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A variational solution to the problem of radiation from a thin-wire antenna is presented. The input impedance is used as the stationary parameter and the current along the antenna as the wave function. Using a suitable trial function, an approximate solution for the case of a symmetrical cylindrical antenna is determined. This solution is shown to be in good agreement with the one obtained by King and Middleton for the same problem.

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VARIATIONAL SOLUTION TO THE PROBLEM OF THE SYMMETRICAL CYLINDRICAL ANTENNA

BY
JAMES E. STORER

FEBRUARY 10, 1950

TECHNICAL REPORT No. 101

CRUFT LABORATORY
HARVARD UNIVERSITY
CAMBRIDGE, MASSACHUSETTS

Office of Naval Research
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Technical Report No. 101

Cruft Laboratory
Harvard University
Cambridge, Massachusetts

TR101

Abstract

The problem of radiation from a thin wire antenna is cast in a variational form. The input impedance is used as the stationary parameter and the current along the antenna as the wave function. Using a suitable trial function an approximate solution for the case of a symmetrical cylindrical antenna is determined. This solution is shown to be in good agreement with the one obtained by King and Middleton for the same problem.

TR101

Variational Solution to the Problem of the Symmetrical Cylindrical Antenna

by James E. Storer*

Cruft Laboratory, Harvard University
Cambridge, Massachusetts

I

Introduction

The problem of the thin wire antenna was first treated by Pocklington,[1] who obtained a one-dimensional integral equation for the total current along a wire excited by an external field. Using this same type of formulation, Hallén[2] derived a similar equation for the current in a driven wire antenna. The principal difference between these equations is the addition of a boundary condition to represent the influence of the generator on the system.

Prior to Hallén's work, this rather complicated integral equation had been solved for only one special case,[3] that of a simple loop antenna. Here the solution is expressed in a complex and slowly convergent series. By investigating the properties of the kernel of the integral equation, Hallén[4] derived an asymptotic expansion for the current. Although this expansion yielded complicated integrals, Hallén obtained some approximate numerical expressions for the current in a cylindrical antenna. Later, King and Middleton,[5] using a modified form of the Hallén technique, computed some fairly accurate solutions for the current and the input impedance of a cylindrical antenna.

The error in the Hallén-King and Middleton solutions would be very difficult to evaluate as the succeeding integrals are extremely complicated. Therefore an independent check of these

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* Atomic Energy Commission Fellow.

results would be worth while. It is the purpose of this paper to present a new solution to the thin wire antenna, that of the variational principle, and to demonstrate by comparison that the King and Middleton results are indeed very close approximations to the true solutions.

II

Hallén's Equation

The Hallén/King-and-Middleton analysis of the thin cylindrical antenna begins with the derivation of an integral equation for the current on the antenna. As this equation is also the starting-point for the variational analysis, an outline of its derivation will be given below.

For the purposes of the analysis, the antenna is assumed to lie along the z-axis, centered at the origin. It is assumed that the antenna is driven by a so-called "slice generator" or discontinuity in scalar potential at the origin. This is obviously a simplification of the actual situation where the antenna is driven by a two-wire line (see Fig. 1) or by some other means. The error resulting from such a simplification has been calculated by King[6] and corrections to the final results can be applied if desired.

The z-component of electric field due to the current in the wire lying along the z-axis may be calculated by means of a Helmholtz integral. Assuming harmonic time-dependence of the form $e^{j\omega t}$ for all electromagnetic quantities, and the dimensions of the antenna to be those indicated in Fig. 2, the expression for the z-component of the electric field becomes

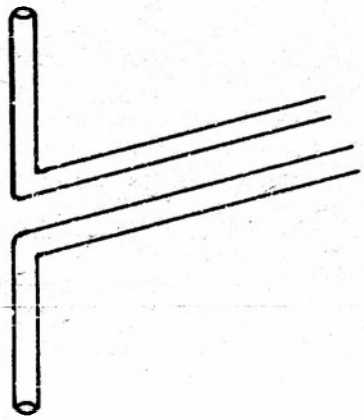


FIGURE 1

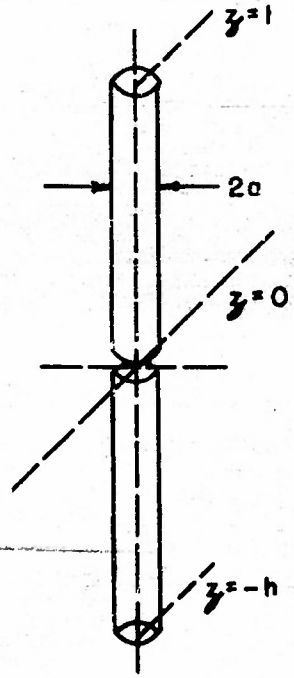


FIGURE 2

$$E_z = \frac{\zeta_o}{4\pi j}\ \beta_o\left(1 + \frac{1}{\beta_o^2}\frac{\partial^2}{\partial z^2}\right)\cdot$$

$$\int_{-h}^{h}\int_{0}^{2\pi}\int_{0}^{a} \frac{e^{-j\beta_o\sqrt{r^2+r'^2-2rr'\cos(\theta-\theta')+(z-z')^2}}}{\sqrt{[r^2+r'^2-2rr'\cos(\theta-\theta')+(z-z')^2]}}\, J_z r'dr'd\theta'dz',$$

where

$$\beta_o = \omega\sqrt{\mu_o\varepsilon_o} = \omega/c$$

$$\zeta_o = \sqrt{\frac{\mu_o}{\varepsilon_o}} \doteq 120\pi \text{ ohms, the "impedance" of free space,}$$

J_z is the z-component of the density of current in the antenna.

In order to reduce this integral to a useful form it is necessary to make some assumptions about J_z. Since the currents and fields are axially symmetric, J_z is independent of θ. Moreover, from physical considerations it is apparent that the radial distribution of J_z must be of the same form as the current distribution in the "skin effect."[7] This is not valid near the ends of the antenna, but end-effects can be shown to be unimportant. Using this form of J_z it is possible to show[8] that to a good approximation

$$[E_z]_{r=a} = \frac{\zeta_o}{4\pi j}\ \beta_o\left(1+\frac{1}{\beta_o^2}\frac{\partial^2}{\partial z^2}\right)\int_{-h}^{h} \frac{e^{-j\beta_o\sqrt{(z-z')^2+a^2}}}{\sqrt{(z-z')^2+a^2}}\, I(z')dz'e^{j\omega t}, \tag{1}$$

where $I(z')e^{j\omega t}$ is the total current at a point z' in the antenna.

The electric field along the surface of the wire $[E_z]_{r=a}$, must equal $z^i I(z)e^{j\omega t}$, $z \neq 0$, where z^i is the internal impedance.[8] At $z = 0$, however, the slice generator is present and a discontinuity in the scalar potential exists. The magnitude of this discontinuity is equal to the driving voltage $Ve^{j\omega t}$. Hence

$$\lim_{\varepsilon \to 0} \int_{-\varepsilon}^{\varepsilon} [E_z]_{r=a}\, dz = -Ve^{j\omega t}.$$

It is apparent that at $z = 0$, $[E_z]_{r=a}$ must be of the form of an impulse function. In terms of the Dirac delta-function defined by

$$\int_{-x}^{x} \delta(t)dt = 1 \qquad x > 0$$

$[E_z]_{r=a}$ in the region of $z = 0$ must be given by

$$[E_z]_{r=a} = -Ve^{j\omega t}\delta(z).$$

Hence, at any place along the antenna

$$[E_z]_{r=a} = z^i I(z)\, e^{j\omega t} - V\, e^{j\omega t}\, \delta(z) \qquad -h \leq z \leq h.$$

Combining this equation and equation (1) the following relationship is obtained:

$$z^i I(z) - V\delta(z) = \frac{\gamma_o}{4\pi j}\, \beta_o\left(1 + \frac{1}{\beta_o^2}\frac{\partial^2}{\partial z^2}\right) \cdot \int_{-h}^{h} \frac{e^{-j\beta_o\sqrt{(z-z')^2+a^2}}}{\sqrt{(z-z')^2+a^2}} I(z')dz' \qquad -h \leq z \leq h. \tag{2}$$

If (2) is treated as an inhomogeneous equation and integrated to remove the $[1+(1/\beta_o^2)(\partial^2/\partial z^2)]$ operator, then the Hallén integral equation is obtained. It is more convenient to use (2) as the starting-point for the variational treatment than the original Hallén equation.

Setting $V = Z_o I(o)$, where Z_o is the input impedance, the final form of the equation is obtained:

$$Z_o\,\delta(z)\,I(o) = \frac{j\eta_o}{4\pi}\,\beta_o\left(1+\frac{1}{\beta_o^2}\frac{\partial^2}{\partial z^2}\right)\int_{-h}^{h}\frac{e^{-j\beta_o\sqrt{a^2+(z-z')^2}}}{\sqrt{a^2+(z-z')^2}}\,I(z')dz' + z^{-1}I(z). \qquad (3)$$

$$-h \le z \le h$$

In order to make this equation yield a unique solution it is necessary to specify some conditions on $I(z)$ at the ends of the antenna. Since in the actual antenna the current vanishes at the ends, it is logical to require $I(z)$ in Hallén's equation to vanish at $z = \pm h$.

III

Variational Formulation

The variational expression is obtained by multiplying equation (3) by $I(z)$, integrating between the limits $-h$ and h, and then dividing through by $I^2(o)$. This yields the expression

$$Z_o = \frac{1}{I^2(o)}\left[z^{-1}\int_{-h}^{h} I^2(z)dz + \frac{j\eta_o}{4\pi}\int_{-h}^{h}\int_{-h}^{h} K(z-z')I(z)I(z')dzdz'\right], \qquad (4)$$

where

$$K(z-z') = K(z'-z)$$

$$= \beta_o\left(1+\frac{1}{\beta_o^2}\frac{\partial^2}{\partial z^2}\right)\frac{e^{-j\beta_o\sqrt{a^2+(z-z')^2}}}{\sqrt{a^2+(z-z')^2}} \qquad (5)$$

Equation (4) may be shown to have many of the usual variational properties. Obviously, if the current is multiplied by some constant, it in no way affects the value of Z_o as given by (4). It is also possible to show that the first variation of Z_o vanishes. Thus, if in (4) an approximate value of the current,

$I(z)+\Delta I(z)$, were used instead of the actual current, $I(z)$, the following expression results.

$$\bar{Z}_o = \frac{1}{[I(o)+\Delta I(o)]^2}\left[z^i \int_{-h}^{h} [I(z)+\Delta I(z)]^2 dz + \frac{j\zeta_o}{4\pi}\int_{-h}^{h}\int_{-h}^{h} K(z-z')(I(z) + \Delta I(z))(I(z')+\Delta I(z'))dzdz' \right]$$

$$= \frac{1}{[I(o)+\Delta I(o)]^2}\left[z^i \int_{-h}^{h} I^2(z)dz + \frac{j\zeta_o}{4\pi}\int_{-h}^{h}\int_{-h}^{h} K(z-z')I(z)I(z')dzdz' \right.$$

$$+ 2z^i\int_{-h}^{h} I(z)\Delta I(z)dz + 2\frac{j\zeta_o}{4\pi}\int_{-h}^{h}\int_{-h}^{h} K(z-z')I(z)\Delta I(z')dzdz'$$

$$\left. + z^i\int_{-h}^{h} [\Delta I(z)]^2 dz + \frac{j\zeta_o}{4\pi}\int_{-h}^{h}\int_{-h}^{h} K(z-z')\Delta I(z)\Delta I(z')dzdz' \right]$$

Using equations (3) and (4), this reduces to

$$\bar{Z}_o = \frac{1}{[I(o)+\Delta I(o)]^2}\left[Z_o I^2(o) + 2\int_{-h}^{h} \Delta I(z)dz \left\{ Z_o I(o)\delta(z) \right\} \right.$$

$$\left. + z^i\int_{-h}^{h} [\Delta I(z)]^2 dz + \frac{j\zeta_o}{4\pi}\int_{-h}^{h}\int_{-h}^{h} K(z-z')\Delta I(z)\Delta I(z')dzdz' \right]$$

Upon performing the delta-function integration and adding and subtracting $Z_o[\Delta I(o)]^2$, this becomes

$$\bar{Z}_o = Z_o + \frac{1}{[I(o)+\Delta I(o)]^2}\left[z^i\int_{-h}^{h} [\Delta I(z)]^2 dz + \frac{j\zeta_o}{4\pi}\int_{-h}^{h}\int_{-h}^{h} K(z-z')\Delta I(z)\Delta I(z')dzdz' - Z_o[\Delta I(o)]^2 \right]. \quad (6)$$

It is apparent from (6) that if $\Delta I(z)$ is small, $\bar{Z}_o$ differs

from Z_o only by terms of second order in smallness, i.e., the first variation is zero. The foregoing procedure may be reversed and it is possible to show that if Z_o in equation (4) is required to be stationary with respect to first-order changes in the current, then the original equation (3) is obtained. Hence, requiring the first variation of Z_o in (4) to be zero is both a necessary and sufficient condition for solution of the original integral equation.

It is apparent that this problem is different from the usual variational problem in that a single complex eigenvalue, Z_o, is involved instead of a denumerable set of real ones. In this respect the problem is similar to the recent work of Levine and Schwinger[9] on the diffraction of a plane wave by an aperture. It is worth while to see whether or not some of the variational properties present with real eigenvalues can be extended to the case of a complex eigenvalue.

In particular, suppose $\bar{Z}_o$ is a function of a parameter $\lambda = \lambda_1 + i\lambda_2$, and it is desired to find the value of λ that produces the optimum value of $\bar{Z}_o$. In the case of real eigenvalues this value of λ is obtained by setting the derivative with respect to λ equal to zero and solving the resulting equation for λ. However, with complex eigenvalues, λ is also in general complex and there are in effect two constants to determine, the real and imaginary parts. Assuming $\bar{Z}_o$ to be an analytic function of λ, then $\bar{Z}_o = \bar{X}(\lambda_1,\lambda_2) + i\bar{Y}(\lambda_1,\lambda_2)$ and the Cauchy-Riemann equations are

$$\frac{\partial}{\partial\lambda_1}\bar{X}(\lambda_1,\lambda_2) = \frac{\partial}{\partial\lambda_2}\bar{Y}(\lambda_1,\lambda_2) \qquad \frac{\partial}{\partial\lambda_2}\bar{X}(\lambda_1,\lambda_2) = -\frac{\partial}{\partial\lambda_1}\bar{Y}(\lambda_1,\lambda_2) \tag{7}$$

The problem is to determine the optimum value of λ. It would be logical to demand that the absolute value of the difference between $\bar{Z}_o$ and Z_o be a minimum. Hence,

$$\frac{\partial}{\partial \lambda_1}\left[\left|Z_o - \bar{Z}_o\right|\right] = 0 \qquad \frac{\partial}{\partial \lambda_2}\left[\left|Z_o - \bar{Z}_o\right|\right] = 0. \tag{8}$$

It can readily be shown that equations (7) and (8) lead to the conditions

$$\frac{\partial}{\partial \lambda_1}\bar{X}(\lambda_1,\lambda_2) = \frac{\partial}{\partial \lambda_2}\bar{X}(\lambda_1,\lambda_2) = \frac{\partial}{\partial \lambda_1}\bar{Y}(\lambda_1,\lambda_2) = \frac{\partial}{\partial \lambda_2}\bar{Y}(\lambda_1,\lambda_2) = 0.$$

These may be combined to yield

$$\frac{\partial}{\partial \lambda_1}\bar{Z}_o = 0, \qquad \frac{\partial}{\partial \lambda_2}\bar{Z}_o = 0$$

In the above equations it is understood that the real and imaginary parts are to be equated to zero separately. Since $\bar{Z}_o$ is assumed to be explicitly a function of λ, these equations could be written as

$$\frac{\partial}{\partial \lambda}\bar{Z}_o = 0. \tag{9}$$

Symbolically this is the same equation for λ as obtained in the case of real eigenvalues. The complex nature of it, however, is quite evident. It must be emphasized that in order to obtain this result it was necessary to assume that $\bar{Z}_o$ was an analytic function of λ and that equation (9), when solved, will yield a complex value for λ.

IV

Approximate Solution of Hallén's Equation

It is apparent that all that remains to be done is to select a suitable trial function, insert it in equation (4), and calculate values of the impedance. By selecting a function with a number of parameters that can be adjusted (for example, a sine series with arbitrary coefficients) it is obviously possible to approximate the current as closely as desired.

However, some sort of compromise must be made between the closeness of approximation and the possibility of numerical evaluation.

It has been shown by King[10] that the current on the antenna can be closely approximated by a sine plus shifted cosine distribution. Hence a logical choice for the trial function is

$$I_T(z) = C_1 + C_2 \sin \beta_c |z| + C_3 \cos \beta_o z.$$

The absolute-value sign in the sine function is inserted to produce symmetry about $z = 0$. This trial function must satisfy either identically or approximately the condition $I(h) = I(-h) = 0$. It is convenient to make the trial function zero at the ends. To accomplish this the trial function may be written in the form

$$I_T(z) = A \sin \beta_o(h - |z|) + B[1 - \cos \beta_o(h - |z|)]. \quad (10)$$

Using this trial function in (4), the value of $\bar{Z}_o$ becomes

$$\bar{Z}_o = \frac{j\zeta_o}{4\pi}[\gamma_{AA} A^2 + 2\gamma_{AB}AB + \gamma_{BB} B^2][\frac{1}{I_T^2(o)}], \quad (11)$$

where

$$\gamma_{AA} = \frac{4\pi}{j\zeta_o} z^i \int_{-h}^{h} \sin^2\beta_o(h-|z|)dz +$$

$$\int_{-h}^{h}\int_{-h}^{h} K(z-z')\sin\beta_o(h-|z|)\sin\beta_o(h-|z'|)dzdz',$$

$$\gamma_{AB} = \frac{4\pi}{j\zeta_o} z^i \int_{-h}^{h} \sin\beta_o(h-|z|)[1-\cos\beta_o(h-|z|)]dz +$$

$$\int_{-h}^{h}\int_{-h}^{h} K(z-z')\sin\beta_o(h-|z|)[1-\cos\beta_o(h-|z'|)]dzdz' ,$$

$$Y_{BB} = \frac{4\pi}{j\zeta_o} z^{1} \int_{-h}^{h} [1 - \cos\beta_o(h-|z|)]^2 dz +$$

$$\int_{-h}^{h}\int_{-h}^{h} K(z-z')[1-\cos\beta_o(h-|z|)][1-\cos\beta_o(h-|z'|)]dzdz' .$$

These integrals may be evaluated in terms of the generalized sine and cosine integrals (see Appendix 1).

$\bar{Z}_o$ is a function of the two parameters, A and B. Actually, $\bar{Z}_o$ is independent of the over-all amplitude of the current and in reality the two parameters are not independent as far as $\bar{Z}_o$ is concerned. Hence it is convenient to fix the amplitude of the current and thereby eliminate one of the parameters. Assuming that a unit voltage is driving the antenna, then the current at $z = 0$ is fixed by the following condition

$$V = 1 = I_T(o)\,\bar{Z}_o \qquad (11)$$

$$\therefore\ I_T(o) = A\sin\beta_o h + B[1-\cos\beta_o h] = 1/\bar{Z}_o. \qquad (12)$$

Using (12) to eliminate B, $\bar{Z}_o$ becomes

$$B = \frac{1}{1-\cos\beta_o h}[1/\bar{Z}_o - A\sin\beta_o h]$$

$$\bar{Z}_o = \frac{j\zeta_o}{4\pi}\left\{Y_{AA}A^2 + 2Y_{AB}A\,\frac{1/\bar{Z}_o - A\sin\beta_o h}{1-\cos\beta_o h} + Y_{BB}\left[\frac{1/\bar{Z}_o - A\sin\beta_o h}{1-\cos\beta_o h}\right]^2\right\} x$$

$$\frac{1}{(1/\bar{Z}_o)^2} .$$

Now, $\bar{Z}_o$ is an analytic function of A. Thus, to obtain the optimum value of $\bar{Z}_o$, A is determined by the condition

$$\frac{\partial \bar{Z}_o}{\partial A} = 0$$

$$\therefore\ 2A\gamma_{AA} + 2\gamma_{AB}\left[\frac{1/Z_o - A\sin\beta_o h}{1-\cos\beta_o h}\right] - 2\gamma_{AB}\frac{A\sin\beta_o h}{1-\cos\beta_o h} +$$

$$2\gamma_{BB}\left[\frac{1/Z_o - A\sin\beta_o h}{1-\cos\beta_o h}\right]\left[\frac{-\sin\beta_o h}{1-\cos\beta_o h}\right] = 0.$$

Setting

$$\Delta = (1-\cos\beta_o h)^2\gamma_{AA} - 2\sin\beta_o h(1-\cos\beta_o h)\gamma_{AB} + \sin^2\beta_o h\gamma_{BB}, \tag{13}$$

the parameter A can be written in the form

$$A = \frac{1}{Z_o}\,\frac{\gamma_{BB}\sin\beta_o h - \gamma_{AB}(1-\cos\beta_o h)}{\Delta} \tag{14}$$

Using (12) and (14) together,

$$B = \frac{1}{Z_o}\,\frac{\gamma_{AA}(1-\cos\beta_o h) - \gamma_{AB}(\sin\beta_o h)}{\Delta} \tag{15}$$

Substituting these values into equation (11), Z_o becomes

$$Z_o = \frac{j\zeta_o}{4\pi}\,\frac{\gamma_{AA}\gamma_{BB} - \gamma_{AB}^2}{\Delta} \tag{16}$$

A, B, and Z_o may now be evaluated by numerical calculations. The details of this process are left to the appendix. It is, however, appropriate at this time to examine the value of Z_o and to determine over what range of $\beta_o h$ it should yield "reasonable values." In equation (11) $I_T^2(o)$ appears in the denominator. Hence for a value of $\beta_o h$ such that $I_T(o) = 0$, Z_o will be infinite. Now $I_T(o) = 0$ at $\beta_o h = 0, 2\pi, 4\pi$, etc. At $\beta_o h = 0$ the infinite value of Z_o is reasonable since the physical antenna does have an increasingly large reactance for short lengths. However, at $\beta_o h = 2\pi$ the antenna certainly has a finite resistance and reactance. This trial function, then, cannot possibly yield correct values in the neighborhood of $\beta_o h = 2\pi$. As a matter of fact, this divergence begins to

affect the values of Z_o shortly after $\beta_o h = 3\pi/2$. A plot of the values of Z_o vs $\beta_o h$ appears in Fig. 3. The King-Middleton computations appear on the same graph for comparison. The above-mentioned divergence is quite evident upon examination of these graphs.

An accurate estimation of the error for any particular value of $\beta_o h$ would be extremely difficult. However, over the range $0 \leq \beta_o h \leq 5.0$, it should be possible to represent the current to 15 per cent accuracy by a sine plus a shifted cosine distribution. Hence the error in Z_o is of the order $(.15)^2$ or 2 per cent. Therefore it is felt that over the range $0 \leq \beta_o h \leq 5.0$, Z_o should be accurate to within 4 per cent.

V

Results

Following other authors on the subject, the computations of the impedance were carried out for a fixed ratio of length to radius of the antenna. For numerical reasons it was decided to choose a ratio of h/a of 904. This corresponds to a value of the parameter $\Omega = 2\ln\frac{2h}{a}$ of 15.

Table 1 contains the computed values of Z_o and the corresponding values obtained by R. W. P. King for comparison. A plot of these functions appears in Fig. 3. Perhaps the most satisfactory test of the agreement between these two sets of data is obtained by comparing the peaks and minimums in the R_o data. Hence

$\beta_o h \simeq 2.85$	R_o(max)	R. W. P. King	2438 ohms
	R_o(max)	Variational	2495 ohms

2.3% Difference

Table 1

Impedance of a Cylindrical Antenna*

$$\Omega = 2\ln\frac{2h}{a} = 15$$

$\beta_o h$	Variational		King-Middleton Second-Order Theory	
0.5	4.85	-j1265.	5.000	-j1256.
0.7	10.03	-j 815.6	10.28	-j 809.3
0.9	17.78	-j 538.1	18.13	-j 533.2
1.0	22.92	-j 430.2		
1.1	29.09	-j 334.9	29.36	-j 330.9
1.2	36.57	-j 247.7	36.73	-j 244.4
1.3	45.65	-j 166.2	45.62	-j 163.5
1.4	56.69	-j 88.06	56.38	-j 86.00
1.5	70.21	-j 11.60	69.46	-j 10.23
1.6	86.97	+j 64.89	85.53	+j 65.50
1.7	108.0	+j 143.0	105.7	+j 142.8
1.8	134.6	+j 224.3	131.5	+j 223.4
1.9	169.0	+j 310.6	165.0	+j 309.1
2.0	214.2	+j 403.6	209.4	+j 401.7
2.1	274.1	+j 505.3	269.8	+j 503.3
2.2	358.6	+j 617.5	354.8	+j 616.3
2.3	477.2	+j 740.5	476.3	+j 740.3
2.4	650.1	+j 870.6	656.3	+j 871.2
2.5	908.0	+j 991.5	929.4	+j 988.5
2.6	1291.	+j1053.	1330.	+j1045.
2.7	1815.	+j 933.2	1878.	+j 860.4
2.8	2341.	+j 443.4	2361.	+j 278.9
2.9	2479.	-j 388.9	2362.	-j 570.8
3.0	2070.	-j1085.	1870.	-j1159.
3.1	1486.	-j1354.	1312.	-j1328.
3.2	1017.	-j1339.	898.8	-j1273.
3.3	700.8	-j1218.	626.4	-j1146.
3.4	494.8	-j1074.	449.8	-j1008.
3.5	359.3	-j 936.0	332.8	-j 881.1
3.6	268.3	-j 811.8	253.2	-j 767.3
3.7	205.8	-j 701.5	197.6	-j 666.1
3.8	162.2	-j 603.2	158.1	-j 574.7
3.9	131.6	-j 514.7	129.9	-j 492.9

*Impedances are measured in ohms.

Table 1
(Continued)

$\beta_o h$	Variational		King-Middleton Second-Order Theory	
4.0	110.3	-j 433.8	110.0	-j 417.0
4.1	95.89	-j 358.8	97.25	-j 345.9
4.2	86.94	-j 288.3	89.70	-j 278.2
4.3	82.43	-j 220.8	85.53	-j 212.5
4.4	81.76	-j 155.2	85.48	-j 148.5
4.5	84.64	-j 90.46	89.91	-j 85.51
4.6	91.00	-j 25.51	99.14	-j 22.67
4.7	101.0	+j 40.77	113.5	+j 41.25
4.8	115.0	+j 109.6	133.9	+j 107.2
4.9	133.6	+j 182.4	161.9	+j 175.7
5.0	158.0	+j 260.9	199.8	+j 247.1

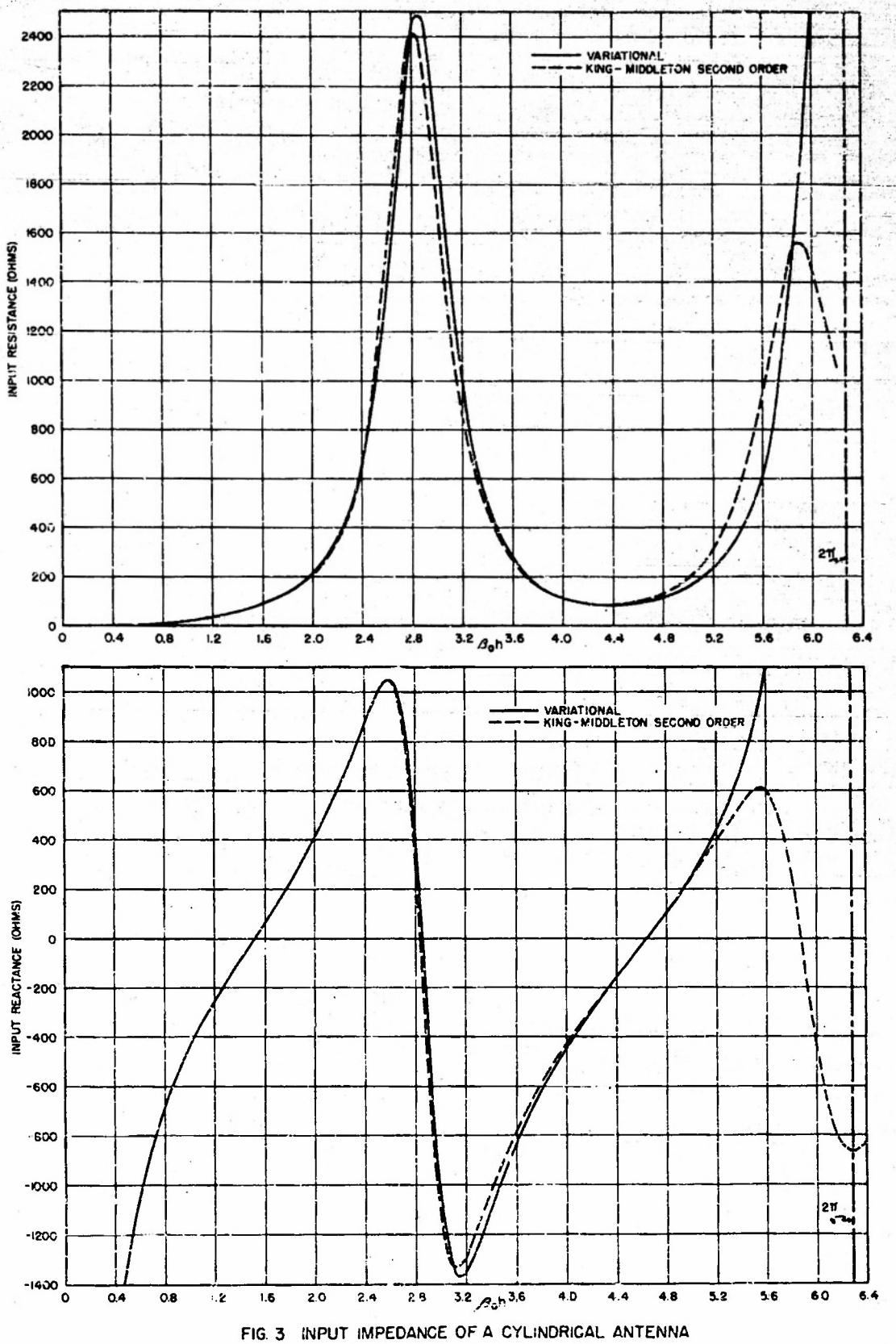


FIG. 3 INPUT IMPEDANCE OF A CYLINDRICAL ANTENNA

$\beta_o h \simeq 4.34$	R_o(min)	R. W. P. King	85.0 ohms
	R_o(min)	Variational	81.6 ohms

4.0% Difference

It is felt by the author that the agreement between these two solutions is extremely satisfactory. The question as to which one has the greater accuracy is quite difficult to answer. It is known that for progressively larger values of $\beta_o h$ the R. W. P. King solution, as well as the variational, becomes less accurate. Over the range $.5 \leq \beta_o h \leq 3.0$, the two solutions appear to be of comparable accuracy. Over the range $3.0 \leq \beta_o h \leq 5.0$, the King solution is probably somewhat better. A discussion of the short antenna, $\beta_o h \leq .5$, will be postponed until the next section.

The variational computation yields a simple and useful expression for the current. Repeating equation (10) for the purposes of the discussion

$$I_T(z) = A \sin\beta_o(h-|z|) + B[1 - \cos\beta_o(h-|z|)].$$

The values of A and B were determined so as to obtain the best value of the impedance. However, it is immediately apparent that these values must also give a good representation of the current. Table 2 contains the values of A and B in the range $0 \leq \beta_o h \leq 5.0$. In this range the approximation should be satisfactory. A graph of these parameters appears in Fig. 4.

It is possible to obtain an expression for the far fields in terms of A and B. Using a standard spherical coordinate system such as that shown in Fig. 5, the far field can be shown[11] to be given by

$$E_R = 0 \qquad B_R = 0$$

$$E_\theta = f(\theta)\frac{e^{-j\beta_o R_o}}{R_o}\epsilon^{j\omega t} \qquad B_\theta = 0$$

$$E_\phi = 0 \qquad B_\phi = f(\theta)\frac{e^{-j\beta_o R_o}}{cR_o}\epsilon^{j\omega t}$$

where

$$f(\theta) = \frac{j\zeta_o}{4\pi}\beta_o \int_{-h}^{h} e^{j\beta_o \cos\theta\, z'} \sin\theta\, I(z')dz'$$

Using the approximation $I_T(z)$ given above for $I(z)$, this integral may be evaluated.

$$f(\theta) = \frac{j\zeta_o}{4\pi}\beta_o \int_{-h}^{h} e^{j\beta_o \cos\theta\, z'} \sin\theta[A\sin\beta_o(h-|z'|) + B[1-\cos\beta_o(h-|z'|)]dz'$$

$$= 60j\,[A\,\frac{\cos[\beta_o h\cos\theta]-\cos\beta_o h}{\sin\theta} + B\,\frac{\sin[\beta_o h\cos\theta] - \cos\theta\sin\beta_o h}{\sin\theta\cos\theta}]$$

VI

The Short Antenna

The case of an antenna short compared to the wavelength, i.e., $\beta_o^2 h^2 \ll 1$, is of some physical interest. The value of the impedance may be obtained by taking the limiting form of equations (11)-(16). Assuming $\beta_o^2 h^2 \ll 1$, the various γ-integrals become

$$\gamma_{AA} = \beta_o h[4 + 4\ln 2 - 2\Omega] + \text{terms of order}(\beta_o h)^2$$

$$\gamma_{AB} = \beta_o^2 h^2[1 + 4\ln 2 - \Omega] + \text{terms of order } (\beta_o h)^3$$

$$\gamma_{BB} = \beta_o^3 h^3[10/9 + 8/3\ln 2 - 2/3\Omega] + \text{terms of order } (\beta_o h)^4$$

$$\Omega = 2\ln\frac{2h}{a}$$

Setting $\Omega = 15$, the limiting forms of A, B, $I_T(z)$, and Z_o become

$$A = j1.2690 \times 10^{-3} \text{ amp/volt} + \text{terms of order } \beta_o h$$

$$B = +j\frac{.3436}{\beta_o h} \times 10^{-3} \text{ amp/volt} + \text{terms of order } (\beta_o h)^0$$

Table 2

The Current Distribution Parameters A and B

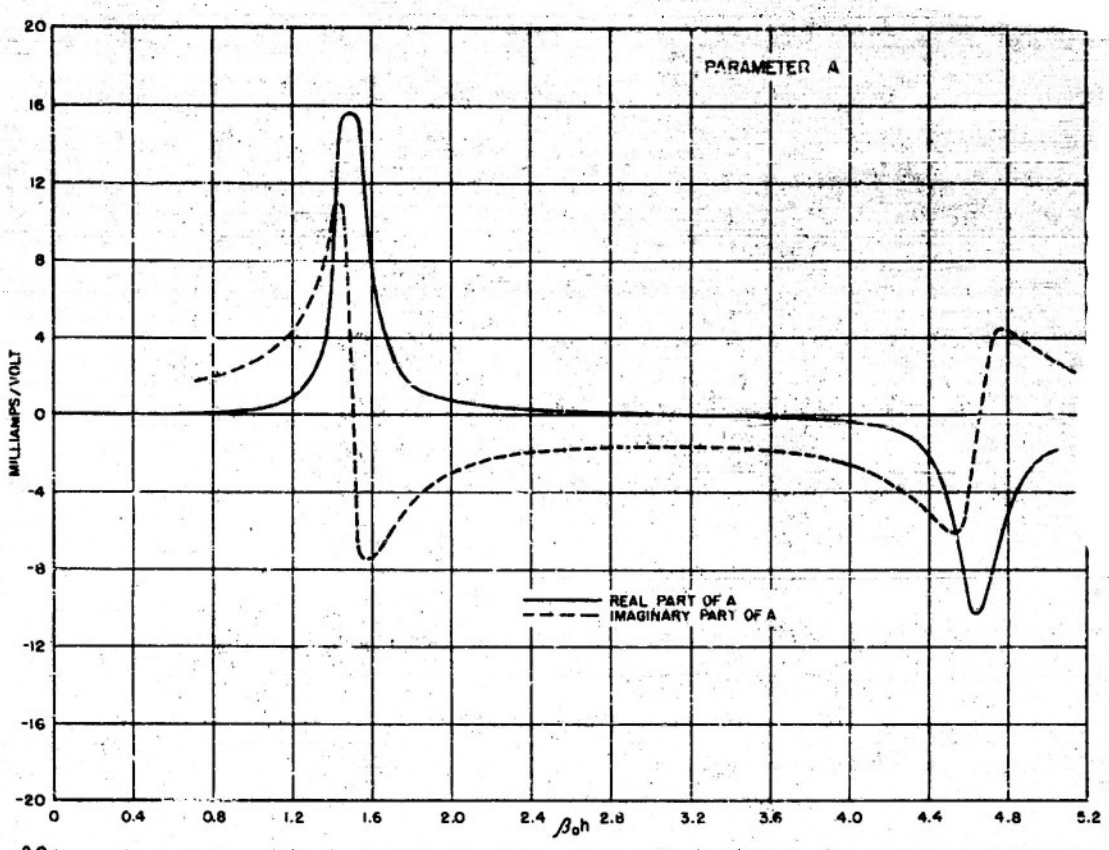
$$I_T(z) = A \sin\beta_o(h-|z|) + B[1-\cos\beta_o(h-|z|)]$$

$$\Omega = 2\ln\frac{2h}{a} = 15$$

$\beta_o h$	A(milliamperes/volt)	B(milliamperes/volt)
0.5	.0126 +j 1.499	- .0244 +j .6647
0.7	.0420 +j 1.800	- .0550 +j .3712
0.9	.1340 +j 2.299	- .1153 +j .1471
1.0	.2383 +j 2.724	- .1677 +j .0553
1.1	.4325 +j 3.332	- .2522 -j .0528
1.2	.8991 +j 4.357	- .3997 -j .1668
1.3	2.135 +j 6.038	- .7081 -j .3006
1.4	7.927 +j10.42	-1.717 -j .5909
1.5	15.74 +j .8847	-1.982 +j1.515
1.6	7.664 -j 7.385	- .2123 +j1.820
1.7	3.146 -j 5.868	.2154 +j1.210
1.8	1.420 -j 4.479	.2768 +j .8839
1.9	1.042 -j 3.604	.2763 +j .6998
2.0	.7241 -j 3.036	.2642 +j .5823
2.1	.5247 -j 2.641	.2511 +j .5005
2.2	.4003 -j 2.361	.2390 +j .4395
2.3	.3139 -j 2.155	.2285 +j .3919
2.4	.2506 -j 2.001	.2195 +j .3534
2.5	.2022 -j 1.884	.2117 +j .3215
2.6	.1636 -j 1.796	.2050 +j .2943
2.7	.1319 -j 1.731	.1992 +j .2709
2.8	.1050 -j 1.685	.1942 +j .2504
2.9	.0815 -j 1.656	.1899 +j .2323
3.0	.0604 -j 1.641	.1862 +j .2162
3.1	.0407 -j 1.641	.1831 +j .2017
3.2	.0216 -j 1.655	.1806 +j .1886
3.3	.0023 -j 1.684	.1787 +j .1767
3.4	- .0180 -j 1.730	.1776 +j .1658
3.5	- .0406 -j 1.794	.1772 +j .1559
3.6	- .0671 -j 1.880	.1778 +j .1468
3.7	- .1002 -j 1.994	.1796 +j .1384
3.8	- .1438 -j 2.144	.1830 +j .1306
3.9	- .2051 -j 2.343	.1884 +j .1232

Table 2
(Continued)

$\beta_o h$	A(milliamperes/volt)	B(milliamperes/volt)
4.0	- .2972 -j 2.608	.1968 +j .1157
4.1	- .4459 -j 2.972	.2097 +j .1075
4.2	- .6343 -j 3.483	.2296 +j .0968
4.3	- 1.221 -j 4.218	.2609 +j .0790
4.4	- 2.366 -j 5.245	.3103 +j .0399
4.5	- 5.182 -j 6.111	.3711 -j .0660
4.6	- 9.913 -j 3.195	.3036 -j .2865
4.7	- 8.477 +j 3.133	.0377 -j .3008
4.8	- 4.618 +j 4.210	- .0479 -j .1646
4.9	- 2.732 +j 3.606	- .0518 -j .0773
5.0	- 1.796 +j 2.891	- .0315 -j .0469



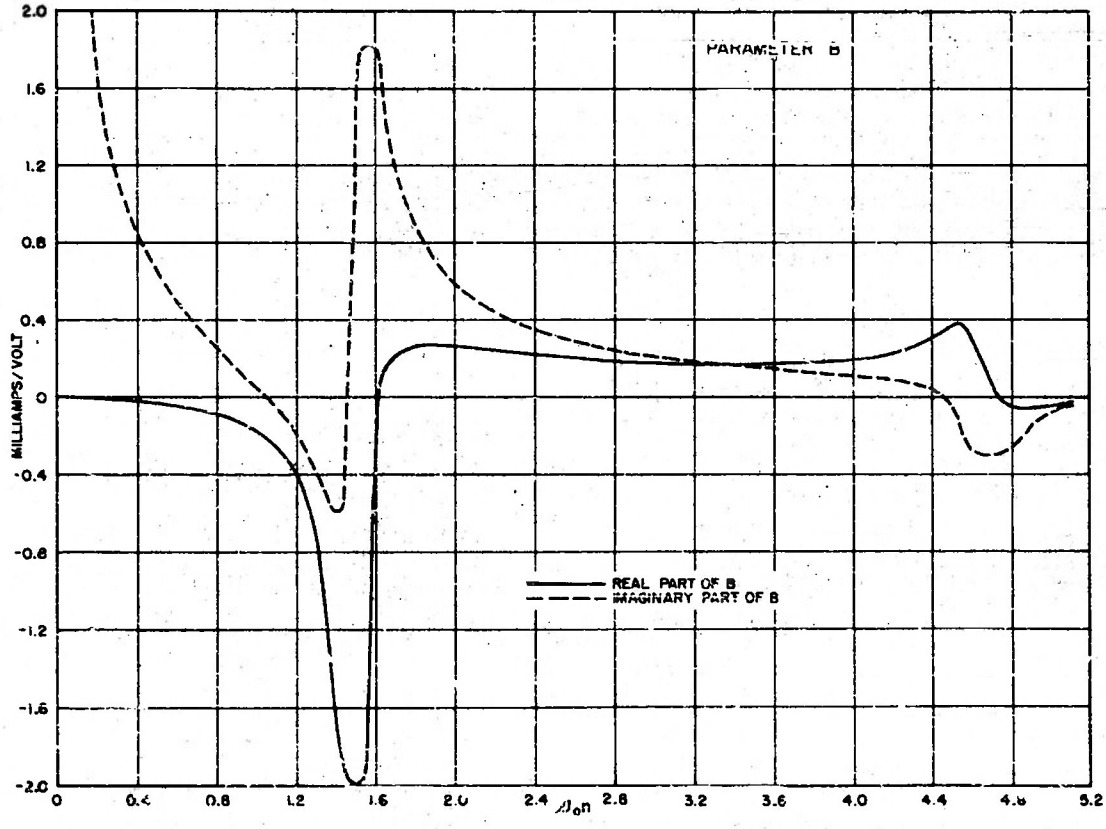


FIG. 4 CURRENT DISTRIBUTION PARAMETERS $I_T(Z) = A \sin \beta_0(h-|Z|) + B\left[1-\cos \beta_0(h-|Z|)\right]$

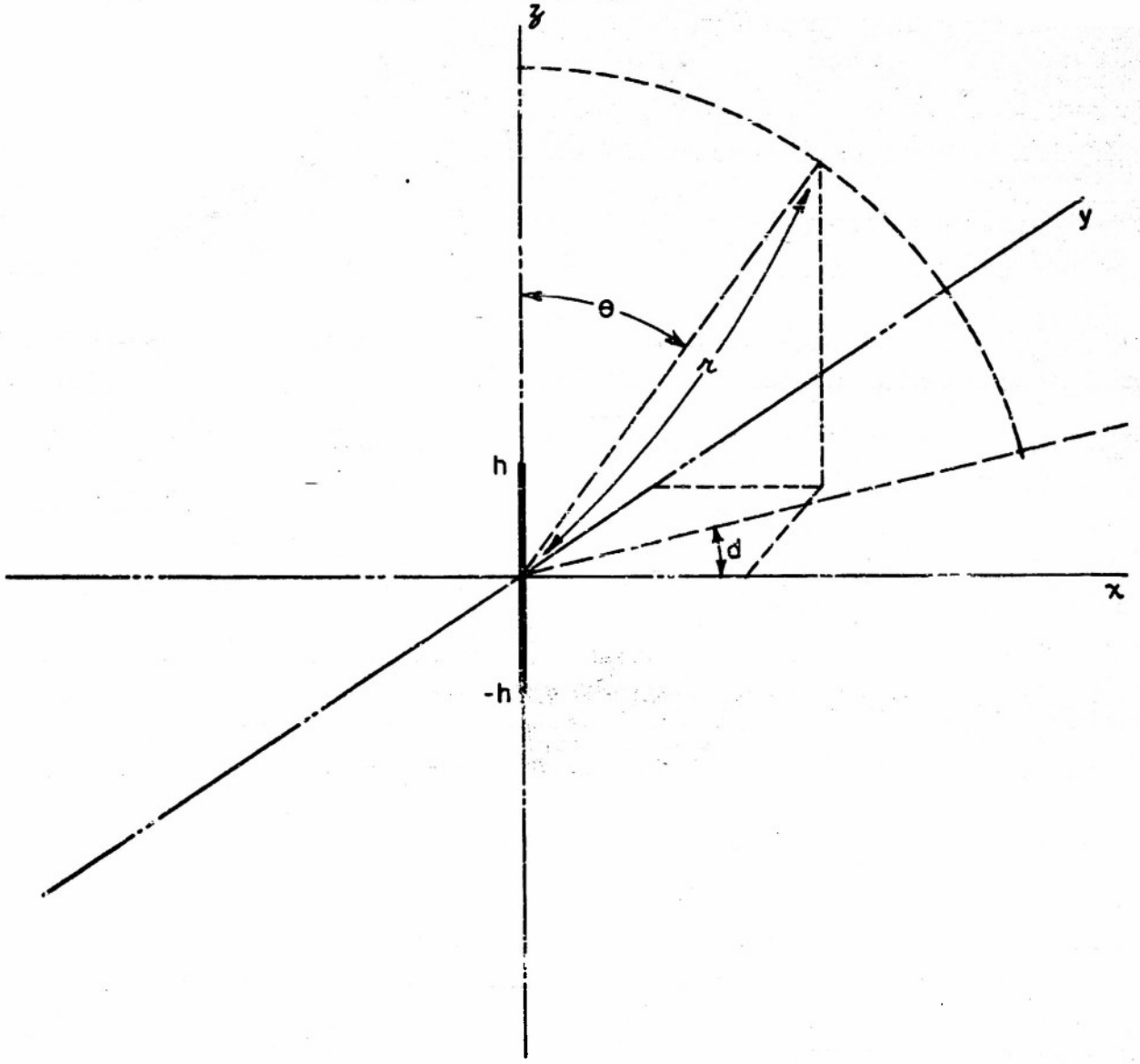


FIGURE 5

$$I_T(z) = j\,\beta_o h[1.2690(1 - \frac{|z|}{h}) + .1718(1 - \frac{|z|}{h})^2] \text{ milliamperes}$$

$$+ \text{ terms of order } (\beta_o h)^3$$

$$Z_o = -j\,\frac{694.05}{\beta_o h} \text{ ohms} \quad + \quad \text{terms of order } (\beta_o h)^0$$

In the first-order theory of King and Middleton the limiting value of the impedance is

$$Z_o = -j\,\frac{734.25}{\beta_o h} \text{ ohms}$$

It is apparent that there exists a sizable discrepancy between this result and that of the variational principle. Probably the limit of the second-order theory (to which previous comparisons in this paper have been made) would bring this into closer agreement.

VII

Conclusion

The variational method provides a new technique for solving the antenna problem. From the point of view of numerical computations it was no more involved than that of the King-Middleton method. It has the advantage of providing convergent series for the solution whereas the King-Middleton solution has asymptotic properties. However, the author feels that its chief utility will come in its application to some of the more complicated antenna configurations and in yielding numerical results. At the present time it is being applied to the solution of the simple loop antenna.

Acknowledgment

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Appendix I

Evaluation of the γ-Integrals

First, a necessary identity involving the Integrals will be derived. Letting $R = \sqrt{(z-z')^2 + a^2}$, and assuming $F(h) = 0$, then

$$\int_0^h K(z-z')F(z)dz = \beta_o \int_0^h F(z)[1 + \frac{1}{\beta_o^2}\frac{\partial^2}{\partial z^2}]\frac{e^{-j\beta_o R}}{R}\,dz$$

$$= \beta_o \int_0^h \frac{e^{-j\beta_o R}}{R}[1 + \frac{1}{\beta_o^2}\frac{\partial^2}{\partial z^2}]F(z)dz +$$

$$\frac{1}{\beta_o}\int_0^h [F(z)\frac{\partial^2}{\partial z^2}\frac{e^{-j\beta_o R}}{R} - \frac{e^{-j\beta_o R}}{R}\frac{\partial^2}{\partial z^2}F(z)]\,dz$$

$$= \beta_o \int_0^h \frac{e^{-j\beta_o R}}{R}[1 + \frac{1}{\beta_o^2}\frac{\partial^2}{\partial z^2}]F(z)dz +$$

$$\frac{1}{\beta_o}\int_0^h \frac{\partial}{\partial z}[F(z)\frac{\partial}{\partial z}\frac{e^{-j\beta_o R}}{R} - \frac{e^{-j\beta_o R}}{R}\frac{\partial}{\partial z}F(z)]\,dz$$

$$= \beta_o \int_0^h \frac{e^{-j\beta_o R}}{R}[1 + \frac{1}{\beta_o^2}\frac{\partial^2}{\partial z^2}]F(z)\,dz +$$

$$\frac{1}{\beta_o}[F(z)\frac{\partial}{\partial z}\frac{e^{-j\beta_o R}}{R} - \frac{e^{-j\beta_o R}}{R}\frac{\partial}{\partial z}F(z)]_o^h$$

Now, $\frac{\partial}{\partial z}\frac{e^{-j\beta_o R}}{R} = -\frac{\partial}{\partial z'}\frac{e^{-j\beta_o R}}{R}$. Letting $R_o = \sqrt{z'^2 + a^2}$, $R_h = \sqrt{(h-z')^2 + a^2}$, this becomes

$$\int_0^h K(z-z')F(z)dz = \beta_o \int_0^h \frac{e^{-j\beta_o R}}{R}\left[1+\frac{1}{\beta_o^2}\frac{\partial^2}{\partial z^2}\right]F(z)dz +$$

$$\frac{1}{\beta_o}\left[F(o)\frac{\partial}{\partial z'}\frac{e^{-j\beta_o R_o}}{R_o} - \frac{e^{-j\beta_o R_h}}{R_h}F'(h) + \frac{e^{-j\beta_o R_o}}{R_o}F'(o)\right].$$

Therefore, assuming $F(z) = F(-z)$ and $G(z) = G(-z)$, the γ-integrals may be written as

$$\int_{-h}^{h}\int_{-h}^{h} K(z-z')F(z)G(z')dzdz' = 2\int_{-h}^{h}\int_0^h K(z-z')F(z)G(z')dzdz'$$

$$= 2\int_{-h}^{h}\int_0^h \beta_o G(z')\frac{e^{-j\beta_o R}}{R}\left[1+\frac{1}{\beta_o^2}\frac{\partial^2}{\partial z^2}\right]F(z)dzdz'$$

$$+ 2\int_{-h}^{h} G(z')dz'\,\frac{1}{\beta_o}F(o)\frac{\partial}{\partial z'}\frac{e^{-j\beta_o R_o}}{R_o} - 2\frac{F'(h)}{\beta_o}\int_{-h}^{h}\frac{e^{-j\beta_o R_h}}{R_h}G(z')dz'$$

$$+ 2\frac{F'(o)}{\beta_o}\int_{-h}^{h} G(z')\frac{e^{-j\beta_o R_o}}{R_o}dz'.$$

Owing to symmetry the integral involving $F(o)$ as a factor vanishes and the others may be reduced to yield

$$\int_{-h}^{h}\int_{-h}^{h} K(z-z')F(z)G(z')dzdz' = 2\int_{-h}^{h}\int_0^h \beta_o G(z')\frac{e^{-j\beta_o R}}{R}\left[1+\frac{1}{\beta_o^2}\frac{\partial^2}{\partial z^2}\right]F(z)dzdz'$$

$$-2\frac{F'(h)}{\beta_o}\int_0^{2h}\frac{e^{-j\beta_o R_o}}{R_o}G(h-z')dz' + \frac{F'(o)}{\beta_o}\int_0^h G(z')\frac{e^{-j\beta_o R_o}}{R_o}dz'$$

This formula may now be used in the γ-integrals. It is apparent that both $\sin\beta_o(h-|z|)$ and $[1-\cos\beta_o(h-|z|)]$ are symmetric

functions of z and vanish at z = h. Hence

Y_{AA} : $F(z) = \sin\beta_o(h - |z|)$ $\quad G(z') = \sin\beta_o(h - |z'|)$

$$z > 0 \qquad [1 + \frac{1}{\beta_o^2}\frac{\partial^2}{\partial z^2}]\, F(z) = 0$$

$$\therefore\ Y_{AA} = \frac{4\pi}{j\zeta_o} z^1 \int_h^h \sin^2\beta_o(h-|z|)dz + 2\int_0^{2h} \frac{e^{-j\beta_o R_o}}{R_o} \sin\beta_o(h-|h-z'|)dz'$$

$$-4\cos\beta_o h \int_0^h \sin\beta_o(h-z')\frac{e^{-j\beta_o R_o}}{R_o}dz'$$

Y_{AB} : $F(z) = \sin\beta_o(h-|z|)$ $\quad G(z') = 1 - \cos\beta_o(h-|z'|)$

$$z > 0 \qquad [1 + \frac{1}{\beta_o^2}\frac{\partial^2}{\partial z^2}]\sin\beta_o(h-|z|) = 0$$

$$Y_{AB} = \frac{4\pi}{j\zeta_o} z^1 \int_h^h \sin\beta_o(h-|z|)[1-\cos\beta_o(h-|z|)]dz +$$

$$2\int_0^{2h} \frac{e^{-j\beta_o R_o}}{R_o}[1 - \cos\beta_o(h-|h-z'|)]dz'$$

$$-4\cos\beta_o h \int_0^h [1 - \cos\beta_o(h-|z'|)]\frac{e^{-j\beta_o R_o}}{R_o}dz'$$

Y_{BB} : $F(z) = 1 - \cos\beta_o(h-|z|)$ $\quad G(z') = 1 - \cos\beta_o(h-|z'|)$

$$z > 0 \qquad [1 + \frac{1}{\beta_o^2}\frac{\partial^2}{\partial z^2}][1 - \cos\beta_o(h-|z|)] = 1$$

$$Y_{B3} = \frac{4\pi}{\eta_o} z^i \int_{-h}^{h} [1 - \cos\beta_o(h-|z|)]^2 dz +$$

$$2\beta_o \int_{-h}^{h}\int_{o}^{h} [1-\cos\beta_o(h-|z'|)]\frac{e^{-j\beta_o R}}{R} dz dz'$$

$$-4 \sin\beta_o h \int_{o}^{h} [1 - \cos\beta_o(h - |z'|)]\frac{e^{-j\beta_o R_o}}{R_o} dz'$$

The double integral in this expression may be reduced by some manipulation and an integration by parts. Setting $t = z-z'$, $dt = dz$,

$$2\beta_o \int_{-h}^{h}\int_{o}^{h} [1 - \cos\beta_o(h-|z'|)]\frac{e^{-j\beta_o R}}{R} dz dz'$$

$$= 2\beta_o \int_{-h}^{h} [1 - \cos\beta_o(h-|z'|)] \int_{-z'}^{h-z'} \frac{e^{-j\beta_o R_o}}{R_o} dt\, dz'$$

$$= 2\beta_o \int_{o}^{h} [1 - \cos\beta_o(h-z')]\left\{ \int_{-z'}^{h-z'} \frac{e^{-j\beta_o R_o}}{R_o} dt + \int_{z'}^{h+z'} \frac{e^{-j\beta_o R_o}}{R_o} dt \right\} dz'$$

$$= 2\beta_o \int_{o}^{h} [1 - \cos\beta_o(h-z')] \int_{-h-z'}^{h-z'} \frac{e^{-j\beta_o R_o}}{R_o} dt\, dz'$$

$$= 2[\beta_o z' + \sin\beta_o(h-z')] \int_{-h-z'}^{h-z'} \frac{e^{-j\beta_o R_o}}{R_o} dt \Bigg|_{z'=o}^{z'=h}$$

$$- \int_{o}^{h} 2[\beta_o z' + \sin\beta_o(h-z')]\left\{ - \frac{e^{-j\beta_o R_h}}{R_h} + \frac{e^{-j\beta_o R_{-h}}}{R_{-h}} \right\} dz'$$

$$= 2[\beta_o h] \int_0^{2h} \frac{e^{-j\beta_o R_o}}{R_o} dt - 2 \sin\beta_o h \int_{-h}^{h} \frac{e^{-j\beta_o R_o}}{R_o} dt$$

$$+2 \int_0^{h} [\beta_o(h-t)+\sin\beta_o t] \frac{e^{-j\beta_o R_o}}{R_o} dt$$

$$-2 \int_h^{2h} [\beta_o(t-h) + \sin\beta_o(2h-t)] \frac{e^{-j\beta_o R_o}}{R_o} dt$$

Substituting this integral back into Y_{BB}, it is seen that several cancellations occur to yield

$$Y_{BB} = \frac{4\pi}{j\eta_o} z^1 \int_{-h}^{h} [1-\cos\beta_o(h-|z|)]^2 dz + 2 \int_0^{h} \sin\beta_o t \frac{e^{-j\beta_o R_o}}{R_o} dt$$

$$- 8 \sin\beta_o h \int_0^{h} \frac{e^{-j\beta_o R_o}}{R_o} dt$$

$$+ 4 \sin\beta_o h \int_0^{h} \cos\beta_o(h-t) \frac{e^{-j\beta_o R_o}}{R_o} dt - 2 \int_h^{2h} \sin\beta_o(2h-t) \frac{e^{-j\beta_o R_o}}{R_o} dt$$

$$+ 2\beta_o \int_0^{2h} (2h-t) \frac{e^{-j\beta_o R_o}}{R_o} dt$$

The integrals that are coefficients of z^1 can readily be evaluated in closed form. The remaining integrals can be expressed in terms of the generalized sine and cosine integrals defined as follows.

$$E(\beta_o z) = \int_0^{\beta_o z} \frac{e^{-j\sqrt{t^2+\beta_o^2 a^2}}}{\sqrt{t^2+\beta_o^2 a^2}} dt = \int_0^{z} \frac{e^{-j\beta_o R_o}}{R_o} dz'$$

$$S(\beta_o z) = \int_0^{\beta_o z} \frac{e^{-j\sqrt{t^2+\beta_o^2 a^2}}}{\sqrt{t^2+\beta_o^2 a^2}} \sin t \, dt = \int_0^z \sin\beta_o z' \frac{e^{-j\beta_o R_o}}{R_o} dz'$$

$$C(\beta_o z) = \int_0^{\beta_o z} \frac{e^{-j\sqrt{t^2+\beta_o^2 a^2}}}{\sqrt{t^2+\beta_o^2 a^2}} \cos t \, dt = \int_0^z \cos\beta_o z' \frac{e^{-j\beta_o R_o}}{R_o} dz'$$

In terms of these, the γ-integrals become

$$Y_{AA} = \frac{4\pi}{j\zeta_o} z^1[h - \frac{1}{2\beta_o}\sin 2\beta_o h] - 4\cos\beta_o h[\sin\beta_o h C(\beta_o h) - \cos\beta_o h S(\beta_o h)] +$$

$$+2\sin 2\beta_o h[C(2\beta_o h) - C(\beta_o h)] - 2\cos 2\beta_o h[S(2\beta_o h) - S(\beta_o h)] +$$

$$2S(\beta_o h)$$

$$Y_{AB} = \frac{4\pi}{j\zeta_o} \frac{z^1}{\beta_o}(1-\cos\beta_o h)^2 + 2E(2\beta_o h) - 2\cos 2\beta_o h[C(2\beta_o h) - C(\beta_o h)]$$

$$-2\sin 2\beta_o h[S(2\beta_o h) - S(\beta_o h)] - 2C(\beta_o h)$$

$$-4\cos\beta_o h E(\beta_o h) + 4\cos^2\beta_o h C(\beta_o h) + 2\sin 2\beta_o h S(\beta_o h)$$

$$Y_{BB} = \frac{4\pi}{j\zeta_o} z^1[3h - 4\sin\beta_o h + \frac{\sin 2\beta_o h}{2}] + 2S(\beta_o h) - 8\sin\beta_o h E(\beta_o h)$$

$$+2\sin 2\beta_o h C(\beta_o h) + 4\sin^2\beta_o h S(\beta_o h) - 2\sin 2\beta_o h[C(2\beta_o h) - C(\beta_o h)]$$

$$+2\cos 2\beta_o h[S(2\beta_o h) - S(\beta_o h)] + 4\beta_o h E(2\beta_o h) - 2j[e^{-j\beta_o\sqrt{4h^2+a^2}} -$$

$$e^{-j\beta_o a}].$$

Appendix II

Numerical Computations

It is apparent that to extend values of the computation over ranges of $\beta_o a$ and $\beta_o h$ would be too large a task. Following the procedure of previous authors it is convenient to define a parameter Ω as follows

$$\Omega = 2 \ln 2\frac{h}{a}.$$

R. W. P. King has tabulated values of the input impedance and current for $\Omega = 10$, $(h/a = 74.2)$, $\Omega = 12.5$ $(h/a = 259)$, $\Omega = 15$ $(h/a = 904)$, and $\Omega = 20$ $(h/a = 11013)$. An antenna with parameter of $\Omega = 20$ is too thin to reproduce physically although the others are not. As this paper is intended only to check this work, it is convenient to select one of these values and perform computations on it. Since for larger values of Ω the computations are simpler, the value $\Omega = 15$ has been chosen. For simplicity, z^1 is taken to be zero, i.e., the antenna is a perfect conductor.

Setting $\Omega = 15$ the various γ-integrals given in Appendix I may be evaluated numerically for different values of $\beta_o h$. The E-, S-, and C-integrals provide the only difficulty. In terms of some functions recently tabulated by the Harvard Computation Laboratory[12] they are given by

$$E(\beta_o z) = \sinh^{-1}\frac{z}{a} - C(\beta_o a, \beta_o z) - j\, S(\beta_o a, \beta_o z)$$

$$S(\beta_o z) = C_s(\beta_o a, \beta_o z) - j\, S_s(\beta_o a, \beta_o z)$$

$$C(\beta_o z) = \sinh^{-1}\frac{z}{a} - C(\beta_o a, \beta_o z) - C_c(\beta_o a, \beta_o z) - j\, S_c(\beta_o a, \beta_o z)$$

where

$$S(\beta_o a, \beta_o z) = \int_0^{\beta_o z} \frac{\sin\sqrt{t^2+\beta_o^2 a^2}}{\sqrt{t^2+\beta_o^2 a^2}}\, dt$$

$$C(\beta_o a, \beta_o z) = \int_0^{\beta_o z} \frac{1-\cos\sqrt{t^2+\beta_o^2 a^2}}{\sqrt{t^2+\beta_o^2 a^2}}\,dt$$

$$S_s(\beta_o a, \beta_o z) = \int_0^{\beta_o z} \frac{\sin\sqrt{t^2+\beta_o^2 a^2}}{\sqrt{t^2+\beta_o^2 a^2}}\,\sin t\,dt$$

$$C_s(\beta_o a, \beta_o z) = \int_0^{\beta_o z} \frac{\cos\sqrt{t^2+\beta_o^2 a^2}}{\sqrt{t^2+\beta_o^2 a^2}}\,\sin t\,dt$$

$$S_c(\beta_o a, \beta_o z) = \int_0^{\beta_o z} \frac{\sin\sqrt{t^2+\beta_o^2 a^2}}{\sqrt{t^2+\beta_o^2 a^2}}\,\cos t\,dt$$

$$C_c(\beta_o a, \beta_o z) = \int_0^{\beta_o z} \frac{\cos\sqrt{t^2+\beta_o^2 a^2}}{\sqrt{t^2+\beta_o^2 a^2}}\,[1-\cos t]\,dt$$

For small $\beta_o a$, the arguments in the table do not extend beyond $\beta_o h = 1.98$ except for the value $\beta_o a = 0$. To circumvent this difficulty and to simplify the computations the following formulas were used:

$$C(\beta_o a, \beta_o z) = C(0, \beta_o z) + [C(\beta_o a, 1.98) - C(0, 1.98)] + \Delta C$$

$$S(\beta_o a, \beta_o z) = S(0, \beta_o z) + [S(\beta_o a, 1.98) - S(0, 1.98)] + \Delta S$$

$$C_s(\beta_o a, \beta_o z) = C_s(\beta_o a, \beta_o z) + [C_s(\beta_o a, 1.98) - C_s(0, 1.98)] + \Delta C_s$$

$$S_s(\beta_o a, \beta_o z) = S_s(\beta_o a, \beta_o z) + [S_s(\beta_o a, 1.98) - S_s(0, 1.98)] + \Delta S_s$$

$$C_c(\beta_o a, \beta_o z) = C_c(\beta_o a, \beta_o z) + [C_c(\beta_o a, 1.98) - C_c(0, 1.98)] + \Delta C_c$$

$$S_c(\beta_o a, \beta_o z) = S_c(\beta_o a, \beta_o z) + [S_c(\beta_o a, 1.98) - S_c(0, 1.98)] + \Delta S_c$$

It can be shown that in the range $.5 \leq \beta_o h \leq 7.0$ the various

delta-terms may be neglected as they contribute less that 1 part in 10^5 to the values of the integrals.

Appendix 3

The King-Middleton Solutions

Since the second order King-Middleton solutions have never been published in tabular form, it was felt worth while to include them with this report. The first table contains values of ψ, the expansion parameter. The next tables contain values of the impedance, magnitude of the impedance, admittance, and magnitude of the admittance for various ratios of length to radius. The last table contains some additional values of the impedances near resonance. The impedances are tabulated to four significant figures, the last digit probably containing a small error.

Table I

The King-Middleton Expansion Parameter, Ψ

$\beta_0 h$	$\Omega = 10$	$\Omega = 12.5$	$\Omega = 15$	$\Omega = 20$
0.5	8.063	10.547		17.998
0.7	8.108	10.589		18.032
0.9	8.167	10.645		18.077
1.1	8.247	10.721		18.143
1.2	8.295	10.767		18.183
1.3	8.349	10.821		18.227
1.4	8.409	10.882		18.278
1.5	8.491	10.952		18.334
$\pi/2$	8.550			
1.6	8.570	11.027		18.398
1.7	8.584	11.033		18.379
1.8	8.545	10.985		18.305
1.9	8.468	10.899		18.194
2.0	8.363	10.785		18.054
2.1	8.235	10.649		17.894
2.2	8.089	10.498		17.723
2.3	7.931	10.335		17.545
2.4	7.766	10.172		17.402
2.5	7.597	10.010	12.436	17.275
2.6	7.429	9.849	12.269	17.108
2.7	7.268	9.690	12.112	16.956
2.8	7.117	9.543	11.967	16.816
2.9	6.980	9.410	11.839	16.699
3.0	6.861	9.296	11.730	16.599
3.1	6.763	9.206	11.647	16.529
π	6.729			
3.2	6.688	9.135	11.583	16.478
3.3	6.636	9.089	11.542	16.448
3.4	6.608	9.108	11.525	16.443
3.5	6.602	9.102	11.530	16.457
3.6	6.618	9.086	11.554	16.491
3.7	6.651	9.122	11.594	16.538
3.8	6.698	9.173	11.647	16.597
3.9	6.756	9.232	11.708	16.660
4.0	6.819	9.296	11.774	16.728

Table I

(Continued)

$\beta_o h$	$\Omega = 10$	$\Omega = 12.5$	$\Omega = 15$	$\Omega = 20$
4.1	6.885	9.362	11.839	16.794
4.2	6.949	9.424	11.899	16.855
4.3	7.008	9.483	11.958	16.907
4.4	7.059	9.532	12.005	16.951
4.5	7.098	9.568	12.038	16.978
4.6	7.124	9.590	12.056	16.988
4.7	7.135	9.597	12.059	16.983
4.8	7.131	9.588	12.046	16.964
4.9	7.110	9.562	12.014	16.919
5.0	7.072	9.519	11.965	16.859
5.1	7.019	9.461	11.902	16.786
5.2	6.951	9.388	11.824	16.697
5.3	6.870	9.302	11.734	16.597
5.4	6.778	9.206	11.634	16.490
5.5	6.678	9.102	11.527	16.377
5.6	6.571	8.994	11.417	16.264
5.7	6.462	8.885	11.308	16.153
5.8	6.354	8.777	11.201	16.047
5.9	6.250	8.675	11.100	15.950
6.0	6.154	8.581	11.009	15.864
6.1	6.068	8.500	10.931	15.794
6.2	5.996	8.433	10.870	15.743
6.3	5.940	8.382	10.823	15.710
6.4	5.901	8.348	10.795	15.689
6.5	5.881	8.334	10.786	15.695
6.6	5.878	8.338	10.799	15.720
6.7	5.893	8.357	10.821	15.750
6.8	5.922	8.390	10.857	15.791
6.9	5.965	8.435	10.906	15.846
7.0	6.019	8.491	10.963	15.908

Table II

King-Middleton Second-Order Impedances

$\Omega = 2\ln\frac{2h}{a} = 10 \quad h/a = 75.206$

$\beta_o h$	Z_o (ohms)	$\lvert Z_o \rvert$	Y_o (mhos x 10^{-3})	$\lvert Y_o \rvert$ x 10^{-3}
0.5	4.988 -j704.8	704.8	0.0100 +j1.419	1.419
0.7	10.30 -j451.3	451.5	0.0506 +j2.214	2.215
0.9	18.29 -j293.8	294.4	0.2110 +j3.390	3.397
1.1	30.02 -j177.4	179.9	0.9277 +j5.481	5.559
1.2	37.84 -j127.1	132.6	2.153 +j7.229	7.543
1.3	47.41 -j 79.76	92.79	5.507 +j9.264	10.78
1.4	59.15 -j 34.27	68.36	12.66 +j7.333	14.63
1.5	73.65 +j 10.30	74.37	13.32 -j1.862	13.45
1.6	91.73 +j 54.72	106.8	8.040 -j4.797	9.363
1.7	114.8 +j 99.67	151.3	4.967 -j4.310	6.608
1.8	145.2 +j145.5	205.5	3.437 -j3.444	4.865
1.9	185.5 +j191.8	266.8	2.606 -j2.693	3.748
2.0	240.2 +j237.1	337.5	2.109 -j2.082	2.963
2.1	314.8 +j277.6	419.8	1.787 -j1.576	2.382
2.2	416.1 +j303.8	515.2	1.568 -j1.145	1.941
2.3	547.5 +j296.8	622.8	1.412 -j0.7652	1.606
2.4	697.1 +j227.3	733.3	1.297 -j0.4228	1.364
2.5	820.4 +j 71.37	823.6	1.210 -j0.1052	1.214
2.6	849.9 -j145.6	862.4	1.143 +j0.1958	1.160
2.7	765.5 -j339.9	837.5	1.091 +j0.4846	1.194
2.8	622.2 -j452.6	769.4	1.051 +j0.7646	1.300
2.9	479.4 -j489.4	685.1	1.021 +j1.043	1.460
3.0	364.5 -j480.4	603.1	1.002 +j1.321	1.658
3.1	280.0 -j450.6	530.5	0.9950 +j1.601	1.885
3.2	219.6 -j414.0	468.6	0.9999 +j1.885	2.134
3.3	175.6 -j376.2	415.2	1.019 +j2.182	2.409
3.4	143.8 -j339.9	369.1	1.056 +j2.495	2.709
3.5	120.9 -j306.1	329.1	1.116 +j2.826	3.038
3.6	103.9 -j274.5	293.5	1.207 +j3.186	3.407
3.7	91.22 -j244.7	261.2	1.337 +j3.588	3.829
3.8	81.84 -j216.4	231.3	1.530 +j4.044	4.323
3.9	75.09 -j188.4	202.8	1.825 +j4.580	4.930

Table II
(Continued)

$\beta_o h$	Z_o (ohms)		$\lvert Z_o \rvert$	Y_o (mhos x 10^{-3})		$\lvert Y_o \rvert$ x 10^{-3}
4.0	70.61	-j160.8	175.6	2.289	+j5.213	5.693
4.1	68.42	-j133.3	149.8	3.049	+j5.938	6.676
4.2	68.61	-j105.4	125.8	4.337	+j6.664	7.951
4.3	71.37	-j 76.93	104.9	6.481	+j6.986	9.529
4.4	77.17	-j 47.77	90.76	9.368	+j5.800	11.02
4.5	86.62	-j 17.91	88.45	11.07	+j2.289	11.31
4.6	100.6	+j 12.49	101.4	9.791	-j1.216	9.867
4.7	120.2	+j 43.02	127.6	7.376	-j2.641	7.834
4.8	147.0	+j 72.76	164.0	5.464	-j2.705	6.097
4.9	182.8	+j 99.87	208.3	4.213	-j2.302	4.801
5.0	229.5	+j121.0	259.4	3.410	-j1.798	3.855
5.1	287.9	+j130.5	316.1	2.882	-j1.306	3.164
5.2	356.3	+j120.5	376.0	2.519	-j0.8516	2.660
5.3	425.9	+j 82.34	433.8	2.263	-j0.4376	2.305
5.4	480.6	+j 12.16	480.8	2.079	-j0.0526	2.080
5.5	502.0	-j 79.40	508.2	1.944	+j0.3074	1.968
5.6	483.1	-j170.1	512.2	1.842	+j0.6486	1.952
5.7	433.4	-j240.3	495.5	1.765	+j0.9786	2.018
5.8	370.0	-j282.3	465.4	1.708	+j1.303	2.148
5.9	307.5	-j299.7	429.4	1.668	+j1.626	2.329
6.0	253.0	-j300.1	392.5	1.642	+j1.948	2.548
6.1	208.4	-j290.3	357.4	1.632	+j2.273	2.798
6.2	173.2	-j275.2	325.2	1.639	+j2.602	3.075
6.3	145.8	-j257.4	295.8	1.666	+j2.941	3.380
6.4	124.7	-j238.8	269.4	1.719	+j3.290	3.712
6.5	108.8	-j220.4	245.8	1.802	+j3.648	4.069
6.6	96.66	-j202.2	224.1	1.924	+j4.026	4.462
6.7	87.46	-j184.4	204.1	2.099	+j4.427	4.899
6.8	80.60	-j167.0	185.4	2.345	+j4.857	5.394
6.9	75.60	-j149.6	167.6	2.690	+j5.324	5.965
7.0	72.19	-j132.0	150.5	3.188	+j5.830	6.645

Table III

King-Middleton Second-Order Impedances

$\Omega = 2\ln\frac{2h}{a} = 12.5 \qquad h/a = 259.01$

$\beta_o h$	Z_o (ohms)	$\lvert Z_o \rvert$	Y_o (mhos x 10^{-3})	$\lvert Y_o \rvert$ x 10^{-3}
0.5	5.005 -j980.6	980.6	0.0052 +j1.020	1.020
0.7	10.29 -j630.3	630.4	0.0259 +j1.586	1.586
0.9	18.18 -j413.6	414.0	0.1061 +j2.413	2.416
1.1	29.60 -j254.2	255.9	0.4520 +j3.881	3.907
1.2	37.13 -j185.7	189.3	1.049 +j5.211	5.315
1.3	46.27 -j121.6	130.1	2.733 +j7.183	7.685
1.4	57.39 -j 60.07	83.08	8.314 +j8.703	12.04
1.5	71.02 +j 0.2681	71.02	14.08 -j0.0532	14.08
1.6	87.90 +j 60.62	106.8	7.718 -j5.322	9.377
1.7	109.2 +j122.2	163.8	4.068 -j4.552	6.104
1.8	136.6 +j186.1	230.8	2.563 -j3.492	4.332
1.9	172.7 +j253.4	306.7	1.837 -j2.694	3.261
2.0	233.2 +j325.0	400.0	1.431 -j2.103	2.544
2.1	287.7 +j400.9	493.5	1.182 -j1.646	2.026
2.2	381.2 +j479.5	612.3	1.016 -j1.278	1.632
2.3	514.5 +j554.2	756.2	0.8997 -j0.9691	1.322
2.4	705.0 +j606.5	930.1	0.8150 -j0.7011	1.075
2.5	965.3 +j594.0	1133.	0.7514 -j0.4624	0.8822
2.6	1268. +j443.0	1344.	0.7025 -j0.2449	0.7440
2.7	1499. +j 93.51	1502.	0.6643 -j0.0414	0.6656
2.8	1490. -j358.7	1533.	0.6343 +j0.1527	0.6524
2.9	1248. -j696.0	1429.	0.6111 +j0.3408	0.6997
3.0	943.6 -j835.5	1260.	0.5941 +j0.5260	0.7934
3.1	689.7 -j841.4	1088.	0.5827 +j0.7108	0.9191
3.2	506.0 -j787.8	936.3	0.5772 +j0.8986	1.068
3.3	378.4 -j714.7	808.7	0.5785 +j1.093	1.236
3.4	289.7 -j640.3	702.8	0.5870 +j1.297	1.424
3.5	227.1 -j569.4	613.0	0.6042 +j1.515	1.631

Table III
(Continued)

$\beta_o h$	Z_o (ohms)		$\lvert Z_o \rvert$	Y_o (mhos x 10^{-3})		$\lvert Y_o \rvert$ x 10^{-3}
3.6	182.0	-j504.4	536.2	0.6331	+j1.754	1.865
3.7	149.0	-j444.2	468.5	0.6787	+j2.023	2.134
3.8	124.6	-j388.3	407.8	0.7493	+j2.335	2.452
3.9	106.7	-j336.0	352.5	0.8592	+j2.704	2.837
4.0	93.91	-j286.1	301.1	1.036	+j3.155	3.321
4.1	85.39	-j237.9	252.8	1.336	+j3.723	3.955
4.2	80.70	-j191.0	207.3	1.877	+j4.443	4.823
4.3	79.63	-j144.7	165.2	2.919	+j5.304	6.054
4.4	82.23	-j 98.56	128.3	4.993	+j5.982	7.792
4.5	88.77	-j 52.14	103.0	8.375	+j4.919	9.713
4.6	99.77	-j 5.109	99.90	9.997	+j0.5119	10.01
4.7	116.0	+j 42.88	123.7	7.584	-j2.802	8.085
4.8	138.8	+j 91.94	166.5	5.006	-j3.316	6.005
4.9	169.9	+j141.9	221.4	3.466	-j2.895	4.516
5.0	211.8	+j192.2	286.1	2.589	-j2.348	3.495
5.1	268.1	+j241.1	360.6	2.062	-j1.854	2.773
5.2	342.9	+j284.5	445.6	1.727	-j1.433	2.244
5.3	440.5	+j315.0	541.5	1.502	-j1.074	1.846
5.4	562.7	+j319.5	647.1	1.344	-j0.7629	1.545
5.5	703.2	+j278.7	756.4	1.229	-j0.4871	1.322
5.6	838.4	+j174.0	856.3	1.143	-j0.2373	1.168
5.7	927.2	+j 5.512	927.2	1.078	-j0.0064	1.078
5.8	933.5	-j191.0	952.8	1.029	+j0.2107	1.050
5.9	856.6	-j361.6	929.8	0.9908	+j0.4183	1.075
6.0	734.5	-j472.9	873.6	0.9625	+j0.6197	1.145
6.1	605.1	-j525.0	801.1	0.9429	+j0.8180	1.248
6.2	490.3	-j534.5	725.3	0.9317	+j1.016	1.378
6.3	396.7	-j519.1	653.3	0.9293	+j1.216	1.530
6.4	323.1	-j490.5	587.3	0.9367	+j1.422	1.703
6.5	266.3	-j455.9	527.9	0.9554	+j1.636	1.894
6.6	222.1	-j419.3	474.5	0.9869	+j1.862	2.108
6.7	187.7	-j382.2	425.8	1.035	+j2.108	2.348
6.8	160.7	-j345.5	381.0	1.107	+j2.379	2.624
6.9	139.6	-j309.5	339.5	1.211	+j2.685	2.945
7.0	123.1	-j273.7	300.1	1.367	+j3.039	3.333

Table IV

King-Middleton Second-Order Impedances

$\Omega = 2\ln \frac{2h}{a} = 15 \quad h/a = 904.02$

$\beta_o h$	Z_o (ohms)	$\lvert Z_o \rvert$	Y_o (mhos x 10^{-3})	$\lvert Y_o \rvert$ x 10^{-3}
0.5	5.000-j1256.	1256.	0.0032 +j0.7960	0.7960
0.7	10.28 -j 809.3	809.3	0.0157 +j1.236	1.236
0.9	18.13 -j 533.2	533.5	0.0637 +j1.873	1.874
1.1	29.36 -j 330.9	332.2	0.2660 +j2.998	3.010
1.2	36.73 -j 244.4	247.1	0.6014 +j4.001	4.046
1.3	45.62 -j 163.5	169.7	1.584 +j5.676	5.892
1.4	56.38 -j 86.00	102.8	5.331 +j8.132	9.724
1.5	69.46 -j 10.23	70.21	14.09 +j2.076	14.24
1.6	85.53 +j 65.50	107.7	7.369 -j5.644	9.282
1.7	105.7 +j 142.8	177.7	3.348 -j4.525	5.629
1.8	131.5 +j 223.4	259.2	1.956 -j3.324	3.857
1.9	165.0 +j 309.1	350.3	1.344 -j2.518	2.854
2.0	209.4 +j 401.7	453.0	1.021 -j1.957	2.208
2.1	269.8 +j 503.3	571.0	0.8273 -j1.543	1.751
2.2	354.8 +j 616.3	711.4	0.7016 -j1.219	1.406
2.3	476.3 +j 740.3	880.3	0.6146 -j0.9553	1.136
2.4	656.3 +j 871.2	1091.	0.5518 -j0.7337	0.9180
2.5	929.4 +j 988.5	1353.	0.5047 -j0.5397	0.7389
2.6	1330. +j1045.	1691.	0.4684 -j0.3639	0.5932
2.7	1878. +j 860.4	2066.	0.4401 -j0.2017	0.4841
2.8	2361. +j 278.9	2378.	0.4177 -j0.0493	0.4206
2.9	2362. -j 570.8	2430.	0.4000 +j0.0967	0.4115
3.0	1870. -j1159.	2200.	0.3864 +j0.2393	0.4545
3.1	1312. -j1328.	1705.	0.3765 +j0.3809	0.5356
3.2	898.8 -j1273.	1558.	0.3702 +j0.5242	0.6417
3.3	626.4 -j1146.	1306.	0.3674 +j0.6720	0.7659
3.4	449.8 -j1008.	1104.	0.3689 +j0.8271	0.9050
3.5	332.8 -j 881.1	941.9	0.3752 +j0.9932	1.062
3.6	253.2 -j 767.3	808.0	0.3878 +j1.175	1.238
3.7	197.6 -j 666.1	694.8	0.4093 +j1.380	1.439
3.8	158.1 -j 574.7	596.0	0.4450 +j1.616	1.676
3.9	129.9 -j 492.9	509.8	0.4998 +j1.897	1.962

Table IV
(Continued)

$\beta_o h$	Z_o(ohms)	$\lvert Z_o \rvert$	Y_o(hmos x 10^{-3})	$\lvert Y_o \rvert$ x 10^{-3}
4.0	110.0 -j 417.0	431.2	0.5913 +j2.242	2.319
4.1	92.25 -j 345.9	359.3	0.7380 +j2.646	2.747
4.2	89.70 -j 278.2	292.3	1.036 +j3.267	3.427
4.3	85.53 -j 212.5	229.1	1.630 +j4.049	4.365
4.4	85.48 -j 148.5	171.3	2.881 +j5.056	5.819
4.5	89.91 -j 85.51	124.1	5.837 +j5.509	8.026
4.6	99.14 -j 22.67	101.7	9.537 +j2.218	9.792
4.7	113.5 +j 41.25	120.7	7.784 -j2.830	8.282
4.8	133.9 +j 107.2	171.5	4.551 -j3.644	5.830
4.9	161.9 +j 175.7	238.9	2.840 -j3.079	4.189
5.0	199.8 +j 247.1	316.4	1.979 -j2.447	3.147
5.1	250.9 +j 321.7	408.0	1.507 -j1.933	2.451
5.2	320.2 +j 399.2	511.7	1.223 -j1.524	1.954
5.3	414.5 +j 476.5	631.6	1.037 -j1.195	1.583
5.4	543.5 +j 547.5	771.5	0.9132 -j0.9199	1.296
5.5	717.7 +j 597.1	933.6	0.8234 -j0.6850	1.071
5.6	944.4 +j 596.8	1117.	0.7566 -j0.4782	0.8951
5.7	1210. +j 500.0	1309.	0.7060 -j0.2919	0.7640
5.8	1452. +j 262.2	1475.	0.6672 -j0.1205	0.6780
5.9	1564. -j 99.34	1567.	0.6368 +j0.0404	0.6381
6.0	1481. -j 469.4	1554.	0.6135 +j0.1944	0.6436
6.1	1258. -j 726.5	1453.	0.5961 +j0.3443	0.6884
6.2	1001. -j 843.2	1308.	0.5839 +j0.4925	0.7639
6.3	775.0 -j 862.1	1159.	0.5767 +j0.6415	0.8626
6.4	598.4 -j 826.7	1020.	0.5746 +j0.7937	0.9798
6.5	465.6 -j 766.8	897.1	0.5784 +j0.9524	1.114
6.6	367.4 -j 698.8	789.5	0.5893 +j1.121	1.266
6.7	293.8 -j 629.8	695.0	0.6083 +j1.304	1.439
6.8	238.5 -j 562.5	611.0	0.6389 +j1.507	1.637
6.9	196.6 -j 497.9	535.3	0.6860 +j1.738	1.868
7.0	164.7 -j 436.0	466.0	0.7584 +j2.007	2.146

Table V

King-Middleton Second-Order Impedances

$\Omega = 2\ln \frac{2h}{a} = 20 \quad h/a = 11013$

$\beta_o h$	Z_o(ohms)	$\lvert Z_o \rvert$	Y_o(mhos x 10^{-3})	$\lvert Y_o \rvert$ x 10^{-3}
0.5	5.030-j1809.	1809.	0.0015 +j0.5527	0.5527
0.9	18.93 -j 885.4	885.6	0.0241 +j1.129	1.129
1.3	44.92 -j 247.3	251.3	0.7110 +j3.914	3.978
1.4	55.22 -j 138.0	148.6	2.499 +j6.246	6.728
1.5	67.65 -j 31.59	74.66	12.14 +j5.667	13.40
1.6	82.98 +j 74.58	111.6	6.666 -j5.991	8.963
1.7	101.9 +j 182.9	209.4	2.325 -j4.172	4.776
1.8	125.3 +j 295.1	320.6	1.219 -j2.871	2.022
1.9	155.5 +j 414.3	442.5	0.7941 -j2.116	2.260
2.0	195.3 +j 544.3	578.3	0.5840 -j1.628	1.729
2.1	247.7 +j 688.9	732.1	0.4537 -j1.285	1.363
2.2	320.0 +j 853.0	911.0	0.3855 -j1.028	1.098
2.3	424.2 +j1046.	1129.	0.3329 -j0.8231	0.8878
2.4	508.8 +j1293.	1399.	0.2966 -j0.6534	0.7176
2.5	815.1 +j1538.	1741.	0.2689 -j0.5075	0.5744
2.6	1210. +j1850.	2211.	0.2475 -j0.3786	0.4523
2.7	1895. +j2150.	2866.	0.2307 -j0.2617	0.3488
2.8	3075. +j2165.	3761.	0.2174 -j0.1531	0.2659
2.9	4567. +j1104.	4699.	0.2068 -j0.0500	0.2128
3.0	4790. -j1191.	4888.	0.1984 +j0.0498	0.2046
3.1	3262. -j2521.	4123.	0.1919 +j0.1483	0.2425
3.2	1944. -j2570.	3223.	0.1872 +j0.2475	0.3103
3.3	1182. -j2240.	2533.	0.1842 +j0.3493	0.3949
3.4	759.3 -j1890.	2037.	0.1831 +j0.4557	0.4911
3.5	514.3 -j1591.	1671.	0.1892 +j0.5693	0.5999
3.6	364.1 -j1343.	1392.	0.1880 +j0.6935	0.7185
3.7	267.5 -j1138.	1170.	0.1956 +j0.8325	0.8552
3.8	203.1 -j 964.9	986.2	0.2089 +j0.9924	1.014
3.9	159.4 -j 814.9	830.4	0.2311 +j1.182	1.204
4.0	139.4 -j 682.2	696.3	0.2637 +j1.398	1.422
4.1	109.3 -j 562.4	573.1	0.3329 +j1.713	1.745
4.2	96.62 -j 452.2	462.4	0.4519 +j2.115	2.163
4.3	89.81 -j 348.6	360.1	0.6929 +j2.690	2.778
4.4	88.04 -j 249.7	264.8	1.256 +j3.562	3.778

Table V
(Continued)

$\beta_o h$	Z_o (ohms)	$\lvert Z_o \rvert$	Y_o (mhos $\times 10^{-3}$)	$\lvert Y_o \rvert \times 10^{-3}$
4.5	90.50 -j 153.5	178.4	2.846 +j4.817	5.596
4.6	98.33 -j 58.47	114.4	7.512 +j4.467	8.740
4.7	110.6 +j 37.10	116.6	8.127 -j2.726	8.572
4.8	128.5 +j 134.8	186.2	3.706 -j3.887	5.371
4.9	153.1 +j 236.2	281.6	1.932 -j2.981	3.552
5.0	186.2 +j 343.4	390.7	1.226 -j2.250	2.560
5.1	230.8 +j 458.5	513.4	0.8761 -j1.740	1.948
5.2	291.1 +j 583.8	652.4	0.6841 -j1.372	1.533
5.3	373.8 +j 721.8	813.1	0.5657 -j1.092	1.230
5.4	489.7 +j 875.0	1003.	0.4871 -j0.8703	0.9973
5.5	656.1 +j1043.	1233.	0.4320 -j0.6868	0.8114
5.6	901.4 +j1220.	1517.	0.3917 -j0.5302	0.6592
5.7	1270. +j1379.	1875.	0.3613 -j0.3925	0.5334
5.8	1815. +j1442.	2318.	0.3378 -j0.2683	0.4314
5.9	2542. +j1222.	2821.	0.3196 -j0.1536	0.3545
6.0	3215. +j 447.8	3247.	0.3053 -j0.0455	0.3086
6.1	3273. -j 651.4	3337.	0.2940 +j0.0585	0.2998
6.2	2663. -j1497.	3055.	0.2854 +j0.1604	0.3274
6.3	1904. -j1786.	2612.	0.2793 +j0.2620	0.3829
6.4	1314. -j1744.	2199.	0.2755 +j0.3658	0.4581
6.5	916.4 -j1582.	1828.	0.2743 +j0.4734	0.5471
6.6	656.2 -j1395.	1542.	0.2760 +j0.5869	0.6485
6.7	482.1 -j1217.	1310.	0.2811 +j0.7100	0.7636
6.8	363.2 -j1056.	1117.	0.2910 +j0.8465	0.8951
6.9	280.2 -j 912.7	954.9	0.3074 +j1.001	1.048
7.0	221.3 -j 786.3	816.4	0.3316 +j1.178	1.224

Table VI

Additional Values of Z_0 Near Antiresonance

β_0	$\Omega = 10$	$\Omega = 12.5$	$\Omega = 15$	$\Omega = 20$
2.30	547.5 +j296.8			
2.35	621.9 +j271.9			
2.40	697.1 +j227.3			
2.45	738.8 +j160.3			
2.50	820.4 +j 71.37	965.3 +j594.0		
2.55	850.3 -j 33.55	1116. +j540.9		
2.60	849.9 -j145.6	1268. +j443.0	1330.+j1045.	
2.65	819.6 -j250.9	1405. +j291.9	1593.+j 987.8	
2.70	765.5 -j339.9	1499. +j 93.51	1878.+j 860.4	1895.+j2150.
2.75	696.6 -j407.3	1531. -j133.6	2153.+j 626.0	2413.+j2228.
2.80	622.2 -j452.6	1490. -j358.7	2361.+j 278.9	3075.+j2165.
2.85		1388. -j552.1	2438.-j 166.7	3847.+j1834.
2.90		1248. -j696.0	2362.-j 570.9	4567.+j1104.
2.95		1094. -j787.9	2149.-j 924.4	4943.-j 5.4
3.00		943.6 -j835.5	1870.-j1159.	4790.-j1191.
3.05			1579.-j1285.	4076.-j2071.
3.10			1313.-j1328.	3262.-j2521.
3.15				2527.-j2639.
3.20				1944.-j2570.

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