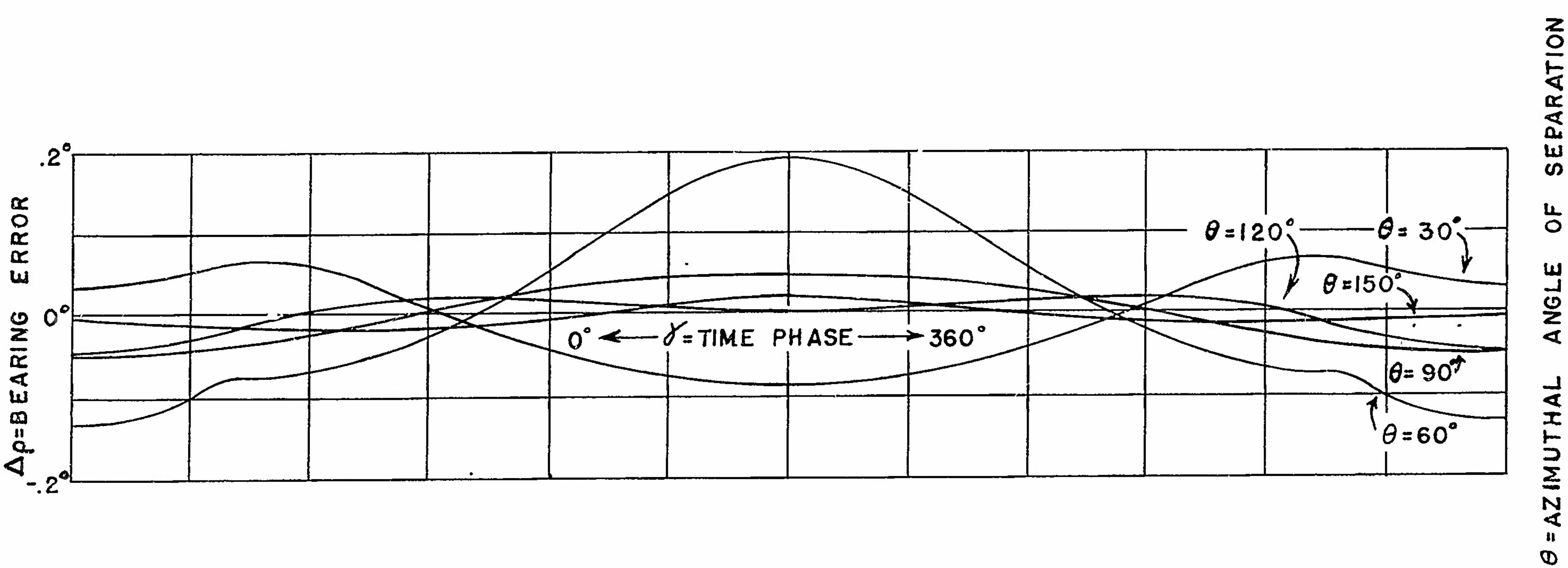
PLATE 15
TECH. RPT. 8



DOPPLER BEARING ERROR

$\beta = 1800°$ $\alpha = .2$

PLATE 16
TECH. RPT. 8

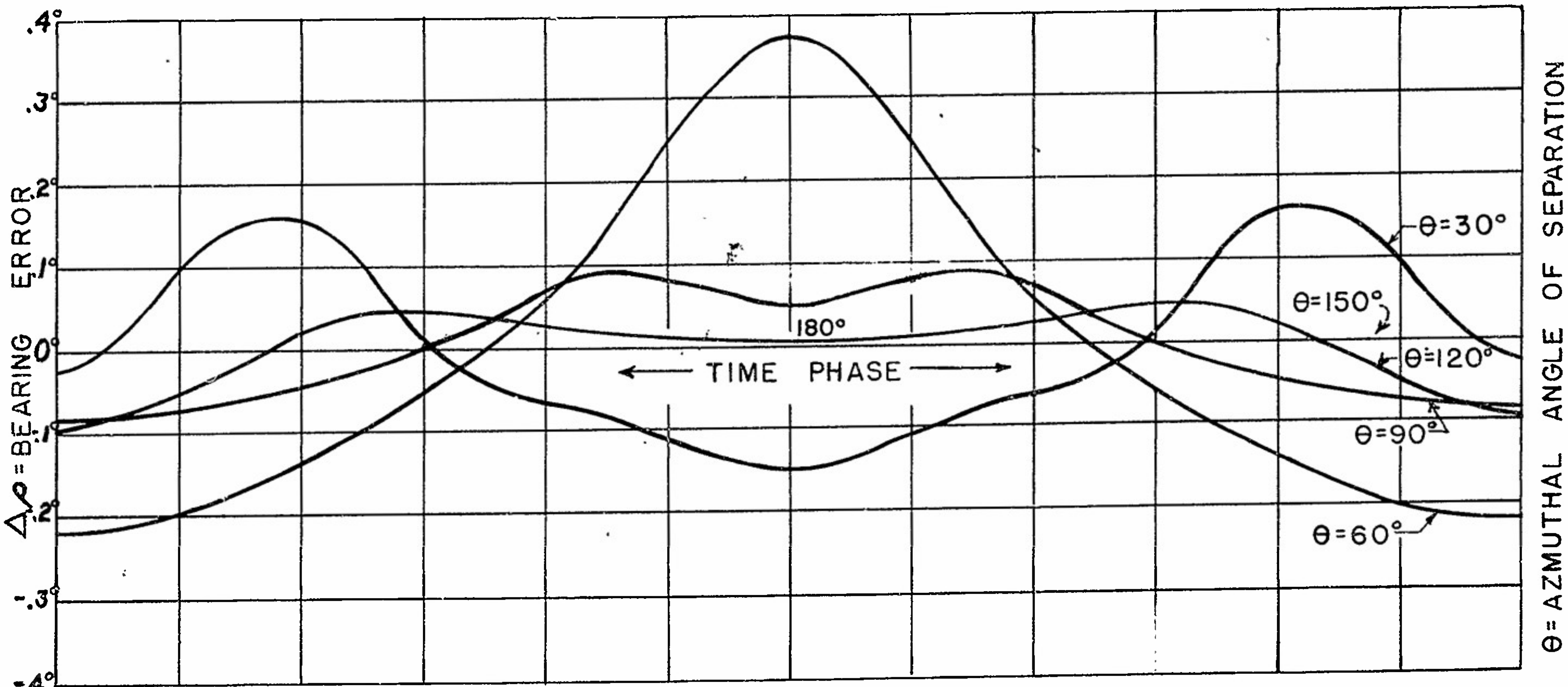


DOPPLER BEARING ERROR

$\beta = 1800°$ $\alpha = .5$

PLATE 17

TECH. RPT. 8



DOPPLER BEARING ERROR

$\alpha = .5 \quad \beta = 1800°$

PLATE 18
TECH. RPT. 8